

ON THE SYMBIOSIS BETWEEN MODEL-THEORETIC AND SET-THEORETIC PROPERTIES OF LARGE CARDINALS

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Abstract. We study some large cardinals in terms of reflection, establishing new connections between the model-theoretic and the set-theoretic approaches.

§1. Introduction. First-order logic alone cannot express important properties such as finiteness or uncountability of the model, well-foundedness of a binary predicate, completeness of a linear order, etc. This led to Mostowski [10] and later Lindström [6] to introduce the concept of a generalised quantifier. This made it possible to compare model-theoretic and set-theoretic definability of various mathematical concepts. It turned out that there is a close connection between the two. Following [14], we call this connection *symbiosis*.

A fundamental property of the universe of sets is *reflection*. Roughly speaking reflection means that every property that holds of the entire universe of sets is permitted already by a set-sized sub-universe. By qualifying what “property” means one can relate reflection closely to large cardinal properties [1]. In model theory the analogue of reflection is the Löwenheim–Skolem Theorem which in its various variants says, roughly speaking, that if a sentence of a logic has a model then the sentence has a “small” (sub)model, e.g. a countable model.

The purpose of this paper is to use symbiosis to relate set-theoretic reflection principles to model-theoretic Löwenheim–Skolem Theorems. Our special interest is in analogues of large cardinals. In [1] strong reflection principles are used to obtain large cardinal properties at supercompactness and above. Here we focus on smaller large cardinals.

By a *logic* \mathcal{L}^* we mean any model-theoretically defined extension of first-order logic, such as infinitary logic $\mathcal{L}_{\kappa\lambda}$, a logic with generalised quantifiers $\mathcal{L}(Q)$ and second order logic \mathcal{L}^2 . It is not important for the purpose of this paper to specify what exactly is the definition of a logic, but such a definition can be found e.g. in [2, Chapter II]. What is important is that for any $\phi \in \mathcal{L}^*$ there is a formula $\Phi(x, y)$ of ordinary first-order set theory such that for all models \mathcal{A} of the vocabulary of ϕ we have:

$$\mathcal{A} \models \phi \iff \Phi(\mathcal{A}, \phi). \quad (1)$$

Received September 12, 2014.

2010 *Mathematics Subject Classification*. Primary: 03E55. Secondary: 03C55.

Key words and phrases. large cardinals, structural reflection, symbiosis, Löwenheim–Skolem Theorems, generalized quantifiers.

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0022-4812/16/8102-0011
DOI:10.1017/jsl.2015.60

For $\phi \in \mathcal{L}_{\infty\omega}$ the formula $\Phi(x, y)$ can be chosen to be Δ_1^{KP} , where KP is the Kripke–Platek axioms of set theory. For $\phi \in \mathcal{L}^2$ we can take $\Phi(x, y)$ to be Δ_2 , but in general not Δ_1 (see Section 7 for details). If ϕ is in the extension of first order logic by the H\"artig-quantifier I (see Section 4.1 for the definition), then $\Phi(x, y)$ can be taken to be $\Delta_1(Cd)$, that is, Δ_1 with respect to the predicate $Cd(x) \iff$ “ x is a cardinal”. This works also in the other direction: If a $\Phi(x, y)$ is given and it is Δ_1^{KP} , $\Delta_1(Cd)$ or Δ_2 , then there is a sentence ϕ in the respective logic such that (1) holds. This is indicative of a tight correspondence for properties of models between expressibility in an extension of first-order logic and definability in set theory. We call this tight correspondence *symbiosis* (see Definition 5.3).

The main reasons for studying symbiosis between model-theoretic and set-theoretic definability are the following.

First of all, the study of strong logics has in general led to a variety of set-theoretical difficulties. Results have turned out to be dependent on set-theoretical assumptions such as $V = L$, CH , \diamond , and large cardinals. It became therefore instrumental to uncover exactly what is the nature of the dependence on set-theoretical hypotheses in each case. Symbiosis pinpoints the position of a given logic in the set-theoretical definability hierarchy and thereby helps us understand better the set-theoretical nature of the logic.

Secondly, strong logics give rise to natural set-theoretical principles. For example, Completeness Theorems of various logics on uncountability (the quantifier Q_1 , Magidor–Malitz quantifiers, stationary logic, etc) can be used as set-theoretical principles which unify certain constructions and give rise to absoluteness results (see e.g. [3]). As we show in this paper, L\"owenheim–Skolem-type results for strong logics give new types of reflection principles in set theory. A good early indication of this is Magidor’s characterisation of supercompactness in terms of a strong L\"owenheim–Skolem theorem for second-order logic [7].

The structure of the paper is the following: After some preliminaries in Section 2 we consider a family of structural reflection principles in Section 3. These principles arise from considering Π_1 -definitions of model classes in the extended vocabulary $\{\in, R\}$ of set theory obtained by adding a particular Π_1 -predicate R to $\{\in\}$.

In Section 4 we recall the Δ -operation on logics and give some examples. Intuitively speaking, the logic $\Delta(\mathcal{L}^*)$ is a minor extension of \mathcal{L}^* obtained by adding explicit definitions of some model classes that would otherwise be merely implicitly definable.

Section 5.3 introduces the key concept of symbiosis and gives a proof of a basic equivalence (Proposition 5.1) between set-theoretic definability and model-theoretic definability. The Structural Reflection principle of set theory is then proved equivalent to a Downward L\"owenheim–Skolem Theorem (Theorem 5.5). The rest of the paper refines and elaborates this basic equivalence.

In Sections 6 and 7 applications of Theorem 6 to particular logics are given.

In Section 8 a weaker “strict” form of Downward L\"owenheim–Skolem Theorem $SLST(\mathcal{L}^*)$ for a strong logic \mathcal{L}^* is formulated and related to large cardinal concepts. For the H\"artig-quantifier logic the least cardinal with $SLST$ is shown to be the first weakly inaccessible cardinal. It is interesting to compare this result with the stronger form of Downward L\"owenheim–Skolem Theorem, LST . The smallest cardinal for which the logic with the H\"artig quantifier satisfies the LST -property can be bigger

than the first measurable cardinal but also equal to the first weakly inaccessible cardinal [8]. We present a logic for which the first cardinal for which *SLST* holds is the first weakly Mahlo, and another logic for which the least cardinal with *SLST* is the least weakly compact cardinal.

§2. Preliminaries. By a *model* we mean a, possibly many-sorted, structure in a language that may have countably-many relation and function symbols of any finite arity, as well as constant symbols. The *vocabulary* of a model \mathcal{A} is the set of nonlogical symbols (including sort symbols) of the language of \mathcal{A} . We usually denote the universe of a model \mathcal{A} by the capital letter A .

The theory ZFC_n^- is *ZFC* minus the Power Set axiom, and with the axiom schemata of Separation and Collection restricted to Σ_n formulas. ZFC_n^- is finitely axiomatizable. We denote the class of all ordinals by OR .

If \mathcal{A} and \mathcal{B} are models with the same vocabulary, then we use the notation $e : \mathcal{B} \preceq \mathcal{A}$ to indicate that e is an *elementary embedding* of \mathcal{B} into \mathcal{A} , i.e., for every formula $\varphi(x_1, \dots, x_n)$ of the first-order language of \mathcal{A} and \mathcal{B} , and every $b_1, \dots, b_n \in B$,

$$\mathcal{B} \models \varphi(b_1, \dots, b_n) \text{ if and only if } \mathcal{A} \models \varphi(e(b_1), \dots, e(b_n)).$$

§3. Small large cardinals from structural reflection. Let \mathcal{R} be a set of Π_1 predicates or relations. A class \mathcal{K} of models in a fixed countable vocabulary is $\Sigma_1(\mathcal{R})$ if it is definable by means of a Σ_1 formula of the first-order language of set theory with additional predicates from \mathcal{R} , but without parameters.

In this section we shall consider the following kind of principles, for \mathcal{R} a set of Π_1 predicates or relations, and κ an infinite cardinal. The notation *SR* stands for *Structural Reflection*.

$(SR)_{\mathcal{R}}(\kappa)$: If \mathcal{K} is a $\Sigma_1(\mathcal{R})$ class of models, then for every $\mathcal{A} \in \mathcal{K}$, there exist $\mathcal{B} \in \mathcal{K}$ of cardinality less than κ and an elementary embedding $e : \mathcal{B} \preceq \mathcal{A}$.

Note that if $(SR)_{\mathcal{R}}$ holds for κ , then it also holds for any cardinal greater than κ . Thus, what is of relevance here is the least cardinal for which $(SR)_{\mathcal{R}}$ holds, hence we shall write $(SR)_{\mathcal{R}} = \kappa$ to indicate that κ is the least such cardinal.

Notation: If $\mathcal{R} = \{R_1, \dots, R_n\}$, then we may write $(SR)_{R_1, \dots, R_n}$ for $(SR)_{\mathcal{R}}$.

We have that $(SR)_{\emptyset} = \aleph_1$ (cf. [1] 4.2). However, if R is the Π_1 relation “ x is an ordinal and $y = V_x$ ”, then $(SR)_R = \kappa$ if and only if κ is the first supercompact cardinal. Moreover, if κ is supercompact, then $(SR)_{\mathcal{R}}$ holds for κ , for any set \mathcal{R} of Π_1 predicates. (See [7], and [1], section 4.)

3.1. Weaker principles. Let *Cd* be the Π_1 predicate “ x is a cardinal”. Magidor and Väänänen [8] show that the principle $(SR)_{Cd}$ implies 0^\sharp , and much more, e.g. there are no good scales. We shall also consider some weaker principles that are consistent with $V = L$. The weakest one is the following.

$(SR)_R^{\bar{\bar{-}}}$: If \mathcal{K} is a nonempty $\Sigma_1(R)$ class of models, then there exists $\mathcal{A} \in \mathcal{K}$ of cardinality less than κ .

PROPOSITION 3.1. $ZFC \vdash \exists \kappa ((SR)_R^{\bar{\bar{-}}} \text{ holds for } \kappa)$.

PROOF. Let $\{\mathcal{K}_n : n < \omega\}$ list all nonempty $\Sigma_1(R)$ model classes. Pick $\mathcal{A}_n \in \mathcal{K}_n$, for each $n < \omega$. Let κ be the supremum of all the cardinalities of the \mathcal{A}_n , for $n < \omega$. Then κ^+ is as required. \dashv

The next stronger principle is more interesting.

3.2. The principle $(SR)_{\mathcal{R}}^-$. Another weakening of $(SR)_{\mathcal{R}}$ is the following.

$(SR)_{\mathcal{R}}^-$: If \mathcal{K} is a $\Sigma_1(\mathcal{R})$ class of models and $\mathcal{A} \in \mathcal{K}$ has cardinality κ , then there exists $\mathcal{B} \in \mathcal{K}$ of cardinality less than κ and an elementary embedding $e : \mathcal{B} \preceq \mathcal{A}$.

As the next theorem shows, the existence of a cardinal κ for which $(SR)_{Cd}^-$ holds implies the existence of a weakly inaccessible cardinal, hence such a κ cannot be proved to exist in ZFC, if ZFC is consistent.

THEOREM 3.2. *If $(SR)_{Cd}^-$ holds for κ , then there exists a weakly inaccessible cardinal $\lambda \leq \kappa$.*

PROOF. Let \mathcal{K} be the class of structures (M, E) such that $(M, E) \models ZFC_n^-$, for a suitable n , and

$$\exists \pi, N (N \text{ is transitive} \wedge \pi : (M, E) \cong (N, \in) \wedge \forall \alpha \in N (Cd^N(\alpha) \rightarrow Cd(\alpha))).$$

Thus, \mathcal{K} is $\Sigma_1(Cd)$ (in fact it is $\Delta_1(Cd)$). Let $\mathcal{A} \preceq H(\kappa^+)$, with $\kappa + 1 \subseteq \mathcal{A}$, be of cardinality κ . We claim that $\mathcal{A} \in \mathcal{K}$. For let N be the transitive collapse of \mathcal{A} . Since the transitive collapsing map π is the identity on $\kappa + 1$, if $\alpha \leq \kappa$ and $N \models Cd(\alpha)$, then $\mathcal{A} \models Cd(\alpha)$, hence α is a cardinal. But if $\alpha \in N$ is greater than κ , then $\alpha = \pi(\beta)$, for some $\beta \geq \alpha$. And since $\mathcal{A} \models \neg Cd(\beta)$, we have that $N \models \neg Cd(\alpha)$.

By $(SR)_{Cd}^-$, let $\mathcal{B} \in \mathcal{K}$ be of cardinality less than κ , and let $e : \mathcal{B} \preceq \mathcal{A}$ be an elementary embedding. Let \tilde{N} be the transitive collapse of \mathcal{B} , with $i : \tilde{N} \rightarrow \mathcal{A}$ being the induced elementary embedding. Since $\mathcal{A} \models \text{“}\kappa \text{ is the largest cardinal”}$, there is $\alpha \in \tilde{N}$ such that $i(\alpha) = \kappa$, so i is not the identity. Let γ be the critical point of i . We claim that γ is regular in \tilde{N} . For suppose $\tilde{N} \models \text{“}f : \alpha \rightarrow \gamma \text{ is cofinal”}$, for some $\alpha < \gamma$. Then $\mathcal{A} \models \text{“}i(f) : \alpha \rightarrow i(\gamma) \text{ is cofinal”}$. Note that $\mathcal{A} \models \text{“}i(f)''\alpha \subseteq \gamma\text{”}$, as for $\beta < \alpha$ we have $f(\beta) < \gamma$, hence $i(f)(\beta) = i(f(\beta)) = f(\beta) < \gamma$. Hence, $\mathcal{A} \models \text{“}\exists \zeta < i(\gamma) (i(f)''\alpha \subseteq \zeta)\text{”}$. Thus, $\tilde{N} \models \text{“}\exists \zeta < \gamma (f''\alpha \subseteq \zeta)\text{”}$, and so $\tilde{N} \models \text{“}f \text{ is not cofinal in } \gamma\text{”}$, yielding a contradiction. So, $\tilde{N} \models \text{“}\gamma \text{ is a regular cardinal”}$, and therefore γ is really a cardinal, although we do not know if it is really regular or not.

We claim that $\tilde{N} \models \text{“}\gamma \text{ is a limit cardinal”}$. For suppose $\alpha < \gamma$. Thus, $\mathcal{A} \models \text{“}\alpha < \gamma < i(\gamma) \text{ and } \gamma \text{ is a cardinal”}$, and so $\tilde{N} \models \text{“}\exists \zeta (\zeta \text{ is a cardinal and } \alpha < \zeta < \gamma)\text{”}$. We have thus shown that $\tilde{N} \models \text{“}\gamma \text{ is a regular limit cardinal”}$, i.e., $\tilde{N} \models \text{“}\gamma \text{ is weakly inaccessible”}$. Hence, $\mathcal{A} \models \text{“}i(\gamma) \text{ is weakly inaccessible”}$, and so $i(\gamma)$ is really weakly inaccessible, and $\leq \kappa$. \dashv

A proof in [8] shows that, starting from a supercompact cardinal, it is consistent that $(SR)_{Cd}$ holds for the first weakly inaccessible cardinal. So, we cannot prove in ZFC that more large-cardinal properties beyond the existence of a weakly inaccessible cardinal hold for some cardinals $\leq \kappa$ just by assuming that $(SR)_{Cd}$ holds at κ .

3.3. The Regularity predicate. Let Rg be the predicate “ x is a regular ordinal”.

THEOREM 3.3. *If κ satisfies $(SR)_{Rg}^-$, then there exists a weakly Mahlo cardinal $\lambda \leq \kappa$.*

PROOF. Let \mathcal{K} now be the class of structures (M, E) such that $(M, E) \models ZFC_n^-$, for some suitable n , and

$$\begin{aligned} &\exists \pi, N(N \text{ is transitive} \wedge \pi : (M, E) \cong (N, \in) \wedge \\ &\forall \alpha \in N (Rg^N(\alpha) \rightarrow Rg(\alpha))). \end{aligned}$$

Thus, \mathcal{K} is $\Sigma_1(Rg)$ (in fact, $\Delta_1(Rg)$). Let $\mathcal{A} \preceq H(\kappa^+)$, with $\kappa + 1 \subseteq \mathcal{A}$, be of cardinality κ . Then $\mathcal{A} \in \mathcal{K}$. For if N is the transitive collapse of \mathcal{A} , then the collapsing map π is the identity on $\kappa + 1$, so if $\alpha \leq \kappa$ and $N \models Rg(\alpha)$, then also $\mathcal{A} \models Rg(\alpha)$, and therefore α is regular. But if $\alpha \in N$ is greater than κ , then $\alpha = \pi(\beta)$, for some $\beta \geq \alpha$, and so $\mathcal{A} \models \neg Rg(\beta)$, which implies $N \models \neg Rg(\alpha)$.

By $(SR)_{Rg}^-$, let $\mathcal{B} \in \mathcal{K}$ of cardinality less than κ , and $e : \mathcal{B} \preceq \mathcal{A}$. Let \bar{N} be the transitive collapse of \mathcal{B} and let $i : \bar{N} \rightarrow \mathcal{A}$ be the induced elementary embedding. Since $\mathcal{A} \models$ “ κ is the largest cardinal”, there is $\alpha \in \bar{N}$ such that $i(\alpha) = \kappa$, so i is not the identity. Let γ be the critical point of i .

Arguing as in the proof of Theorem 3.2, γ is weakly inaccessible in \bar{N} . To show that γ is weakly Mahlo in \bar{N} , let C be a club subset of γ in \bar{N} . Then $i(C)$ is a club subset of $i(\gamma)$ in V . Since γ is a limit point of $i(C)$, as C is unbounded in γ and i is the identity function below γ , and since $i(C)$ is closed, $\gamma \in i(C)$. Since \bar{N} thinks that γ is regular, γ is really regular, and thus $\mathcal{A} \models$ “ $i(C)$ contains a regular cardinal”. Hence, $\bar{N} \models$ “ C contains a regular cardinal”. This shows $\bar{N} \models$ “ γ is weakly Mahlo”. Hence $\mathcal{A} \models$ “ $i(\gamma)$ is weakly Mahlo”, and so $i(\gamma)$ is weakly Mahlo and $\leq \kappa$. ⊣

Let us observe that Cd is $\Delta_1(Rg)$, and therefore $(SR)_{Cd, Rg}$ is equivalent to $(SR)_{Rg}$. By its definition, Cd is clearly Π_1 . And it is also $\Sigma_1(Rg)$, because we have:

$$Cd(\alpha) \leftrightarrow Rg(\alpha) \vee \exists A(\alpha = \bigcup A \wedge \forall \beta \in A Rg(\beta)).$$

A result in [8] shows that we cannot hope to get from $(SR)_{Rg}$ more than one weakly Mahlo cardinal $\leq \kappa$. Indeed, starting from a weakly Mahlo cardinal the authors obtain a model in which $(SR)_{Rg}$ holds for the least weakly Mahlo cardinal. We cannot hope either to obtain from $(SR)_{Rg}$ that κ is strongly inaccessible, for in [13] it is shown that one can have $(SR)_{Rg}$ for $\kappa = 2^{\aleph_0}$.

3.4. The Weakly Inaccessible predicate. There is a condition between $(SR)_{Cd}^-$ and $(SR)_{Rg}^-$, namely $(SR)_{Cd, WI}^-$, where WI is the Π_1 predicate “ x is weakly inaccessible”.

PROPOSITION 3.4. *If κ satisfies $(SR)_{Cd, WI}^-$, then there exists a 2-weakly inaccessible cardinal $\lambda \leq \kappa$.*

PROOF. Similarly as before. ⊣

We may also consider predicates α - WI , for α an ordinal. That is, the predicate “ x is α -weakly inaccessible”. Then, similar arguments would show that the principle $(SR)_{Cd, \alpha-WI}^-$ implies that there is an $(\alpha + 1)$ -weakly inaccessible cardinal $\leq \kappa$.

3.5. Weak compactness. Let $WC(x, \alpha)$ be the Π_1 relation “ α is a limit ordinal and x is a partial ordering with no chain of order-type α ”.

THEOREM 3.5. *If κ satisfies $(SR)_{Cd,WC}^-$, then there exists a weakly compact cardinal $\lambda \leq \kappa$.*

PROOF. Let \mathcal{K} be the class of structures (M, E) such that $(M, E) \models ZFC_n^-$, for some suitable n , and

$$\begin{aligned} &\exists \pi, N(N \text{ is transitive} \wedge \pi : (M, E) \cong (N, \in) \wedge \\ &\forall x, \alpha \in N((Cd^N(\alpha) \rightarrow Cd(\alpha)) \wedge (WC^N(x, \alpha) \rightarrow WC(x, \alpha))). \end{aligned}$$

Thus, \mathcal{K} is $\Sigma_1(Cd, WC)$ (in fact, $\Delta_1(Cd, WC)$). Let $\mathcal{A} \preceq H(\kappa^+)$ be of cardinality κ and such that $OR \cap \mathcal{A} \in OR$. We claim that $\mathcal{A} \in \mathcal{K}$. For suppose N is the transitive collapse of \mathcal{A} via the transitive collapsing map π . As in the proof of Theorem 3.2, if $\alpha \in N$ and $Cd^N(\alpha)$, then α is a cardinal. Now suppose α is an ordinal in N , $x \in N$, and $N \models WC(x, \alpha)$. Since π is the identity on $OR \cap \mathcal{A}$, we have that $\mathcal{A} \models WC(\pi^{-1}(x), \alpha)$. So, $\pi^{-1}(x)$ is a partial ordering with no chain of length α . But since $\pi^{-1} \upharpoonright x : x \rightarrow \pi^{-1}(x)$ yields a partial-ordering embedding, it follows that x has no chain of length α either.

By $(SR)_{Cd,WC}^-$, let $\mathcal{B} \in \mathcal{K}$ be of cardinality less than κ , and let $e : \mathcal{B} \preceq \mathcal{A}$. Let \bar{N} be the transitive collapse of \mathcal{B} and let $i : \bar{N} \rightarrow \mathcal{A}$ be the induced elementary embedding. Since $\mathcal{A} \models “\kappa$ is the largest cardinal”, there is $\alpha \in \bar{N}$ such that $i(\alpha) = \kappa$, and so i is not the identity. Let γ be the critical point of i .

From Theorem 3.2 we know that $\bar{N} \models “\gamma$ is weakly inaccessible”. We will show that $\bar{N} \models “\gamma$ is weakly compact”. For this it is sufficient to show that in \bar{N} every tree of height γ such that $|T_\lambda| \leq 2^{|\lambda|}$, for all $\lambda < \gamma$, has a branch of length γ (see [5], IX, 2.35). So, suppose that T is such a tree in \bar{N} . Without loss of generality, T is a tree on γ . Then $i(T)$ is a tree of height $i(\gamma)$, so it has a node t of height γ . The set of predecessors of t in $i(T)$ form a chain of length γ . Since γ is the critical point of i , $i(T)_\gamma = T$, and so the set of predecessors of t form a chain of T of length γ . Since \bar{N} is correct about the pair (T, γ) satisfying the WC relation, if $\bar{N} \models “T$ has no chain of length $\gamma”$, then T has really no chain of length γ . So, it follows that $\bar{N} \models “T$ has a chain of length $\gamma”$. And this shows that $\bar{N} \models “\gamma$ is weakly compact”. Hence, $H(\kappa^+) \models “i(\gamma)$ is weakly compact”, and therefore $i(\gamma)$ is really weakly compact, and $i(\gamma) \leq \kappa$. ⊣

Since the first weakly Mahlo cardinal can satisfy $(SR)_{Rg}$ ([8]), we cannot get a weakly compact cardinal $\leq \kappa$ just from $(SR)_{Rg}$. Hence, $(SR)_{Cd,WC}$ is stronger than $(SR)_{Rg}$.

In the next two sections we shall see how to formulate a *model-theoretic* condition equivalent to $(SR)_R$, where R is a Π_1 predicate or relation.

§4. Definable model classes. Suppose \mathcal{K} is a class of models with vocabulary L , and suppose $L' \subseteq L$. Note that vocabularies can be *many-sorted*, so L' may have fewer *sorts* than L . Then we can take the *projection* of \mathcal{K} to L' , that is

$$\mathcal{K} \upharpoonright L' := \{\mathcal{A} \upharpoonright L' : \mathcal{A} \in \mathcal{K}\}.$$

Suppose \mathcal{L}^* is a *logic*. E.g.,

- First-order logic ($\mathcal{L}_{\omega\omega}$).
- Infinitary logic ($\mathcal{L}_{\kappa\lambda}$).
- Higher-order logic ($\mathcal{L}^n, n \geq 2$).

possibly extended with generalized quantifiers. In all cases of logics under consideration, isomorphism of models implies \mathcal{L}^* -equivalence.

A model class \mathcal{K} (i.e., a class of models in some fixed vocabulary) is said to be \mathcal{L}^* -definable if there is a sentence $\varphi \in \mathcal{L}^*$ such that $\mathcal{K} = \text{Mod}(\varphi)$, i.e., $\mathcal{K} = \{A : A \models \varphi\}$.

Sometimes, for some logic \mathcal{L}^* , a model class is a projection of an \mathcal{L}^* -definable model class, and at the same time the complement of the model class is also a projection of an \mathcal{L}^* -definable model class. Then we say that the model class is $\Delta(\mathcal{L}^*)$ -definable [9].

The Δ -operation became popular in the 70s when it turned out that adding for example the generalised quantifier Q_1 (“there exist uncountably many x such that . . .”) to first-order logic does not lead to an extension with the Craig Interpolation or the Beth Definability Theorem. So the Δ -operation was introduced to “fill obvious gaps” in logics. For example, the class of equivalence relations with uncountably many uncountable equivalence classes is definable in $\Delta(\mathcal{L}(Q_1))$ but not in $\mathcal{L}(Q_1)$. The Δ -operation preserves many properties (compactness, Löwenheim–Skolem, axiomatisation, etc.) of logics (see [9] for details).

4.1. A paradigm example. The model class of structures $(M, <)$, where $<$ well-orders M is $\Delta(\mathcal{L}^*)$, where \mathcal{L}^* is $\mathcal{L}_{\omega\omega}(I)$, i.e., first-order logic with the additional quantifier I , known as the *Härtig quantifier*, given by

$$Ixy\varphi(x)\psi(y) \leftrightarrow |\varphi(\cdot)| = |\psi(\cdot)|.$$

To see why this is so, look first at the model class \mathcal{K}_0 of models $(M, <, X)$, where $<$ is a linear ordering and X is a subset of M that has no $<$ -least element (a first-order property). The projection $\mathcal{K}_0 \upharpoonright \{<\}$ is the class of non-well-ordered structures. Now we represent the class of well-ordered structures as the projection of a model class that is definable using the generalized quantifier I . This “trick” is due to Per Lindström [6]. The point is that a linear order $(M, <)$ is a well-order if and only if there are sets A_a , for $a \in M$, such that $a <_M b$ if and only if $|A_a| < |A_b|$. So let \mathcal{K}_1 be the class of 2-sorted structures $(A, M, <, R)$ such that (we denote the two sorts by s_0 and s_1):

- (1) A has sort s_0 , M has sort s_1 ,
- (2) $M \subseteq A$,
- (3) $(M, <)$ is a linear order,
- (4) $R \subseteq M \times A$,
- (5) $a <_M b$ implies $|R(a, \cdot)| < |R(b, \cdot)|$.

So, the class of well-ordered models is the projection $\mathcal{K}_1 \upharpoonright \{s_1, <\}$. As a result, both the class \mathcal{W} of well-ordered $(M, <)$ and the class of non-well-ordered $(M, <)$ are projections of $\mathcal{L}_{\omega\omega}(I)$ -definable model classes, i.e., \mathcal{W} is $\Delta(\mathcal{L}_{\omega\omega}(I))$ -definable.

4.2. Another example. The class of well-founded models (M, E) such that $(M, E) \models ZFC_n^-$, for some n , and if \bar{M} is the transitive collapse of M , then $\bar{M} \models$ “ α is a cardinal” if and only if α is really a cardinal, is $\Delta(\mathcal{L}_{\omega\omega}(I))$. Indeed, similarly as

in the previous example, we can see that the class of well-founded models of ZFC_n is $\Delta(\mathcal{L}_{\omega\omega}(I))$. If $\alpha \in \bar{M}$ is a cardinal, then, of course, $\bar{M} \models \text{“}\alpha \text{ is a cardinal”}$. On the other hand, if $\bar{M} \models \text{“}\alpha \text{ is a cardinal”}$, then we can say, using I , that α is a cardinal in V , as follows:

$$\forall x < \alpha (\neg Iyz(y \in x)(z \in \alpha)).$$

That is, the set of elements of x has smaller cardinality than the set of elements of α .

§5. Symbiosis. Given a definable (but not necessarily Π_1) n -ary predicate R of set theory¹, let

$$Q_R := \{A : A \cong (M, \in, \bar{a}_1, \dots, \bar{a}_n), M \text{ transitive, and } R(\bar{a}_1, \dots, \bar{a}_n)\}.$$

The class Q_R yields a *generalized quantifier* (in the sense of Lindström [6]). Namely,

$$\begin{aligned} A \models Q_R uvx_1 \dots x_n (uEv)(x_1 = c_1) \dots (x_n = c_n) \\ \text{if and only if} \\ \langle A, E^A, c_1^A, \dots, c_n^A \rangle \in Q_R. \end{aligned}$$

PROPOSITION 5.1. *Suppose R is an n -ary predicate of set theory. The following are equivalent for any logic \mathcal{L}^* that contains first-order logic as a sublogic:*

- (1) *Every $\Delta_1(R)$ -definable model class that is closed under isomorphisms is $\Delta(\mathcal{L}^*)$ -definable.*
- (2) *The model class Q_R is $\Delta(\mathcal{L}^*)$ -definable.*

PROOF. (1) \Rightarrow (2): Notice first that the class Q_R is $\Sigma_1(R)$ -definable: Suppose $\mathcal{A} = (A, E, a_1, \dots, a_n)$. Then $\mathcal{A} \in Q_R$ iff

$$\begin{aligned} \exists (M, \in, \bar{a}_1, \dots, \bar{a}_n) (M \text{ is transitive} \wedge \\ \mathcal{A} \cong (M, \in, \bar{a}_1, \dots, \bar{a}_n) \wedge R(\bar{a}_1, \dots, \bar{a}_n)). \end{aligned}$$

Also, the complement of Q_R is $\Sigma_1(R)$ -definable: $\mathcal{A} \notin Q_R$ iff

$$\begin{aligned} \mathcal{A} \text{ is not a well-founded extensional structure} \vee \\ [\exists (M, \in, \bar{a}_1, \dots, \bar{a}_n) (M \text{ is transitive} \wedge \mathcal{A} \cong (M, \in, \bar{a}_1, \dots, \bar{a}_n) \wedge \\ \neg R(\bar{a}_1, \dots, \bar{a}_n))]. \end{aligned}$$

Hence, Q_R is $\Delta_1(R)$ -definable. Since Q_R is (trivially) closed under isomorphism, (1) yields (2).

(2) \Rightarrow (1): Suppose \mathcal{K} is a $\Delta_1(R)$ -definable model class that is closed under isomorphisms. Suppose the vocabulary of \mathcal{K} is L_0 which we assume for simplicity to have just one sort s_1 and one binary predicate P . The predicate P , which a priori could be n -ary for any n , should not be confused with our set theoretical predicate R . Let $\Phi(x)$ be a $\Sigma_1(R)$ formula of set theory such that $\mathcal{A} \in \mathcal{K}$ if and only if $\Phi(\mathcal{A})$. Let s_0 be a new sort, E a new binary predicate symbol of sort s_0 , F a new function symbol from sort s_1 to sort s_0 , and c a new constant symbol of sort s_0 . Consider the class \mathcal{K}_1 of models

$$\mathcal{N} := (N, B, E^{\mathcal{N}}, R^{\mathcal{N}}, c^{\mathcal{N}}, F^{\mathcal{N}}, P^{\mathcal{N}}),$$

¹To avoid trivialities, we assume $ZFC \vdash \exists x_1 \dots x_n R(x_1, \dots, x_n)$.

where N is the universe of sort s_0 and B the universe of sort s_1 , that satisfy the sentence φ given by the conjunction of the following sentences:

- (i) ZFC_n^- (for suitable $n \geq 1$) written in the vocabulary $\{E\}$ (instead of \in) in sort s_0 .
- (ii) $\Phi(c)$ written in the vocabulary $\{E\}$ in sort s_0 .
- (iii) $\forall x_1 \dots x_n (R(x_1 \dots x_n) \leftrightarrow Q_R uvw_1 \dots w_n (uEv)(w_1 = x_1) \dots (w_n = x_n))$ with the predicate R written in the vocabulary $\{E\}$ and everything in sort s_0 .
- (iv) c is a pair (a, b) , where $b \subseteq a \times a$, all written in the vocabulary E in sort s_0 .
- (v) F is an isomorphism between the s_1 -part (B, P^N) of the model and the structure $(\{x \in N : xE^N a\}, \{(x, y) \in N^2 : (x, y)E^N b\})$.

Note that φ is a sentence in the extension of first-order logic by the generalized quantifier Q_R .

CLAIM 5.2. $\mathcal{A} \in \mathcal{K}$ if and only if $\mathcal{A} = \mathcal{N} \upharpoonright \{s_1, P\}$, for some $\mathcal{N} \in \mathcal{K}_1$.

PROOF OF THE CLAIM. Suppose first $\mathcal{A} \in \mathcal{K}$. Pick m so that R is Σ_m -definable. Let $V_\alpha \preceq_{m+n} V$, with $\mathcal{A} \in V_\alpha$. Then, $R^N = R \cap V_\alpha$, and so $\mathcal{A} = \mathcal{N} \upharpoonright \{s_1, P\}$, where

$$\mathcal{N} := (V_\alpha, A, \in, R \cap V_\alpha, A, id, P^A) \in \mathcal{K}_1.$$

Conversely, suppose $\mathcal{N} := (N, B, E^N, R^N, c^N, F^N, P^N) \in \mathcal{K}_1$ with $\mathcal{A} = \mathcal{N} \upharpoonright \{s_1, P\}$. Clearly, the structure (N, E^N) is extensional and well-founded (by condition (iii)). Moreover, $(N, E^N) \models \Phi(c^N)$. Since \mathcal{K} is closed under isomorphisms, we may assume, w.l.o.g., that N is a transitive set and $E^N = \in \upharpoonright N$. Now, R is absolute for N : for every a_1, \dots, a_n in N , we have that $R(a_1, \dots, a_n)$ iff $(N, \in, a_1, \dots, a_n) \in Q_R$ iff

$$(N, \in, a_1, \dots, a_n) \models Q_R uvx_1 \dots x_n (uEv)(x_1 = a_1) \dots (x_n = a_n)$$

iff, by (iii), $N \models R(a_1, \dots, a_n)$. Since $(N, \in) \models \Phi(c^N)$ and N is transitive, and since Φ is $\Sigma_1(R)$, we have that $\Phi(c^N)$ is true, i.e., it holds in V . Thanks to condition (v), c^N is a binary structure isomorphic to \mathcal{A} . Since \mathcal{K} is closed under isomorphism, $\mathcal{A} \in \mathcal{K}$. ⊣

Since, by assumption (2), the class Q_R is $\Delta(\mathcal{L}^*)$ -definable, the model class \mathcal{K}_1 is $\Delta(\mathcal{L}^*)$ -definable. Hence, since by the claim above \mathcal{K} is a projection of \mathcal{K}_1 , \mathcal{K} is a projection of an \mathcal{L}^* -definable model class. We can do the same for the complement of \mathcal{K} . Hence \mathcal{K} is $\Delta(\mathcal{L}^*)$ -definable. ⊣

The following notion of symbiosis, between an abstract logic and a predicate of set theory, is due to Väänänen [14].

DEFINITION 5.3. A (finite set of) n -ary relation(s) \mathcal{R} and a logic \mathcal{L}^* are *symbiotic* if the following conditions are satisfied:

- (1) Every \mathcal{L}^* -definable model class is $\Delta_1(\mathcal{R})$ -definable.
- (2) Every $\Delta_1(\mathcal{R})$ -definable model class closed under isomorphisms is $\Delta(\mathcal{L}^*)$ -definable.

By the proposition above, for a suitable \mathcal{L}^* , (2) may be replaced by: The model class $Q_{\mathcal{R}}$, defined as an appropriate product of the models classes Q_R , for $R \in \mathcal{R}$, is $\Delta(\mathcal{L}^*)$ -definable.

5.1. Examples of symbiosis. Let us see some examples of symbiosis.

PROPOSITION 5.4. *The following pairs $(\mathcal{R}, \mathcal{L}^*)$ are symbiotic.*

- (1) \mathcal{R} : Cd
 \mathcal{L}^* : $\mathcal{L}_{\omega\omega}(I)$, where $Ixy\varphi(x)\psi(y) \leftrightarrow |\varphi| = |\psi|$ is the H\"artig quantifier.
- (2) \mathcal{R} : Cd
 \mathcal{L}^* : $\mathcal{L}_{\omega\omega}(R)$, where $Rxy\varphi(x)\psi(y) \leftrightarrow |\varphi| \leq |\psi|$ is the Rescher quantifier.
- (3) \mathcal{R} : Cd
 \mathcal{L}^* : $\mathcal{L}_{\omega\omega}(W^{Cd})$, where $W^{Cd}xy\varphi(x, y) \leftrightarrow \varphi(\cdot, \cdot)$ is a well-ordering of the order-type of a cardinal.
- (4) \mathcal{R} : Cd, WI
 \mathcal{L}^* : $\mathcal{L}_{\omega\omega}(I, W^{WI})$, where $W^{WI}x\varphi(x) \leftrightarrow |\varphi(\cdot)|$ is weakly-inaccessible.
- (5) \mathcal{R} : Rg
 \mathcal{L}^* : $\mathcal{L}_{\omega\omega}(I, W^{Rg})$, where $W^{Rg}xy\varphi(x, y) \leftrightarrow \varphi(\cdot, \cdot)$ has the order-type of a regular cardinal.
- (6) \mathcal{R} : Cd, WC
 \mathcal{L}^* : $\mathcal{L}_{\omega\omega}(I, Q_{Br})$, where $Q_{Br}xy\varphi(x, y) \leftrightarrow \varphi(\cdot, \cdot)$ is a tree order of height some α and has no branch of length α [12].
- (7) \mathcal{R} : Cd, WC
 \mathcal{L}^* : $\mathcal{L}_{\omega\omega}(I, \bar{Q}_{Br})$, where $\bar{Q}_{Br}xyuv\varphi(x, y)\psi(u, v) \leftrightarrow \varphi(\cdot, \cdot)$ is a partial order with a chain of order-type $\psi(\cdot, \cdot)$.

PROOF. Notice that in all the examples \mathcal{L}^* contains first-order logic. Also, it is easy to see that every \mathcal{L}^* -definable model class is $\Delta_1(\mathcal{R})$ -definable. Let us check example (1): suppose \mathcal{K} is the class of models that satisfy a fixed sentence $\varphi \in \mathcal{L}_{\omega\omega}(I)$. Then, $\mathcal{A} \in \mathcal{K}$ if and only if

$$\exists M(M \models ZFC_n^- \wedge M \text{ transitive} \wedge \mathcal{A} \in M \wedge \forall \alpha \in M (Cd^M(\alpha) \rightarrow Cd(\alpha)) \wedge M \models \text{“}\mathcal{A} \models \varphi\text{”}).$$

And also, $\mathcal{A} \in \mathcal{K}$ if and only if

$$\forall M([M \models ZFC_n^- \wedge M \text{ transitive} \wedge \mathcal{A} \in M \wedge \forall \alpha \in M (Cd^M(\alpha)) \rightarrow Cd(\alpha)] \rightarrow M \models \text{“}\mathcal{A} \models \varphi\text{”}).$$

Since the two displayed formulas above are $\Sigma_1(Cd)$ and $\Pi_1(Cd)$, respectively, \mathcal{K} is $\Delta_1(Cd)$. The other examples are similar.

So, by Proposition 5.1, it is sufficient to see that, in each case, the corresponding model classes $Q_{\mathcal{R}}$ are $\Delta(\mathcal{L}^*)$ -definable. In the case of the Cd predicate, (1) is done in Example 4.2. The other cases are easy. ⊣

THEOREM 5.5. *Suppose \mathcal{L}^* and R are symbiotic. Then the following are equivalent:*

- (i) $(SR)_R$ holds for κ .
- (ii) For any $\varphi \in \mathcal{L}^*$ and any \mathcal{A} that satisfies φ , there exists $\mathcal{B} \subseteq \mathcal{A}$ of cardinality less than κ that also satisfies φ .

PROOF. (i) \Rightarrow (ii): Suppose $\varphi \in \mathcal{L}^*$ and $\mathcal{A} \models \varphi$. Let $\mathcal{K} = Mod(\varphi)$. By symbiosis (1), \mathcal{K} is $\Delta_1(R)$. In particular, \mathcal{K} is $\Sigma_1(R)$. By $(SR)_R$, there is $\mathcal{B} \in \mathcal{K}$ of cardinality less than κ and an embedding $e : \mathcal{B} \preceq \mathcal{A}$. Note that since $\mathcal{B} \in \mathcal{K}$, $\mathcal{B} \models \varphi$.

Then, the pointwise image of \mathcal{B} under the embedding e is a substructure of \mathcal{A} that has cardinality less than κ and which, being isomorphic to \mathcal{B} , also satisfies φ .

(ii) \Rightarrow (i): Suppose \mathcal{K} is a $\Sigma_1(R)$ class of models in a fixed vocabulary. Thus, $\mathcal{A} \in \mathcal{K}$ if and only if $\Phi(\mathcal{A})$, for some $\Sigma_1(R)$ formula Φ . Suppose $\Phi(\mathcal{A})$ holds, i.e., $\mathcal{A} \in \mathcal{K}$. We will find $\bar{\mathcal{A}}$ of cardinality less than κ such that $\Phi(\bar{\mathcal{A}})$ holds and there is an elementary embedding $e : \bar{\mathcal{A}} \preceq \mathcal{A}$.

For each formula $\psi(x, y_1, \dots, y_n)$ without quantifiers, in the language $\{E, c\}$, where E is a binary relation symbol and c is a constant symbol, let $f_{\psi(x, \vec{y})}$ be an n -ary function symbol.

Consider the class \mathcal{K}^* of well-founded models

$$(M, E^M, R^M, c^M, \langle f_{\psi(x, \vec{y})}^M \rangle_{\psi(x, \vec{y})})$$

that satisfy the following sentences (with \in interpreted as E^M):

- (1) ZFC_n , for some suitable n .
- (2) $\Phi(c)$.
- (3) $\forall a_1, \dots, a_n (R(a_1, \dots, a_n) \leftrightarrow \mathcal{Q}_R uvx_1 \dots x_n (uEv)(x_1 = a_1) \dots (x_n = a_n))$.
- (4) $\forall \vec{z} (\exists x \psi(x, \vec{z}) \rightarrow \psi(f_{\psi(x, \vec{y})}(\vec{z}), \vec{z}))$, for each quantifier-free formula $\psi(x, \vec{y})$.

Since the class of well-founded models is $\Delta_1(R)$ -definable, and \mathcal{Q}_R is also $\Delta_1(R)$ -definable (see the proof of Proposition 5.1), the class \mathcal{K}^* is $\Delta_1(R)$ -definable. Since \mathcal{K}^* is closed under isomorphisms, by symbiosis it is $\Delta(\mathcal{L}^*)$ -definable. Let φ be the \mathcal{L}^* -formula that defines a model class of which \mathcal{K}^* is a projection.

Suppose R , as a set-theoretic predicate, is Σ_m . By Levy’s Reflection Theorem, let α be such that $\mathcal{A} \in V_\alpha$ and $V_\alpha \preceq_{m+n} V$. Note that $R^{V_\alpha} = R \cap V_\alpha$.

For each quantifier-free formula $\psi(x, \vec{y})$ in the language $\{E, c\}$, choose a Skolem function $f_{\psi(x, \vec{y})}^{V_\alpha} : V_\alpha \rightarrow V_\alpha$ for $(V_\alpha, \in, \mathcal{A})$, i.e., a function so that for every $\vec{b} \in V_\alpha$, if $(V_\alpha, \in, \mathcal{A}) \models \exists x \psi(x, \vec{b})$, where E is interpreted as \in and c as \mathcal{A} , then

$$(V_\alpha, \in, \mathcal{A}) \models \psi(f_{\psi(x, \vec{y})}^{V_\alpha}(\vec{b}), \vec{b}).$$

Then $(V_\alpha, \in, R \cap V_\alpha, \mathcal{A}, \langle f_{\psi(x, \vec{y})}^{V_\alpha} \rangle_{\psi(x, \vec{y})}) \in \mathcal{K}^*$, and hence some model expansion $(V_\alpha, \in, R \cap V_\alpha, \mathcal{A}, \langle f_{\psi(x, \vec{y})}^{V_\alpha} \rangle_{\psi(x, \vec{y})}, \dots)$ satisfies φ . By (ii), there is $\mathcal{B} \subseteq (V_\alpha, \in, R \cap V_\alpha, \mathcal{A}, \langle f_{\psi(x, \vec{y})}^{V_\alpha} \rangle_{\psi(x, \vec{y})}, \dots)$ of cardinality less than κ such that $\mathcal{B} \models \varphi$. Let $(M, \in, \bar{R}, \bar{\mathcal{A}})$ be the transitive collapse of the corresponding projection of \mathcal{B} . Thus, the transitive collapse map $\pi : \mathcal{B} \rightarrow M$ is an \in -isomorphism that sends \mathcal{A} to $\bar{\mathcal{A}}$. It follows that $(M, \in, \bar{R}, \bar{\mathcal{A}}) \models \Phi(\bar{\mathcal{A}})$. Moreover, R is absolute for $(M, \in, \bar{R}, \bar{\mathcal{A}})$: for every a_1, \dots, a_n in M , we have that $R(a_1, \dots, a_n)$ iff $(M, \in, a_1, \dots, a_n) \in \mathcal{Q}_R$ iff

$$(M, \in, a_1, \dots, a_n) \models \mathcal{Q}_R uvx_1 \dots x_n (uEv)(x_1 = a_1) \dots (x_n = a_n)$$

iff, by (3), $(M, \in, \bar{R}, \bar{\mathcal{A}}) \models R(a_1, \dots, a_n)$. So, since Φ is $\Sigma_1(R)$, the sentence $\Phi(\bar{\mathcal{A}})$ is true, hence $\bar{\mathcal{A}} \in \mathcal{K}$.

Since we added to the structure $(V_\alpha, \in, \mathcal{A})$ Skolem functions for quantifier-free formulas in the language $\{E, c\}$, the structure $(\mathcal{B}, \in, \mathcal{A})$ is in fact a Σ_1 -elementary substructure of $(V_\alpha, \in, \mathcal{A})$. Now, for every $\vec{a} \in \mathcal{A}$, and every formula $\theta(\vec{x})$ in the language of \mathcal{A} , the set-theoretic sentence $\mathcal{A} \models \theta(\vec{a})$ is Σ_1 , in the parameters \mathcal{A} and \vec{a} . So, if $\vec{a} \in \mathcal{A} \cap \mathcal{B}$, then we have that $\mathcal{A} \models \theta(\vec{a})$ iff $(V_\alpha, \in, \mathcal{A}) \models \text{“}\mathcal{A} \models \theta(\vec{a})\text{”}$ iff $(\mathcal{B}, \in, \mathcal{A}) \models \text{“}\mathcal{A} \models \theta(\vec{a})\text{”}$ iff $(M, \in, \bar{\mathcal{A}}) \models \text{“}\bar{\mathcal{A}} \models \theta(\pi(\vec{a}))\text{”}$ iff $\bar{\mathcal{A}} \models \theta(\pi(\vec{a}))$.

Thus, the inverse transitive collapsing map π^{-1} yields an elementary embedding $e : \tilde{\mathcal{A}} \preceq \mathcal{A}$. ⊣

§6. The Löwenheim–Skolem–Tarski property. The Löwenheim–Skolem–Tarski property of cardinals, for a logic \mathcal{L}^* , denoted by $LST(\mathcal{L}^*)$, is defined as follows.

DEFINITION 6.1. A cardinal κ has the $LST(\mathcal{L}^*)$ property if for any \mathcal{L}^* -definable model class \mathcal{K} and any $\mathcal{A} \in \mathcal{K}$, there is $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{B} \in \mathcal{K}$ and $|\mathcal{B}| < \kappa$.

Notice that if κ has the $LST(\mathcal{L}^*)$ property, then any larger cardinal also has it. We call the least cardinal κ that has the $LST(\mathcal{L}^*)$ property, provided it exists, the $LST(\mathcal{L}^*)$ -number, and we write $LST(\mathcal{L}^*) = \kappa$ to indicate this.

EXAMPLES 6.2.

- $LST(\mathcal{L}_{\omega\omega}) = LST(\mathcal{L}_{\omega_1\omega}) = \aleph_1$.
- $LST(\mathcal{L}_{\omega\omega}(MM_{\aleph_1}^n)) = \aleph_2$, where $MM_{\aleph_1}^n$ is the Magidor–Malitz quantifier. Namely,

$$MM_{\aleph_1}^n x_1, \dots, x_n \varphi(x_1, \dots, x_n, \vec{y})$$

if and only if there exists X such that $|X| \geq \aleph_1$ and $\varphi(a_1, \dots, a_n, \vec{y})$ holds for all $a_1, \dots, a_n \in X$.

- $LST(\mathcal{L}_{\omega\omega}(W)) = \aleph_1$, where

$$Wxy\varphi(x, y, \vec{z}) \text{ iff } \varphi(\cdot, \cdot) \text{ is a well-ordering.}$$

By Theorem 5.5, if \mathcal{L}^* and R are symbiotic, then $LST(\mathcal{L}^*)$ holds for κ if and only if $(SR)_R$ holds for κ . So, $LST(\mathcal{L}^*) = \kappa$ if and only if $(SR)_R = \kappa$. Thus, writing \equiv to indicate that the corresponding cardinals are the same, assuming they exist, we have:

- (1) $(SR)_{Cd} \equiv LST(\mathcal{L}_{\omega\omega}(I))$.
- (2) $(SR)_{Cd, WI} \equiv LST(\mathcal{L}_{\omega\omega}(I, W^{WI}))$.
- (3) $(SR)_{Rg} \equiv LST(\mathcal{L}_{\omega\omega}(W^{Rg}))$.
- (4) $(SR)_{Cd, WC} \equiv LST(\mathcal{L}_{\omega\omega}(I, Q_{Br}))$.

Thus, the $LST(\mathcal{L}^*)$ -number yields a hierarchy of logics, and in the case of symbiotic R and \mathcal{L}^* it also yields a hierarchy of $(SR)_R$ principles.

§7. The case of second-order logic. Let $PwSet$ be the Π_1 relation $\{(x, y) : y = \mathcal{P}(x)\}$. Let \mathcal{L}^2 be second-order logic. Then we have the following.

LEMMA 7.1. *The $PwSet$ relation and \mathcal{L}^2 are symbiotic.*

PROOF. Suppose $\mathcal{K} = Mod(\varphi)$, for some \mathcal{L}^2 -sentence φ . Then $\mathcal{A} \in \mathcal{K}$ if and only if $\langle V_\alpha, \in, \mathcal{A} \rangle \models \text{“}\mathcal{A} \models \varphi\text{”}$, for some (any) α greater than the rank of \mathcal{A} . Since the predicate $x = V_\alpha$ is $\Delta_1(PwSet)$, \mathcal{K} is $\Delta_1(PwSet)$ -definable.

By Proposition 5.1 it only remains to show that the class Q_{PwSet} is $\Delta(\mathcal{L}^2)$ -definable. But $\mathcal{A} \in Q_{PwSet}$ if and only if $\mathcal{A} = (A, E, a, b)$ satisfies the \mathcal{L}^2 sentence asserting that E is well-founded and $\pi(a) = \mathcal{P}(\pi(b))$, where π is the function on A given by $\pi(x) = \{\pi(y) : yEx\}$. ⊣

THEOREM 7.2 ([7]). $\kappa = LST(\mathcal{L}^2)$ iff $\kappa = (SR)_{PwSet}$ iff κ is the first supercompact cardinal.

PROOF. The first equivalence follows from theorem 5.5 and the lemma above. For the second equivalence, see [1]. ⊣

§8. The strict Löwenheim–Skolem–Tarski property. For symbiotic R and \mathcal{L}^* , the weaker principles $(SR)_R^-$ and $(SR)_{R^-}^-$ are also equivalent to weaker forms of $LST(\mathcal{L}^*)$.

DEFINITION 8.1. We say that a cardinal κ has the $LS(\mathcal{L}^*)$ property if for every nonempty \mathcal{L}^* -definable model class \mathcal{K} there exists $\mathcal{A} \in \mathcal{K}$ of cardinality less than κ .

DEFINITION 8.2. We say that a cardinal κ has the *strict Löwenheim–Skolem–Tarski property* for \mathcal{L}^* , written $SLST(\mathcal{L}^*)$, if whenever \mathcal{A} is a model and $\varphi \in \mathcal{L}^*$ is such that $\mathcal{A} \models \varphi$, and $|\mathcal{A}| = \kappa$, then there is $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{B} \models \varphi$ and $|\mathcal{B}| < \kappa$.

Using similar (in fact, simpler) arguments to those in the proof of Theorem 5.5, one can prove the following.

THEOREM 8.3. *If R and \mathcal{L}^* are symbiotic, then*

- (1) κ has the $LS(\mathcal{L}^*)$ property if and only $(SR)_{R^-}^-$ holds for κ .
- (2) κ has the $SLST(\mathcal{L}^*)$ property if and only $(SR)_R^-$ holds for κ .

Thus, from our results in Section 3 we have the following:

- (1)[−] $(SR)_{Cd}^- \equiv SLST(\mathcal{L}_{\omega\omega}(I))$.
- (2)[−] $(SR)_{Cd, WI}^- \equiv SLST(\mathcal{L}_{\omega\omega}(I, W^{WI}))$.
- (3)[−] $(SR)_{Rg}^- \equiv SLST(\mathcal{L}_{\omega\omega}(W^{Rg}))$.
- (4)[−] $(SR)_{Cd, WC}^- \equiv SLST(\mathcal{L}_{\omega\omega}(I, Q_{Br}))$.

We shall devote most of the rest of the paper to showing that the cardinals corresponding to items (1)[−] – (4)[−] above are precisely the first weakly-inaccessible, the first 2-weakly inaccessible, the first weakly Mahlo, and the first weakly compact.

The proof of Theorem 3.2, together with Example 4.2, shows that if the property $SLST(\mathcal{L}_{\omega\omega}(I))$ holds at κ , then there exists a weakly inaccessible cardinal less than or equal to κ . The following theorem, which follows from a result of A. G. Pinus ([11], Theorem 3), shows that $SLST(\mathcal{L}_{\omega\omega}(I))$ holds at every κ weakly inaccessible. Although the ideas are quite similar, our proof bears some differences with that of [11], e.g., it uses elementary submodels. We provide all details as they will be of further use in the proofs of the last two theorems of this section.

THEOREM 8.4. *If κ is weakly inaccessible, then $SLST(\mathcal{L}_{\omega\omega}(I))$ holds at κ .*

PROOF. We start with some little tricks due essentially to G. Fuhrken [4]. For any infinite model \mathcal{A} in a countable vocabulary L , we add two predicates $S(x)$ and $R(x, y)$, as follows:

- (1) $S \subseteq \mathcal{A}$ is arbitrary, except that

$$|S| = |\{\lambda : \lambda \text{ is an infinite cardinal, and } \lambda \leq |\mathcal{A}|\}|.$$

- (2) $R \subseteq S \times \mathcal{A}$ is arbitrary, except that

$$\{|R(a, \cdot)| : a \in S\} = \{\lambda : \lambda \text{ is a cardinal and } \lambda \leq |\mathcal{A}|\}.$$

- (3) If $a, b \in S$ are distinct, then $|R(a, \cdot)| \neq |R(b, \cdot)|$.

From (1) – (3) it follows that if $X, Y \subseteq A$, then the following are equivalent:

- a) $|X| = |Y|$.
- b) For all $a, b \in S$, if $|X| = |R(a, \cdot)|$ and $|Y| = |R(b, \cdot)|$, then $a = b$.

We also add new functions $F(a)$, for every $a \in A$, so that letting L_1 be the expansion of L that contains S, R , and all the functions $F(a)$, the following holds:

- (*) For each $\mathcal{L}_{\omega\omega}(I)$ formula $\varphi(x, \vec{z})$ in the vocabulary L_1 , and every $\vec{b} \in A$, there is some $a \in A$ and $c \in S$ such that

$$F(a) : \varphi(\cdot, \vec{b}) \rightarrow R(c, \cdot)$$

is a bijection.

Thus, for all φ and ψ , and all $\vec{b}, \vec{b}' \in A$, we have that

$$|\varphi(\cdot, \vec{b})| = |\psi(\cdot, \vec{b}')|$$

if and only if there exist $a, a' \in A$ and $c \in S$ such that the functions

$$F(a) : \varphi(\cdot, \vec{b}) \rightarrow R(c, \cdot),$$

$$F(a') : \psi(\cdot, \vec{b}') \rightarrow R(c, \cdot)$$

are bijective.

We define a (Führken-)translation $\varphi \mapsto \varphi^*$ from $\mathcal{L}_{\omega\omega}(I)$, in the vocabulary L , into $\mathcal{L}_{\omega\omega}$, in the vocabulary L_1 , as follows:

- (i) $\varphi^* = \varphi$, if φ is atomic.
- (ii) $(\neg\varphi)^* = \neg\varphi^*$.
- (iii) $(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^*$.
- (iv) $(\exists x\varphi(x, \vec{z}))^* = \exists x\varphi^*(x, \vec{z})$.
- (v) $(Ixy\varphi(x, \vec{z})\psi(y, \vec{z}))^* = \exists u\exists v\exists w(S(w) \wedge$
 $“F(u) : \varphi^*(\cdot, \vec{z}) \rightarrow R(w, \cdot) \text{ is bijective}” \wedge$
 $“F(v) : \psi^*(\cdot, \vec{z}) \rightarrow R(w, \cdot) \text{ is bijective}”)$.

If \mathcal{A}^* an expansion of \mathcal{A} to an L_1 -model satisfying all occurrences of (*), then for all $\mathcal{L}_{\omega\omega}(I)$ formulas φ , in the vocabulary L , and all $\vec{a} \in A$,

$$\mathcal{A} \models \varphi(\vec{a}) \text{ if and only if } \mathcal{A}^* \models \varphi^*(\vec{a}).$$

Notice that if $\mathcal{B}^* \subseteq \mathcal{A}^*$ is a model of the relevant cases of (*) (i.e., for subformulas of φ^*) and, in addition, \mathcal{B}^* satisfies (3) above (but not necessarily (1) or (2)), then for the L -reduct \mathcal{B} of \mathcal{B}^* , all L -subformulas ψ of φ , and all $\vec{a} \in B$,

$$\mathcal{B} \models \psi(\vec{a}) \text{ if and only if } \mathcal{B}^* \models \psi^*(\vec{a}).$$

To prove the Theorem, suppose $\mathcal{A} \models \varphi$, where $|A| = \kappa$. We expand \mathcal{A} to \mathcal{A}^* satisfying all occurrences of (*), so that $\mathcal{A}^* \models \varphi^*$. We need to find $\mathcal{B}^* \subseteq \mathcal{A}^*$ such that $\mathcal{B}^* \models \varphi^*$ and $|\mathcal{B}^*| < \kappa$. Then $\mathcal{B} = \mathcal{B}^* \upharpoonright L$ is the required model, provided that \mathcal{B}^* satisfies all the relevant instances of (*), and (3).

In the sequel we shall be taking elementary substructures $\mathcal{B} \preceq \mathcal{A}^*$ of cardinality $< \kappa$. By this we mean that \mathcal{B} is an elementary substructure of \mathcal{A}^* for the language L together with S, R and $F(a)$, for all $a \in B$. Also, we shall always assume that

\mathcal{B} is closed under the operation that takes every $\mathcal{L}_{\omega\omega}(I)$ formula $\varphi(x, \vec{z})$ in the vocabulary of \mathcal{B} , and every $\vec{b} \in \mathcal{B}$, to some $a \in \mathcal{A}$ and $c \in \mathcal{S}$ such that

$$F(a) : \varphi(\cdot, \vec{b}) \rightarrow R(c, \cdot).$$

Thus, $\mathcal{B} \preceq \mathcal{A}^*$ automatically implies that \mathcal{B} satisfies all instances of $(*)$.

We start with any $\mathcal{B}^0 \preceq \mathcal{A}^*$ countable. We shall produce, in fact, a continuous chain $\langle \mathcal{B}^\alpha : \alpha < \kappa \rangle$ of elementary submodels of \mathcal{A}^* , each of size $< \kappa$, such that any \mathcal{B}^α , for $\alpha > 0$, could be taken as our \mathcal{B}^* . This is, of course, an overkill for the purposes of the present proof, but the construction of the chain will be also useful in the proof of the next two theorems.

If $\mathcal{B}^\alpha \preceq \mathcal{A}^*$ is already defined for $\alpha < \beta = \bigcup \beta < \kappa$, then we let $\mathcal{B}^\beta = \bigcup_{\alpha < \beta} \mathcal{B}^\alpha$. Now suppose $\mathcal{B}^\alpha \preceq \mathcal{A}^*$ is defined; we define $\mathcal{B}^{\alpha+1}$ as follows.

First, let us see what could go wrong with (3). There can be $a, b \in \mathcal{S}^{\mathcal{B}^\alpha}$ such that $a \neq b$ but

$$|R(a, \cdot)^{\mathcal{B}^\alpha}| = |R(b, \cdot)^{\mathcal{B}^\alpha}|.$$

Of course,

$$|R(a, \cdot)^{\mathcal{A}^*}| \neq |R(b, \cdot)^{\mathcal{A}^*}|$$

say

$$|R(a, \cdot)^{\mathcal{A}^*}| < |R(b, \cdot)^{\mathcal{A}^*}|.$$

Note that since $\mathcal{B}^\alpha \preceq \mathcal{A}^*$,

$$|R(a, \cdot)^{\mathcal{B}^\alpha}| \leq |R(a, \cdot)^{\mathcal{A}^*}|$$

hence

$$|R(a, \cdot)^{\mathcal{B}^\alpha}| < |R(b, \cdot)^{\mathcal{A}^*}|.$$

As κ is a limit cardinal, let $X \subseteq R(b, \cdot)^{\mathcal{A}^*}$ be such that

$$|R(a, \cdot)^{\mathcal{A}^*}| < |X| < \kappa.$$

Then we choose $\mathcal{B} \preceq \mathcal{A}^*$ such that $(\mathcal{B}^\alpha \cup X) \subseteq \mathcal{B}$, and we have that

$$|R(a, \cdot)^{\mathcal{B}}| < |R(b, \cdot)^{\mathcal{B}}|.$$

Since κ is regular, by doing the same, repeatedly, with all $a, b \in \mathcal{S}^{\mathcal{B}^\alpha}$ such that $a \neq b$, in the end we have \mathcal{B}_1 , still of size $< \kappa$, such that $\mathcal{B}^\alpha \subseteq \mathcal{B}_1 \preceq \mathcal{A}^*$, and

$$|R(a, \cdot)^{\mathcal{B}_1}| < |R(b, \cdot)^{\mathcal{B}_1}|$$

for all $a, b \in \mathcal{S}^{\mathcal{B}^\alpha}$ such that $|R(a, \cdot)^{\mathcal{A}^*}| < |R(b, \cdot)^{\mathcal{A}^*}|$.

Now, starting with \mathcal{B}_1 , we repeat the process to get \mathcal{B}_2 , still of size $< \kappa$ and such that $\mathcal{B}_1 \preceq \mathcal{B}_2 \preceq \mathcal{A}^*$, so that (3) holds for all $a \neq b$ in $\mathcal{S}^{\mathcal{B}_1}$. And so on. Finally, let $\mathcal{B}^{\alpha+1} = \bigcup_n \mathcal{B}_n$. So, $\mathcal{B}^{\alpha+1}$ has size less than κ . And if $a, b \in \mathcal{S}^{\mathcal{B}^{\alpha+1}}$, then $a, b \in \mathcal{S}^{\mathcal{B}_n}$, for some n , and so if $a \neq b$, then

$$|R(a, \cdot)^{\mathcal{B}^{\alpha+1}}| \neq |R(b, \cdot)^{\mathcal{B}^{\alpha+1}}|. \tag*{\dashv}$$

COROLLARY 8.5. *SLST($\mathcal{L}_{\omega\omega}(I)$) = κ if and only if κ is the first weakly inaccessible cardinal.*

We also have the following theorem, using similar arguments.

THEOREM 8.6. *If κ is 2-weakly inaccessible, then SLST($\mathcal{L}_{\omega\omega}(I, W^{WI})$) holds at κ .*

COROLLARY 8.7. $SLST(\mathcal{L}_{\omega\omega}(I, W^{WI})) = \kappa$ if and only if κ is the first 2-weakly inaccessible cardinal.

Let us consider next the cases of a weakly Mahlo and a weakly compact cardinal, which use ideas similar to the previous proof, and so we will only provide the relevant details.

THEOREM 8.8. *If κ is weakly Mahlo, then $SLST(\mathcal{L}_{\omega\omega}(I, W^{Rg}))$ holds for κ .*

PROOF. For any infinite model \mathcal{A} in a countable vocabulary L , let $S(x)$ and $R(x, y)$ be as in the proof of Theorem 8.4. Thus,

(*) If $a, b \in S$ are distinct, then $|R(a, \cdot)| \neq |R(b, \cdot)|$.

Also, let $\bar{S} = \{a \in S : |R(a, \cdot)| \text{ is regular}\}$, and let $\bar{R} \subseteq \bar{S} \times A \times A$ be such that for each $a \in \bar{S}$, $\bar{R}(a, \cdot, \cdot)$ is a well-order of order-type $|R(a, \cdot)|$. We also add new functions $F(a)$ and $\bar{F}(a)$, for every $a \in A$, so that, letting L_1 be the expansion of L by adding S, \bar{S}, R, \bar{R} , and the functions $F(a)$ and $\bar{F}(a)$, for every $a \in A$, the following holds:

(**) For each $\mathcal{L}_{\omega\omega}(I, W^{Rg})$ formula $\varphi(x, \vec{z})$ in the vocabulary L_1 , and every $\vec{b} \in A$, there are some $a \in A$ and $c \in S$ such that

$$F(a) : \varphi(\cdot, \vec{b}) \rightarrow R(c, \cdot)$$

is a bijection.

(***) For each $\mathcal{L}_{\omega\omega}(I, W^{Rg})$ formula $\varphi(x, y, \vec{z})$ in the vocabulary L_1 , and every $\vec{b} \in A$, if $\varphi(\cdot, \cdot, \vec{b})$ is a well-order whose order-type is an infinite regular cardinal, then there are some $a \in A$ and $c \in \bar{S}$ such that

$$\bar{F}(a) : \text{field}(\varphi(\cdot, \cdot, \vec{b})) \rightarrow \text{field}(\bar{R}(c, \cdot, \cdot))$$

is an order-isomorphism.

We define a translation $\varphi \mapsto \varphi^*$ from $\mathcal{L}_{\omega\omega}(I, W^{Rg})$, in the vocabulary L , into $\mathcal{L}_{\omega\omega}$, in the vocabulary L_1 , as follows:

- (i) $\varphi^* = \varphi$, if φ is atomic.
- (ii) $(\neg\varphi)^* = \neg\varphi^*$.
- (iii) $(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^*$.
- (iv) $(\exists x\varphi(x, \vec{z}))^* = \exists x\varphi^*(x, \vec{z})$.
- (v) $(\exists x\varphi(x, \vec{z})\psi(y, \vec{z}))^* = \exists u\exists v\exists w(S(w) \wedge$
 $“F(u) : \varphi^*(\cdot, \vec{z}) \rightarrow R(w, \cdot) \text{ is bijective}” \wedge$
 $“F(v) : \psi^*(\cdot, \vec{z}) \rightarrow R(w, \cdot) \text{ is bijective}”)$.
- (vi) $(W^{Rg}xy\varphi(x, y, \vec{z}))^* = \exists u\exists w(\bar{S}(w) \wedge “\bar{F}(u) : \text{field}(\varphi^*(\cdot, \cdot, \vec{z})) \rightarrow$
 $\text{field}(\bar{R}(w, \cdot, \cdot)) \text{ is an order-isomorphism}”)$.

If \mathcal{A}^* an expansion of \mathcal{A} to an L_1 -model satisfying all occurrences of (*), (**), and (***), then for all $\mathcal{L}_{\omega\omega}(I, W^{Rg})$ formulas φ in the vocabulary L , and all $\vec{a} \in A$,

$$\mathcal{A} \models \varphi(\vec{a}) \text{ if and only if } \mathcal{A}^* \models \varphi^*(\vec{a}).$$

To prove the Theorem, suppose $\mathcal{A} \models \varphi$, where $|A| = \kappa$. We first expand \mathcal{A} to an L_1 -model \mathcal{A}^* that satisfies all occurrences of (*), (**), and (***), so that $\mathcal{A}^* \models \varphi^*$. We need to find $\mathcal{B}^* \subseteq \mathcal{A}^*$ such that $\mathcal{B}^* \models \varphi^*$ and $|\mathcal{B}^*| < \kappa$, for then $\mathcal{B} = \mathcal{B}^* \upharpoonright L$ is the required model, provided \mathcal{B}^* also satisfies all the necessary instances of (*),

(**), and (***)). We can ensure that \mathcal{B}^* satisfies (**) by taking $\mathcal{B}^* \preceq \mathcal{A}^*$ (see the proof of theorem 8.4). In order to ensure that \mathcal{B}^* satisfies (*) and (***) we first proceed as in the proof of Theorem 8.4. As in that proof, we can produce a continuous chain $\langle \mathcal{B}^\alpha : \alpha < \kappa \rangle$ of elementary substructures of \mathcal{A}^* of size less than κ , with $\mathcal{A}^* = \bigcup_{\alpha < \kappa} \mathcal{B}^\alpha$, and such that every \mathcal{B}^α , for $\alpha > 0$, satisfies all the relevant instances of (*).

Now, given any $\mathcal{L}_{\omega\omega}(I, W^{Rg})$ formula $\psi(x, y, \vec{z})$, in the vocabulary L_1 , the set T_ψ of all $\alpha < \kappa$ such that for every $\vec{b} \in B^\alpha$, if $\psi(\cdot, \cdot, \vec{b})^{B^\alpha}$ is a well-order whose order-type is an infinite regular cardinal, then $\psi(\cdot, \cdot, \vec{b})^{A^*}$ is also a well-order whose order-type is an infinite regular cardinal, is unbounded in κ . The reason is that, given B^α , a witness to non-well-foundedness, or to nonregularity, is a subset of A^* of size less than κ , and therefore can be added to B^α , while keeping it of size less than κ . So, by closing off under this operation of adding witnesses, we eventually obtain a desired $B^{\alpha'} \supseteq B^\alpha$ in T_ψ . Now suppose that α is a regular limit of ordinals in T_ψ such that $\alpha = |B^\alpha|$. We claim that α is also in T_ψ . For suppose $\vec{b} \in B^\alpha$ is such that $\psi(\cdot, \cdot, \vec{b})^{B^\alpha}$ is a well-order whose order-type is an infinite regular cardinal γ . If $\gamma < \alpha$, then by regularity of α , $\text{field}(\psi(\cdot, \cdot, \vec{b})^{B^\alpha})$ is contained in some $B^{\alpha'}$, for some $\alpha' \in T_\psi$ smaller than α . But then $\psi(\cdot, \cdot, \vec{b})^{B^\alpha} = \psi(\cdot, \cdot, \vec{b})^{B^{\alpha'}}$, hence $\psi(\cdot, \cdot, \vec{b})^{A^*}$ is a regular cardinal. Now suppose $\gamma = \alpha$. Let $a \in B^\alpha$ and $c \in \vec{S}$ be such that B^α satisfies that

$$\vec{F}(a) : \text{field}(\psi(\cdot, \cdot, \vec{b})) \rightarrow \text{field}(\vec{R}(c, \cdot, \cdot))$$

is an order-isomorphism. Then B^α also satisfies that, for some a' ,

$$F(a') : \{x : x = x\} \rightarrow R(c, \cdot)$$

is a bijection. But by elementarity, the same holds in A^* . Hence, $\psi(\cdot, \cdot, \vec{b})^{A^*}$ must have order-type κ , which is regular.

Since κ is weakly-Mahlo, there is some α regular that is a limit point of T_ψ , for all the (finitely-many) relevant formulas ψ . Then we can take $\mathcal{B}^* = B^\alpha$, as it satisfies φ^* (because $B^\alpha \preceq \mathcal{A}_1$), has size less than κ , and satisfies all the necessary instances of (***)). ⊣

The proof of Theorem 3.3, together with a version of Example 4.2 using both the I and W^{Rg} quantifiers, shows that if the property $SLST(\mathcal{L}_{\omega\omega}(I, W^{Rg}))$ holds at κ , then there exists a weakly Mahlo cardinal less than or equal to κ . Thus, the last theorem yields the following corollary.

COROLLARY 8.9. *$SLST(\mathcal{L}_{\omega\omega}(I, W^{Rg})) = \kappa$ if and only if κ is the first weakly Mahlo cardinal.*

THEOREM 8.10. *If κ is weakly compact, then $SLST(\mathcal{L}_{\omega\omega}(I, Q_{Br}))$ holds for κ .*

PROOF. For any infinite model \mathcal{A} in a countable vocabulary L , let S and R be as in the previous two proofs. Thus,

$$(*) \text{ If } a, b \in S \text{ are distinct, then } |R(a, \cdot)| \neq |R(b, \cdot)|.$$

Let $\lambda^{\mathcal{A}}$ be the supremum of all the heights of tree orders definable in \mathcal{A} by formulas of the language $\mathcal{L}_{\omega\omega}(I, Q_{Br})$ in the vocabulary L , with parameters. Let $\vec{R} \subseteq A \times A$ be a well-ordering of A of order-type some $\delta^{\mathcal{A}} \geq \lambda^{\mathcal{A}} + 1$.

Also, let $F(a)$ and $\bar{F}(a)$, for every $a \in A$, be new functions so that, letting L_1 be the expansion of L by adding S, R, \bar{R} , and the functions $F(a)$ and $\bar{F}(a)$, for every $a \in A$, the following holds:

- (**) For each $\mathcal{L}_{\omega\omega}(I, Q_{Br})$ formula $\varphi(x, \vec{z})$ in the vocabulary L_1 , and every $\vec{b} \in A$, there are some $a \in A$ and $c \in S$ such that

$$F(a) : \varphi(\cdot, \vec{b}) \rightarrow R(c, \cdot)$$

is a bijection.

- (***) For each $\mathcal{L}_{\omega\omega}(I, Q_{Br})$ formula $\varphi(x, y, \vec{z})$, in the vocabulary L_1 , and every $\vec{b} \in A$, if $\varphi(\cdot, \cdot, \vec{b})$ is a tree order with a chain of length some $\beta < \delta^A$, then there are some $a, c \in A$ such that $\bar{R}(\cdot, c)$ has order-type β and

$$\bar{F}(a) : \bar{R}(\cdot, c) \rightarrow \text{field}(\varphi(\cdot, \cdot, \vec{b}))$$

is an order-homomorphism (i.e., $\bar{F}(a)$ witnesses that the tree order has a chain of length β).

We define a translation $\varphi \mapsto \varphi^*$ from $\mathcal{L}_{\omega\omega}(I, Q_{Br})$, in the vocabulary L , into $\mathcal{L}_{\omega\omega}$, in the vocabulary L_1 , as follows:

- (i) $\varphi^* = \varphi$, if φ is atomic.
- (ii) $(\neg\varphi)^* = \neg\varphi^*$.
- (iii) $(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^*$.
- (iv) $(\exists x\varphi(x, \vec{z}))^* = \exists x\varphi^*(x, \vec{z})$.
- (v) $(Ixy\varphi(x, \vec{z})\psi(y, \vec{z}))^* = \exists u\exists v\exists w(S(w) \wedge$
 $“F(u) : \varphi^*(\cdot, \vec{z}) \rightarrow R(w, \cdot)$ is bijective” \wedge
 $“F(v) : \psi^*(\cdot, \vec{z}) \rightarrow R(w, \cdot)$ is bijective”).
- (vi) $(Q_{Br}xy\varphi(x, y, \vec{z}))^* = \varphi^*(\cdot, \cdot, \vec{z})$ is a tree order² \wedge
 $\exists u(\forall w([w \neq u \wedge (\bar{R}(\cdot, w) \subseteq \bar{R}(\cdot, u)]) \rightarrow$
 $\exists v(“\bar{F}(v) : \bar{R}(\cdot, w) \rightarrow \text{field}(\varphi^*(\cdot, \cdot, \vec{z}))$
 $\text{is an order-homomorphism}”)) \wedge$
 $\forall x((\bar{R}(\cdot, u) \subseteq \bar{R}(\cdot, x)) \rightarrow$
 $\forall y(“\bar{F}(y) : \bar{R}(\cdot, x) \rightarrow \text{field}(\varphi^*(\cdot, \cdot, \vec{z}))$
 $\text{is not an order-homomorphism}”)))))$.

Suppose \mathcal{A}^* is an expansion of \mathcal{A} to an L_1 -model satisfying all occurrences of $(*)$, $(**)$, and $(***)$. Then for all $\mathcal{L}_{\omega\omega}(I, Q_{Br})$ formulas φ in the vocabulary L , and every $\vec{a} \in A$,

$$\mathcal{A} \models \varphi(\vec{a}) \text{ if and only if } \mathcal{A}^* \models \varphi^*(\vec{a}).$$

To prove the Theorem, suppose $\mathcal{A} \models \varphi$, where $|A| = \kappa$. Expand \mathcal{A} to an L_1 -model $\mathcal{A}^* \models \varphi^*$ satisfying all occurrences of $(*)$, $(**)$, and $(***)$, so that $\mathcal{A}^* \models \varphi^*$.

Let us first observe that if $\mathcal{B}^* \preceq \mathcal{A}^*$ is a model of the relevant cases of $(*)$, $(**)$, and $(***)$ (i.e., for subformulas of φ^*) and, in addition, \mathcal{B}^* satisfies that for every subformula $\psi(x, y, \vec{z})$ of φ^* and every $\vec{b} \in B$, if $\psi(\cdot, \cdot, \vec{b})^{\mathcal{A}^*}$ is a tree order of height some limit ordinal but with no cofinal chain (i.e., a chain of order-type the height of the tree), then $\psi(\cdot, \cdot, \vec{b})^{\mathcal{B}^*}$ is also a tree order of height some limit ordinal with

²This can be expressed.

no cofinal chain, then for the L -reduct \mathcal{B} of \mathcal{B}^* , all L -subformulas ψ of φ , and all $\vec{a} \in B$,

$$\mathcal{B} \models \psi(\vec{a}) \text{ if and only if } \mathcal{B}^* \models \psi^*(\vec{a}).$$

This can be easily checked by induction on the complexity of φ . So it will be sufficient to find some such $\mathcal{B}^* \preceq \mathcal{A}^*$ of cardinality less than κ , for then the L -reduct of \mathcal{B}^* will be the required model of φ .

To ensure that \mathcal{B}^* satisfies $(*)$ we first produce, as in the proof of Theorem 8.4, a continuous chain $\langle \mathcal{B}^\alpha : \alpha < \kappa \rangle$ of elementary substructures of \mathcal{A}^* of size less than κ , with $\mathcal{A}^* = \bigcup_{\alpha < \kappa} \mathcal{B}^\alpha$, and such that every \mathcal{B}^α satisfies the relevant instances of $(*)$. Note that since $\mathcal{B}^\alpha \preceq \mathcal{A}^*$, all $\alpha < \kappa$, every \mathcal{B}^α satisfies φ^* , and all the instances of $(**)$ are also satisfied (see the proof of theorem 8.4).

Now, given any $\mathcal{L}_{\omega\omega}(I, Q_{Br})$ formula $\psi(x, y, \vec{z})$, in the vocabulary L_1 , the set T_ψ of ordinals $\alpha < \kappa$ such that for every $\vec{b} \in B^\alpha$, if $\psi(\cdot, \cdot, \vec{b})^{B^\alpha}$ is a tree-order of height some limit ordinal, then $\psi(\cdot, \cdot, \vec{b})^{A^*}$ is a also a tree-order of height some limit ordinal, is a club subset of κ . The reason is that, given \mathcal{B}^α and \vec{b} such that $\psi(\cdot, \cdot, \vec{b})^{B^\alpha}$ is a tree-order, any witness to the non-well-foundedness of the relation $\psi(\cdot, \cdot, \vec{b})^{A^*}$ is a countable subset of A , which can be added to \mathcal{B}^α . So we can close off under the operation of adding witnesses to non-well-foundedness and obtain a desired $\mathcal{B}^{\alpha'} \supseteq \mathcal{B}^\alpha$ with $\alpha' \in T_\psi$.

Suppose $\alpha \in T_\psi$ and $\vec{b} \in B^\alpha$ are such that $\psi(\cdot, \cdot, \vec{b})^{B^\alpha}$ is a tree order of height some limit ordinal γ . Note that $\gamma < \kappa \leq \delta^{A^*}$. Then $\psi(\cdot, \cdot, \vec{b})^{A^*}$ is a also a tree-order of height some limit ordinal $\geq \gamma$. So, since $(***)$ holds for \mathcal{A}^* , for every $\beta < \gamma$ there are some $a = a_\beta$ and c in A such that $\bar{R}(\cdot, c)^{A^*}$ has order-type β and

$$\bar{F}(a)^{A^*} : \bar{R}(\cdot, c)^{A^*} \rightarrow \text{field}(\varphi(\cdot, \cdot, \vec{b})^{A^*})$$

is an order-homomorphism. Since κ is strongly inaccessible, we may close \mathcal{B}^α under the operation of adding to it c and a_β , for all $\beta < \gamma$, so that we obtain a club $C_\psi \subseteq T_\psi$ with the property that if $\alpha \in C_\psi$, then for every $\vec{b} \in B^\alpha$, if for any given $\beta < \delta^{B^\alpha}$ there are some a, c witnessing $(***)$ for $\psi(\cdot, \cdot, \vec{b})$ in \mathcal{A}^* , then there are some such a, c in \mathcal{B}^α . Then if α belongs to the club $C := \bigcap \{C_\psi : \psi \text{ is an } \mathcal{L}_{\omega\omega}(I, Q_{Br}) \text{ formula in the vocabulary } L_1\}$, we have that \mathcal{B}^α satisfies all the instances of $(*)$, $(**)$, and $(***)$.

Finally, since $|A| = \kappa$, we may as well assume that $A \subseteq V_\kappa$. Moreover, since L_1 has also size κ , we may view \mathcal{A}^* as being in fact a subset of V_κ . The assertion that some tree order defined in \mathcal{A}^* by some subformula of φ^* does not have a cofinal branch is Π_1^1 over the structure $\langle V_\kappa, \in, \mathcal{A}^* \rangle$. And since κ is weakly-compact, hence Π_1^1 -indescribable, this must be also true in $\langle V_\alpha, \in, \mathcal{A}^* \cap V_\alpha \rangle$ for a stationary set of α 's. Now, since the set of α 's such that $\mathcal{B}^\alpha = \mathcal{A}^* \cap V_\alpha$ is a club, we can find stationary many α 's in C with \mathcal{B}^α satisfying that for every subformula $\psi(x, y, \vec{z})$ of φ^* and every $\vec{b} \in B^\alpha$, if $\psi(\cdot, \cdot, \vec{b})^{A^*}$ is a tree order of height some limit ordinal without a cofinal chain, then $\psi(\cdot, \cdot, \vec{b})^{B^\alpha}$ is also a tree order of height some limit ordinal without a cofinal chain. Thus we can take as \mathcal{B}^* any such \mathcal{B}^α . \dashv

The proof of Theorem 3.5, together with a version of Example 4.2 using both the I and Q_{Br} quantifiers, shows that if $SLST(\mathcal{L}_{\omega\omega}(I, Q_{Br}))$ holds at κ , then there exists a weakly compact cardinal less than or equal to κ . Thus, last theorem yields the following.

COROLLARY 8.11. $SLST(\mathcal{L}_{\omega\omega}(I, Q_{Br})) = \kappa$ if and only if κ is the first weakly compact cardinal.

Let us conclude by observing that Scott's method (i.e., the use of an elementary embedding of V into the transitive collapse of the ultrapower of V given by a κ -complete nonprincipal measure on κ) shows that if κ is a measurable cardinal, then $SLST(\mathcal{L}^2)$ holds for κ . However, it is still an open question if $SLST(\mathcal{L}^2)$ is exactly the first measurable cardinal, or even if $SLST(\mathcal{L}^2)$ implies the existence of an inner model with a measurable cardinal.

§9. Acknowledgments. The research work of the first author was partially supported by the Spanish Government under grant MTM2011-25229, and by the Generalitat de Catalunya (Catalan Government) under grant 2009 SGR 187 and 2014 SGR 437. The research of the second author was supported by the Academy of Finland grant number 40734.

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