

PARAMETER-FREE SCHEMES IN SECOND-ORDER ARITHMETIC

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Abstract. Lyubetsky and Kanovei showed in [8] that there is a second-order arithmetic model of Z_2^p , (comprehension for all second-order formulas without parameters), in which Σ_2^1 -CA (comprehension for all Σ_2^1 -formulas with parameters) holds, but Σ_4^1 -CA fails. They asked whether there is a model of $Z_2^p + \Sigma_2^1$ -CA with the optimal failure of Σ_3^1 -CA. We answer the question positively by constructing such a model in a forcing extension by a tree iteration of Jensen's forcing. Let Coll^p be the parameter-free collection scheme for second-order formulas and let AC^p be the parameter-free choice scheme. We show that there is a model of $Z_2^p + \text{AC}^p + \Sigma_2^1$ -CA with a failure of Σ_4^1 -CA. We also show that there is a model of $Z_2^p + \text{Coll}^p + \Sigma_2^1$ -CA with a failure of Σ_4^1 -CA and a failure of AC^p , so that, in particular, the schemes Coll^p and AC^p are not equivalent over Z_2^p .

§1. Introduction. The strength of a second-order arithmetic theory depends mostly on the amount of comprehension that the theory admits. A comprehension scheme specifies which formulas give collections that turn out to be sets in the model. At the lower extreme, the second-order theory ACA_0 includes comprehension for all first-order formulas and at the higher extreme, full second-order arithmetic Z_2 includes comprehension for all (second-order) formulas. In between, we have Σ_n^1 -comprehension schemes for second-order formulas of complexity Σ_n^1 (n alternations of set quantifiers followed by a first-order formula). A precise formulation of these comprehension schemes should mention that the specified formulas are allowed to use set parameters from the model. But how significant really are parameters in comprehension? What if we formulate the comprehension schemes without parameters, do we get equivalent theories, perhaps, equiconsistent theories?

Let Z_2^p be the version of full second-order arithmetic Z_2 , where we do not allow parameters in the comprehension scheme. Friedman showed in [3, Lemma 3.1.7] that given a model of Z_2^p , we can appropriately restrict its sets to get a model of Z_2 . Thus, the theories Z_2 and Z_2^p are equiconsistent. In a recent work [8], Lyubetsky and Kanovei separated the theories Z_2 and Z_2^p by constructing models of Z_2^p with various failures of comprehension with parameters. They constructed a model of Z_2^p in which the sets were not even closed under complement, showing that comprehension without parameters does not even give comprehension with parameters for Σ_0^0 -assertions. They next considered whether it was possible to have a model of Z_2^p with enough comprehension with parameters to carry out most standard constructions, but still have comprehension with parameters fail

Received August 6, 2024.

2020 *Mathematics Subject Classification.* Primary 03C62.

Key words and Phrases. second-order arithmetic, parameter-free comprehension in second-order arithmetic, parameter-free choice scheme in second-order arithmetic.

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0022-4812/00/0000-0000
DOI:10.1017/jsl.2024.81



somewhere above. They argued that a reasonable amount of comprehension with parameters for general mathematical purposes is the Σ_2^1 -comprehension scheme and constructed a model of Z_2^{-p} with the Σ_2^1 -comprehension scheme and a failure of Σ_4^1 -comprehension. The model was constructed in a forcing extension by a (non-linear) tree iteration of Sacks forcing. Lyubetsky and Kanovei asked whether the result can be improved to obtain the optimal failure of Σ_3^1 -comprehension [8, Problem 8.2].

In this article, we answer their question positively by constructing a model with the required properties, similarly to how the model of [8] was constructed, in a forcing extension by a tree iteration of Jensen's forcing. Jensen's forcing is a subposet of Sacks forcing constructed in L using the \diamond -principle. Jensen's forcing has the ccc and adds a unique generic real whose singleton is Π_2^1 -definable.¹ Jensen originally introduced the forcing because it gives a forcing extension of L in which there is a Π_2^1 -definable non-constructible singleton real [5].

THEOREM 1.1. *It is consistent that there is a model of Z_2^{-p} together with Σ_2^1 -comprehension and a failure of Σ_3^1 -comprehension.*

Theorem 1.1 is proved in Section 3.

If the first-order part of a second-order arithmetic model satisfies PA, then via Gödel's pairing function, we can view any set in the model as a collection of n -tuples, for some $n < \omega$, of elements of the model. Among these collections are orderings of the first-order domain of the model, which the model believes are well-orders. In a model of Z_2 , these well-orders are comparable in that given any two of them, there is a set coding an isomorphism of one of them with an initial segment of the other. Using coding, a model of Z_2 can construct an initial segment of the L -hierarchy along any of its well-orders, and these initial segments are similarly coherent (for details of the construction, see [10], Chapter VII). We will call a real of a model of Z_2 *constructible* if it appears in an initial segment of the L -hierarchy of the model. Similarly, in a model of Z_2^{-p} , we can show that any two definable without parameters well-orders are comparable and we can construct an initial segment of the L -hierarchy along any such well-order so that the resulting structures are coherent. We will call a real of a model of Z_2^{-p} *constructible* if it appears in an initial segment of the L -hierarchy constructed along a definable without parameters well-order.

The theory Z_2 can be further strengthened by adding the choice scheme AC, which is a set choice principle asserting for every second-order formula $\varphi(n, X, A)$ (with set parameter A) that if for every n , there is a set X such that $\varphi(n, X, A)$ holds, then there is a single set Y whose n -th slice Y_n is a witness for n . It is not difficult to see that the scheme AC actually implies Z_2 over ACA_0 . The collection of the constructible reals of a model of Z_2 satisfies $Z_2 + \text{AC}$ (see [10, Section VII.6]), giving that the two theories are equiconsistent. Friedman's sub-collection of a model of Z_2^{-p} , which he showed satisfies Z_2 , was precisely the constructible reals of the model, as defined above, coming from the initial segments of the L -hierarchy constructed along definable without parameters well-orders. His argument additionally showed that the collection of the constructible reals satisfies AC. Thus, the theories Z_2^{-p} and $Z_2 + \text{AC}$ are equiconsistent.

¹A similar forcing can be constructed in any universe in which the \diamond -principle holds, not just in L , and we retain its properties except for, possibly, Π_2^1 -definability of the generic singleton.

Let AC^{-p} denote the parameter-free choice scheme. Because standard arguments that the choice scheme AC implies comprehension over ACA_0 require parameters, it is probably not the case that AC^{-p} implies Z_2^{-p} over ACA_0 . The reals of the Feferman–Lévy model of ZF (a symmetric submodel of a forcing extension), in which ω_1^L is countable for every $n < \omega$ and ω_ω^L is the first uncountable cardinal [9], is a model of Z_2 in which AC^{-p} fails. In this model (the code of) $L_{\omega_n^L}$ exists for every $n < \omega$, but (the code of) $L_{\omega_\omega^L}$ does not exist, because ω_ω^L is uncountable in the Feferman–Lévy model, and therefore cannot be coded by a real. Guzicki constructed a model of ZF (again a symmetric submodel of a forcing extension) whose reals are a model satisfying $Z_2 + AC^{-p}$ in which AC fails [4], separating the two principles.²

Vladimir Kanovei suggested to try to construct a model of $Z_2^{-p} + AC^{-p}$ together with Σ_2^1 -comprehension in which there is a failure of Σ_3^1 -comprehension. However, the best we were able to get is the following theorem.

THEOREM 1.2. *It is consistent that there is a model of $Z_2^{-p} + AC^{-p}$ together with Σ_2^1 -comprehension in which there is a failure of Σ_4^1 -comprehension.*

Theorem 1.3 is proved in Section 5.

A set existence principle closely related to the choice scheme is the collection scheme Coll, which asserts for every second-order formula $\varphi(n, X, A)$ (with set parameter A) that if for every n , there is a set X such that $\varphi(n, X, A)$ holds, then there is a set Y whose slices are a collecting set for the witnesses X . A slice of Y may not be one of the witnesses and a witness for n may be on some slice $m \neq n$. It is easy to see that over Z_2 , Coll is equivalent to AC. Over ACA_0 , Coll implies Z_2 , and hence, over ACA_0 , Coll is equivalent to AC. Let $Coll^{-p}$ be the parameter-free collection scheme. I separate the schemes $Coll^{-p}$ and AC^{-p} over Z_2^{-p} .

THEOREM 1.3. *It is consistent that there is a model of $Z_2^{-p} + Coll^{-p}$ together with Σ_2^1 -comprehension in which there is a failure of Σ_4^1 -comprehension and a failure of AC^{-p} .*

Theorem 1.3 is proved in Section 5. Thus, we get

THEOREM 1.4. *The schemes $Coll^{-p}$ and AC^{-p} are not equivalent over Z_2^{-p} .*

§2. Preliminaries.

2.1. Second-order arithmetic. Second-order arithmetic has two types of objects: numbers and sets of numbers, which we think of as the reals of the model. It is formalized in a two-sorted logic, which has two separate sorts of variables and corresponding quantifiers. By convention, we use lower-case variables for numbers and upper-case variables for sets. The language of second-order arithmetic consists of the language of PA together with a membership relation \in used to determine which numbers belong to which sets. A model of second-order arithmetic is a structure $\mathcal{M} = \langle M, +, \times, <, 0, 1, \in, \mathcal{S} \rangle$, where M is the universe of numbers and \mathcal{S} is the universe of sets. We associate each set A with a subset of M consisting of those $a \in M$ such that $a \in A$. This interpretation is formalized by the extensionality

²Recently Lyubetsky and Kanovei showed how to obtain these results involving AC^{-p} starting with a model of Z_2 instead of ZFC [7].

axiom for sets. The second-order arithmetic theories we consider include the PA axioms for the first-order part together with the extensionality axiom for sets and a single induction axiom

$$\forall X ((0 \in X \wedge \forall n (n \in X \rightarrow n + 1 \in X)) \rightarrow \forall n n \in X)$$

asserting that induction holds for every set. The theories then differ by which sets are posited to exist. The most common set existence axioms are the comprehension schemes. The Σ_n^1 -comprehension scheme, Σ_n^1 -CA, asserts that every Σ_n^1 -formula with parameters defines a set. Formally, Σ_n^1 -CA asserts for every Σ_n^1 -formula $\varphi(n, A)$, with set parameter A , that there is a set $X \in \mathcal{S}$ such that $X = \{n \mid \varphi(n, A)\}$.³ One of the strongest second-order arithmetic theories is full second-order arithmetic Z_2 , which includes the comprehension scheme asserting that every second-order formula with parameters defines a set, namely Σ_n^1 -CA for every $n < \omega$. If we exclude parameters from the comprehension scheme, we get the theory Z_2^p , which is the focus of this article.

Other common set existence principles are the set choice principles. The choice scheme AC asserts for every second-order formula $\varphi(n, X, A)$ that if $\forall n \exists X \varphi(n, X)$ holds, then there is a set Y such that $\forall n \varphi(n, Y_n, A)$ holds, where

$$Y_n = \{m \mid \langle m, n \rangle \in Y\}$$

is the n -th slice of Y . We will denote by AC^{-p} the parameter-free choice scheme. Closely related to the choice scheme is the collection scheme, which asserts for every second-order formula $\varphi(n, X, A)$ that if $\forall n \exists X \varphi(n, X)$ holds, then there is a set Y such that $\forall n \exists m \varphi(n, Y_m, A)$ holds. We will denote by $Coll^{-p}$ the parameter-free collection scheme.

2.2. Tree iterations of Jensen's forcing. Recall that a *perfect tree* is a subtree of $2^{<\omega}$ in which every node has a splitting node above it. Also recall that given a tree T and a node $s \in T$, we denote by T_s the subtree of T consisting of all nodes compatible with s . If T and S are perfect trees whose intersection includes a perfect tree, then there is a maximal such perfect tree, which we call the *meet* of T and S and denote by $T \wedge S$. A *perfect poset* \mathbb{P} is a collection of perfect trees ordered by the subtree relation satisfying the following closure properties:

- (1) $(2^{<\omega})_s \in \mathbb{P}$ for every $s \in 2^{<\omega}$,
- (2) if $T, S \in \mathbb{P}$, then $T \wedge S \in \mathbb{P}$,
- (3) finite unions of trees in \mathbb{P} are in \mathbb{P} .

Perfect posets are subsets of Sacks forcing and they share many of the properties of Sacks forcing such as that the generic filter is determined by a single generic real that is precisely the intersection of all the trees in the generic filter. While Sacks forcing is the largest perfect poset, the smallest perfect poset, which I will denote by \mathbb{P}_{\min} , is the closure under finite unions of the collection of all trees $(2^{<\omega})_s$ for some $s \in 2^{<\omega}$ (it is easy to see that this collection is closed under meets).

Given a perfect poset \mathbb{P} , let $\mathbb{Q}(\mathbb{P})$ be the poset whose elements are pairs (T, n) , where $T \in \mathbb{P}$ and $n < \omega$, ordered so that $(S, m) \leq (T, n)$ whenever S is a subtree

³A single set parameter suffices since we can code finitely many sets by a single set using Gödel's pairing function.

of T such that S and T agree up to level n and $m \geq n$. Forcing with $\mathbb{Q}(\mathbb{P})$ adds a generic perfect tree. Let $\mathbb{Q}(\mathbb{P})^{<\omega}$ be the finite-support ω -length product of the $\mathbb{Q}(\mathbb{P})$ and let $G \subseteq \mathbb{Q}(\mathbb{P})^{<\omega}$ be V -generic. In the forcing extension $V[G]$, let $\{\mathcal{T}_n \mid n < \omega\}$ be the ω -many generic perfect trees added by G . Finally, let \mathbb{P}^* be the closure of \mathbb{P} and $\{\mathcal{T}_n \mid n < \omega\}$ under finite meets and unions. In $V[G]$, the perfect poset \mathbb{P}^* has the following two key properties:

- (1) $\{\mathcal{T}_n \mid n < \omega\}$ is a maximal antichain,
- (2) every maximal antichain of \mathbb{P} from V remains maximal in \mathbb{P}^*

(see [2] for details). Let us now work in the constructible universe L . Let $\vec{D} = \langle D_\xi \mid \xi < \omega_1 \rangle$ be the canonical \diamond -sequence obtained by letting $D_{\alpha+1}$ be the least counterexample if one exists. Let us call a countable level L_α of L suitable if

$$L_\alpha \models \text{ZFC}^- + P(\omega) \text{ exists.}$$

For example, the collapse of any countable elementary substructure of L_{ω_2} is suitable. Observe also that, by canonicity of the \diamond -sequence \vec{D} , if L_α is suitable and $\delta = (\omega_1)^{L_\alpha}$, then $\vec{D} \upharpoonright \delta \in L_\alpha$. We will use suitable models L_α handed to us by \vec{D} to construct Jensen’s forcing \mathbb{J} in ω_1 -many steps as a continuous chain of length ω_1 of countable perfect posets. Let $\mathbb{P}_0 = \mathbb{P}_{\min}$. At limit ordinals, we will take unions. So suppose we have constructed a countable perfect poset \mathbb{P}_ξ . We let $\mathbb{P}_{\xi+1} = \mathbb{P}_\xi$ unless the following happens. Suppose that D_ξ codes a suitable model L_{α_ξ} such that $\mathbb{P}_\xi \in L_{\alpha_\xi}$ and $(\omega_1)^{L_{\alpha_\xi}} = \xi$. Then we let G_ξ be the L -least L_{α_ξ} -generic filter for $\mathbb{Q}(\mathbb{P}_\xi)^{<\omega}$ and let $\mathbb{P}_{\xi+1} = \mathbb{P}_\xi^*$ as constructed in $L_{\alpha_\xi}[G_\xi]$. Notice that by our observations above, we seal maximal antichains ensuring that the resulting union poset will have the ccc. We let $\mathbb{J} = \bigcup_{\xi < \omega_1} \mathbb{P}_\xi$. Jensen showed that the poset \mathbb{J} has the ccc and adds a unique generic real, whose singleton is Π^1_2 -definable [5]. Products and iterations of Jensen’s forcing also turned out to have “unique generics” properties.

Lyubetsky and Kanovei showed that in a forcing extension by a finite-support α -length product of Jensen’s forcing the only L -generic reals for \mathbb{J} are those explicitly added by the $(\alpha$ -many) slices of the generic filter [6].

As mentioned in footnote 1, a forcing with the same properties as \mathbb{J} , minus the low complexity of the generic singleton, can be constructed in any model of the \diamond -principle. In particular, since ccc posets of size ω_1 preserve the \diamond -principle, any forcing extension of L by \mathbb{J} continues to satisfy the \diamond -principle, and we can construct a Jensen-like poset again in it. In this way, we can make sense of iterating Jensen’s forcing and define finite n -length iterations \mathbb{J}_n of \mathbb{J} . The iterations \mathbb{J}_n are constructed, analogously to Jensen’s forcing, as a chain of length ω_1 of iterations $\mathbb{P}_n^\xi = \mathbb{Q}_0^\xi * \dot{\mathbb{Q}}_1^\xi * \dots * \dot{\mathbb{Q}}_{n-1}^\xi$ for $\xi < \omega_1$, where \mathbb{P}_0^ξ is a countable perfect poset, and for $i < n - 1$,

$$\mathbb{1} \Vdash \dot{\mathbb{Q}}_i^\xi \text{ is a countable perfect poset.}$$

$$\mathbb{P}_n^\xi \upharpoonright i$$

We start with \mathbb{P}_n^0 in which $\mathbb{Q}_0^0 = \mathbb{P}_{\min}$ and for $i < n - 1$, $\dot{\mathbb{Q}}_i^0 = \check{\mathbb{P}}_{\min}$. At stages $\xi + 1$ at which the canonical \diamond -sequence \vec{D} codes a suitable model L_{α_ξ} , we construct $\mathbb{P}_n^{\xi+1}$ in a generic extension of L_{α_ξ} (by a poset generalizing $\mathbb{Q}(\mathbb{P})^{<\omega}$ which adds ω -many generic perfect trees to every iterate in \mathbb{P}_n^ξ). As in the case of the construction of

Jensen’s forcing \mathbb{J} , every maximal antichain of \mathbb{P}_n^ξ remains maximal in \mathbb{J}_n (see [2] for details). Abraham showed that the finite iterations \mathbb{J}_n have the ccc and add a unique n -length generic sequence of reals [1, Lemmas 2.3 and 2.4].

In [2], together with Friedman and Kanovei, we studied the properties of the (non-linear) tree iterations of \mathbb{J} . Suppose that \mathcal{T} is a tree of height at most ω . We will call the following poset $\mathbb{P}(\mathbb{J}, \mathcal{T})$, the *tree iteration of \mathbb{J} along \mathcal{T}* . Elements of $\mathbb{P}(\mathbb{J}, \mathcal{T})$ are functions p from a finite subtree D_p of \mathcal{T} into $\bigcup_{n < \omega} \mathbb{J}_n$ such that for every node $s \in D_p$, $p(s) \in \mathbb{J}_{\text{len}(s)}$ and whenever $s \subseteq t$ in D_p , then $p(s) = p(t) \upharpoonright \text{len}(s)$. In other words, a condition is a tree isomorphic to a finite subtree of \mathcal{T} whose level n nodes are elements of the n -length iteration \mathbb{J}_n and the assignment is coherent in the sense that conditions on the lower nodes are restrictions of the conditions on the higher nodes. The elements are ordered so that $q \leq p$ if $D_p \subseteq D_q$ and for every node $s \in D_p$, $q(s) \leq p(s)$. A generic filter G for $\mathbb{P}(\mathbb{J}, \mathcal{T})$ adds a tree \mathcal{T}^G isomorphic to \mathcal{T} whose nodes on level n are mutually generic sequences for \mathbb{J}_n .

THEOREM 2.1 [2]. *Suppose that $\mathcal{T} = \omega^{<\omega}$ or $\mathcal{T} = \omega_1^{<\omega}$. Then the tree iteration $\mathbb{P}(\mathbb{J}, \mathcal{T})$ has the ccc.*

The tree iterations of \mathbb{J} also have a “uniqueness of generics” property generalizing Lyubetsky and Kanovei’s result for finite-support products of \mathbb{J} .

THEOREM 2.2 [2]. *Suppose that $\mathcal{T} = \omega^{<\omega}$ or $\mathcal{T} = \omega_1^{<\omega}$ and $G \subseteq \mathbb{P}(\mathbb{J}, \mathcal{T})$ is L -generic. In $L[G]$:*

- (1) *The only L -generic sequences of reals for \mathbb{J}_n are those coming from the nodes of \mathcal{T}^G on level n .*
- (2) *The collection of all L -generic n -length sequences of reals for \mathbb{J}_n is Π_2^1 -definable.*

The following two easy propositions will be useful in later arguments. Given a condition $p \in \mathbb{P}(\mathbb{J}, \mathcal{T})$ and a subtree T of \mathcal{T} , let $p \upharpoonright T$ be the restriction of p to the domain $D_p \cap T$.

PROPOSITION 2.3. *Suppose that T is a subtree of \mathcal{T} , $a \in \mathbb{P}(\mathbb{J}, T)$, $p \in \mathbb{P}(\mathbb{J}, \mathcal{T})$, and a is compatible with $p \upharpoonright T$ in $\mathbb{P}(\mathbb{J}, T)$, then a is compatible with p in $\mathbb{P}(\mathbb{J}, \mathcal{T})$.*

PROOF. Let $q \in \mathbb{P}(\mathbb{J}, T)$ be such that $q \leq a, p \upharpoonright T$. Define a condition q' with domain $D_{q'} = D_q \cup D_p$ as follows. If $s \in T$, then $q'(s) = q(s)$. If $s \in D_{q'} \setminus T$, then let i be largest such that $s \upharpoonright i \in D_p$ and let $q'(s)$ be $q(s \upharpoonright i)$ concatenated with the tail of $p(s)$ from i . It is easy to see that $q' \leq p, a$ in $\mathbb{P}(\mathbb{J}, \mathcal{T})$. ⊢

Suppose that $G \subseteq \mathbb{P}(\mathbb{J}, \mathcal{T})$ is L -generic. Given a subtree T of \mathcal{T} , let G_T consist of all conditions $p \in G$ with domain $D_p \subseteq T$.

PROPOSITION 2.4. *Suppose that T is a subtree of \mathcal{T} . Then G_T is L -generic for $\mathbb{P}(\mathbb{J}, T)$.*

PROOF. Suppose that $p \in G_T$ and $q \geq p$ in $\mathbb{P}(\mathbb{J}, T)$. Then $q \geq p$ in $\mathbb{P}(\mathbb{J}, \mathcal{T})$. So $q \in G$, and hence $q \in G_T$. Next, suppose that $p, q \in G_T$. Since $p, q \in G$, there is $r \in G$ such that $r \leq p, q$, and hence $r \upharpoonright T \leq p, q$. Observe that $r \upharpoonright T$ is compatible with every condition in G because r is compatible with every condition in G . Thus, $r \upharpoonright T \in G$, and hence $r \upharpoonright T \in G_T$. Finally, suppose that \mathcal{A} is a maximal antichain of $\mathbb{P}(\mathbb{J}, T)$. Let’s argue that \mathcal{A} is also a maximal antichain of $\mathbb{P}(\mathbb{J}, \mathcal{T})$. Let $p \in \mathbb{P}(\mathbb{J}, \mathcal{T})$.

Then there is $a \in \mathcal{A}$ that is compatible with $p \upharpoonright T$ in $\mathbb{P}(\mathbb{J}, T)$, but then a is also compatible with p in $\mathbb{P}(\mathbb{J}, T)$ by Proposition 2.3. Thus, there is $p \in G \cap \mathcal{A}$. This completes the argument that G_T is L -generic for $\mathbb{P}(\mathbb{J}, T)$. \dashv

2.3. Automorphisms of forcing notions. Suppose that \mathbb{P} is a partial order, π is an automorphism of \mathbb{P} , and $G \subseteq \mathbb{P}$ is V -generic. We can apply π recursively to any \mathbb{P} -name σ to obtain the \mathbb{P} -name $\pi(\sigma)$ defined by $\langle \tau, p \rangle \in \sigma$ if and only if $\langle \pi(\tau), \pi(p) \rangle \in \pi(\sigma)$. Standard arguments then show that for a condition $p \in \mathbb{P}$, $p \Vdash \varphi(\sigma)$ if and only if $\pi(p) \Vdash \varphi(\pi(\sigma))$. It is also easy to see that $\pi \restriction G$ is V -generic. We will use the following standard proposition later, and we include the proof here for completeness.

PROPOSITION 2.5. *Suppose that σ is a \mathbb{P} -name. Then $\pi(\sigma)_G = \sigma_{\pi^{-1} \restriction G}$.*

PROOF. We can assume recursively that we already showed the conclusion for all \mathbb{P} -names τ of rank less than σ . Suppose that $b \in \sigma_{\pi^{-1} \restriction G}$. Then there is a \mathbb{P} -name \dot{b} such that $b = \dot{b}_{\pi^{-1} \restriction G}$, $\langle \dot{b}, p \rangle \in \sigma$, and $p \in \pi^{-1} \restriction G$. Let $\pi^{-1}(\bar{p}) = p$. Then $\langle \pi(\dot{b}), \bar{p} \rangle \in \pi(\sigma)$ and $\bar{p} \in G$. Thus, $\pi(\dot{b})_G \in \pi(\sigma)_G$, and by our recursive assumption, since \dot{b} has rank less than σ , $\pi(\dot{b})_G = \dot{b}_{\pi^{-1} \restriction G}$, and so, $b \in \pi(\sigma)_G$. Reversing the steps of this argument shows that if $b \in \pi(\sigma)_G$, then $b \in \sigma_{\pi^{-1} \restriction G}$, verifying the equality. \dashv

§3. A model of $Z_2^P + \Sigma_2^1\text{-CA} + \neg \Sigma_3^1\text{-CA}$. In this section, we construct a model of $Z_2^P + \Sigma_2^1\text{-CA}$ in which $\Sigma_3^1\text{-CA}$ fails.

Let $G \subseteq \mathbb{P}(\mathbb{J}, \omega^{<\omega})$ be L -generic. We work in $L[G]$. Let $\dot{\varphi}$ be the canonical name for the isomorphism between $\omega^{<\omega}$ and the tree, added by G , whose nodes on level n are mutually generic n -length sequences of reals for \mathbb{J}_n .

Observe that any automorphism π of the tree $\omega^{<\omega}$ gives rise to the obvious corresponding automorphism π^* of the poset $\mathbb{P}(\mathbb{J}, \omega^{<\omega})$.

THEOREM 3.1. *Suppose that T is a finite subtree of $\omega^{<\omega}$. Then the only L -generic sequences for \mathbb{J}_n in $L[G_T]$ are the sequences $\dot{\varphi}_G(s)$ for $s \in T$.*

PROOF. Let $r \in L[G_T]$ be an n -length sequence of reals L -generic for \mathbb{J}_n . Then by Theorem 2.2 (1), $r = \dot{\varphi}_G(s)$ for some node s on level n of $\omega^{<\omega}$. Suppose towards a contradiction that $s \notin T$. Let $i \geq 1$ be least such that $s \upharpoonright i \notin T$. Let \dot{r} be a $\mathbb{P}(\mathbb{J}, T)$ -name for r . Let $p \in G$ be such that $p \Vdash \dot{r} = \dot{\varphi}(\dot{s})$. Fix any condition $q \leq p$. Let $t \in \omega^{<\omega}$ be a node on the same level as s such that $t \upharpoonright i - 1 = s \upharpoonright i - 1$ and $t \notin D_q \cup T \cup \{s\}$. Many such nodes t must exist since D_q and T are both finite. Let π be an automorphism of $\omega^{<\omega}$ which maps $(\omega^{<\omega})_s$ onto $(\omega^{<\omega})_t$, while fixing everything outside these subtrees. In particular, π fixes T and $\pi(s) = t$. Let $D_{\bar{q}} = D_q \cup \pi \restriction D_q$ and let \bar{q} be the condition defined so that for $a \in D_q$, $\bar{q}(a) = \bar{q}(\pi(a)) = q(a)$. In other words, conditions on nodes in $D_q \cap (\omega^{<\omega})_s$ are copied over to $(\omega^{<\omega})_t$. This means, in particular, that $\pi^*(\bar{q}) = \bar{q}$. We have just argued that such conditions \bar{q} are dense below p , and so some such $\bar{q} \in G$. Let $H = \pi^* \restriction G$, which is also L -generic for $\mathbb{P}(\mathbb{J}, \omega^{<\omega})$, and note that $\pi^*(\bar{q}) = \bar{q} \in H$. Observe that $\dot{r}_H = \dot{r}_G = r$ since the name \dot{r} only mentioned conditions with domain contained in T , and π fixes T . Since $p \geq \bar{q}$, $p \in H$. So it must be the case that $r = \dot{\varphi}_H(s)$. But this is impossible because $\dot{\varphi}_H(s) = \dot{\varphi}_G(t)$ and, by genericity, $\dot{\varphi}_G(s) \neq \dot{\varphi}_G(t)$. \dashv

It is not clear how to make the argument above work for infinite subtrees $T \in L$, and I suspect that the result will turn out to be false for these.

We now follow the ideas in [8] to define the following subtree \mathcal{T} of $\omega^{<\omega}$ in $L[G]$. For $1 \leq m < \omega$, let $\vec{0}_m$ denote the m -length sequence of zeroes. We have for every $n < \omega$ and $1 \leq m < \omega$:

- (1) $\langle n \rangle \in \mathcal{T}$,
- (2) $\langle n \rangle \widehat{\vec{0}}_m \in \mathcal{T}$,
- (3) $\langle n \rangle \widehat{\vec{0}}_{m+1} \widehat{\langle 1 \rangle} \in \mathcal{T}$ if and only if $\dot{\varphi}_G(\langle n, 1 \rangle)(1)(m) = 1$.

Condition (1) puts all level 1 nodes of $\omega^{<\omega}$ into \mathcal{T} . Condition (2) puts all left-most branches of $\omega^{<\omega}$ above level 1 nodes into \mathcal{T} . Condition (3) says that a node on level $m + 1$ of the left-most branch above $\langle n \rangle$ splits in \mathcal{T} if and only if the m th bit of the second real in the pair of reals $\dot{\varphi}_G(\langle n, 1 \rangle)$ is 1. Condition (3) is coding the reals $\dot{\varphi}_G(\langle n, 1 \rangle)(1)$ into the tree \mathcal{T} . Let \vec{T} be the canonical $\mathbb{P}(\mathbb{J}, \omega^{<\omega})$ -name for \mathcal{T} .

Let

$$\mathcal{S} = \{A \in P^{L[G_T]}(\omega) \mid T \subseteq \mathcal{T} \text{ is a finite tree}\}$$

be the union of the powersets of ω from all submodels $L[G_T]$ with T a finite subtree of \mathcal{T} , and let

$$\mathcal{M} = \langle \omega, +, \times, <, 0, 1 \in, \mathcal{S} \rangle$$

be a model of second-order arithmetic in $L[G]$. We will argue that the model \mathcal{M} has all the required properties.

PROPOSITION 3.2. $\mathcal{S} = \{A \in P^{L[\{\dot{\varphi}_G(s) \mid s \in X\}]}(\omega) \mid X \subseteq \mathcal{T} \text{ is finite}\}$.

PROOF. Suppose that $A \in P^{L[\{\dot{\varphi}_G(s) \mid s \in X\}]}(\omega)$ for some finite $X \subseteq \mathcal{T}$. Let $X \subseteq T$, where T is a finite subtree of \mathcal{T} . Then $A \in L[\{\dot{\varphi}_G(s) \mid s \in X\}] \subseteq L[G_T]$. Next, suppose that $A \in L[G_T]$ for some finite subtree T of \mathcal{T} . The generic G_T is clearly inter-definable with the collection $\{\dot{\varphi}_G(s) \mid s \in T\}$, which means that $A \in L[G_T] = L[\{\dot{\varphi}_G(s) \mid s \in T\}]$. \dashv

In other words, \mathcal{S} depends only on the collection $\vec{T} = \{\dot{\varphi}_G(s) \mid s \in \mathcal{T}\}$. Let $\dot{\mathcal{S}}$ be the canonical $\mathbb{P}(\mathbb{J}, \omega^{<\omega})$ -name for \mathcal{S} . Since, via coding, we can view L -generic sequences for \mathbb{J}_n as subsets of ω , we can talk about these sequences internally to the model \mathcal{M} . In particular, clearly $\vec{T} \subseteq \mathcal{S}$.

LEMMA 3.3. *For every n -length sequence of reals $r \in \mathcal{S}$, we have that r is L -generic for \mathbb{J}_n if and only if $r \in \vec{T}$. Hence $\dot{\varphi}_G(\langle 0, 1 \rangle) \notin \mathcal{S}$.*

PROOF. Suppose that $r \in \mathcal{S}$ is an n -length sequence L -generic for \mathbb{J}_n . Then $r \in L[G_T]$ for some finite tree $T \subseteq \mathcal{T}$. But then, by Theorem 3.1, $r = \dot{\varphi}_G(s)$ for some $s \in T$. Since $\langle 0, 1 \rangle \notin \mathcal{T}$, it follows that $\dot{\varphi}_G(\langle 0, 1 \rangle) \notin \mathcal{S}$. \dashv

Observe that any permutation $f \in L$ of ω gives rise to the automorphism π_f of the tree $\omega^{<\omega}$ which correspondingly permutes the trees $(\omega^{<\omega})_{\langle n \rangle}$ for $n < \omega$. More precisely $\pi_f(\langle n_0, n_1, \dots, n_{\text{len}(s)-1} \rangle) = \langle f(n_0), n_1, \dots, n_{\text{len}(s)-1} \rangle$. Given any two conditions $p, q \in \mathbb{P}(\mathbb{J}, \omega^{<\omega})$, there is a permutation $f \in L$ of ω such that p and $\pi_f^*(q)$ are compatible, namely, let f be any permutation such that for every $\langle n \rangle \in D_p \cap D_q$, $f(n) = m$ for some m such that $\langle m \rangle \notin D_p$. We will argue that the collection \vec{T} is not affected by applying automorphisms π_f^* to the generic filter G , and hence neither is the collection \mathcal{S} .

LEMMA 3.4. For any permutation $f \in L$ of ω , if $\pi_f^* \text{ " } G = H$, then $\dot{\varphi}_G \text{ " } \mathcal{T} = \dot{\varphi}_H \text{ " } \dot{\mathcal{T}}_H$, and hence $\mathcal{S} = \dot{\mathcal{S}}_H$.

PROOF. Suppose that $s = \langle n \rangle$ for some $n < \omega$ and let $r = \dot{\varphi}_G(s)$. Then $r = \dot{\varphi}_H(\langle f(n) \rangle)$ by the definition of π_f , and hence $r \in \dot{\varphi}_H \text{ " } \dot{\mathcal{T}}_H$. Next, suppose that $s = \langle n \rangle \widehat{\bar{0}}_m$ for some $n < \omega$ and $1 \leq m < \omega$, and $r = \dot{\varphi}_G(s)$. Then $r = \dot{\varphi}_H(\langle f(n) \rangle \widehat{\bar{0}}_m)$ by the definition of π_f , and hence $r \in \dot{\varphi}_H \text{ " } \dot{\mathcal{T}}_H$. Finally, suppose that $s = \langle n \rangle \widehat{\bar{0}}_{m+1} \widehat{\langle 1 \rangle} \in \mathcal{T}$ for some $n < \omega$ and $1 \leq m < \omega$, and $r = \dot{\varphi}_G(s)$. Then $\dot{\varphi}_G(\langle n, 1 \rangle)(1)(m) = 1$ by the definition of \mathcal{T} , and hence $\dot{\varphi}_H(\langle f(n), 1 \rangle)(1)(m) = 1$ by the definition of π_f . Thus, $r = \dot{\varphi}_G(\langle n \rangle \widehat{\bar{0}}_{m+1} \widehat{\langle 1 \rangle}) = \dot{\varphi}_H(\langle f(n) \rangle \widehat{\bar{0}}_{m+1} \widehat{\langle 1 \rangle}) \in \dot{\mathcal{T}}_H$ by the definition of $\dot{\mathcal{T}}$. Since f^{-1} is also a permutation of ω , the same argument shows that $\dot{\varphi}_H \text{ " } \dot{\mathcal{T}}_H \subseteq \dot{\varphi}_G \text{ " } \mathcal{T}$. Since \mathcal{S} depends only on $\dot{\varphi}_G \text{ " } \mathcal{T}$, it follows immediately that $\mathcal{S} = \dot{\mathcal{S}}_H$. ⊣

COROLLARY 3.5. For any permutation $f \in L$ of ω , $\mathbb{1}_{\mathbb{P}(\mathbb{J}, \omega^{<\omega})} \dot{\mathcal{S}} = \pi_f^*(\dot{\mathcal{S}})$.

PROOF. We need to show that for any L -generic filter \bar{G} for $\mathbb{P}(\mathbb{J}, \omega^{<\omega})$, $\dot{\mathcal{S}}_{\bar{G}} = \pi_f^*(\dot{\mathcal{S}})_{\bar{G}}$. By Lemma 3.4, we know that $\dot{\mathcal{S}}_{\bar{G}} = \dot{\mathcal{S}}_{\pi_{f^{-1}}^* \text{ " } \bar{G}}$. Since, clearly, $\pi_{f^{-1}} = \pi_f^{-1}$, we have $\dot{\mathcal{S}}_{\bar{G}} = \dot{\mathcal{S}}_{\pi_f^* \text{ " } \bar{G}}$. By Proposition 2.5, we have $\dot{\mathcal{S}}_{\bar{G}} = \dot{\mathcal{S}}_{\pi_f^* \text{ " } \bar{G}} = \pi_f^*(\dot{\mathcal{S}})_{\bar{G}}$. ⊣

LEMMA 3.6. The model $\mathcal{M} = \langle \omega, +, \times, <, 0, 1, \mathcal{S} \rangle \models \mathcal{Z}_2^p$.

PROOF. Suppose that $\psi(x)$ is a second-order arithmetic formula. We need to argue that

$$A = \{n \in \omega \mid \mathcal{M} \models \psi(x)\} \in \mathcal{S}.$$

Indeed, we will argue that $A \in L$ by showing that if $\mathcal{M} \models \psi(n)$ for some $n < \omega$, then

$$\mathbb{1}_{\mathbb{P}(\mathbb{J}, \omega^{<\omega})} \Vdash \langle \omega, +, \times, <, 0, 1, \dot{\mathcal{S}} \rangle \models \psi(n).$$

Suppose toward a contradiction that $p, q \in \mathbb{P}(\mathbb{J}, \omega^{<\omega})$ and $n < \omega$ such that

$$p \Vdash \langle \omega, +, \times, <, 0, 1, \dot{\mathcal{S}} \rangle \models \psi(n)$$

and

$$q \Vdash \langle \omega, +, \times, <, 0, 1, \dot{\mathcal{S}} \rangle \not\models \psi(n).$$

Let $f \in L$ be a permutation of ω such that p and $\pi_f^*(q)$ are compatible. We have

$$\pi_f^*(q) \Vdash \langle \omega, +, \times, <, 0, 1, \pi_f^*(\dot{\mathcal{S}}) \rangle \not\models \psi(n).$$

But since by Corollary 3.5, $\mathbb{1}_{\mathbb{P}(\mathbb{J}, \omega^{<\omega})} \Vdash \dot{\mathcal{S}} = \pi_f^*(\dot{\mathcal{S}})$,

$$\pi_f^*(q) \Vdash \langle \omega, +, \times, <, 0, 1, \dot{\mathcal{S}} \rangle \not\models \psi(n).$$

Let $r \leq p, \pi_f^*(q)$. Then

$$r \Vdash \langle \omega, +, \times, <, 0, 1, \dot{\mathcal{S}} \rangle \models \psi(n)$$

and

$$r \Vdash \langle \omega, +, \times, <, 0, 1, \dot{\mathcal{S}} \rangle \not\models \psi(n),$$

which is the desired contradiction. \dashv

LEMMA 3.7. *The collection $\bar{\mathcal{T}} = \{\dot{\varphi}_G(s) \mid s \in \mathcal{T}\}$ is definable in \mathcal{M} by a Π_2^1 -formula.*

PROOF. By Lemma 3.3, for every n -length sequence $r \in \mathcal{S}$, $r \in \bar{\mathcal{T}}$ if and only if r is an n -length sequence of reals L -generic for some \mathbb{J}_n . Thus, it suffices to argue that \mathcal{M} can recognize in a Π_2^1 -way whether an n -length sequence of reals is L -generic for \mathbb{J}_n . Recall, from Section 2.2, that the iteration \mathbb{J}_n is constructed as the union of the ω_1 -length chain of iterations \mathbb{P}_n^ξ for $\xi < \omega_1$, and every maximal antichain of \mathbb{P}_n^ξ remains maximal in \mathbb{J}_n (for the precise result, see Lemma 6.11 (4) in [2]). Thus, any n -length L -generic sequence of reals for \mathbb{J}_n is also L_{α_ξ} -generic for \mathbb{P}_n^ξ , where $\xi + 1$ is a non-trivial stage in the construction of \mathbb{J}_n . Since each iteration \mathbb{J}_n has the ccc, the converse holds as well: if an n -length sequence of reals is L_{α_ξ} -generic for \mathbb{P}_n^ξ for all non-trivial stages $\xi + 1$ in the construction of \mathbb{J}_n , then it is fully L -generic for \mathbb{J}_n . Next, observe that whether an n -length sequence of reals r is L_{α_ξ} -generic for \mathbb{P}_n^ξ can be verified in any $L_\alpha[r] \models \text{ZFC}^-$ with $\alpha > \xi$. Notice also that any $L_\alpha \models \text{ZFC}^-$ can verify whether $\xi + 1 < \alpha$ is a non-trivial stage in the construction of \mathbb{J}_n . Putting it all together, we get that an n -length sequence r of reals is L -generic for \mathbb{J}_n if and only if for every $\alpha < \omega_1$ such that $L_\alpha[r] \models \text{ZFC}^-$, if $\xi < \alpha$ is such that $\xi + 1$ is a non-trivial stage in the construction of \mathbb{J}_n , then $L_\alpha[r]$ satisfies that r is L_{α_ξ} -generic for \mathbb{P}_n^ξ . Because \mathcal{S} is the union powersets of ω for models of the form $L[A]$, given any $r \in \mathcal{S}$, \mathcal{S} has (a code for) $L_\alpha[r]$ for any countable α and $r \in \mathcal{S}$. Thus, an n -length sequence $r \in \mathcal{S}$ of reals is L -generic for \mathbb{J}_n if and only if \mathcal{M} satisfies that for every class X, Y, Z , if X is a well-order and Y codes $L_X[r]$ and Z is a truth predicate for $L_X[r]$ and $L_X[r] \models \text{ZFC}^-$ (according to Z), then r is L_{α_ξ} -generic for \mathbb{P}_n^ξ for every non-trivial successor stage $\xi + 1$ in L_X . The formula is Π_2^1 because being a well-order is Π_1^1 . \dashv

LEMMA 3.8. *$\dot{\varphi}_G(\langle 0, 1 \rangle)$ is definable in \mathcal{M} from the parameter $\dot{\varphi}_G(\langle 0 \rangle)$ by a Σ_3^1 -formula. Thus, Σ_3^1 -CA fails in \mathcal{M} .*

PROOF. We have that $m \in \dot{\varphi}_G(\langle 0, 1 \rangle)(1)$ if and only if there are two L -generic $m + 1$ -length sequences of reals generic for \mathbb{J}_{m+1} whose first coordinate is $\dot{\varphi}_G(\langle 0 \rangle)$. By Lemma 3.7, this is a Σ_3^1 -formula with parameter $\dot{\varphi}_G(\langle 0 \rangle)$. Since $\dot{\varphi}_G(\langle 0, 1 \rangle) \notin \mathcal{S}$, Σ_3^1 -CA fails in \mathcal{M} . \dashv

LEMMA 3.9. *Σ_2^1 -CA holds in \mathcal{M} .*

PROOF. Since \mathcal{S} is clearly closed under complement, it suffices to show that Π_2^1 -CA holds in \mathcal{M} . Let $A \in \mathcal{S}$ and fix a formula $\psi(x, A) := \forall X \exists Y \varphi(x, X, Y, A)$, where φ is a first-order formula. We will argue that

$$\{n \in \omega \mid L[A] \models \psi(n, A)\} = \{n \in \omega \mid \mathcal{M} \models \psi(n, A)\}.$$

This suffices since if $A \in \mathcal{S}$, then $A \in L[G_T]$ for some finite subtree T of \mathcal{T} , and hence the set $\{n \in \omega \mid L[A] \models \psi(n, A)\}$ is in $L[G_T]$ as well.

Suppose $L[A] \models \psi(n, A)$ for some $n < \omega$. Let $C \in \mathcal{S}$. By Shoenfield's absoluteness,

$$L[A, C] \models \psi(n, A).$$

Thus, there is $B \in L[A, C]$ such that

$$L[A, C] \models \varphi(n, C, B, A).$$

Since $A, C \in \mathcal{S}$, $P^{L[A, C]}(\omega) \subseteq \mathcal{S}$, and so $B \in \mathcal{S}$. Thus, $\mathcal{M} \models \varphi(n, C, B, A)$. Since $C \in \mathcal{S}$ was arbitrary, $\mathcal{M} \models \psi(n, A)$. Next, suppose that $\mathcal{M} \models \psi(n, A)$. Fix $C \in L[A]$. Then $\mathcal{M} \models \exists Y \varphi(n, C, Y, A)$. Fix $B \in \mathcal{S}$ such that $\mathcal{M} \models \varphi(n, C, B, A)$. Then $L[A, C, B] \models \varphi(n, C, B, A)$, and so $L[A, C, B] \models \exists Y \varphi(n, C, Y, A)$. But then by Shoenfield's absoluteness, $L[A] \models \exists Y \varphi(n, C, Y, A)$. Since $C \in L[A]$ was arbitrary, $L[A] \models \psi(n, A)$. \dashv

This finishes the argument that $\mathcal{M} \models Z_2^p + \Sigma_2^1\text{-CA} + \neg\Sigma_3^1\text{-CA}$. Finally, let's observe that the parameter-free collection scheme fails in \mathcal{M} .

PROPOSITION 3.10. *Coll^{-p} fails in \mathcal{M} .*

PROOF. The model \mathcal{M} satisfies that for every n , there is an n -length L -generic sequence of reals for \mathbb{J}_n . But there cannot be a single set $X \in \mathcal{S}$ collecting on its slices n -length L -generic sequences of reals for \mathbb{J}_n for every $n < \omega$ because X has to come from some $L[G_T]$, where T is a finite tree. Since T is finite, by the uniqueness Theorem 3.1, there is some $n < \omega$ such that $L[G_T]$ cannot have an L -generic sequence of reals for \mathbb{J}_n . \dashv

§4. A model of $Z_2^{-p} + \text{Coll}^{-p} + \Sigma_2^1\text{-CA} + \neg\Sigma_4^1\text{-CA} + \neg\text{AC}^{-p}$. In this section, we construct a model of $Z_2^p + \text{Coll}^{-p} + \Sigma_2^1\text{-CA}$ in which $\Sigma_4^1\text{-CA}$ fails and AC^{-p} fails. We do this by using a more complicated version of the construction in Section 3 working in a forcing extension of L by $\mathbb{P}(\mathbb{J}, \omega_1^{<\omega})$ instead of $\mathbb{P}(\mathbb{J}, \omega^{<\omega})$.

Let $G \subseteq \mathbb{P}(\mathbb{J}, \omega_1^{<\omega})$ be L -generic. We work in $L[G]$. Let $\dot{\varphi}$ be the canonical name for the isomorphism between $\omega_1^{<\omega}$ and the tree, added by G , whose nodes on level n are mutually generic sequences of reals for \mathbb{J}_n .

We start with the following generalization of Theorem 3.1.

THEOREM 4.1. *Suppose that $T \in L$ that is a countable subtree of $\omega_1^{<\omega}$. Then the only L -generic sequences for \mathbb{J}_n in $L[G_T]$ are the sequences $\dot{\varphi}_G(s)$ for $s \in T$.*

The proof is analogous to the proof of Theorem 3.1, using that $\omega_1^{<\omega}$ is uncountable and the tree T is countable to have enough room to find the required node t for the automorphism.

We define the following subtree \mathcal{T} of $\omega_1^{<\omega}$ in $L[G]$. We have for every $\xi < \omega_1$, $1 \leq m < \omega$, $n < \omega$:

- (1) $\langle \xi \rangle \in \mathcal{T}$,
- (2) $\langle \xi \rangle \widehat{\sim} \vec{0}_m \in \mathcal{T}$,
- (3) $\langle \xi \rangle \widehat{\sim} \vec{0}_{m+1} \widehat{\sim} \langle n \rangle \in \mathcal{T}$,
- (4) $\langle \xi \rangle \widehat{\sim} \vec{0}_{m+1} \widehat{\sim} \langle \eta \rangle \in \mathcal{T}$ for all $\eta < \omega_1$ if and only if $\dot{\varphi}_G(\langle \xi, 1 \rangle)(1)(m) = 1$.

Conditions (3) and (4) say that whether a node $\langle \xi \rangle \widehat{\vec{0}}_{m+1}$ of a left-most branch above a level 1 node splits into ω -many or ω_1 -many nodes depends on whether $\dot{\varphi}_G(\langle \xi, 1 \rangle)(1)(m) = 1$.

Let

$$S = \{A \in P^{L[G_T]}(\omega) \mid T \subseteq \mathcal{T} \text{ is a countable tree and } T \in L\}$$

be the union of the powersets of ω from all submodels $L[G_T]$ with T a countable subtree of \mathcal{T} from L , and let

$$\mathcal{M} = \langle \omega, +, \times, <, 0, 1 \in, S \rangle.$$

We will argue that the model \mathcal{M} has all the required properties. As before, let $\vec{T} = \{\dot{\varphi}_G(s) \mid s \in \mathcal{T}\}$ and let \dot{S} be the canonical $\mathbb{P}(\mathbb{J}, \omega_1^{<\omega})$ -name for S . Using Theorem 4.1, we show

LEMMA 4.2. *For every n -length sequence of reals $r \in S$, we have that r is L -generic for \mathbb{J}_n if and only if $r \in \vec{T}$. Hence $\dot{\varphi}_G(\langle 0, 1 \rangle) \notin S$.*

Observe that any permutation $f \in L$ of ω_1 gives rise to the automorphism π_f of the tree $\omega_1^{<\omega}$ which correspondingly permutes the trees $(\omega_1^{<\omega})_{\langle \xi \rangle}$ for $\xi < \omega_1$. More precisely $\pi_f(\langle \xi_0, \xi_1, \dots, \xi_{\text{len}(s)-1} \rangle) = \langle f(\xi_0), \xi_1, \dots, \xi_{\text{len}(s)-1} \rangle$. As before, given any two conditions $p, q \in \mathbb{P}(\mathbb{J}, \omega_1^{<\omega})$, there is a permutation $f \in L$ of ω_1 such that p and $\pi_f^*(q)$ are compatible. We will argue that the collection S is not affected by applying automorphisms π_f^* to the generic filter G .

LEMMA 4.3. *For any permutation $f \in L$ of ω_1 , we have $S = \dot{S}_{\pi_f^*} \upharpoonright_G$. Thus, $\mathbb{1}_{\mathbb{P}(\mathbb{J}, \omega_1^{<\omega})} \Vdash \dot{S} = \pi_f^*(\dot{S})$.*

PROOF. Let $\pi_f^* \upharpoonright G = H$. Suppose that $T \in L$ is a countable subtree of \mathcal{T} . Let $T' = \pi_f \upharpoonright T$. Let's argue that $T' \subseteq \vec{T}_H$. It suffices to focus on nodes $s = \langle \xi \rangle \widehat{\vec{0}}_{m+1} \widehat{\langle \eta \rangle}$, where $\eta \notin \omega$. So suppose $s \in T$. It follows that $\dot{\varphi}_G(\langle \xi, 1 \rangle)(1)(m) = 1$, and so $\dot{\varphi}_H(\langle f(\xi), 1 \rangle)(m) = 1$, which means that $\pi_f(s) = \langle f(\xi) \rangle \widehat{\vec{0}}_{m+1} \widehat{\langle \eta \rangle} \in \vec{T}_H$. Since f^{-1} is also a permutation of ω_1 , we have that if $T' \in L$ is a countable subtree of \vec{T}_H , then $\pi_{f^{-1}} \upharpoonright T' = T$ is a subtree of \mathcal{T} . Next, observe that given such trees T and T' , we have that $L[G_T] = L[H_{T'}]$ because the trees T and T' are isomorphic by π_f , making the posets $\mathbb{P}(\mathbb{J}, T)$ and $\mathbb{P}(\mathbb{J}, T')$ isomorphic by π_f^* , and the isomorphism takes G_T to $H_{T'}$. It follows that $S = \dot{S}_{\pi_f^*} \upharpoonright_G$. \dashv

Analogs of Lemma 3.6 and Lemma 3.9 in Section 3 give that \mathcal{M} satisfies $Z_2^p + \Sigma_2^1$ -CA.

LEMMA 4.4. *$\dot{\varphi}_G(\langle 0, 1 \rangle)$ is definable in \mathcal{M} from the parameter $\dot{\varphi}_G(\langle 0 \rangle)$ by a Π_4^1 -formula. Thus, Σ_4^1 -CA fails in \mathcal{M} .*

PROOF. Recall that for $m < \omega$, $\dot{\varphi}_G(\langle 0, 1 \rangle)(1)(m) = 1$ if and only if \mathcal{T} splits ω_1 -many times above the node $\langle 0 \rangle \widehat{\vec{0}}_m$. In this case, S does not have a maximal set of L -generic sequences for \mathbb{J}_{m+1} whose first coordinate is $\dot{\varphi}_G(\langle 0 \rangle)$. More formally, there is no set $A \in S$ such that each slice A_n is an L -generic $m + 1$ -length sequence for \mathbb{J}_{m+1} whose first coordinate is $\dot{\varphi}_G(\langle 0 \rangle)$ and whenever R is an L -generic $m + 1$ -length sequence for \mathbb{J}_{m+1} whose first coordinate is $\dot{\varphi}_G(\langle 0 \rangle)$, then $R = A_n$ for some

$n < \omega$. The assertion that each slice A_n is an L -generic $m + 1$ -length sequence for \mathbb{J}_{m+1} whose first coordinate is $\dot{\varphi}_G(\langle 0 \rangle)$ is Π_2^1 and the assertion that every L -generic $m + 1$ -length sequence R for \mathbb{J}_{m+1} , whose first coordinate is $\dot{\varphi}_G(\langle 0 \rangle)$, is A_n for some $n < \omega$ is Π_3^1 . Thus, the assertion that A is maximal as above is Π_3^1 , and therefore, the assertion that there is no such A is Π_4^1 . \dashv

Next, we verify that $\mathcal{M} \models \text{Coll}^p$.

We will call a countable subtree $T \in L$ of \mathcal{T} full if whenever it contains a node $\langle \xi \rangle$, then it contains all nodes $\langle \xi \rangle \frown \vec{0}_m$ for $1 \leq m < \omega$ and all nodes $\langle \xi \rangle \frown \vec{0}_{m+1} \frown \langle n \rangle$ for $n < \omega$ and $1 \leq m < \omega$. We will call a countable tree $T \in L$ undecided if the only nodes in T are of the form $\langle \xi \rangle$ for some $\xi < \omega_1$ or $\langle \xi \rangle \frown \vec{0}_m$ for some $m < \omega$ or $\langle \xi \rangle \frown \vec{0}_{m+1} \frown \langle n \rangle$ for some $n < \omega$ and $1 \leq m < \omega$. Observe that every undecided tree T is automatically contained in \mathcal{T} . As the name suggests, we cannot tell from an undecided tree whether $\dot{\varphi}_G(\langle \xi, 1 \rangle)(1)(m) = 1$ for any $\xi < \omega_1$ and $m < \omega$.

LEMMA 4.5. *Suppose that $T \in L$ is a full countable subtree of \mathcal{T} , $\varphi(x, X)$ is a second-order formula, and there is $A \in L[G_T]$ such that $\mathcal{M} \models \varphi(n, A)$. Then there is a full undecided countable tree $\tilde{T} \in L$ and $\tilde{A} \in L[G_{\tilde{T}}]$ such that $\mathcal{M} \models \varphi(n, \tilde{A})$.*

PROOF. Let $p \in G$ be such that

$$p \Vdash \exists X \in L[\dot{G}_T] \langle \omega, +, \times, <, 0, 1, \dot{S} \rangle \models \varphi(n, X).$$

Let's argue that there can only be finitely many $\xi < \omega_1$ such that $\langle \xi \rangle \frown \vec{0}_{m+1} \frown \langle \eta \rangle \in T$ for some $\eta \notin \omega$. Since $T \subseteq \mathcal{T}$, there is some condition $q \in G$ such that $q \Vdash T \subseteq \dot{\mathcal{T}}$. Thus, for every node $\langle \xi \rangle \frown \vec{0}_{m+1} \frown \langle \eta \rangle \in T$ with $\eta \notin \omega$,

$$q \Vdash \dot{\varphi}_G(\langle \xi, 1 \rangle)(1)(m) = 1.$$

But this is clearly only possible for ξ such that $\langle \xi \rangle \in D_q$, and D_q is finite. So let $\{\xi_0, \dots, \xi_{n-1}\}$ be a finite set such that if $\langle \xi \rangle \frown \vec{0}_{m+1} \frown \langle \eta \rangle \in T$ with $\eta \notin \omega$, then $\xi = \xi_i$ for some $i < n$. By strengthening the condition p , we can assume that

- (1) $\{\langle \xi_i \rangle \mid i < n\}$ are exactly the level 1 nodes in D_p ,
- (2) for $i < n$, $\langle \xi_i, 1 \rangle \in D_p$,
- (3) for every $\langle \xi_i \rangle \frown \vec{0}_{m+1} \frown \langle \eta \rangle \in T$ with $\eta \notin \omega$, $p(\langle \xi_i \rangle) \Vdash p(\langle \xi_i, 1 \rangle)(1)(m) = 1$.

Fix some $\vec{\eta} = \{\eta_i \mid i < n\}$ such that for $i < n$, $\langle \eta_i \rangle \notin T \cup \{\langle \xi_i \rangle \mid i < n\}$, which is possible since T is countable. Let $f : \omega_1 \rightarrow \omega_1$ such that $f(\xi_i) = \eta_i$, $f(\eta_i) = \xi_i$ for $i < n$ and $f(\xi) = \xi$ otherwise. For each $i < n$ and $1 \leq m < \omega$, let $f_{i,m} : \omega_1 \rightarrow \omega_1$ be a bijection which maps the set

$$\{\eta < \omega_1 \mid \langle \xi_i \rangle \frown \vec{0}_{m+1} \frown \langle \eta \rangle \in T\}$$

onto ω . Let $\pi_{\vec{\eta}}$ be the following automorphism of $\omega_1^{<\omega}$. If $t = \langle \alpha_0, \alpha_1, \dots, \alpha_n \rangle$ does not end-extend a node $s = \langle \alpha \rangle \frown \vec{0}_{m+1} \frown \langle \eta \rangle$ with $\alpha = \xi_i$ or $\alpha = \eta_i$ for some $i < n$, then $\pi_{\vec{\eta}}(t) = \langle f(\alpha_0), \alpha_1, \dots, \alpha_n \rangle$. Suppose $s = \langle \alpha \rangle \frown \vec{0}_{m+1} \frown \langle \eta \rangle$, where $\alpha = \xi_i$ or $\alpha = \eta_i$. If $\alpha = \xi_i$, then $\pi_{\vec{\eta}}(s) = \langle f(\alpha) \rangle \frown \vec{0}_{m+1} \frown \langle f_{i,m}(\eta) \rangle$ and if $\alpha = \eta_i$, then $\pi_{\vec{\eta}}(s) = \langle f(\alpha) \rangle \frown \vec{0}_{m+1} \frown \langle f_{i,m}^{-1}(\eta) \rangle$. To summarize, $\pi_{\vec{\eta}}$ switches the tree $(\omega_1^{<\omega})_{\xi_i}$ with the tree $(\omega_1^{<\omega})_{\eta_i}$ for $i < n$, keeping everything the same other than mapping nodes of the form $s = \langle \xi_i \rangle \frown \vec{0}_{m+1} \frown \langle \eta \rangle$ to nodes of the form $t = \langle \eta_i \rangle \frown \vec{0}_{m+1} \frown \langle \vec{\eta} \rangle$ such that if $s \in T$, then $\vec{\eta} < \omega$. This ensures that the image of T under $\pi_{\vec{\eta}}$ is an undecided tree.

By density, we can find such a set of nodes $\bar{\eta}$ and a condition $q \in G$ with $D_q = D_p \cup \pi_{\bar{\eta}} \text{'' } D_p$ such that for nodes $s \in D_p$, $q(s) = p(s)$ and $q(\pi_{\bar{\eta}}(s)) = p(s)$. In particular, $q \leq p$. By the definition of $\pi_{\bar{\eta}}$, this means that $\pi_{\bar{\eta}}^*(q) = q$. Let $\pi_{\bar{\eta}}^* \text{'' } G = H$. It is easy to see that if $S \in L$ is a subtree of \mathcal{T} , then $S' = \pi_{\bar{\eta}} \text{'' } S$ is a subtree of $\dot{\mathcal{T}}_H$. So let's argue that if $S' \in L$ is a subtree of $\dot{\mathcal{T}}_H$, then $S = \pi_{\bar{\eta}}^{-1} \text{'' } S'$ is a subtree of \mathcal{T} . The main concern is to show that if $\dot{\varphi}_H(\langle \eta_i, 1 \rangle)(m)(1) = 0$, then for every $t = \langle \eta_i \rangle \hat{\ } \bar{0}_{m+1} \hat{\ } \langle n \rangle \in S'$ with $n < \omega$, $\pi_{\bar{\eta}}^{-1}(t) = \langle \xi_i \rangle \hat{\ } \bar{0}_{m+1} \hat{\ } \langle k \rangle$ with $k < \omega$. But this is true because, according to $\pi_{\bar{\eta}}$, the only situation in which k might end up outside of ω is when $\pi_{\bar{\eta}}^{-1}(t) \in T$, and in this case, by our assumption on p , $q(\langle \eta_i \rangle) = p(\langle \xi_i \rangle) \Vdash p(\langle \xi_i, 1 \rangle)(1)(m) = 1$, and $p(\langle \xi_i, 1 \rangle) = q(\langle \eta_i, 1 \rangle)$, which means that $\dot{\varphi}_H(\langle \eta_i, 1 \rangle)(1)(m) = 1$.

Thus, by our usual arguments, the automorphism $\pi_{\bar{\eta}}^*$ does not affect \mathcal{S} . Next, observe that $\pi_{\bar{\eta}}^* \text{'' } T = T'$ is a full undecided subtree of $\dot{\mathcal{T}}_H$. Since $q \leq p$,

$$q \Vdash \exists X \in L[\dot{G}_T] \langle \omega, +, \times, <, 0, 1, \dot{S} \rangle \Vdash \varphi(n, X).$$

Since $q = \pi_{\bar{\eta}}^*(q) \in H$ and $\pi_{\bar{\eta}}^*$ does not affect \mathcal{S} , there is $A \in L[H_T]$ such that $\mathcal{M} \Vdash \varphi(n, A)$. But now since T is isomorphic to T' via $\pi_{\bar{\eta}}$, the posets $\mathbb{P}(\mathbb{J}, T)$ and $\mathbb{P}(\mathbb{J}, T')$ are isomorphic via $\pi_{\bar{\eta}}^*$, which means that $L[H_T] = L[G_{T'}]$. Since T' is undecided, we have proved what we promised. \dashv

LEMMA 4.6. $\mathcal{M} \Vdash \text{Coll}^{-p}$.

PROOF. Suppose that $\mathcal{M} \Vdash \forall n \exists X \varphi(n, X)$ for some second-order arithmetic formula $\varphi(n, X)$. Given $n < \omega$, by Lemma 4.5, there is a full undecided countable subtree $T_n \in L$ and $X \in L[G_{T_n}]$ such that $\mathcal{M} \Vdash \varphi(n, X)$. By extending T_n to a larger tree, if necessary, we can assume that $\{\xi < \omega_1 \mid \langle \xi \rangle \in T_n\}$ is an ordinal α_n . Let $\alpha = \bigcup_{n < \omega} \alpha_n$ and let T be the full undecided tree whose level 1 nodes are precisely $\langle \xi \rangle$ for $\xi < \alpha$. Then each T_n is contained in T . Thus, for every $n < \omega$, there is some $X \in L[G_T]$ such that $\mathcal{M} \Vdash \varphi(n, X)$.

Since $\mathbb{P}(\mathbb{J}, \omega_1^{<\omega})$ has the ccc by Theorem 2.1, all $\mathbb{P}(\mathbb{J}, \omega_1^{<\omega})$ -nice names for reals are in L_{ω_1} . Thus, there is a countable ordinal β such that L_β already has, for every $n < \omega$, a $\mathbb{P}(\mathbb{J}, T)$ -name \dot{X}_n such that $\mathcal{M} \Vdash \varphi(n, (\dot{X}_n)_G)$. It follows that $\{\dot{X}_{G_T} \mid \dot{X} \in L_\beta \text{ is a } \mathbb{P}(\mathbb{J}, T)\text{-name}\}$ is a countable set of reals in $L[G_T]$ having a witness to $\mathcal{M} \Vdash \varphi(n, X)$ for every $n < \omega$, and this set is in \mathcal{S} . \dashv

This finishes the argument that $\mathcal{M} \Vdash \mathbf{Z}_2^p + \text{Coll}^{-p} + \Sigma_2^1\text{-CA} + \neg \Sigma_4^1\text{-CA}$.

Finally, let's observe that AC^{-p} fails in \mathcal{M} .

LEMMA 4.7. AC^{-p} fails in \mathcal{M} .

PROOF. The model \mathcal{M} satisfies that for every $n < \omega$, there is an L -generic real Y for \mathbb{J} and $s \in 2^n$ such that for $m < n$, $s(m) = 1$ if and only if for every set coding ω -many $m + 1$ -length L -generic sequences for \mathbb{J}_{m+1} with Y as the first coordinate, there is another $m + 1$ -length L -generic sequence for \mathbb{J}_{m+1} not in the set. In other words, $Y = \dot{\varphi}_G(\langle \xi \rangle)$ for some $\xi < \omega_1$ and $s = \dot{\varphi}_G(\langle \xi, 1 \rangle) \upharpoonright n$.

Suppose that $\mathcal{M} \Vdash \text{AC}^{-p}$. Then \mathcal{S} has a set A such that A_n codes Y_n and s_n witnessing the statement above. Let $T \in L$ be a countable subtree of \mathcal{T} such that $A \in L[G_T]$. First, let's observe that there cannot be cofinally many $n < \omega$ such that

A_n codes the same real Y because if this was the case and $Y = \dot{\varphi}_G(\langle \xi \rangle)$, then using Σ_2^1 -CA, we could argue that $\dot{\varphi}_G(\langle \xi, 1 \rangle) \in \mathcal{S}$. Thus, there are countably many different reals Y coded on the slices of A . Next, we fix a condition p and a $\mathbb{P}(\mathbb{J}, T)$ -name \dot{A} such that p forces that for all $n < \omega$, \dot{A}_n codes, for some $\xi < \omega_1$, $\dot{\varphi}(\langle \xi \rangle)$ and $\dot{\varphi}(\langle \xi, 1 \rangle) \upharpoonright n$. Choose some $Y_n = \dot{\varphi}_G(\langle \alpha \rangle)$ with $\langle \alpha \rangle \notin D_p$, which is possible since there are infinitely many different Y and p is finite. Next choose some $1 \leq \eta < \omega_1$ such that $\dot{\varphi}_G(\langle \alpha, \eta \rangle) \upharpoonright n \neq \dot{\varphi}_G(\langle \alpha, 1 \rangle) \upharpoonright n$, which must exist by density. Let π be an automorphism of $\omega_1^{<\omega}$ which switches the trees $(\omega_1^{<\omega})_{\langle \alpha, 1 \rangle}$ and $(\omega_1^{<\omega})_{\langle \alpha, \eta \rangle}$, while preserving everything else, and note that π fixes T and p by our choice of α and η . Let $H = \pi^* \text{'' } G$, and note that $\pi^*(p) = p \in H$. Also, $\pi^*(p) = p$ forces that for all $n < \omega$, $\pi^*(\dot{A})_n = \dot{A}_n$ codes, for some $\xi < \omega_1$, $\dot{\varphi}(\langle \xi \rangle)$ and $\dot{\varphi}(\langle \xi, 1 \rangle) \upharpoonright n$. Now we have that $\dot{A}_G = \dot{A}_H$, but $\dot{\varphi}_H(\langle \alpha, 1 \rangle) = \dot{\varphi}_G(\langle \alpha, \eta \rangle)$, and, by construction, $\dot{\varphi}_G(\langle \alpha, 1 \rangle) \upharpoonright n \neq \dot{\varphi}_G(\langle \alpha, \eta \rangle) \upharpoonright n$. Thus, we have reached a contradiction, showing that such a set A cannot exist. \dashv

COROLLARY 4.8. *The schemes AC^{-p} and $Coll^{-p}$ are not equivalent over Z_2^{-p} or even over $Z_2^{-p} + \Sigma_2^1$ -CA.*

§5. A model of $Z_2^{-p} + AC^{-p} + \Sigma_2^1$ -CA + $\neg\Sigma_4^1$ -CA. In this section, we construct a model of $Z_2^{-p} + AC^{-p} + \Sigma_2^1$ -CA in which Σ_4^1 -CA fails. We will again be working in a forcing extension of L by $\mathbb{P}(\mathbb{J}, \omega_1^{<\omega})$, but we are going to change the tree \mathcal{T} to code in a different set to witness the failure of comprehension with parameters in our model.

Let $G \subseteq \mathbb{P}(\mathbb{J}, \omega_1^{<\omega})$ be L -generic. We work in $L[G]$. Let $\dot{\varphi}$ be the canonical name for the isomorphism between $\omega_1^{<\omega}$ and the tree, added by G , whose nodes on level n are mutually generic sequences of reals for \mathbb{J}_n .

In this construction, we are going to define a tree \mathcal{T} similarly to how we defined it in the previous sections, but instead of coding, for every $\xi < \omega_1$, the sets $\dot{\varphi}_G(\langle \xi, 1 \rangle)(1)$ into \mathcal{T} , we are going to code in, for every $\xi < \omega_1$, the finite fragments $\dot{\varphi}_G(\langle \xi, n \rangle)(1) \upharpoonright n$ for $n < \omega$. Let p_n , for $n < \omega$, denote the n -th prime. For $n < \omega$, we will code the n -length fragment $\dot{\varphi}_G(\langle \xi, n \rangle)(1) \upharpoonright n$ so that $\dot{\varphi}_G(\langle \xi, n \rangle)(1)(m) = 1$ if and only if the node $\langle \xi \rangle \frown \vec{0}_{p_n^m}$ splits ω_1 -many times in \mathcal{T} .

We define the following subtree \mathcal{T} of $\omega_1^{<\omega}$ in $L[G]$. We have for every $\xi < \omega_1$, $1 \leq m < \omega$, $n < \omega$:

- (1) $\langle \xi \rangle \in \mathcal{T}$,
- (2) $\langle \xi \rangle \frown \vec{0}_m \in \mathcal{T}$,
- (3) $\langle \xi \rangle \frown \vec{0}_{m+1} \frown \langle n \rangle \in \mathcal{T}$,
- (4) $\langle \xi \rangle \frown \vec{0}_{p_n^m} \frown \langle \eta \rangle \in \mathcal{T}$ for all $\eta < \omega_1$ if and only if $\dot{\varphi}_G(\langle \xi, n \rangle)(1)(m) = 1$ and $m < n$.

Let

$$\mathcal{S} = \{A \in P^{L[G]}(\omega) \mid T \subseteq \mathcal{T} \text{ is a countable tree and } T \in L\},$$

and let

$$\mathcal{M} = \langle \omega, +, \times, <, 0, 1 \in, \mathcal{S} \rangle.$$

We will argue that the model \mathcal{M} has all the required properties.

As before, let \dot{S} be the canonical $\mathbb{P}(\mathbb{J}, \omega_1^{<\omega})$ -name for S . An analogous proof to the proof of Lemma 4.3 gives

LEMMA 5.1. *For any permutation $f \in L$ of ω_1 , we have $S = \dot{S}_{\pi_f^*} \ast G$. Thus, $\mathbb{1}_{\mathbb{P}(\mathbb{J}, \omega_1^{<\omega})} \Vdash \dot{S} = \pi_f^*(\dot{S})$.*

Analog of Lemma 3.6 and Lemma 3.9 in Section 3 give that \mathcal{M} satisfies $Z_2^{-p} + \Sigma_2^1$ -CA.

Let

$$C = \{ \langle n, i \rangle \mid i < n, \dot{\varphi}_G(\langle 0, n \rangle)(m) = i \}.$$

It is easy to see using the proof of Lemma 4.4 that C is Π_4^1 -definable in \mathcal{M} from the parameter $\dot{\varphi}_G(\langle 0 \rangle)$.

LEMMA 5.2. *The set $C \notin \mathcal{S}$.*

PROOF. Suppose towards a contradiction that $C \in \mathcal{S}$, and fix a countable subtree $T \in L$ of \mathcal{T} such that $C \in L[G_T]$. Let \dot{C} be a $\mathbb{P}(\mathbb{J}, T)$ -name such that $\dot{C}_G = C$ and fix a condition

$$p \Vdash \dot{C} = \{ \langle n, i \rangle \mid i < n, \dot{\varphi}(\langle 0, n \rangle)(m) = i \}.$$

Fix some $1 \leq k < \omega$ such that $\langle 0, k \rangle \notin D_p$, which is possible since D_p is finite. Fix some $1 \leq \eta < \omega_1$ such that $\dot{\varphi}_G(\langle 0, \eta \rangle) \upharpoonright k \neq \dot{\varphi}_G(\langle 0, k \rangle) \upharpoonright k$, which must exist by density. Let π be an automorphism of the tree $\omega_1^{<\omega}$ which switches the trees $(\omega_1^{<\omega})_{\langle 0, k \rangle}$ and $(\omega_1^{<\omega})_{\langle 0, \eta \rangle}$ and fixes everything else. Observe that π fixes every node in the tree T . By construction, $\pi^*(p) = p$, and since π fixes the tree T , $\pi^*(\dot{C}) = \dot{C}$. Let $H = \pi^* \ast G$. Since $\pi^*(p) = p$, $p \in H$. Thus,

$$C = \dot{C}_G = \dot{C}_H = \{ \langle n, i \rangle \mid i < n, \dot{\varphi}_H(\langle 0, n \rangle)(m) = i \}$$

because of the statement forced by p . But, we have

$$C_k = (\dot{C}_G)_k = \dot{\varphi}_G(\langle 0, k \rangle) \upharpoonright k,$$

while

$$(\dot{C}_H)_k = \dot{\varphi}_H(\langle 0, k \rangle) \upharpoonright k = \dot{\varphi}_G(\langle 0, \eta \rangle) \upharpoonright k,$$

which is impossible by our choice of η . Thus, we have reached the desired contradiction showing that $C \notin \mathcal{S}$. \dashv

Combining our observation about the complexity of the definition of the set C in the model \mathcal{M} with the fact that $C \notin \mathcal{S}$, we obtain the following lemma.

LEMMA 5.3. *The set C is definable in \mathcal{M} from the parameter $\dot{\varphi}_G(\langle 0 \rangle)$ by a Π_4^1 -formula. Thus, Σ_4^1 -CA fails in \mathcal{M} .*

It remains to verify that AC^{-p} holds in \mathcal{M} .

We will use the definitions of a *full* and *undecided* tree from Section 4 verbatim. An analogous proof to that of Lemma 4.5 gives

LEMMA 5.4. *Suppose that $T \in L$ is a full countable subtree of \mathcal{T} , $\varphi(x, X)$ is a second-order formula, and there is $A \in L[G_T]$ such that $\mathcal{M} \models \varphi(n, A)$. Then there is a full undecided countable tree $\bar{T} \in L$ and $\bar{A} \in L[G_{\bar{T}}]$ such that $\mathcal{M} \models \varphi(n, \bar{A})$.*

LEMMA 5.5. *Suppose that $T \in L$ is a full undecided countable subtree of \mathcal{T} and $p, q \in \mathbb{P}(\mathbb{J}, \omega_1^{<\omega})$ are such that $q \leq p \upharpoonright T$. Then there is an automorphism π of $\omega_1^{<\omega}$ fixing all nodes in T such that p is compatible with $\pi^*(q)$ and $\mathcal{S} = \dot{\mathcal{S}}_{\pi^* \text{''} G}$. Thus, $\mathbb{1}_{\mathbb{P}(\mathbb{J}, \omega_1^{<\omega})} \Vdash \dot{\mathcal{S}} = \pi^*(\dot{\mathcal{S}})$.*

PROOF. We define the automorphism π as follows. Suppose that $s \in D_p \cap D_q$ and $s \notin T$. We first handle nodes of the form $s = \langle \xi \rangle$. In this case, π switches the tree $(\omega_1^{<\omega})_{\langle \xi \rangle}$ with a tree $(\omega_1^{<\omega})_{\langle \eta \rangle}$ for some $\eta < \omega_1$ such that $\langle \eta \rangle \notin T \cup D_p \cup D_q$. Next, we handle nodes of the form $s = \langle \xi, n \rangle$, for some $n < \omega$, which weren't handled in the previous case (this must have been because $\langle \xi \rangle \in T$). In this case, π switches the tree $(\omega_1^{<\omega})_s$ with a tree $(\omega_1^{<\omega})_t$, where $t = \langle \xi, \beta \rangle \notin T \cup D_p \cup D_q$, $\beta \geq \omega$ and

$$\dot{\varphi}_G(\langle \xi, n \rangle)(1) \upharpoonright n = \dot{\varphi}_G(\langle \xi, \beta \rangle) \upharpoonright n.$$

Note that such a node t must exist by density. Finally, if a node $s = \langle \alpha_0, \dots, \alpha_n \rangle$ was not handled by one of the above cases, then π switches the tree $(\omega_1^{<\omega})_s$ with a tree $(\omega_1^{<\omega})_t$, where $t = \langle \alpha_0, \dots, \beta \rangle \notin T \cup D_p \cup D_q$ was not yet placed in the image of π by the above cases. All other nodes are fixed by π . In particular, all nodes in T are fixed by π .

Suppose that $H = \pi^* \text{''} G$. Then it should be clear that $T \in L$ is a countable subtree of \mathcal{T} if and only if $\pi^* \text{''} T$ is a countable subtree of $\dot{\mathcal{T}}_H$, which implies that $\mathcal{S} = \dot{\mathcal{S}}_{\pi^* \text{''} G}$. ⊣

LEMMA 5.6. $\mathcal{M} \models \text{AC}^{-P}$.

PROOF. Suppose that $\mathcal{M} \models \forall n \exists X \varphi(n, X)$ for some second-order arithmetic formula $\varphi(n, X)$. Given $n < \omega$, by Lemma 5.4, there is a full undecided countable subtree $T_n \in L$ and $X \in L[G_{T_n}]$ such that $\mathcal{M} \models \varphi(n, X)$. By extending T_n to a larger tree, if necessary, we can assume that $\{\xi < \omega_1 \mid \langle \xi \rangle \in T_n\}$ is an ordinal α_n . Let $\alpha = \bigcup_{n < \omega} \alpha_n$ and let T be the full undecided tree whose level 1 nodes are precisely $\langle \xi \rangle$ for $\xi < \alpha$. Then each T_n is contained in T . Thus, for every $n < \omega$, there is some $X \in L[G_T]$ such that $\mathcal{M} \models \varphi(n, X)$.

Fix a condition

$$p \Vdash \psi(\dot{G}_T, \dot{\mathcal{S}}) := \forall n \exists X \in L[\dot{G}_T] \langle \omega, +, \times, <, 0, 1, \dot{\mathcal{S}} \rangle \models \varphi(n, X).$$

Let's argue that this statement is already forced by $p \upharpoonright T$. Suppose towards a contradiction that this is not the case. Then there is a condition $q \leq p \upharpoonright T$ such that $q \Vdash \neg \psi(\dot{G}_T, \dot{\mathcal{S}})$. By Lemma 5.5, there is an automorphism π of $\omega_1^{<\omega}$ fixing all nodes in T such that p is compatible with $\pi^*(q)$ and $\mathbb{1}_{\mathbb{P}(\mathbb{J}, \omega_1^{<\omega})} \Vdash \dot{\mathcal{S}} = \pi^*(\dot{\mathcal{S}})$. Observe that the condition $\pi^*(q) \Vdash \psi(\dot{G}_T, \dot{\mathcal{S}})$ because π^* fixes \dot{G}_T and $\mathbb{1}_{\mathbb{P}(\mathbb{J}, \omega_1^{<\omega})} \Vdash \dot{\mathcal{S}} = \pi^*(\dot{\mathcal{S}})$. Thus, we have reached the desired contradiction showing that $p \upharpoonright T \Vdash \psi(\dot{G}_T, \dot{\mathcal{S}})$. We can now assume without loss of generality that $D_p \subseteq T$.

Fix $n < \omega$. We will construct a mixed $\mathbb{P}(\mathbb{J}, T)$ -name \dot{X}_n such that

$$p \Vdash \langle \omega, +, \times, <, 0, 1, \dot{\mathcal{S}} \rangle \models \varphi(n, \dot{X}_n).$$

For every $r \leq p$, there is a condition $q \leq r$ and a $\mathbb{P}(\mathbb{J}, T)$ -name \dot{X}_n^q such that

$$q \Vdash \langle \omega, +, \times, <, 0, 1, \dot{\mathcal{S}} \rangle \models \varphi(n, \dot{X}_n^q).$$

An analogous automorphism argument to above shows that actually

$$q \upharpoonright T \Vdash \langle \omega, +, \times, <, 0, 1, \dot{S} \rangle \models \varphi(n, \dot{X}_n^q),$$

and so we can assume without loss of generality that $D_q \subseteq T$. Thus, we can construct a maximal antichain \mathcal{A}_n of such conditions q with $D_q \subseteq T$ below p . By Proposition 2.3, it easy to see that \mathcal{A}_n is a maximal antichain of $\mathbb{P}(\mathbb{J}, \omega_1^{<\omega})$ below p .

Let

$$\dot{X}_n = \bigcup_{q \in \mathcal{A}_n} \{(\tau, r) \mid r \leq q, D_r \subseteq T, r \Vdash \tau \in \dot{X}_n^q, \tau \in \text{dom}(\dot{X}_n^q)\}.$$

Clearly, \dot{X}_n is a $\mathbb{P}(\mathbb{J}, T)$ -name. Since \mathcal{A}_n is an antichain, if $(\tau, r) \in \dot{X}_n$, then there is a unique $q \in \mathcal{A}_n$ such that $r \leq q$. Fix an L -generic filter \bar{G} for $\mathbb{P}(\mathbb{J}, \omega_1^{<\omega})$ with $p \in \bar{G}$. Let $q \in \bar{G} \cap \mathcal{A}_n$. We will argue that $(\dot{X}_n)_{\bar{G}} = (\dot{X}_n^q)_{\bar{G}}$. Suppose that $a \in (\dot{X}_n)_{\bar{G}}$. Then $(\tau, r) \in \dot{X}_n$ such that $\tau_{\bar{G}} = a$ and $r \in \bar{G}$. It follows that r is compatible with q , which, since \mathcal{A}_n is an antichain, means that $r \leq q$. By definition of \dot{X}_n , $r \Vdash \tau \in \dot{X}_n^q$, which implies that $a \in (\dot{X}_n^q)_{\bar{G}}$. Next, suppose that $a \in (\dot{X}_n^q)_{\bar{G}}$. Then $(\tau, r) \in \dot{X}_n^q$ such that $a = \tau_{\bar{G}}$ and $r \in \bar{G}$. Let $r' \in \bar{G}$ such that $r' \leq r, q$. Observe that $r' \upharpoonright T \leq r, q$ and $r' \upharpoonright T \in \bar{G}$ because it is above r' . Thus, we can assume without loss of generality that $D_{r'} \subseteq T$. Also, since $r' \leq r, r' \Vdash \tau \in \dot{X}_n^q$. Thus, $(\tau, r') \in \dot{X}_n$, and so $a \in (\dot{X}_n)_{\bar{G}}$. Since \mathcal{A}_n was a maximal antichain below p , it follows that

$$p \Vdash \langle \omega, +, \times, <, 0, 1, \dot{S} \rangle \models \varphi(n, \dot{X}_n).$$

Now, let \dot{Y} be a $\mathbb{P}(\mathbb{J}, T)$ -name for the set whose n -th slice is \dot{X}_n . Then clearly,

$$p \Vdash \forall n < \omega \langle \omega, +, \times, <, 0, 1, \dot{S} \rangle \models \varphi(n, \dot{Y}_n),$$

and $\dot{Y}_G \in L[G_T]$. Hence $\dot{Y}_G \in S$. +

§6. Questions. In Sections 3 and 4, we obtained models $Z_2^{-p} + \Sigma_2^1\text{-CA}$ with a failure of $\Sigma_4^1\text{-CA}$ in which Coll^{-p} and AC^{-p} holds respectively. Is it possible to obtain such models with an optimal failure of $\Sigma_3^1\text{-CA}$?

QUESTION 6.1. *Can we obtain a model of $Z_2^{-p} + \text{Coll}^{-p} + \Sigma_2^1\text{-CA}$ in which, optimally, $\Sigma_3^1\text{-CA}$ fails?*

QUESTION 6.2. *Can we obtain a model of $Z_2^{-p} + \text{AC}^{-p} + \Sigma_2^1\text{-CA}$ in which, optimally, $\Sigma_3^1\text{-CA}$ fails?*

Since standard arguments showing that AC implies Z_2 over ACA_0 rely on parameters, it is most likely not the case that AC^{-p} implies Z_2^{-p} over ACA_0 .

QUESTION 6.3. *Can we obtain a model $\text{ACA}_0 + \text{AC}^{-p}$ in which Z_2^{-p} fails?*

This is not directly relevant to the results in this article, but for the complete understanding of forcing extensions by $\mathbb{P}(\mathbb{J}, \omega^{<\omega})$, it is interesting to know whether the uniqueness result of Theorem 3.1 holds for countable trees and an analogous question applies to the forcing $\mathbb{P}(\mathbb{J}, \omega_1^{<\omega})$.

QUESTION 6.4. *Suppose that $G \subseteq \mathbb{P}(\mathbb{J}, \omega^{<\omega})$ is L -generic and that $T \in L$ is a countable subtree of $\omega^{<\omega}$. Is it true that the only L -generic sequences for \mathbb{J}_n in $L[G_T]$*

are the sequences $\dot{\varphi}_G(s)$ for $s \in T$? What about an analogous question for uncountable subtrees and forcing extensions by $\mathbb{P}(\mathbb{J}, \omega_1^{<\omega})$?

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