Canad. Math. Bull. Vol. **17** (2), 1974

LINEAR FUNCTIONALS AND SUMMABILITY INVARIANTS

BY
M. S. MACPHAIL AND A. WILANSKY

1. **Introduction.** The purpose of this paper is to continue the study of certain "distinguished" subsets of the convergence domain of a matrix, as developed by A. Wilansky [6] and G. Bennett [1]. We also consider continuous linear functionals on the domain, and the extent to which their representation is unique; this turns out to be connected with the behaviour of the subsets.

As in [7], we use s, m, c, c_0, E^∞, respectively, for the set of all sequences, bounded sequences, convergent sequences, null sequences, and sequences with almost all terms zero. If A is a matrix (a_{nk}) and x a sequence $\langle x_k\rangle$, we put $(Ax)_n = \sum_k a_{nk}x_k$, $Ax = \langle (Ax)_n\rangle$, $d_A = \{x : (Ax)_n$ exists for $n = 1, 2, \ldots\}$, $c_A = \{x : Ax \in c\}$, and $c_A^0 = \{x : Ax \in c_0\}$. We assume A conservative, that is, $c \subset c_A$. We use the FK topology on c_A, as described in [7]. We put 1 for $\langle 1, 1, \ldots\rangle$, δ^k for $\langle 0, 0, \ldots, 0, 1, 0, \ldots\rangle$ (1 in the k-th place), and Δ for the set $\{\delta^k\}$. For any letter, say y, denoting a sequence, we use $y_1, y_2, \ldots$ for the terms of y.

The primary subsets are

$$\begin{aligned} S &= \{x \in c_A : \textstyle\sum x_k\delta^k = x\}, \\ W &= \{x \in c_A : \textstyle\sum x_k f(\delta^k) = f(x) \text{ for all } f \in c_A'\}, \\ F &= \{x \in c_A : \textstyle\sum x_k f(\delta^k) \text{ converges for all } f \in c_A'\}, \\ B &= \{x \in c_A : \textstyle\sum_1^p x_k\delta^k \text{ is bounded in } c_A\}. \end{aligned}$$

We can write equivalently ([6], [3])

$$B = \left\{x \in c_A : \text{there exists } M = M(x) \text{ such that } \left|\sum_{k=1}^p a_{nk}x_k\right| < M \text{ for all } p, n = 1, 2, \ldots\right\},$$

or again

$$B = \left\{x \in c_A : \sum_k \sum_n t_n a_{nk} x_k \text{ exists for all } t \in l\right\},$$

where $t \in l$ means as usual $\sum |t_n| < \infty$. It is also known ([6], p. 331) that

$$\sum_k \sum_n t_n a_{nk} x_k = \sum_n \sum_k t_n a_{nk} x_k \tag{1}$$

for all $x \in B$, $t \in l$.

When dependence on a matrix is in question, we write S_A, and so forth. With a_k denoting the k-th column limit of A, we define

$$I = \{x \in c_A : \sum a_k x_k \text{ converges}\}.$$

On I we define $\Lambda(x) = \lim_A x - \sum a_k x_k = \lim (Ax)_n - \sum a_k x_k$; we then define $\Lambda^\perp = \{x : \Lambda(x) = 0\}$. We have the relations

$$S \subset W \subset F \subset B,$$

but I, $\Lambda^\perp$ and also $m \cap c_A$ cut across S, W, F, B in an apparently capricious way, as the matrix A varies. Examples are given in [1] and [6].

The general form of a continuous linear functional f on c_A is [7, page 230]

$$f(x) = \alpha \lim_A x + t(Ax) + \beta x \tag{2}$$

where $t \in l$, and by a product of two sequences such as βx we understand $\sum \beta_k x_k$. The sequence β is such that βx converges for all $x \in d_A$. Sometimes we shall let β be such that βx converges for all $x \in c_A$; this also defines a continuous linear functional on c_A. We shall call β *restricted* or *unrestricted* in the two cases, respectively.

The representation (2) is far from unique, as α, t, β are interrelated; for example, we could change any one term t_k and adjust β accordingly. If A is row-finite we have $d_A = s$, and so $\beta \in E^\infty$ (restricted), while if A is a triangle (i.e. $a_{nk} = 0$ for $k > n$, but $a_{nn} \neq 0$ for all n) there is a representation with $\beta = 0$, though other representations are also possible.

In this connection the most interesting question is whether α is *unique*, that is, uniquely determined by f for each $f \in c_A'$. This was briefly considered in [6]. We define $\chi = \lim_n \sum_k a_{nk} - \sum a_k$, and call A *coregular* if $\chi \neq 0$, *conull* if $\chi = 0$. It is known [6, page 329] that α is unique if A is coregular. If A is conull, α may or may not be unique, and our first objective is to give certain classes of conull matrices for which α is unique. We also consider α for other matrices D with $c_D = c_A$. When necessary we write $\alpha(f)$ for α.

We then present some new results, mostly connected with invariance and replaceability ([4], [6]) for I, $\Lambda^\perp$, and for the set P defined in section 4.

2. **The coefficient** α. To clarify the ideas, we start with some examples.

Example 1. Let
$$A = \begin{matrix} c_1 & c_2 & c_3 & c_4 & \cdots \\ 0 & c_2 & c_3 & c_4 & \cdots \\ 0 & 0 & c_3 & c_4 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{matrix}$$

with $\sum |c_k| < \infty$. Then $\lim_A x = 0$ for every $x \in c_A$, and so for any given $f \in c_A'$, α may have any value.

Example 2. Let $A = \begin{matrix} c_1 & 0 & 0 & 0 & \cdots \\ c_1 & c_2 & 0 & 0 & \cdots \\ c_1 & c_2 & c_3 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$

with $\sum |c_k| < \infty$, $c_k \neq 0$ for all n. Then $\lim_A x = \sum c_k x_k$, so with β unrestricted we may take $\alpha(\lim_A)$ to be 1 or 0, or indeed any value, by adjusting β. Any function $f \in c_A'$ has a representation

$$f(x) = \alpha \lim_A x + t(Ax),$$

and if we insist on this form, α is unique. See, moreover, Theorem 2.1 below.

Example 3. Let $A = \begin{matrix} 1 & -1 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & -1 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{matrix}$

Here the equation $\lim_A x = t(Ax) + \beta x$ cannot hold for any choice of t and β, restricted or not, for if it did we would find by considering $x = \delta^1, \delta^2, \ldots$ and $\langle(-1)^{k+1}\rangle$ that $t_n \to -2$, which contradicts $t \in l$. So in this case α is unique, with β unrestricted.

We recall that a matrix A is said to be *reversible* if the equation $y = Ax$ has a unique solution x for each $y \in c$. It is well known [6, page 229, Theorem 4] that in this case each mapping $y \mapsto x_k$ is continuous, so we may write

$$x_k = v_k \lim y + \sum_n c_{kn} y_n$$

or

$$x = v \lim y + Cy \tag{3}$$

with $\langle c_{k1}, c_{k2}, \ldots \rangle \in l$.

THEOREM 2.1. *Let A be row-finite and reversible. Then with β restricted, α is unique.*

Proof. Suppose α is not unique. Then for some t, β we have

$$\lim_A x = t(Ax) + \beta x$$

or

$$\lim y = ty + \beta x,$$

with $t \in l$, $\beta \in E^\infty$. Now with A row-finite we have $v = 0$ in (3) [5, Lemma 4], and each member of the finite set $\{\beta_k x_k\}$ can be expressed in terms of y and combined with ty; thus

$$\lim y = \tau y$$

for each $y \in c$, which is impossible.

The row-finiteness condition cannot be dropped; for example, the transformation defined by

$$y_{2r} = \sum_{p=1}^{r} 2^{-2p} x_{2p},$$

$$y_{2r-1} = 2^{-2r+1} x_{2r-1} + \sum_{p=1}^{\infty} 2^{-2p} x_{2p}$$

is reversible and has

$$x_{2r-1} = 2^{2r-1}(y_{2r-1} - \lim y_n)$$

for each $y \in c$; thus (with $P_k(x) = x_k$) we have $\alpha(P_{2r-1}) = 0$ or -2^{2r-1}.

In the rest of this section A need not be reversible, except in 2.4, and β is unrestricted.

A property or set, associated with a matrix A, which remains unaltered for any matrix D with $c_D = c_A$ is called *invariant for A*. If it is invariant for each conservative matrix A, it is called simply *invariant*. In particular the FK topology on c_A is invariant, and the subsets S, F, W, B, being defined in terms of this topology, are invariant.

It is well known that if A is the Cesàro matrix,

$$A = \begin{matrix} 1 & & \\ \frac{1}{2} & \frac{1}{2} & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \cdot & \cdot & \cdot\ \cdot\ \cdot \end{matrix}$$

then $I_A = c_A$, and $I_B = c_B$ for every matrix B with $c_B = c_A$ ([4], Theorem 2). But for

$$D = \begin{matrix} 1 & & & \\ -1 & 1 & & \\ 0 & -1 & 1 & \\ 0 & 0 & -1 & 1 \\ \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot \end{matrix}$$

I is not invariant ([6], Example 5). Thus I is invariant for A, but not invariant in the unqualified sense.

THEOREM 2.2. *If A has $W \neq B$, then α is unique.*

PROOF. If α is not unique, we can find t and β such that $\lim_A x + t(Ax) + \beta x = 0$. Then [9, Satz 5.3] there is a matrix D such that $c_D = c_A$ and $\lim_D = 0$. In particular the column limits of D are all zero, and $\Lambda_D^{\perp} = c_D$. By [6, Theorem 5.4], $W_D = B_D \cap \Lambda_D^{\perp} = B_D$, and by invariance, $W_A = B_A$.

We remark that if A is coregular we have $F = W \oplus u$ for some $u \in c_A \backslash W$, while if A is conull F may be either W or $W \oplus u$ [6, Theorem 5.4]. In either case $B \supset F$, and B may or may not equal F. Thus 2.2 extends the known uniqueness of α for the coregular matrices to a class of conull matrices.

https://doi.org/10.4153/CMB-1974-046-2 Published online by Cambridge University Press

According to standard definitions, A is *multiplicative* if there is a constant M such that $\lim_A x = M \lim x$ for all $x \in c$. A necessary and sufficient condition for this is that $\langle a_k \rangle = 0$ and $\lim_n \sum_k a_{nk} = M$. If A is conull, M must be zero. A matrix is called *replaceable* if there is a multiplicative matrix with the same convergence domain.

THEOREM 2.3. *If A is not replaceable, then α is unique.*

The proof is contained in the opening lines of 2.2.

We return briefly to the study of reversible matrices, and make the following remark.

COROLLARY 2.4. *Let A be reversible, and assume either $W \neq B$ or A not replaceable. Then $v=0$ in* (3).

This follows at once from 2.2 and 2.3. It generalizes the corresponding result for reversible coregular matrices [8, Theorem 7], and as in that theorem leads to the conclusion that A^{-1} exists and is the matrix of the inverse of the transformation defined by A.

Theorem 2.3 can be strengthened as follows.

THEOREM 2.5. *If A is not replaceable, and $f=g$ on Δ, then $\alpha(f)=\alpha(g)$.*

Proof. Suppose if possible there is a function f which vanishes on Δ, but has a representation (2) with $\alpha \neq 0$. Then as in 2.2 there is a matrix D with $c_D = c_A$, and $\lim_D = f$. Then $d_k = f(\delta^k) = 0$, and A is replaceable.

There is a similar strengthening of 2.2, namely,

THEOREM 2.6. *If A has $W \neq B$, and $f=g$ on B, then $\alpha(f)=\alpha(g)$.*

Proof. Suppose $f=B$ on B. For $x \in B$ we have, using (1),

$$\begin{aligned} f(x) &= \alpha \lim_A x + t(Ax) + \beta x \\ &= \alpha \lim_A x + (tA+\beta)x \\ &= \alpha \lim_A x + \gamma x, \text{ say.} \end{aligned}$$

By putting $x = \delta^k$ we find $\alpha a_k + \gamma_k = 0$, whence $f(x) = \alpha(\lim_A x - \sum a_k x_k) = \alpha \Lambda(x) = 0$ on B. Now $W = B \cap \Lambda^{\perp}$ [6, Theorem 5.4] so from $W \neq B$ we get $B \not\subset \Lambda^{\perp}$, whence $\alpha = 0$.

The theorem of Zeller [9, Satz 5.3] referred to in the proof of our Theorem 2.2 states that if f has a representation (1) with $\alpha \neq 0$, there is a matrix D with $c_D = c_A$, $\lim_D = f$. It is left open whether a function f with α uniquely zero could have such a matrix representation. Our next theorem will show that if the uniqueness arises from $W \neq B$, this cannot occur.

THEOREM 2.7. *Let A have $W \neq B$, and let D be such that $c_D = c_A$. Then with* $\lim_D$ *regarded as a functional on c_A, we have $\alpha(\lim_D) \neq 0$.*

6

https://doi.org/10.4153/CMB-1974-046-2 Published online by Cambridge University Press

Proof. By 2.2, α is unique. Suppose $\alpha(\lim_D)=0$. Then $\lim_D x=t(Ax)+\beta x$. For $x\in B$ we have as before $t(Ax)=(tA)x$, and so $\lim_D x=\gamma x$, say. By putting $x=\delta^k$ we find $\gamma_k=d_k$, so $\lim_D x=\sum d_k x_k$, that is, $B_D\subset\Lambda_D^\perp$. As noted in 2.2, $W_A=W_D=B_D\cap\Lambda_D^\perp=B_D=B_A$.

We now define α to be *invariantly unique* if α is unique for every D with $c_D=c_A$. Any invariant condition that implies α is unique obviously implies α is invariantly unique, for example, A coregular, $W\neq B$, or A not replaceable. But the matrix in Example 2 has α unique (with β restricted), while the matrix in Example 1 has the same convergence domain, but α not unique.

If α is invariantly unique, and D is any matrix with $c_D=c_A$, and f is a continuous linear functional on c_A (or c_D), we write $\alpha_A(f)$, $\alpha_D(f)$ for the values of α when f is expressed in the form (2) with respect to A or D. We put $\alpha_A^\perp=\{f\in c_A':\alpha_A(f)=0\}$, and similarly for $\alpha_D^\perp$. If $\alpha_D^\perp=\alpha_A^\perp$ for every D with $c_D=c_A$, we say that $\alpha^\perp$ *is invariant.*

THEOREM 2.8. *If A has $W\neq B$, then $\alpha^\perp$ is invariant.*

Proof. Suppose $\alpha^\perp$ is not invariant. Without loss of generality we may assume that for some D with $c_D=c_A$ we have $\lim_A x+t(Ax)+\beta x=u(Dx)+\gamma x$. For $x\in B$ this reduces to $\lim_A x=\zeta x$, say. Setting $x=\delta^k$ we find $a_k=\zeta_k$, whence $\Lambda(x)=0$. Thus $B\subset\Lambda^\perp$, and since $W=B\cap\Lambda^\perp$ we obtain $W=B$.

The following questions are left open.

A. Does α invariantly unique imply $\alpha^\perp$ invariant? We observe that α is not invariantly unique if and only if there exists D with $c_D=c_A$, $\lim_D=0$, and that $\alpha^\perp$ is not invariant if and only if there exists D with $c_D=c_A$, $\alpha(\lim_D)=0$.

B. Does A not-replaceable imply $\alpha^\perp$ invariant?

C. If A is a matrix for which α is unique, must $\alpha(\lim_D)\neq 0$ for all D with $c_D=c_A$?

D. Does α not-unique imply $\Lambda^\perp=c_A$? (By 2.2, $\Lambda^\perp\supset B$.) Or possibly $W=c_A$?

3. **The subsets I and $\Lambda^\perp$.** In this section we consider the relations between $\Lambda^\perp$ and the other subsets of c_A, and also the question of invariance of I and $\Lambda^\perp$. They are certainly not invariant in the general sense, but it may happen that for a particular matrix A every matrix D with $c_D=c_A$ has $I_D=I_A$ or $\Lambda_D^\perp=\Lambda_A^\perp$ or both.

We observe first that W and $m\cap c_A$ are about the same "size", meaning that they both lie between $m\cap\Lambda^\perp$ and F, but are ordinarily of different "shapes": they usually cut across one another, though inclusion relations are possible.

Now $\Lambda^\perp\supset W$ always [6, Theorem 5.4], but $\Lambda^\perp\supset m\cap c_A$ implies A conull, since $1\in m\cap c_A$ and $\chi=\Lambda(1)$. Some but not all conull matrices have $\Lambda^\perp\supset m\cap c_A$; if it holds, then also $W\supset m\cap c_A$ [2]. The inclusion $\Lambda^\perp\subset m\cap c_A$ is possible, but implies $\Lambda^\perp=c_0$, as we shall show.

THEOREM 3.1. *If $\Lambda^\perp\subset m\cap c_A$, then $\Lambda^\perp=c_0$.*

Proof. We consider first the case $\langle a_k\rangle=0$, so that $\Lambda^\perp=c_A^0$, and we are assuming $c_A^0\subset m$. It can be proved by adapting [9, Satz 7.1] that if A sums to zero a bounded

https://doi.org/10.4153/CMB-1974-046-2 Published online by Cambridge University Press

sequence which does not tend to zero, then A also sums an unbounded sequence to zero. That is, $c_A^0 \subset m$ implies $c_A^0 \subset c_0$, or $\Lambda^\perp \subset c_0$, whence $\Lambda^\perp = c_0$.

If not all a_k are zero, define

$$D = \begin{matrix} a_1 & a_2 & \cdots \\ a_{11}-a_1 & a_{12}-a_2 & \cdots \\ a_{21}-a_1 & a_{22}-a_2 & \cdots \\ \cdots & \cdots & \cdots \end{matrix}$$

Then $m \cap c_D = m \cap c_A$, and $\Lambda_A^\perp = c_D^0$. Finally,

$$\Lambda_A^\perp \subset m \cap c_A \Rightarrow c_D^0 \subset m \cap c_D \Rightarrow c_D^0 \subset c_0 \Rightarrow \Lambda_A^\perp = c_0.$$

As to the invariance of I and $\Lambda^\perp$, we collect some results which are already known, or easily proved. It is familar that, for certain matrices A, I may equal c_A and be invariant [6, Corollary 5.9]. For an example where I is invariant but not equal to c_A, see [1, Example 3]. If I is invariant, it must equal F, since $F = \bigcap \{I_D : c_D = c_A\}$ [6, page 332].

If I is invariant, then $\Lambda^\perp$ is invariant [1, Prop. 4]. The converse holds if A is coregular [1, Prop. 5], or indeed if we assume only $W \neq F$; this can be seen from the relations $W = B \cap \Lambda^\perp$, $F = B \cap I$, $F = W \oplus u$ [6, pages 332–333].

We note also that if $\Lambda^\perp$ is invariant, then $S = W$. For $W = \bigcap \{\Lambda_D^\perp : c_D = c_A\}$ (this is proved by the same method as the corresponding result for F, [6, page 332]), so if $\Lambda^\perp$ is invariant we have $W = \Lambda^\perp$. Then by a theorem of Zeller [10, 8.2] it follows that $S = W$.

We leave the following question open:

E. If $\Lambda_A^\perp = I_A$, must $\Lambda_D^\perp = I_D$ for every matrix D with $c_D = c_A$? (Compare [6] and [1], Question VI).

4. **The sets T and P.** A set P was introduced in [6, Section 6]; it is most conveniently described by first setting

$$T = \{t \in l : (tA)x \text{ exists for all } x \in c_A\},$$

then

$$P = \{x \in c_A : (tA)x = t(Ax) \text{ for all } t \in T\}.$$

Obviously $T = l$ if and only if $B = c_A$ (see Introduction). We shall consider conditions on A and f under which the sequence t in (2) belongs to T. It is easy to see that if f has the form $f(x) = t(Ax) + \beta x$, and $f = 0$ on Δ, then $t \in T$. It then follows from 2.5 that if A is not replaceable, and $f = 0$ on Δ, then $t \in T$. If $I = c_A$, and $f = 0$ on Δ, then $t \in T$; this can be seen by writing (2) in the form [6, equation (4)]:

$$f(x) = \alpha \lim_A x + t(Ax) + \sum_k \left\{ f(\delta^k) - \alpha a_k - \sum_n t_n a_{nk} \right\} x_k.$$

However, the condition $f = 0$ on Δ is not by itself sufficient to ensure $t \in T$. For let $\chi(A) = 1$, $I \neq c_A$, $f(1) = 1$, and $f = 0$ on Δ. Then we can calculate from (2) that $(tA)_k = -a_k - \beta_k$, so $t \notin T$, since $\sum a_k x_k$ diverges for some $x \in c_A$.

https://doi.org/10.4153/CMB-1974-046-2 Published online by Cambridge University Press

It will appear in the course of an example given later that T is not invariant in general.

The question of the invariance of P was raised in [6, Question VIII], and studied in [1]. It is known that P is invariant for A except when A satisfies the three conditions: A replaceable, $W=F$, $\bar{B}\neq c_A$, simultaneously, in which case the invariance remains in doubt. The bar denotes closure.

To illustrate these ideas, we consider the example

$$A = \begin{matrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & -1 & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{matrix}$$

As shown in [6, Example 5], we have $B=m \cap c_A$, and obviously $I=c_A$, $\Lambda^{\perp}=c_A^0$. Then $F=B \cap I=m \cap c_A$, $W=B \cap \Lambda^{\perp}=m \cap c_A^0$, and it can be checked that $W=F$. Next, let $v=\langle 1, 2, \ldots\rangle$; with $\varepsilon<1$ it can be verified that the ball of radius ε centred at v in c_A consists entirely of unbounded sequences, so $\bar{B}\neq c_A$. Also A is multiplicative, so we have the doubtful situation described in the preceding paragraph. We have not decided whether P is invariant for A. We shall show that T is not invariant, but that $P_H=P_A$ for $H=JA$, where J is any matrix of the type:

$$J = \begin{matrix} 1 & & & \\ b_1 & 1 & & \\ b_1 & b_2 & 1 & \\ b_1 & b_2 & b_3 & 1 \\ \cdots & \cdots & \cdots & \cdots \end{matrix}$$

with $b \in l$. (It is well known that $c_J=c$, so $c_H=c_A$). We shall show that $T_H\neq T_A$ if J is properly chosen. Let

$$R = R(r, t, x) = \sum_{k=1}^{r}(tH)_k x_k - \sum_{k=1}^{r} t_n (Hx)_n.$$

With $H=(h_{nk})$, $\lambda_r = \sum_{n=r}^{\infty} t_n$, we find

$$R = \sum_{k=1}^{r} \sum_{n=r+1}^{\infty} t_n h_{nk} x_k$$
$$= \lambda_{r+1} \sum_{k=1}^{r} (b_k - b_{k+1}) x_k + t_{r+1}(b_{r+1}-1)x_r.$$

Now let $y=Ax$, that is, $y_n=x_n-x_{n-1}$. Then

$$\sum_{k=1}^{r}(b_k-b_{k+1})x_k = \sum_{k=1}^{r} b_k y_k - b_{r+1}x_r,$$

and

$$R = \lambda_{r+1}\sum_{k=1}^{r} b_k y_k - \lambda_{r+1} b_{r+1} x_r + t_{r+1} b_{r+1} x_r - t_{r+1} x_r$$
$$= o(1) - \mu_r x_r,$$

when $\mu_r = t_{r+1} + \lambda_{r+2} b_{r+1}$. Now $t \in T_A$ if and only if $t_{r+1} = o(r)$ [6, p. 345], while $t \in T_H$ if and only if $\langle \mu_r x_r \rangle$ converges for all $x \in c_H$. Choose $t = \langle r^{-3/2} \rangle$. Then $t \in T_A$, but with $x = \langle 1, 2, \ldots \rangle \in c_H$ we can find a sequence $b \in l$ (using terms of a convergent series suitably diluted with zeros) such that $\langle \mu_r x_r \rangle$ diverges, and so $t \notin T_H$.

Now $P_A = c_A$ ([6], p. 345), and we shall show that although $T_H \neq T_A$, we have $P_H = c_A = P_A$. Let $M = \operatorname{diag} \mu_n$. Then for $x \in c_A$, $t \in T_H$, we have as before $R = o(1) - \mu_r x_r$, and now $\mu_r x_r = (Mx)_r = (MA^{-1}Ax)_r$. We find

$$MA^{-1} = \begin{matrix} \mu_1 & & \\ \mu_1 & \mu_2 & \\ \mu_1 & \mu_2 & \mu_3 \\ \cdot & \cdot & \cdot \; \cdot \end{matrix}$$

Since $\mu \in l$ and MA^{-1} is conservative, it must be multiplicative-0, so $R \to 0$, and $x \in P_H$.

It was indicated earlier that if A is not replaceable, P is invariant. We now give a more precise result.

THEOREM 4.1. *If A is not replaceable, then $P = \bar{c}_0$.*

This is Theorem 9.1 of [6].

THEOREM 4.2. *If A is multiplicative, then $P = \bar{c}_0$* or $\bar{c}_0 \oplus u$ for some $u \in c_A$.

Proof. Assume $f = 0$ on c_0; then with A multiplicative we have $f(\delta^k) = (tA)_k + \beta_k = 0$, $(tA)_k = -\beta_k$, so $(tA)x$ exists for all $x \in c_A$, which gives $t \in T$. Then for $x \in P$ we have $f(x) = \alpha \lim_A x + (tA)x + \beta x = \alpha \lim_A x + \gamma x$, say. Again using $f(\delta^k) = 0$ we find $\gamma = 0$, so $f(x) = \alpha \lim_A x$ on P.

If $\lim_A = 0$ on P we have $f = 0$ on P, and $P \subset \bar{c}_0$. Otherwise let $u \in P$, $\lim_A u = 1$. Now assume $f = 0$ on $c_0 \oplus u$. Let $x \in P$ and put $y = x - (\lim_A x)u$. Then $y \in P$ and as before $f(y) = \alpha \lim_A y = 0$, whence $f(x) = 0$. We now have $P \subset \bar{c}_0 \oplus u$; but by [6, Theorem 6.3] $P \supset \bar{c}_0$, so $P = \bar{c}_0$ or $\bar{c}_0 \oplus u$.

COROLLARY 4.3. *Let A be any conservative matrix, and let $P^i = \bigcap \{P_D : c_D = c_A\}$. Then $P^i = \bar{c}_0$ or $\bar{c}_0 \oplus u$.*

Proof. If A is not replaceable, we have $P^i = \bar{c}_0$ by 4.1. If A is replaceable, let D be multiplicative, with $c_D = c_A$. Then by 4.2, $P_D = \bar{c}_0$ or $\bar{c}_0 \oplus u$, for some $u \in c_A$. If $P_D = \bar{c}_0$, then $P^i = \bar{c}_0$. If $P_D = \bar{c}_0 \oplus u$, and among the matrices E with $c_E = c_A$ there is one such that P_E does not contain u, then $P^i = \bar{c}_0$. But if for every matrix E with $c_E = c_A$, P_E contains u, then $P^i = \bar{c}_0 \oplus u$.

THEOREM 4.4. *Let A have $I = c_A$. Then $P = \bar{c}_0$ or $\bar{c}_0 \oplus u$, for some $u \in c_A$; moreover, $P = \bar{c}_0$ if and only if $P \subset \Lambda^\perp$.*

Proof. With $I = c_A$ and $f = 0$ on c_0 we find $f(x) = \alpha \Lambda(x)$ on P, and conclude as in 4.2 that $P = \bar{c}_0$ or $\bar{c}_0 \oplus u$. We conclude also that

$$P \subset \Lambda^\perp \Rightarrow P \subset \bar{c}_0 \Rightarrow P = \bar{c}_0.$$

https://doi.org/10.4153/CMB-1974-046-2 Published online by Cambridge University Press

But $I=c_A$ makes Λ continuous, and as Λ vanishes on c_0 we have $\Lambda^\perp \supset \bar{c}_0$, so

$$P = \bar{c}_0 \Rightarrow P \subset \Lambda^\perp.$$

This completes the proof.

Added in proof. While this paper was in press, it was shown by W. Beekman, J. Boos and K. Zeller [Math. Z. **130** (1973), 287–290] that our Theorem 4.2 holds for any conservative matrix, and that P is invariant.

References

1. G. Bennett, *Distinguished subsets and summability invariants*. Studia Math. **40** (1971), 225–234.

2. S.-C. Chang, *Conull FK spaces belonging to the class O*, Math. Z. **113** (1970), 249–254.

3. John J. Sember, *The associative part of a convergence domain is invariant*, Canad. Math. Bull. **13** (1970), 147–148.

4. A. Wilansky, *Summability: the inset, replaceable matrices, the basis in summability space*, Duke Math. J. **19** (1952), 647–660.

5. ——, *Convergence fields of row-finite and row-infinite reversible matrices*, Proc. Amer. Math. Soc. **3** (1952), 389–391.

6. ——, *Distinguished subsets and summability invariants*, J. Analyse Math. **12** (1964), 327–350.

7. ——, *Functional analysis*, Blaisdell, New York, 1964.

8. A. Wilansky and K. Zeller, *Inverses of matrices and matrix-transformations*, Proc Amer. Math. Soc. **6** (1955), 414–420.

9. K. Zeller, *Allgemeine Eigenschaften von Limitierungsverfahren*, Math. Z. **53** (1951), 463–487.

10. ——, *Faktorfolgen bei Limitierungsverfahren*, Math. Z. **56** (1952), 134–151.

Carleton University, Ottawa, Canada,
Lehigh University, Bethlehem, Pennsylvania

https://doi.org/10.4153/CMB-1974-046-2 Published online by Cambridge University Press