Proceedings of the Edinburgh Mathematical Society (2022) **65**, 303–310
doi:10.1017/S0013091521000687

ON THE DERIVATIVE OF A POLYNOMIAL

PRASANNA KUMAR

Department of Mathematics, Birla Institute of Technology and Science Pilani, K K Birla Goa Campus, Goa 403726, India (prasannak@goa.bits-pilani.ac.in)

(Received 20 August 2020; first published online 24 May 2022)

Abstract In this paper, we prove the well-known Erdős–Lax inequality [4] in a sharpened form. As a consequence, another widely used inequality due to Ankeny and Rivlin [1] gets sharpened. These results may be useful in various applications that required the Erdős–Lax and the Ankeny–Rivlin inequalities.

Keywords: polynomials; zeros; inequalities

2020 *Mathematics subject classification:* Primary: 30A10

1. Introduction

The well-known Bernstein's inequalities [3] on polynomials state that if $P(z)$ is a polynomial of degree n, then

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)| \tag{1.1}$$

and

$$\max_{|z|=R} |P(z)| \leq R^n \max_{|z|=1} |P(z)|, \tag{1.2}$$

whenever $R \geq 1$.

The inequality (1.1) is a direct consequence of Bernstein's Theorem on the derivative of a trigonometric polynomial [8] and the inequality (1.2) follows from the maximum modulus theorem (see [7, Corollary 12.1.3]). For the class of polynomials having no zeros inside the unit circle, Erdős [4] conjectured, and Lax [6] proved that, if $P(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.3}$$

Equality holds in (1.3) if all zeros of $P(z)$ lie on the circle $|z| = 1$.

The inequality (1.3) appears to be the best inequality for the class of polynomials having no zeros in the unit disc, but the equality in (1.3) holds when all the zeros of $P(z)$ are on $|z| = 1$. Definitely, the bound $\frac{n}{2}$ given in inequality (1.3) does not depend on how far the zeros lie outside the unit circle. Aziz and Dawood [2] made an attempt to address this issue to some extent and improved the inequality (1.3) by proving that if $P(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \left(\max_{|z|=1} |P(z)| - \min_{|z|=1} |P(z)| \right). \tag{1.4}$$

Even though the inequality (1.4) sharpens inequality (1.3) but it has a drawback that if there is even one zero on $|z| = 1$, then $\min_{|z|=1} |P(z)| = 0$, and so the inequality (1.4) fails to give any improvement over (1.3). Now naturally a question arises; is there any way to refine the inequality (1.3) for the class of polynomials satisfying the hypothesis of the Erdős–Lax inequality, by capturing the information on the moduli of zeros? Can we obtain a bound in terms of the extreme coefficients of $P(z)$ whose ratio is informative about the distance of zeros from the origin? In this paper, we approach this side of the Erdős–Lax inequality and obtain a bound which sharpens the inequality (1.3) significantly. Let us state the result below.

Theorem 1.1. *If $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ is a polynomial of degree n having no zeros in $|z| < 1$, then*

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \left[1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)} \right] \max_{|z|=1} |P(z)|. \tag{1.5}$$

The result is best possible and equality holds in (1.5) for the polynomial $P(z) = z^n + az^{n-1} + z + a$, where $a \geq 1$.

It is a straightforward fact that the term $\frac{|a_0|-|a_n|}{n(|a_0|+|a_n|)} \geq 0$ for any polynomial satisfying the hypothesis of Theorem 1.1, and hence (1.5) clearly sharpens (1.3). One can observe that $\frac{|a_0|-|a_n|}{n(|a_0|+|a_n|)}$ is a function of the modulus of the product of the zeros of $P(z)$, which is $\frac{|a_0|}{|a_n|}$.

There are a few reasons why we deem Theorem 1.1 interesting. Firstly, the inequality (1.5) sharpens the inequality (1.3) strictly for the class of polynomials having no zeros in the open unit disc with at least one zero lying outside the closed unit disc, or more precisely whenever $|a_0| \neq |a_n|$. One can observe that, as the zeros go farther and farther from the circle $|z| = 1$, the value of the term $\frac{|a_0|-|a_n|}{n(|a_0|+|a_n|)}$ increases considerably and hence the bound given in (1.5) will be much closer to the value of $\max_{|z|=1} |P'(z)|$ than the one given in (1.3), which does not take into account, the distance of zeros from the unit circle. Secondly, it can be used to get the proofs of Bernstein-type inequalities for the class of polynomials having no zeros in an open disc of any radius greater than or equal to one and related extensions to polar derivative of the polynomials. Thirdly, Lemma 2.2 can be thought of as a fundamental alternative for Laguerre's Theorem [5] on the study of Geometry of polynomials in the circular regions. Fourthly, the polynomial associated with a given polynomial introduced in Lemma 2.2 is obtained by an operator similar to the polar derivative operator, which might open up several extensions of results from the ordinary derivative operator to this operator on the class of polynomials.

It was Ankeny and Rivlin [1], who improved the inequality (1.2) for the class of polynomials having no zeros in the unit disk by proving that, if $P(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then

$$\max_{|z|=R} |P(z)| \leq \frac{1 + R^n}{2} \max_{|z|=1} |P(z)|, \tag{1.6}$$

for any $R \geq 1$. Ankeny and Rivlin used the Erdős–Lax inequality (1.3) and Bernstein's inequality (1.2), and some simple integral properties [1, p. 849–850] to establish (1.6). Instead of (1.3) if we use the sharpened version of it given in (1.5) and proceeding similarly as in the proof of the result due to Ankeny and Rivlin [1], without any difficulty we can arrive at the following refinement of (1.6).

Corollary 1.2. *If $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ is a polynomial of degree n having no zeros in $|z| < 1$ then*

$$\max_{|z|=R} |P(z)| \leq \frac{(1 + R^n) - \lambda(R^n - 1)}{2} \max_{|z|=1} |P(z)|, \tag{1.7}$$

for any $R \geq 1$, and $\lambda = \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}$.

Since the proof does not deviate much from that of Ankeny and Rivlin, we suggest the readers to understand the proof by referring the paper [1] with our Theorem 1.1. With the refinement to the Ankeny–Rivlin inequality, Corollary 1.2 may attract researchers to obtain generalizations and extensions, like we see in the literature on various results obtained by the Ankeny–Rivlin inequality (1.6).

2. Lemmas

The following few results bring attention to some well-known facts about polynomials having no zeros in the unit disc. Our first result describes an estimate for the real value of the logarithmic derivative of a complex polynomial having no zeros in the open unit disc.

Lemma 2.1. *If $P(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n$ is a polynomial of degree $n \geq 1$ having no zeros in $|z| < 1$, then for all z on $|z| = 1$ for which $P(z) \neq 0$*

$$\operatorname{Re}\left(\frac{zP'(z)}{P(z)}\right) \leq \frac{n}{2} - \frac{|a_0| - |a_n|}{2(|a_0| + |a_n|)}. \tag{2.1}$$

Proof. To prove (2.1), it suffices to establish its equivalent form

$$\operatorname{Re}\left(\frac{zP'(z)}{P(z)}\right) \leq \frac{n-1}{2} + \frac{1}{1 + \frac{|a_0|}{|a_n|}}. \tag{2.2}$$

Clearly, without loss of generality, we can assume $a_n = 1$. We will prove the above inequality (2.2) with the assumption $a_n = 1$, by the use of the principle of mathematical induction on the degree n, and for this, we first verify the result for $n = 1$.

https://doi.org/10.1017/S0013091521000687 Published online by Cambridge University Press

If $n = 1$, then $P(z) = z - w$ with $|w| \geq 1$, and therefore for $|z| = 1$ and $z \neq w$, we have

$$\operatorname{Re}\left(\frac{zP'(z)}{P(z)}\right) = \operatorname{Re}\left(\frac{z}{z-w}\right) \leq \frac{1}{1+|w|},$$

which is nothing but (2.2) when $n = 1$.

Let $Q(z) := (z - w)P(z)$ with $|w| \geq 1$, where $P(z) = \sum_{\gamma=0}^{n-1} a_\gamma z^\gamma + z^n$ is a polynomial of degree n having no zeros in $|z| < 1$. Then for all z on $|z| = 1$ where $Q(z) \neq 0$, we get by using the induction hypothesis

$$\begin{aligned}\operatorname{Re}\left(\frac{zQ'(z)}{Q(z)}\right) &= \operatorname{Re}\left(\frac{z}{z-w}\right) + \operatorname{Re}\left(\frac{zP'(z)}{P(z)}\right)\\ &\leq \frac{1}{1+|w|} + \frac{n-1}{2} + \frac{1}{1+|a_0|}.\end{aligned}$$

To complete the induction step, we need to show that on $|z| = 1$,

$$\operatorname{Re}\left(\frac{zQ'(z)}{Q(z)}\right) \leq \frac{n}{2} + \frac{1}{1+|w||a_0|}. \tag{2.3}$$

Clearly, the inequality (2.3) holds if

$$\frac{1}{1+|w|} + \frac{n-1}{2} + \frac{1}{1+|a_0|} \leq \frac{n}{2} + \frac{1}{1+|w||a_0|},$$

which is equivalent to

$$\frac{1}{1+|w|} - \frac{1}{2} + \frac{1}{1+|a_0|} - \frac{1}{1+|w||a_0|} \leq 0. \tag{2.4}$$

But

$$\frac{1}{1+|w|} - \frac{1}{2} + \frac{1}{1+|a_0|} - \frac{1}{1+|w||a_0|} = \frac{(1-|w|)(1-|a_0|)(1-|wa_0|)}{2(1+|w|)(1+|a_0|)(1+|wa_0|)} \leq 0,$$

since $|w| \geq 1$, and $|a_0| \geq 1$. Hence (2.3) is also true, and with this, the proof becomes complete on using the induction hypothesis. □

Lemma 2.1 appears to be best possible and equality holds for some special class of polynomials. By considering the circle or line onto which $|z| = 1$ is mapped by the Möbius transformation $T(z) = \frac{z}{z-w}$ one may easily check that if $|w| \geq 1$ and $|z| = 1$, then $\operatorname{Re}\left(\frac{z}{z-w}\right) \leq \frac{1}{1+|w|}$ as presented in the proof of Lemma 2.1 with equality if and only if either $|w| = 1$ or $z = \frac{-w}{|w|}$. In view of this, in the inequality (2.1) equality holds only when all the zeros of $P(z)$ lie on the unit circle or all zeros of $P(z)$ lie on the unit circle apart from one simple zero say a such that $|a| > 1$, and $z = \frac{-a}{|a|}$. Therefore, it is possible to have the equality in (2.1) for the polynomial $P(z) = z^n + az^{n-1} + z + a$ at $z = 1$ whenever $a \geq 1$.

The well-known Theorem of Laguerre [5, 9] states that, if $P(z)$ is a polynomial of degree n having no zeros in the disc $|z| < 1$, then the polynomial

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$$

has no zeros in $|z| < 1$ for every α with $|\alpha| < 1$. The next result is conceptually in line with Theorem of Laguerre, but includes leading and constant coefficients of the underlying polynomial $P(z)$, thereby revealing information on the role of coefficients of $P(z)$ in refining $D_\alpha P(z)$.

Lemma 2.2. *If $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ is a polynomial of degree n having no zeros in the disc $|z| < 1$, then the polynomial*

$$n\left(1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\right) P(z) + (\alpha - z)P'(z)$$

has no zeros in $|z| < 1$ for every α with $|\alpha| < 1$.

Proof. Since $P(z)$ has no zeros in the disc $|z| < 1$, it follows from Lemma 2.1 that, for all z on $|z| = 1$ for which $P(z) \neq 0$, we have

$$\operatorname{Re}\left(\frac{zP'(z)}{n\left(1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\right) P(z)}\right) \leq \frac{1}{2},$$

and hence

$$\left|1 - \left(\frac{zP'(z)}{n\left(1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\right) P(z)}\right)\right| \geq \left|\left(\frac{zP'(z)}{n\left(1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\right) P(z)}\right)\right|$$

for all z on $|z| = 1$, for which $P(z) \neq 0$. Therefore,

$$\left|n\left(1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\right) P(z) - zP'(z)\right| \geq |P'(z)| \tag{2.5}$$

on $|z| = 1$. If $P(z)$ has no zeros in $|z| \leq 1$ then $\operatorname{Re}(\frac{zP'(z)}{P(z)})$ is harmonic on $|z| < 1$, and continuous on $|z| \leq 1$. Therefore, it follows from Lemma 2.1 and the Maximum Modulus Principle that

$$\operatorname{Re}\left(\frac{zP'(z)}{P(z)}\right) \leq \frac{n}{2} - \frac{|a_0| - |a_n|}{2(|a_0| + |a_n|)}$$

holds for all z such that $|z| \leq 1$. But then

$$\frac{zP'(z)}{P(z)} \neq n\left(1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\right)$$

for all z such that $|z| \leq 1$, which further implies that

$$n\left(1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\right) P(z) - zP'(z) \neq 0$$

https://doi.org/10.1017/S0013091521000687 Published online by Cambridge University Press

for all z such that $|z| \leq 1$. Therefore

$$\frac{P'(z)}{n\left(1-\frac{|a_0|-|a_n|}{n(|a_0|+|a_n|)}\right)P(z)-zP'(z)}$$

is analytic on $|z|<1$ and continuous on $|z|\leq 1$. Now from (2.5) and applying Maximum Modulus Principle, we get

$$\left|\frac{P'(z)}{n\left(1-\frac{|a_0|-|a_n|}{n(|a_0|+|a_n|)}\right)P(z)-zP'(z)}\right|\leq 1$$

on $|z|\leq 1$. In other words, if $P(z)$ has no zeros in $|z|\leq 1,$ then (2.5) holds for all z such that $|z|\leq 1$. The continuity of the function $g(z)$ with respect to the polynomial $P(z)$, where

$$g(z)=\left|n\left(1-\frac{|a_0|-|a_n|}{n(|a_0|+|a_n|)}\right)P(z)-zP'(z)\right|-|P'(z)|\geq 0$$

on $|z|=1$ ensures that (2.5) continue to hold in $|z|\leq 1$, even if we restrict $P(z)$ to have no zeros in the open unit disc and allow $P(z)$ to have zeros on $|z|=1$. Hence (2.5) holds for all z such that $|z|\leq 1$ whenever $P(z)$ has no zeros in $|z|<1$ and thus for any α with $|\alpha|<1$ and $|z|<1$, we have

$$\left|n\left(1-\frac{|a_0|-|a_n|}{n(|a_0|+|a_n|)}\right)P(z)-zP'(z)\right|>|\alpha P'(z)|.$$

Therefore for any α with $|\alpha|<1$ and $|z|<1$, we have

$$n\left(1-\frac{|a_0|-|a_n|}{n(|a_0|+|a_n|)}\right)P(z)-zP'(z)+\alpha P'(z)\neq 0,$$

and hence the proof is complete. □

Lemma 2.3. *Let D be the open unit disc and $P(D)=\{P(z): z\in D\}$, where $P(z)=a_0+a_1z+\cdots+a_nz^n$. Then for any $\alpha\in D$ and $z\in D$,*

$$\frac{(\alpha-z)P'(z)}{n\left(1-\frac{|a_0|-|a_n|}{n(|a_0|+|a_n|)}\right)}+P(z)\in P(D).$$

Proof. Suppose δ is outside $P(D)$. Then $P(z)\neq\delta$ for any $z\in D$. Now applying Lemma 2.2 to the polynomial $P(z)-\delta$, one can deduce that

$$(\alpha-z)P'(z)+n\left(1-\frac{|a_0|-|a_n|}{n(|a_0|+|a_n|)}\right)P(z)\neq n\left(1-\frac{|a_0|-|a_n|}{n(|a_0|+|a_n|)}\right)\delta,$$

which is equivalent to

$$\frac{(\alpha-z)P'(z)}{n\left(1-\frac{|a_0|-|a_n|}{n(|a_0|+|a_n|)}\right)}+P(z)\neq\delta$$

for all $z\in D$, $\alpha\in D$ and any $\delta\notin P(D)$. This completes the proof. □

3. Proof of theorem 1.1

In the proof of Lemma 2.2, we have shown that, if $P(z) \neq 0$ in $|z| < 1$ then

$$|P'(z)| \leq \left| zP'(z) - n\left(1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\right) P(z) \right| \tag{3.1}$$

for any z such that $|z| \leq 1$. On the other hand, from Lemma 2.3, for any α, $z \in D$,

$$\frac{(\alpha - z)P'(z)}{n\left(1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\right)} + P(z) \in P(D). \tag{3.2}$$

Thus, we have

$$n\left(1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\right) \max_{|z|=1} |P(z)| = n\left(1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\right) \sup_{z \in D} |P(z)|$$

or equivalently

$$n\left(1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\right) \max_{|z|=1} |P(z)| = n\left(1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\right) \sup_{w \in P(D)} |w|. \tag{3.3}$$

Now from (3.2), it is evident that

$$\sup_{w \in P(D)} |w| \geq \sup_{|\alpha|<1} \left\{ \frac{(\alpha - z)P'(z)}{n\left(1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\right)} + P(z) \right\}.$$

Using this in the above equation (3.3), we get

$$n\left(1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\right) \max_{|z|=1} |P(z)|$$
$$\geq \sup_{|\alpha|<1} \left| (\alpha - z)P'(z) + n\left(1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\right) P(z) \right|$$

on $|z| = 1$. Now choosing an appropriate argument of α, we obtain

$$n\left(1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\right) \max_{|z|=1} |P(z)|$$
$$\geq |P'(z)| + \left| zP'(z) - n\left(1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\right) P(z) \right| \quad \textit{on } |z| = 1. \tag{3.4}$$

The inequalities (3.1) and (3.4) together yield

$$n\left(1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\right) \max_{|z|=1} |P(z)| \geq 2|P'(z)|,$$

as required and hence the proof is complete.

https://doi.org/10.1017/S0013091521000687 Published online by Cambridge University Press

Acknowledgements. The author is extremely grateful to the referee for his or her careful reading of the paper, and for the comments, examples and detailed suggestions for developing the proofs which enhanced the quality of the paper immensely. The work in the paper was supported partially by research grants from the National Board for Higher Mathematics, Government of India.

References

1. N. C. Ankeny and T. J. Rivlin, On a theorem of S. Bernstein, *Pacific J. Math.* **5** (1955), 849–852.
2. A. Aziz and Q. M. Dawood, Inequalities for a polynomial and its derivative, *J. Approx. Theory* **54** (1988), 306–313.
3. S. N. Bernstein, *Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réelle* (Gauthier-Villars, Paris, 1926).
4. P. Erd ős, On extremal properties of derivatives of polynomials, *Ann. Math.* **41** (1940), 310–313.
5. E. Laguerre, Oeuvres Vol. 1, *Nouvelles Ann. de Math.* **17**(2) (1878), 20–25.
6. P. D. Lax, Proof of a conjecture due to Erd ős on the derivative of a polynomial, *Bull. Amer. Math. Soc.* **50** (1944), 509–513.
7. Q. I. Rahman and G. Schmeisser, analytic theory of polynomials, LMS monographs new series, Vol. 26 (Clarendon Press, Oxford, 2002).
8. A. C. Schaeffer, Inequalities of A. Markoff and S. Bernstein for polynomials and related functions, *Bull. Amer. Math. Soc.* **47** (1941), 565–579.
9. G. Szeg ő, Bemerkungen zu einem satz von J.H. Grace etc., *Math. Zeitschr.* **13** (1922), 22–55.