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VARIATION ON A THEME BY KISELEV AND NAZAROV: HÖLDER ESTIMATES FOR NONLOCAL TRANSPORT-DIFFUSION, ALONG A NON-DIVERGENCE-FREE BMO FIELD

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Abstract We prove uniform Hölder regularity estimates for a transport-diffusion equation with a fractional diffusion operator and a general advection field in of bounded mean oscillation, as long as the order of the diffusion dominates the transport term at small scales; our only requirement is the smallness of the negative part of the divergence in some critical Lebesgue space. In comparison to a celebrated result by Silvestre, our advection field does not need to be bounded. A similar result can be obtained in the supercritical case if the advection field is Hölder continuous. Our proof is inspired by Kiselev and Nazarov and is based on the dual evolution technique. The idea is to propagate an atom property (i.e., localisation and integrability in Lebesgue spaces) under the dual conservation law, when it is coupled with the fractional diffusion operator.

Keywords: Transport-diffusion, Hölder regularity; fractional diffusion equation; nonlocal operator; BMO drift; dual equation; conservation law; functional analysis; atoms

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In this article, we are interested in the following transport-diffusion equation:

$$\begin{cases} \partial_t\theta + (-\Delta)^{\alpha/2}\theta = (v\cdot\nabla)\theta \\ \theta(0,x) = \theta_0(x). \end{cases} \tag{1}$$

We consider this Cauchy problem where $\theta : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ is unknown. The vector field v is given and is of bounded mean oscillation; that is, it belongs to the space $\mathrm{BMO}(\mathbb{R}^d)$, i.e. the space of bounded mean oscillation functions. In what follows, we will **not** assume that $\operatorname{div} v = 0$. We will also consider the periodic problem on $\mathbb{T}^d$.

We are interested in the whole range of parameters $0 < \alpha < 2$ and, in particular, the critical nonlocal diffusion (i.e., $\alpha = 1$) where diffusion and transport are of similar order.

We will, in passing, consider the classical local case $\alpha = 2$; however, the local case is much better understood and we refer the reader, for example, to the recent monograph [3] and the references therein.

For $\alpha > 1$, the equation is called *subcritical* because the drift is then of lower order than the diffusion, which means that the diffusion will be stronger at the smallest scales. On the contrary, for $\alpha < 1$, the drift will be stronger than the diffusion at smaller scales and the equation is called supercritical. However, because the diffusion operator remains invariant under Galilean transforms, one can still expect a mild smoothing effect, as long as $\alpha > 0$, because putative advected singularities cannot ride along a 'defect' of the diffusion operator, simply because the homogeneous and isotropic operator does not have any.

The fractional derivative $(-\Delta)^{\alpha/2}$ is the Fourier multiplier by $|\xi|^\alpha$. It admits the following kernel expansion on $\mathbb{R}^d$ (see, e.g., [8, §2]):

$$(-\Delta)^{\alpha/2}\theta(x) = -c_{d,\alpha}\,\mathrm{pv}\left(\int_{\mathbb{R}^d} \frac{\theta(y)-\theta(x)}{|y-x|^{d+\alpha}}\,dy\right) = -c_{d,\alpha}\lim_{\varepsilon\to 0}\left(\int_{|y-x|>\varepsilon} \frac{\theta(y)-\theta(x)}{|y-x|^{d+\alpha}}\,dy\right) \quad (2)$$

with $c_{d,\alpha} > 0$. As $(-\Delta)^{\alpha/2}$ is a nonlocal derivative of order α that does not induce a phase shift; that is, it performs a sort of 'graphical' interpolation between the graph of θ and that of $-\Delta\theta$.

The goal is to establish uniform Hölder regularity estimates of the solution of (1) at a later time $t > 0$. As demonstrated by Kiselev and Nazarov [16] and Dabkowski [9] for $\alpha = 1$, an elegant but powerful technique includes using the atomic characterisation of Hölder classes (see §1). Multiplying the equation by a test function $\psi(t-s)$ where ψ solves the dual evolution equation, one can then exchange an atomic estimate of $\theta(t)\psi_0$ against an atomic estimate of $\theta_0\psi(t)$. Provided that the dual evolution propagates the atomic property in a controlled way, one then gets the desired Hölder regularity.

Another powerful approach, used, for example, by Caffarelli and Vasseur [4] and Silvestre [20], includes using the parabolic De Giorgi method. The idea is to first establish a set of energy estimates. Then, one uses the rigidity given by the equation to deduce scalable uniform bounds. More precisely, one computes the energy cost of one oscillation of the solution between its maximum and minimum values. Given that the total amount of energy available is finite, this limits the size of the oscillations that the solution can perform at a given scale, which ultimately translates as regularity in the Hölder classes.

Finally, the method of the modulus of continuity was successfully introduced by Kiselev et al. [17] for SQG (the surface quasi-geostrophic equation) and later used by Silvestre and Vicol [21] to study related partial differential equations.

The case of a divergence-free transport field in (1) is of particular interest because it closely relates to the quasi-geostrophic equation where $d = 2$ and $u = \nabla^\perp(-\Delta)^{-1/2}\theta$; see [5], [4], [17]. At a technical level, the computations of [16], [9] rely in a crucial way on the assumption that $\operatorname{div} v = 0$. First, because they use the nonconservative form for the dual evolution equation

$$\partial_s\psi + (-\Delta)^{\alpha/2}\psi = -(v(t-s)\cdot\nabla)\psi,$$

each integration by part requires a divergence-free field. Second, they invoke the maximum principle and the decay of the L^p-norms for this kind of equation, which was established by Cordoba and Cordoba [8], but either for the particular vector field of SQG or for a divergence-free transport field.

The key point of our article is that we allow $\operatorname{div} v \neq 0$. The sign of $\operatorname{div} v$ then becomes crucial because convergent characteristics will tend to create shocks and break down the regularity and divergent characteristics (rarefaction waves) have a natural smoothing effect and tend to reduce steep gradients. The presence of the diffusion operator fixes the issues of either nonexistence or nonuniqueness that the pure transport would induce. In that context, the pertinent question is thus to establish quantitative estimates of the regularity norms.

Our main result is the following statement that provides uniform Hölder bounds for solutions of (1).

Theorem 1 *We consider either $\mathbb{R}^d$ or $\mathbb{T}^d$ as the ambiant space, which is of dimension $d > \alpha$, and an advection field*

$$\begin{cases} v \in C^{1-\alpha} & \text{if} \quad 0 < \alpha < 1, \\ v \in \mathrm{BMO} & \text{if} \quad 1 \leq \alpha \leq 2. \end{cases} \tag{3}$$

In both cases, one requires

$$\|(\operatorname{div} v)_-\|_{L^{d/\alpha}} \leq S_{\alpha/2} = \sup_{f \in \dot{H}^{\alpha/2}} \left(\frac{\displaystyle\int |(-\Delta)^{\alpha/4} f|^2}{\|f\|^2_{L^{2d/(d-\alpha)}}} \right), \tag{4}$$

where $(\operatorname{div} v)_-$ is the negative part of the divergence (see (8) below) and $\dot{H}^{\alpha/2}$ denotes the homogeneous Sobolev space on $\mathbb{R}^d$ or the average-free space on $\mathbb{T}^d$. There exist $\beta > 0$ and $C > 0$ that depend solely on d, α and respectively on either $\|v\|_{\mathrm{BMO}}$ or $\|v\|_{C^{1-\alpha}}$ such that, for any $\theta_0 \in L^q$ with $2 \leq q \leq \infty$, the corresponding solution of (1) satisfies

$$\forall t \in (0, 1], \qquad \|\theta(t, \cdot)\|_{C^\beta} \leq C t^{-(\beta + \frac{d}{q})/\alpha} \|\theta_0\|_{L^q}. \tag{5}$$

Moreover, if $\theta_0 \in C^\beta$, then

$$\forall t \geq 0, \qquad \|\theta(t, \cdot)\|_{C^\beta} \leq C' \|\theta_0\|_{C^\beta} \tag{6}$$

for some constant C'.

Among the results noted in this introduction, that of Silvestre [20] is the only one that allows for non-divergence-free fields, so we will focus on this one. For $\alpha \geq 1$, he assumes that $v \in L^\infty(\mathbb{R}^d)$ but requires no other size constraint on v; for $\alpha < 1$, he assumes that $v \in C^{1-\alpha}$ in order to compensate for the supercriticality of the equation. In both cases, Silvestre proves the Hölder continuity of the solution of (1) at positive times, namely, that

$$\|\theta(t)\|_{C^\beta(\mathbb{R}^d)} \leq C t^{-\beta/\alpha} \|\theta\|_{L^\infty([0,T] \times \mathbb{R}^d)},$$

and he also establishes a similar $C^{\beta/\alpha}$-regularity estimate time-wise.

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Our result is complementary and does not quite compare to that of Silvestre for two reasons. First, when $\alpha \geq 1$, we do not assume that v is bounded. Instead, we only assume that v belongs to BMO; however, we require the negative part of its divergence to be small. In particular, we do not constrain the size of $(\operatorname{div} v)_+$ and, in particular, any BMO divergence-free field is admissible in our result. Second, for all values of $\alpha \in (0,2]$, we do not assume an a priori uniform space–time bound for $\theta \in L^\infty_{t,x}$; instead, the right-hand side of (5) is finite as soon as $\theta_0 \in L^q$.

It is likely that a 'universal' result holds true for a general advection field $v = v_1 + v_2$ with $v_1 \in L^\infty$ and $v_2 \in \mathrm{BMO}$ with $(\operatorname{div} v_2)_-$ small. Results in this spirit are known for general diffusion operators of order $\alpha = 2$ (see [3]). However, mixing the DeGiorgi and atomic methods is not obvious.

Remark 2 The critical value in (4) can be slightly relaxed but the constants β and C will then also depend on the size of $(\operatorname{div} v)_-$; see Remark 26. When $\alpha = 2$, the dimension $d = 2$ is excluded from our statement (and similarly the case $\alpha = d = 1$) because the corresponding Sobolev space $H^{\alpha/2}(\mathbb{R}^d)$ fails to be included in L^∞.

Our proof is inspired by Kiselev and Nazarov [16] and we use the dual evolution technique. Following Dabkowski [9], we also use L^p-based atoms for more flexibility. In order to deal with non-divergence-free transport fields, the key is to replace the dual evolution equation by a transport-diffusion equation expressed in a *conservative* form; that is, as a conservation law. The presence of a general exponent of diffusion $\alpha \neq 1$ also brings about some technicalities that need to be addressed (see the end of Section 2).

There are many other takes on the question of regularity for transport-diffusion equations. For example, Bresch and Jabin [2] have studied the regularity of weak solutions to the advection equation set in conservative form. In particular, they investigated the case where both v and $\operatorname{div} v$ belong to some Lebesgue space, which allows pointwise unbounded variations of the field and of its compressibility. Danchin [10] contributed to the study of the case of less regular coefficients by using Littlewood-Paley techniques. Hmidi and Zerguine [14] used a similar approach for the Euler-Boussinesq system with fractional dissipation.

Another related study, albeit slightly more distant from (1), is that of the kinetic Fokker-Planck equation. On that mater, we refer the reader to Mouhot [19] and references therein, and, for example, to the regularity result by Golse and Vasseur [13].

The role of fractional diffusion operators in physical models is growing (see, e.g., Vaźquez's book [24]). As for our motivations, we are interested in studying variants of a nonlocal Burgers' equation introduced by Imbert et al. in [15]. The regularity of the solution for unsigned data remains an open problem that seems to share some nontrivial similarities with hydrodynamic turbulence.

The present article is structured as follows. In Section 1, we recall how Hölder classes can be characterised in terms of atoms. We draft the general ideas on how regularity can be obtained by studying the dual conservation law in Section 2. A weak version (i.e., in Lebesgue spaces) of the maximum principle is established in Section 3. It is then put to use in Section 4 to quantify how the atomic property is propagated. The proof of

Theorem 1 and, in particular, how the choices of structural constants should be made is detailed in Section 5.

Notations. In this article, we use the following common notations for $a, b, x \in \mathbb{R}$:

$$a \wedge b = \min\{a, b\}, \qquad a \vee b = \max\{a, b\} \tag{7}$$

$$x_+ = a \vee 0, \qquad x_- = (-x)_+, \qquad x = x_+ - x_-, \qquad |x| = x_+ + x_-. \tag{8}$$

Balls are denoted by $\mathcal{B}(x_0, r) = \left\{x \in \mathbb{R}^d ; |x - x_0| < r\right\}$ where $|\cdot|$ denotes the Euclidian distance on $\mathbb{R}^d$.

1. Atomic characterisation of Hölder classes

Throughout the article, the constant $A \gg 1$ is fixed but is chosen arbitrarily large, and $\omega \in (0,1)$. One can check that none of the final estimates actually depend on the particular values of A or ω.

Definition 3 For $r > 0$ and $p \in (1, \infty]$, the atomic class $\mathcal{A}_r^p(\mathbb{R}^d)$ is defined as a subset of $C^\infty(\mathbb{R}^d)$ by the following three conditions:

$$\int_{\mathbb{R}^d} \varphi(x)\, dx = 0, \tag{9}$$

$$\|\varphi\|_{L^1} \le 1 \qquad \text{and} \qquad \|\varphi\|_{L^p} \le A r^{-d(1-\frac{1}{p})}, \tag{10}$$

$$\exists x_0 \in \mathbb{R}^d, \qquad \int_{\mathbb{R}^d} |\varphi(x)| \Omega(x - x_0)\, dx \le r^\omega, \tag{11}$$

where $\Omega(z) = |z|^\omega \wedge 1 \in L^\infty(\mathbb{R}^d)$. If $\lambda^{-1}\varphi(x) \in \mathcal{A}_r^p(\mathbb{R}^d)$ for some $\lambda > 0$, then one says that

$$\varphi \in \lambda \cdot \mathcal{A}_r^p(\mathbb{R}^d). \tag{12}$$

Remark 4 A typical example of an atom $\mathcal{A}_r^p$ of radius $r < 1$ is the function $\varphi = \varphi_r * \rho_\epsilon$ where ρ_ϵ is an standard mollifier supported in $\mathcal{B}(0, \epsilon)$ and

$$\varphi_r(x) = \begin{cases} -Cr^{-d} & \text{if } |x - x_0| \le r 2^{-1/d} \\ +Cr^{-d} & \text{if } r2^{-1/d} < |x - x_0| \le r \\ 0 & \text{if } |x - x_0| > r \end{cases}$$

for small enough constants $C < \min\{|\mathcal{B}(0,1)|^{-1}; A|\mathcal{B}(0,1)|^{-1/p}; |\mathcal{B}(0,1)|/(d+\omega)\}$ and $\epsilon > 0$. The volume of the unit ball is $|\mathcal{B}(0,1)| = \pi^{d/2} / \Gamma\left(\frac{d}{2} + 1\right)$.

One can control the L^q-norm of atoms for $q \in [1, p]$ by a real interpolation estimate.

Proposition 5 *If* $\varphi \in \mathcal{A}_r^p(\mathbb{R}^d)$, *then for any* $1 \le q \le p$:

$$\|\varphi\|_{L^q} \le A^{\frac{1-1/q}{1-1/p}} r^{-d(1-\frac{1}{q})}. \tag{13}$$

Proof Apply the interpolation inequality $\|f\|_{L^q} \le \|f\|_{L^1}^{1-\theta}\|f\|_{L^p}^{\theta}$ with $\theta = (1-\frac{1}{q})/(1-\frac{1}{p}) \in [0,1]$. □

Atoms are the 'poor man's wavelet' and offer a very comfortable characterisation of Hölder's classes.

Proposition 6 *For $0<\beta<1$, a bounded function f belongs to $C^\beta(\mathbb{R}^d)$ if and only if*

$$\sup_{\substack{0<r\le 1\\ \varphi\in\mathcal{A}_r^p(\mathbb{R}^d)}} r^{-\beta}\left|\int_{\mathbb{R}^d} f(x)\varphi(x)\,dx\right| < \infty \tag{14}$$

for some $p\in(1,\infty]$. Moreover, the left-hand side of (14) is equivalent to the usual semi-norm on $C^\beta(\mathbb{R}^d)$.

Proof The proof is a classical exercise. See [16], [9] (respectively for $p=\infty$ and $p<\infty$) for a short proof that relies on the Littlewood-Paley theory [22]. The key is the equivalent control of $\sup_{j\in\mathbb{N}} 2^{\beta j}\|\Delta_j f\|_{L^\infty}$ where Δ_j is a smooth projection on the frequency scale of order 2^j. □

2. Regularity through the dual conservation law

As explained in the previous section, in order to obtain estimates of the Hölder regularity of the solution of (1) at a time $t>0$, one needs to control

$$r^{-\beta}\int_{\mathbb{R}^d}\theta(t,x)\psi_0(x)\,dx,$$

where $\psi_0\in\mathcal{A}_r^p(\mathbb{R}^d)$, and this control needs to be uniform in $0<r\le 1$. Let us consider the following test function that solves the dual evolution problem, set in a *conservative* form:

$$\begin{cases}\partial_s\psi(s)+(-\Delta)^{\alpha/2}\psi(s) = -\operatorname{div}(v(t-s)\psi(s))\\ \psi(0,x)=\psi_0(x).\end{cases} \tag{15}$$

One can then immediately check the following result.

Proposition 7 *If θ is a smooth solution of (1) and ψ is a smooth solution of (15), then one has*

$$\forall t\ge 0,\qquad \int_{\mathbb{R}^d}\theta(t,x)\psi_0(x)\,dx = \int_{\mathbb{R}^d}\theta_0(x)\psi(t,x)\,dx. \tag{16}$$

Proof Let us indeed use $\psi(t-s)$ as a test function for (1). One gets

$$\iint \partial_s\theta(s)\cdot\psi(t-s) + \iint((-\Delta)^{\alpha/2}\theta(s))\cdot\psi(t-s) = \iint (v(s)\cdot\nabla)\theta(s)\cdot\psi(t-s).$$

Here and below, all double integrals are computed for $(s,x)\in[0,t]\times\mathbb{R}^d$ unless stated otherwise and the x-variable is not made explicit unless it is absolutely necessary.

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Integrating by part time-wise in the first integral and space-wise in the other two gives

$$\iint \theta(s)\cdot(\partial_s\psi)(t-s)+\left[\int_{\mathbb{R}^d}\theta(s)\psi(t-s)\right]_0^t+\iint \theta(s)\cdot((-\Delta)^{\alpha/2}\psi)(t-s)$$
$$=-\iint \theta(s)\cdot \operatorname{div}(v(s)\psi(t-s)).$$

Thanks to (15), the double integrals cancel each other out and one is left with (16). □

Remark 8 In [16], the authors used the nonconservative dual form $-v(t-s)\cdot\nabla\psi(s)$. This choice was harmless and perfectly adapted to their purpose because they assumed v to be divergence free. Here, on the contrary, it is crucial that the right-hand side of (15) takes the form of a conservation law.

For the sake of the argument, let us assume that one will be able to show subsequently (which is indeed the case if $\alpha=1$) the following infinitesimal propagation property for (15):

$$\psi_0\in\mathcal{A}_r^p \qquad\Longrightarrow\qquad \forall s\in[0,\gamma r],\qquad \psi(s)\in(1-h(r)s)\mathcal{A}_{r+Ks}^p$$

for a given value of $p\in(1,\infty]$, with universal constants γ, K that do not depend on r or ψ_0 and a universal function h. One can then immediately infer a global propagation property:

$$\psi_0\in\mathcal{A}_r^p \qquad\Longrightarrow\qquad \forall s\in\left[0,\frac{1-r}{K}\right],\qquad \psi(s)\in f_r(s)\mathcal{A}_{r+Ks}^p$$

with $f_r'(s)\geq -h(r+Ks)f_r(s)$. Let us introduce a function H such that $H'(z)=h(z)$. Then

$$f_r(s)=\exp\left(\frac{H(r)-H(r+Ks)}{K}\right)$$

is an acceptable bound for the global propagation property. Coupled with (16), this means the following: For any solution of (1) and $\psi_0\in\mathcal{A}_r^p$, one has

$$\left|\int_{\mathbb{R}^d}\theta(t,x)\psi_0(x)\,dx\right|=\left|\int_{\mathbb{R}^d}\theta_0(x)\psi(t,x)\,dx\right|\leq\|\theta_0\|_{C^\beta}(r+Kt)^\beta f_r(t).$$

One is thus able to propagate $C^\beta(\mathbb{R}^d)$ bounds of θ if $f_r(s)\leq C(\frac{r}{r+Ks})^\beta$. This is the case if, for example, $h(r)=\delta/r$ with $\delta=K\beta$. The regularisation estimate is obtained in the same fashion.

Dealing with a general exponent α requires a slightly more careful computation. The fundamental idea remains that the dual conservation law propagates atoms and that a small gain on the amplitude of the atoms can be obtained as a trade-off with a slight increase in the size of the atoms' radii.

The main technical difficulty is that the radii now grow as $(r^\alpha+Ks)^{1/\alpha}$, which is not linear in s anymore, at least not when $s\gg r^\alpha$. This nonlinear region invades any neighbourhood of $r=s=0$ and the corresponding correction of amplitude will be $h(r)=\delta/r^\alpha$. We found that the simplest workaround is to forfeit the to forfeit the point of view of ordinary differential equations presented here point of view presented here for

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$\alpha = 1$ and to use direct estimates on the corresponding rate of change during a finite increment of an Euler scheme (see Section 4, estimate (53)).

3. Weak maximum principle for the dual conservation law

In this section, we establish the weak maximum principle; that is, the decay of the L^p-norms for a nonlocal transport-diffusion equation written in a conservative form. In this section, one considers thus the following general problem for $0 < \alpha \le 2$:

$$\begin{cases} \partial_s \psi(s) + (-\Delta)^{\alpha/2}\psi(s) = -\operatorname{div}(\mathbf{v}(s)\psi(s)) \\ \psi(0,x) = \psi_0(x) \end{cases} \tag{17}$$

and we will assume, when necessary, that $\int_{\mathbb{R}^d} \psi_0 = 0$. Subsequently, the results of this section will be applied to (15) at a given time $t > 0$ by choosing $\mathbf{v}(s) = v(t-s) \in \mathrm{BMO}(\mathbb{R}^d)$.

3.1. A brief note on the well-posedness theory

For smooth $\mathbf{v}(s,x)$, the well-posedness theory of the scalar conservation law

$$\partial_s \psi(s) = -\operatorname{div}(\mathbf{v}(s)\psi(s))$$

was established by Kružkov [18], in the setting of entropy solutions. The theory was refined and generalised to the nonconservative convective form by DiPerna and Lions [12]; their theory ensures that assuming a transport field $\mathbf{v} \in L^1([0,T]; W^{1,1})$ with $(\operatorname{div}\mathbf{v})_- \in L^1([0,T]; L^\infty)$ is enough to guarantee the existence, uniqueness and stability in the proper function spaces. The key idea is a celebrated commutation lemma:

$$\rho_\varepsilon * (\mathbf{v}\cdot\nabla)\psi - \mathbf{v}\cdot\nabla(\rho_\varepsilon * \psi) \to 0 \quad \text{in } L^1([0,T]; L^\beta_{\mathrm{loc}}).$$

On $\mathbb{R}^d$, an additional assumption of mild growth at infinity is required; for example, $\mathbf{v} \in (1+|x|)\cdot(L^1 + L^\infty)$. To handle unbounded data, the idea is to use renormalisation; that is, to consider $\Phi(\psi)$ for suitable smooth and bounded Φ. For a review of the fundamental ideas and the last developments of the theory, we refer the reader to the monograph [3] by Le Bris and Lions and references therein. See also the lecture notes by Ambrosio and Trevisan [1] or those of De Lellis [11].

Adding the coercive diffusion term $(-\Delta)^{\alpha/2}\psi$ in (17) with $0 < \alpha \le 2$ obviously does not alter these results. On the contrary, the assumptions on the transport field can even be relaxed. For example, for $\alpha = 2$ and even with a fully general second-order elliptic operator, one can accept a field $\mathbf{v} \in L^2 + W^{1,1}$ with $(\operatorname{div}\mathbf{v})_- \in L^\infty$, as mentioned in [3, §3.2].

The local well-posedness of (17) is thus classical; see, for example, [2].

Remark 9 If the transport term takes the conservative form, the equation is called a *conservation law*; if not, it is referred to as a general *convection*. When the Laplace operator has variable coefficients, the term *conservative* is preferred to describe the equation with the operator written in divergence form $-\partial_i(a_{ij}\partial_j)$, regardless of whether the transport part is a convection or a conservation law. In our present case, however, the fractional power $(-\Delta)^\alpha$ is obviously a conservative operator, so our use of the adjective *conservative* concerns only the form of the advection term.

3.2. Propagation of positivity

The classical positivity result for $\alpha = 2$ can be generalised for fractional diffusions.

Proposition 10 *If ψ is a solution to (17), stemming from $\psi_0 \geq 0$, then $\psi(s) \geq 0$ for any $s \geq 0$.*

Proof Let us sketch the argument first. If a solution of (17) is smooth and positive, then at a first contact point with zero, say (s_0, x_0), it reaches a global minimum. One thus has $\psi(s_0, x_0) = 0$ and $\nabla\psi(s_0, x_0) = \mathbf{0}$, and therefore

$$\operatorname{div}(\mathbf{v}\psi) = (\operatorname{div}\mathbf{v})\psi(s_0, x_0) + (\mathbf{v}\cdot\nabla)\psi(s_0, x_0) = 0.$$

Moreover, for $0 < \alpha < 2$, by (2), there exists a positive kernel $K_{d,\alpha}$ such that

$$(-\Delta)^{\alpha/2}\psi(s_0, x_0) = -\int_{\mathbb{R}^d} (\psi(s_0, y) - \psi(s_0, x_0))\, K_{d,\alpha}(y - x_0)\, dy \leq 0$$

and the inequality is strict, unless $\psi(s_0, \cdot) \equiv 0$. The equation ensures that $\partial_s\psi(s_0, x_0) \geq 0$ and, in particular, the solution remains positive forever. To make the proof fully rigorous, one proceeds, for example, as in [15, §2.1]: For given $T, R > 0$ and $\psi_0 > 0$, one considers the approximation ψ_R where the kernel $K_{d,\alpha}$ is restricted to $\mathcal{B}(0, R)$ and

$$s_0 = \inf\{s \in (0, T)\,;\, \exists x_0 \in \mathcal{B}(0, R), \quad \psi_R(s, x_0) = 0\}\,.$$

By compactness, s_0 is attained and $s_0 > 0$. Because $\psi_R(s, \cdot)$ is not identically zero, the previous computation ensures that $\partial_s\psi_R(s_0, x_0) > 0$ and thus ψ_R had to be negative in the neighbourhood of x_0 a short time before s_0, which is contradictory. For a general $\psi_0 \geq 0$, the data can be approximated by a strictly positive mollification, whose strict positivity propagates downstream. Passing to the limit at a later time $s > 0$ therefore ensures that $\psi(s) \geq 0$. □

3.3. Propagation of the L^1 norm and conservation of momentum

The simple structure of (17) inherited from the underlying conservation law plays in our favour.

Proposition 11 *Let ψ be a solution to (17). Then*

$$\|\psi(s, \cdot)\|_{L^1} \leq \|\psi_0\|_{L^1} \tag{18}$$

and

$$\int_{\mathbb{R}^d} \psi(s, x)\, dx = \int_{\mathbb{R}^d} \psi_0(x)\, dx. \tag{19}$$

Proof For the first statement, let us decompose $\psi_0 = \psi_0^+ - \psi_0^-$, where both ψ_0^+ and ψ_0^- are positive and have disjoint supports. Let ψ^+ and ψ^- be the solutions to the equation

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with initial data ψ_0^+ and ψ_0^- correspondingly. Then, by linearity, $\psi(s) = \psi^+(s) - \psi^-(s)$, and therefore

$$\|\psi(s,\cdot)\|_{L^1} \le \|\psi^+(s,\cdot)\|_{L^1} + \|\psi^-(s,\cdot)\|_{L^1}.$$

Equation (17) and an integration by part ensure that

$$\frac{d}{ds}\int_{\mathbb{R}^d} \psi^\pm(s,x)\,dx = -\int_{\mathbb{R}^d} (-\Delta)^{\alpha/2}\psi^\pm - \int_{\mathbb{R}^d} \mathrm{div}(\mathbf{v}\psi^\pm)\,dx = 0.$$

The term $(-\Delta)^{\alpha/2}\psi^\pm$ is average free because the integration of the singular integral (2) vanishes by anti-symmetry between the roles of x and y. Because the propagation of positivity yields that $\psi^\pm \ge 0$, one gets $\|\psi^\pm(s,\cdot)\|_{L^1} = \|\psi_0^\pm\|_{L^1}$ and finally $\|\psi(s,\cdot)\|_{L^1} \le \|\psi_0^+\|_{L^1} + \|\psi_0^-\|_{L^1} = \|\psi_0\|_{L^1}$, and hence (18). The identity (19) is immediate. □

Remark 12 Note that, because of the diffusion, the functions $\psi^\pm$ of the previous proof will not coincide, in general, with the positive and negative parts $\psi_\pm$ of ψ.

3.4. Estimate of the L^p norm

For $h < d/2$ let us introduce the constant in the Sobolev embedding $\dot{H}^h(\mathbb{R}^d) \subset L^{2d/(d-2h)}$ (see, e.g., [23]):

$$S_h(\mathbb{R}^d)^{-1} = \sup\left\{ \|f\|^2_{L^{2d/(d-2h)}}\,;\, f \in \dot{H}^h(\mathbb{R}^d),\quad \int_{\mathbb{R}^d} |(-\Delta)^{h/2} f|^2 = 1 \right\} > 0. \tag{20}$$

The idea is to relax the uniform control given by the maximum principle for (17) into a weaker one in the scale of Lebesgue spaces.

Proposition 13 *For any $\alpha \in (0,2]$ and any dimension $d > \alpha$, if the transport field satisfies*

$$(p-1)\|(\mathrm{div}\,\mathbf{v})_-\|_{L^\infty_t L^{d/\alpha}_x} \le S_{\alpha/2}(\mathbb{R}^d) \tag{21}$$

for some $p \ge 2$ (eventually restricted to $p = 2^n$ with $n \in \mathbb{N}$ if $\alpha < 2$), then any solution of (17) satisfies

$$\|\psi(s)\|^p_{L^p} + S_{\alpha/2}(\mathbb{R}^d)\int_0^s \|\psi(\tau)\|^p_{L^\sigma}\,d\tau \le \|\psi_0\|^p_{L^p} \qquad \text{with} \qquad \sigma = \frac{dp}{d-\alpha} \tag{22}$$

and, in particular,

$$\forall q \in [1,p], \qquad \forall s \ge 0, \qquad \|\psi(s)\|_{L^q} \le \|\psi_0\|_{L^q} \tag{23}$$

for any $\psi_0 \in L^1 \cap L^p$. In particular, when $\mathrm{div}(\mathbf{v}) \ge 0$, *the estimate* (23) *holds for* $1 \le q \le \infty$.

Remark 14 The following proof also establishes that all solutions of (17) satisfy

$$\|\psi(s)\|_{L^p} \le \|\psi_0\|_{L^p}\, e^{t(1-\frac{1}{p})\|(\mathrm{div}\,\mathbf{v})_-\|_{L^\infty_{t,x}}} \tag{24}$$

regardless of the diffusion term and independent of (21). For what follows, we are, however, interested in getting a better (i.e., nonincreasing) control of the L^p-norm as given by (22)–(23).

Proof Using $p|\psi|^{p-2}\psi$ as a multiplier for the equation leads to

$$\frac{d}{ds}\int_{\mathbb{R}^d}|\psi(s,x)|^p\,dx + p\int_{\mathbb{R}^d}|\psi|^{p-2}\psi\cdot(-\Delta)^{\alpha/2}\psi = -p\int_{\mathbb{R}^d}\operatorname{div}(\mathbf{v}\psi)\,|\psi|^{p-2}\psi.$$

For the integral on the right-hand side, an integration by part gives

$$\begin{aligned}\int_{\mathbb{R}^d}\operatorname{div}(\mathbf{v}\psi)\,|\psi|^{p-2}\psi &= -(p-1)\int_{\mathbb{R}^d}|\psi|^{p-2}\psi(\mathbf{v}\cdot\nabla)\psi\\ &= -(p-1)\int_{\mathbb{R}^d}\operatorname{div}(\mathbf{v}\psi)\,|\psi|^{p-2}\psi + (p-1)\int_{\mathbb{R}^d}|\psi|^p\operatorname{div}\mathbf{v}.\end{aligned}$$

One thus has this identity:

$$\int_{\mathbb{R}^d}\operatorname{div}(\mathbf{v}\psi)\,|\psi|^{p-2}\psi = \left(1-\frac{1}{p}\right)\int_{\mathbb{R}^d}|\psi|^p\operatorname{div}\mathbf{v} \tag{25}$$

and thus

$$\frac{d}{ds}\int_{\mathbb{R}^d}|\psi(s,x)|^p\,dx + p\int_{\mathbb{R}^d}|\psi|^{p-2}\psi\cdot(-\Delta)^{\alpha/2}\psi = -(p-1)\int_{\mathbb{R}^d}|\psi|^p\operatorname{div}\mathbf{v}. \tag{26}$$

For the integral on the left-hand side of (26) and when $\alpha=2$, the following identity holds:

$$p\int_{\mathbb{R}^d}|\psi|^{p-2}\psi\cdot(-\Delta)\psi = p(p-1)\int_{\mathbb{R}^d}|\psi|^{p-2}|\nabla\psi|^2 = 4\left(1-\frac{1}{p}\right)\int_{\mathbb{R}^d}|\nabla(|\psi|^{p/2})|^2\geq 0. \tag{27a}$$

For $0<\alpha<2$, one needs to replace (27a) because the Leibniz formula is no longer valid; instead, one follows the ideas of [8]. The key is the pointwise inequality [8, Prop. 2.3]:

$$2\psi\cdot(-\Delta)^{\alpha/2}\psi \geq (-\Delta)^{\alpha/2}(|\psi|^2),$$

which follows immediately from the kernel representation (2) of $(-\Delta)^{\alpha/2}$. Applied recursively $n-1$ times when $p=2^n$ and $n\geq 1$ is an integer, it provides for $1\leq k\leq n-1$ (or without intermediary k if $n=1$):

$$p\int_{\mathbb{R}^d}|\psi|^{p-2}\psi\cdot(-\Delta)^{\alpha/2}\psi \geq \frac{p}{2^k}\int_{\mathbb{R}^d}|\psi|^{p-2^k}(-\Delta)^{\alpha/2}(|\psi|^{2^k}) \geq 2\int_{\mathbb{R}^d}|(-\Delta)^{\alpha/4}(\psi^{p/2})|^2. \tag{27b}$$

Compared to [8, Lemma 2.4], the inequality (27b) is improved by a factor of 2. Overall, for $p\geq 2$ (restricted to exact powers of 2 if $0<\alpha<2$), the evolution of the L^p-norm of smooth solutions of (17) obeys the following inequality:

$$\frac{d}{ds}\|\psi\|_{L^p}^p + 2\int_{\mathbb{R}^d}|(-\Delta)^{\alpha/4}(\psi^{p/2})|^2 \leq -(p-1)\int_{\mathbb{R}^d}|\psi|^p\operatorname{div}\mathbf{v}. \tag{28}$$

Obviously, only the focussing zones (i.e., regions where $\operatorname{div}\mathbf{v}<0$) of the transport field can contribute to an increase of the L^p norm; the other just tends to spread ψ out. Using the notation (8) for the negative part, one thus has the following estimate:

$$\frac{d}{ds}\|\psi\|_{L^p}^p + 2\int_{\mathbb{R}^d}|(-\Delta)^{\alpha/4}(\psi^{p/2})|^2 \leq (p-1)\int_{\mathbb{R}^d}|\psi|^p(\operatorname{div}\mathbf{v})_-. \tag{29}$$

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In dimension $d \geq 2$ and for $0 < \alpha < 2$, one uses the Sobolev embedding (20). For $\sigma = dp/(d-\alpha)$, one thus has

$$\|\psi\|_{L^\sigma}^p = \|\psi^{p/2}\|_{L^{2d/(d-\alpha)}}^2 \leq \frac{1}{S_{\alpha/2}(\mathbb{R}^d)} \int_{\mathbb{R}^d} |(-\Delta)^{\alpha/4}\psi^{p/2}|^2.$$

The estimate (29) becomes

$$\frac{d}{ds}\|\psi\|_{L^p}^p + 2S_{\alpha/2}(\mathbb{R}^d) \cdot \|\psi\|_{L^\sigma}^p \leq (p-1) \int_{\mathbb{R}^d} |\psi|^p (\operatorname{div} \mathbf{v})_-. \tag{30}$$

Finally, because the conjugate exponent of $q = d/\alpha > 1$ satisfies $pq' = \sigma$, one splits the right-hand side in the following way:

$$\int_{\mathbb{R}^d} |\psi|^p (\operatorname{div} \mathbf{v})_- \leq \|\psi\|_{L^\sigma}^p \|(\operatorname{div} \mathbf{v})_-\|_{L^{d/\alpha}}.$$

Thanks to the smallness assumption (21), it is then possible to bootstrap the Lebesgue norm into the left-hand side. In that case, (30) ensures that $\frac{d}{ds}\|\psi\|_{L^p} + S_{\alpha/2}(\mathbb{R}^d) \cdot \|\psi\|_{L^\sigma}^p \leq 0$, which gives (22). One can then interpolate with (18) to control all L^q norms for $1 \leq q \leq p$. □

Remark 15 If $C_{\alpha,p}(\mathbf{v}) = 2S_{\alpha/2}(\mathbb{R}^d) - (p-1)\|(\operatorname{div} \mathbf{v})_-\|_{L_t^\infty L_x^{d/\alpha}} > 0$, then (22) still holds but with the constant $S_{\alpha/2}(\mathbb{R}^d)$ replaced by $C_{\alpha,p}(\mathbf{v})$, which is not uniform in $\mathbf{v}$ anymore.

Remark 16 An improved version of (27b) valid for average-free functions is found in [6] or [7, Prop. 2.4]:

$$\int_{\mathbb{R}^d} |\psi|^{p-2}\psi \cdot (-\Delta)^{\alpha/2}\psi \geq \frac{1}{p}\|(-\Delta)^{\alpha/2}(\psi^{p/2})\|_{L^2}^2 + C\|\psi\|_{L^p}^p.$$

However, in our case, using a Sobolev embedding for $\psi^{p/2}$ provides some additional integrability and in particular a control of the L^σ-norm with $\sigma > p$. This gain will be crucial in what follows. It also allows us to put a restriction on the $L^{d/\alpha}$-norm of $(\operatorname{div} \mathbf{v})_-$, instead of requiring smallness in L^∞.

Note that on $\mathbb{T}^d$, the Sobolev embedding $\dot{H}^h(\mathbb{T}^d) \subset L^{2d/(d-2h)}$ (20) is only valid for average-free functions. However, $\psi^{p/2}$ is not average free in general (i.e., $p \neq 2$), even if ψ is so. For the periodic case, one will use the following simpler result, whose proof is also contained above.

Proposition 17 *If ψ is an average-free solution of (17) on $\mathbb{T}^d$ with $\alpha \in (0,2]$ and $d > \alpha$ and*

$$\|(\operatorname{div} \mathbf{v})_-\|_{L_t^\infty L_x^{d/\alpha}} \leq S_{\alpha/2}(\mathbb{T}^d),$$

then

$$\|\psi(s)\|_{L^2}^2 + S_{\alpha/2}(\mathbb{T}^d) \int_0^s \|\psi(\tau)\|_{L^\sigma}^2 \, d\tau \leq \|\psi_0\|_{L^2}^2 \qquad \textit{with} \qquad \sigma = \frac{2d}{d-\alpha}.$$

4. Propagation of the atom property by the dual conservation law

As long as the advection field has mildly convergent characteristics (expressed precisely by (59)), the weak maximum principle implies that the (nonlocal) diffusion propagates the properties of atoms. It is possible to trade a slow increase in each atomic radius to gain some decay in amplitude.

4.1. Local propagation

Proposition 18 *Let us assume that* $1 \le \alpha \le 2$ *and* $d > \alpha$ *and that the velocity field* $\mathbf{v} \in \mathrm{BMO}$ *satisfies*

$$(p-1)\|(\operatorname{div}\mathbf{v})_-\|_{L^\infty_t L^{d/\alpha}_x} \le S_{\alpha/2}(\mathbb{R}^d) \tag{31}$$

for some $p \ge 2$ *(eventually restricted to* $p = 2^n$ *with* $n \in \mathbb{N}$ *if* $\alpha < 2$*) such that*

$$p > \frac{d}{d-(\alpha-\omega)} \quad \text{with} \quad 0 < \omega < 1.$$

Then there exist constants δ, K *and* γ*, depending only on* d*,* p*,* α *and* $\|\mathbf{v}\|_{\mathrm{BMO}}$*, such that for all* $r \in (0,1]$*, the following implication holds:*

$$\psi_0 \in \mathcal{A}^p_r \qquad \Longrightarrow \qquad \forall s \in [0, \gamma r^\alpha], \quad \psi(s,\cdot) \in \left(1 - \frac{\delta s}{r^\alpha}\right) \mathcal{A}^p_{(r^\alpha + Ks)^{1/\alpha}}, \tag{32}$$

where ψ *denotes the solution of the Cauchy problem* (17)*. The constant* A*, which is implicit in the definition of* $\mathcal{A}^p_r$*, has to be chosen large enough. The admissible threshold for* A*, which also depends only on* d*,* p*,* α *and* $\|\mathbf{v}\|_{\mathrm{BMO}}$*, is specified in Remark 21.*

Remark 19 The proposition holds with $p = 2$ if $d > 2(\alpha - \omega)$, which is always possible if one chooses ω such that $\alpha - 1 < \omega < 1$ when $\alpha < 2$ (and $\omega > 1/2$ when $\alpha = 2$ and $d \ge 3$); in this case, (59) is also the least restrictive. Thanks to Proposition 17, the result then also holds, mutatis mutandis, on $\mathbb{T}^d$.

Proof The proof of Proposition 18 is inspired by [16] and [9], though the fractional derivative requires some additional care. Thanks to (19), the zero-average property of atoms is obviously propagated by (17).

Let $x(s)$ be the solution to the following ODE, which tracks the average flow on a ball of size r. It is obviously well defined for $\mathbf{v} \in L^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)$:

$$\begin{cases} x'(s) = \bar{\mathbf{v}}_{\mathcal{B}(x(s),r)} \\ x(0) = x_0 \end{cases} \qquad \text{where} \qquad \bar{\mathbf{v}}_{\mathcal{B}(x,r)}(s) = \frac{1}{|\mathcal{B}(x,r)|}\int_{\mathcal{B}(x,r)} \mathbf{v}(s,y)\,dy. \tag{33}$$

Step 1. Strict decay of the L^1-norm. One introduces $S = \psi(s,\cdot)^{-1}(\{0\})$ and $D_\pm = \operatorname{supp}\psi_\pm(s)$. Arguing as in [16, §4] and taking advantage of the conservative form of (17):

$$\frac{d}{ds}\|\psi(s)\|_{L^1} = -\int_{D_+\cup D_-} \frac{\psi}{|\psi|}(-\Delta)^{\alpha/2}\psi + \int_S |(-\Delta)^{\alpha/2}\psi|. \tag{34}$$

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The kernel formula (2) allows us to improve (18) and gives

$$\frac{d}{ds}\|\psi(s)\|_{L^1} \le -\frac{c_{d,\alpha}}{2}\iint_{(D_+\cup D_-)^2}\left(\frac{\psi(y)}{|\psi(y)|}-\frac{\psi(x)}{|\psi(x)|}\right)\frac{\psi(y)-\psi(x)}{|y-x|^{d+\alpha}}\,dydx \tag{35}$$

because (if S is a set of strictly positive measure)

$$\int_S |(-\Delta)^{\alpha/2}\psi(y)|\,dy \le c_{d,\alpha}\int_{x\in D_+\cup D_-}\int_{y\in S}\frac{|\psi(x)|}{|y-x|^{d+\alpha}}\,dydx.$$

The right-hand side of (35) is negative:

$$\begin{aligned}\frac{d}{ds}\|\psi(s)\|_{L^1} \le -c_{d,\alpha}\Bigg[&\int_{D_+}\left(\int_{D_-}\frac{dy}{|x-y|^{d+\alpha}}\right)\psi_+(s,x)\,dx\\ &+\int_{D_-}\left(\int_{D_+}\frac{dy}{|x-y|^{d+\alpha}}\right)\psi_-(s,x)\,dx\Bigg].\end{aligned}$$

As explained in [16], the idea is that the domains of integration $D_\pm$ can be reduced to $D_\pm\cap\mathcal{B}(x(s),100r)$ and that, provided that most of the L^1-mass of ψ is localised in $\mathcal{B}(x(s),100r)$, which is ensured by the third part of the proof, it ends up giving

$$\frac{d}{ds}\|\psi(s)\|_{L^1} \lesssim -r^{-\alpha};$$

that is, for δ and $\gamma>0$ small enough,

$$\forall s\in[0,\gamma r^\alpha],\qquad \|\psi(s)\|_{L^1}\le 1-\frac{\delta s}{r^\alpha}\cdot \tag{36}$$

For the sake of clarity, let us give a few more details about the computation that converts (34) into (36). The definition of S and the kernel formula (2) for $(-\Delta)^{\alpha/2}$ give

$$\int_S |(-\Delta)^{\alpha/2}\psi| \le M_1 = c_{d,\alpha}\int_S\left|\int_{D_+}\frac{\psi(y)\,dy}{|x-y|^{d+\alpha}}+\int_{D_-}\frac{\psi(y)\,dy}{|x-y|^{d+\alpha}}\right|dx$$

and, similarly,

$$-\int_{D_+\cup D_-}\frac{\psi}{|\psi|}(-\Delta)^{\alpha/2}\psi = M_2+M_3^+ + M_3^-$$

with

$$M_2=-c_{d,\alpha}\int_{D_+\cup D_-}\int_S\frac{|\psi(x)|}{|x-y|^{d+\alpha}}\,dydx \quad\text{and}\quad M_3^\pm=\pm c_{d,\alpha}\int_{D_\pm}\int_{D_\mp}\frac{\psi(y)-\psi(x)}{|x-y|^{d+\alpha}}\,dydx.$$

Discussing which of the contributions to M_1 on $D_\pm$ is the biggest in absolute value, one checks immediately that one has

$$M_1+M_2\le 2c_{d,\alpha}\int_S \max\left(-\int_{D_+}\frac{\psi(y)\,dy}{|x-y|^{d+\alpha}},\int_{D_-}\frac{\psi(y)\,dy}{|x-y|^{d+\alpha}}\right)dx<0.$$

At this point, note that one can suppose that $9/10\le\|\psi(s)\|_{L^1}\le 1$ because otherwise the strict decay of the L^1-norm would already be achieved once δ was small enough to have $10\delta s<r^\alpha$. Similarly, thanks to the subsequent part 3 of the proof, one can also assume

that $\int_{\mathbb{R}^d} \Omega(x-x(s))|\psi(x,s)|dx \leq \frac{11}{10} r^{\omega}$ provided the time interval $[0, \gamma r^{\alpha}]$ is short enough. This anticipation on the estimate (49) is logically acceptable because one will only rely on Proposition 13 to prove it. These bounds imply

$$\int_{B(x(s),100r)} |\psi(x,s)|dx \geq \frac{4}{5}$$

and, thanks to the mean-zero property, also

$$\int_{D_+ \cap B(x(s),100r)} |\psi(x,s)|dx \geq \frac{3}{10} \quad \text{and} \quad \int_{D_- \cap B(x(s),100r)} |\psi(x,s)|dx \geq \frac{3}{10}.$$

As a consequence, we deduce that

$$\int_{D_+} \frac{\psi(y)\,dy}{|x-y|^{d+\alpha}} \geq \int_{\widetilde{D}_+} \frac{\psi(y)\,dy}{|x-y|^{d+\alpha}} \geq Cr^{-d-\alpha}$$
$$-\int_{D_-} \frac{\psi(y)\,dy}{|x-y|^{d+\alpha}} \geq -\int_{\widetilde{D}_-} \frac{\psi(y)\,dy}{|x-y|^{d+\alpha}} \geq Cr^{-d-\alpha},$$

where $\widetilde{D}_\pm = D_\pm \cap B(x(s),100r)$ and $\widetilde{S} = S \cap B(x(s),100r)$. Hence,

$$M_1 + M_2 \leq 2c_{d,\alpha} \int_S \max\left(-\int_{\widetilde{D}_+} \frac{\psi(y)\,dy}{|x-y|^{d+\alpha}}, \int_{\widetilde{D}_-} \frac{\psi(y)\,dy}{|x-y|^{d+\alpha}}\right) dx \leq -Cr^{-d-\alpha}|\widetilde{S}|.$$

On the other hand, using the antisymmetric roles of x and y and observing that the sign of ψ works in our favour in order to control the last terms, one gets

$$\begin{aligned} M_3^+ + M_3^- &\leq -2c_{d,\alpha} \int_{\widetilde{D}_+} \psi(x) \int_{\widetilde{D}_-} \frac{dy}{|x-y|^{d+\alpha}} dx + 2c_{d,\alpha} \int_{\widetilde{D}_-} \psi(y) \int_{\widetilde{D}_+} \frac{dx}{|x-y|^{d+\alpha}} dy \\ &\leq -r^{-d-\alpha}|\widetilde{D}_-| \int_{\widetilde{D}_+} \psi + r^{-d-\alpha}|\widetilde{D}_+| \int_{\widetilde{D}_-} \psi \leq -Cr^{-d-\alpha}(|\widetilde{D}_+| + |\widetilde{D}_-|). \end{aligned}$$

Finally, because $|\widetilde{S}| + |\widetilde{D}_-| + |\widetilde{D}_+| \geq Cr^d$, one gets, as announced:

$$\begin{aligned} \frac{d}{ds}\|\psi(s)\|_{L^1} &= -\int_{D_+ \cup D_-} \frac{\psi}{|\psi|}(-\Delta)^{\alpha/2}\psi + \int_S |(-\Delta)^{\alpha/2}\psi| \leq M_1 + M_2 + M_3^+ + M_3^- \\ &\leq -Cr^{-d-\alpha}(|\widetilde{S}| + |\widetilde{D}_-| + |\widetilde{D}_+|) \leq -Cr^{-\alpha}. \end{aligned}$$

Step 2. Strict decay of the L^p norm. We have already proven that, under the smallness assumption (59), the right-hand side of (30) can be resorbed within the elliptic term; that is,

$$\frac{d}{ds}\|\psi\|_{L^p}^p \leq -S_{\alpha/2}(\mathbb{R}^d) \cdot \|\psi\|_{L^\sigma}^p < 0.$$

Next, as $\sigma = \frac{dp}{d-\alpha} > p$, one can use an elegant idea of [8, p. 517], which is to combine the interpolation inequality $\|\psi\|_{L^p} \leq \|\psi\|_{L^1}^{1-\theta}\|\psi\|_{L^\sigma}^{\theta}$ for $\theta = \frac{(p-1)d}{(p-1)d+\alpha}$ with the propagation of the L^1-norm (18). Because ψ_0 is an atom $\mathcal{A}_r^p$, it ensures that

$$\frac{d}{ds}\|\psi\|_{L^p}^p \leq -S_{\alpha/2}(\mathbb{R}^d) \cdot \|\psi_0\|_{L^1}^{-\frac{\alpha p}{(p-1)d}} \|\psi\|_{L^p}^{p/\theta} \leq -S_{\alpha/2}(\mathbb{R}^d) \cdot \|\psi\|_{L^p}^{p/\theta}. \tag{37}$$

This is a Riccati-type ODE that can be solved explicitly:

$$\|\psi(s)\|_{L^p}^p \le \left(\|\psi_0\|_{L^p}^{-\frac{\alpha p}{(p-1)d}} + \frac{\alpha S_{\alpha/2}(\mathbb{R}^d)}{(p-1)d}\cdot s\right)^{-\frac{(p-1)d}{\alpha}}.$$

Using the atom property $\|\psi_0\|_{L^p} \le Ar^{-d(1-\frac{1}{p})}$ and rearranging the terms, one gets

$$\|\psi(s)\|_{L^p} \le Ar^{-d(1-\frac{1}{p})}\left(1+\frac{\alpha S_{\alpha/2}(\mathbb{R}^d)}{(p-1)d}\cdot A^{\frac{\alpha p}{(p-1)d}}r^{-\alpha}s\right)^{-\frac{d}{\alpha}(1-\frac{1}{p})};$$

that is, $\|\psi(s)\|_{L^p} \le Ar(s)^{-d(1-\frac{1}{p})}$ with

$$r(s) = \left(r^\alpha + C_{d,p,\alpha}A^\mu s\right)^{1/\alpha} \quad \text{and} \quad \mu = \frac{\alpha}{d(1-\frac{1}{p})}. \tag{38}$$

One chooses $\delta > 0$ small enough, then

$$0 < K < C_{d,p,\alpha}A^\mu - \frac{2\delta}{\frac{d}{\alpha}\left(1-\frac{1}{p}\right)}. \tag{39}$$

Thanks to the reversed Bernoulli inequality $(1-x)^{-1/\beta} \le 1+2x/\beta$ for $\beta > 1$ and $x \in [0,1/2]$, this choice ensures that

$$\forall t \in [0,\gamma], \qquad \frac{1+C_{d,p,\alpha}A^\mu t}{1+Kt} \ge 1+\frac{2\delta t}{\frac{d}{\alpha}(1-\frac{1}{p})} \ge (1-\delta t)^{-1/[\frac{d}{\alpha}(1-\frac{1}{p})]}$$

with $\gamma = \frac{\frac{d}{\alpha}(1-\frac{1}{p})}{2\delta}\left(\frac{C_{d,p,\alpha}A^\mu - 2\delta/[\frac{d}{\alpha}(1-\frac{1}{p})]}{K} - 1\right)$ and thus, after substituting $t = s/r^\alpha$:

$$\|\psi(s)\|_{L^p} \le A\left(1-\frac{\delta s}{r^\alpha}\right)(r^\alpha + Ks)^{-\frac{d}{\alpha}(1-\frac{1}{p})} \tag{40}$$

for $s \in [0, \gamma r^\alpha]$.

Step 3. Propagation of the concentration. With $x(s)$ defined by (33), one considers

$$\chi(s) = \int_{\mathbb{R}^d} \psi(s,x)\Omega(x-x(s))\,dx. \tag{41}$$

Using Equation (17) and the fact that $(-\Delta)^{\alpha/2}$ is self-adjoint, the derivative of χ satisfies

$$\chi'(s) = \int_{\mathbb{R}^d} (\mathbf{v} - \bar{\mathbf{v}}_{\mathcal{B}(x(s),r)}) \cdot \nabla\Omega(x-x(s))\psi(s,x)\,dx - \int_{\mathbb{R}^d} \psi(s,x)\cdot(-\Delta)^{\alpha/2}\Omega(x-x(s))\,dx. \tag{42}$$

Let us collect obvious estimates for the derivatives of Ω:

$$|\nabla\Omega(z)| \lesssim |z|^{-(1-\omega)} \cdot \mathbf{1}_{\mathcal{B}(0,2)}(z), \tag{43a}$$

$$|(-\Delta)^{\alpha/2}\Omega(z)| \lesssim |z|^{-(\alpha-\omega)_+} \cdot \mathbf{1}_{\mathcal{B}(0,2)}(z) + |z|^{-2-\alpha} \cdot \mathbf{1}_{\mathcal{B}(0,2)^c}(z). \tag{43b}$$

They follow easily from the scaling properties of the Fourier transform (and thus of $(-\Delta)^{\alpha/2}$) and from the kernel representation (2). Recall that we assume $\omega < \min\{\alpha, 1\}$ throughout the proof.

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3a. Transport term in χ'. Let us introduce $E_k(s) := \{x \in \mathbb{R}^d : |x - x(s)| \in [2^{k-1}r, 2^k r)\}$ to estimate

$$I_1 = \left| \int_{\mathbb{R}^d} (\mathbf{v} - \bar{\mathbf{v}}_{\mathcal{B}(x(s),r)}) \cdot \nabla\Omega(x - x(s))\psi(s,x)\,dx \right|.$$

One has $I_1 \lesssim J_0 + \sum_{k=1}^{\infty} J_k$ with

$$J_0 = \int_{\mathcal{B}(x(s),r)} |\mathbf{v} - \bar{\mathbf{v}}_{\mathcal{B}(x(s),r)}||x - x(s)|^{-(1-\omega)}|\psi|$$

and

$$J_k = \left(\int_{E_k(s)} |\mathbf{v} - \bar{\mathbf{v}}_{\mathcal{B}(x(s),r)}||\psi| \right) r^{-(1-\omega)} 2^{-k(1-\omega)}.$$

For J_0, we use the Hölder inequality with $a^{-1} + b^{-1} + c^{-1} = 1$ and the BMO property

$$\begin{aligned} J_0 &\le \|\mathbf{v} - \bar{\mathbf{v}}_{\mathcal{B}(x(s),r)}\|_{L^a(\mathcal{B}(x(s),r))} \|\psi_0\|_{L^b} \||x - x(s)|^{-(1-\omega)}\|_{L^c(\mathcal{B}(x(s),r))} \\ &\lesssim \|\mathbf{v}\|_{\mathrm{BMO}} \cdot r^{d/a} \cdot A^{b_*} r^{-d(1-\frac{1}{b})} \cdot r^{\frac{d}{c} - (1-\omega)} \\ &\lesssim r^{-(1-\omega)} A^{b_*} \|\mathbf{v}\|_{\mathrm{BMO}} \end{aligned}$$

with

$$b_* = \frac{1 - \frac{1}{b}}{1 - \frac{1}{p}} = \frac{p'}{b'}. \tag{44}$$

Here, one should comment on the choice of the powers a, b, c. Obviously, we have to take $c < d/(1-\omega)$ for local integrability reasons. Second, because we used the decay of the L^b norm of ψ given by Proposition 13 followed by Proposition 5 on ψ_0, one needs $p \ge b > d/(d - (1-\omega))$. Because a can be chosen arbitrary large, it is always possible to find a proper triplet (a, b, c) as soon as

$$p > \frac{d}{d - (1-\omega)}. \tag{45}$$

For J_k with $k \ge 1$, we apply the Hölder inequality with a pair of conjugate powers q_1 and q_1', with $q_1 > d/(1-\omega)$. Thanks to (45), one thus has $q_1' < \frac{d}{d-(1-\omega)} < p$, which ensures again that we have propagation of the $L^{q_1'}$ norm of ψ and that Proposition 5 may be used liberally on ψ_0. One also uses that for BMO functions, the averages of adjacent dyadic balls are comparable and that $\|\psi\|_{L^1(E_k)} \le 2^{kd/q_1} \|\psi\|_{L^{q_1'}(E_k)}$ uniformly for $r \in (0,1]$. One thus gets

$$\begin{aligned} \int_{E_k(s)} |\mathbf{v} - \bar{\mathbf{v}}_{\mathcal{B}(x(s),r)}||\psi| &\le \int_{E_k(s)} |\mathbf{v} - \bar{\mathbf{v}}_{\mathcal{B}(x(s),2^k r)}||\psi| + \int_{E_k(s)} |\bar{\mathbf{v}}_{\mathcal{B}(x(s),2^k r)} - \bar{\mathbf{v}}_{\mathcal{B}(x(s),r)}||\psi| \\ &\lesssim \|\mathbf{v} - \bar{\mathbf{v}}_{\mathcal{B}(x(s),2^k r)}\|_{L^{q_1}(\mathcal{B}(x(s),2^k r))} \|\psi_0\|_{L^{q_1'}} + k\|\mathbf{v}\|_{\mathrm{BMO}} \|\psi\|_{L^1(E_k)} \\ &\lesssim (1+k) 2^{kd/q_1} A^{p'/q_1} \|\mathbf{v}\|_{\mathrm{BMO}}. \end{aligned}$$

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Because we choose $d/q_1 < 1-\omega$, the geometric series in $k \geq 1$ is convergent and $p'/q_1 < b_*$, and thus

$$I_1 \lesssim r^{-(1-\omega)} A^{b_*} \|\mathbf{v}\|_{\mathrm{BMO}}. \tag{46}$$

Let us observe that, owing to the admissible range for b, the value of b_* can be chosen arbitrarily within the interval

$$\frac{1-\omega}{d(1-\frac{1}{p})} < b_* \leq 1.$$

In the next part of this proof, we will chose b_* to be as close as possible to the lowest bound.

Remark 20 As $\operatorname{supp} \nabla\Omega \subset \mathcal{B}(0,2)$ the series of J_k terms is limited to $k \lesssim |\log r|$. However, this upper bound becomes arbitrarily large when $r \to 0$. Our previous estimate is uniform for $r \in (0,1]$. □

3b. Nonlocal viscous term in χ'. Let us now consider the second term of (42):

$$I_2 = \left| \int_{\mathbb{R}^d} \psi(s,x) \cdot (-\Delta)^{\alpha/2} \Omega(x - x(s))\, dx \right|.$$

Recall that we assume $\alpha > \omega$. Thanks to (43b), for any $0 < \rho \leq r < 1$, one has

$$I_2 \lesssim \int_{\mathcal{B}(x(s),\rho)} |x - x(s)|^{-(\alpha-\omega)} |\psi(s,x)|\, dx + \rho^{-(\alpha-\omega)} \|\psi_0\|_{L^1}.$$

We apply the Hölder inequality with another pair of conjugate powers q_2 and q_2', with $1 \leq q_2 < \frac{d}{\alpha-\omega}$, which is always possible. One also needs $q_2' \leq p$ to ensure the propagation of the $L^{q_2'}$ norm by Proposition 13; that is, $\frac{1}{q_2} \leq 1 - \frac{1}{p}$. Such a choice is possible if

$$p > \frac{d}{d-(\alpha-\omega)}. \tag{47}$$

Because $\alpha \geq 1$, this restriction on p supersedes (45). One gets

$$I_2 \leq \| |x - x(s)|^{-(\alpha-\omega)} \|_{L^{q_2}(\mathcal{B}(x(s),\rho))} \|\psi_0\|_{L^{q_2'}} + \rho^{-(\alpha-\omega)} \|\psi_0\|_{L^1};$$

that is,

$$I_2 \lesssim \rho^{\frac{d}{q_2}-(\alpha-\omega)} A^{p'/q_2} r^{-d/q_2} + \rho^{-(\alpha-\omega)}.$$

The optimal choice for ρ is given by $\rho = rA^{-p'/d}$, which belongs indeed to $(0,r]$ as $A \gg 1$. Substituting this value in the previous estimate of I_2 gives

$$I_2 \lesssim r^{-(\alpha-\omega)} A^{\mu_*} \qquad \text{with} \qquad \mu_* = \frac{\alpha-\omega}{d(1-\frac{1}{p})}. \tag{48}$$

3c. Conclusion. Putting together (42) with (46) and (48), one gets

$$|\chi'(s)| \le I_1 + I_2 \lesssim r^{-(1-\omega)} A^{b_*} \|\mathbf{v}\|_{\mathrm{BMO}} + r^{-(\alpha-\omega)} A^{\mu_*}.$$

After integration and considering that $\chi(0) \le r^{\omega}$ and $1 \le r^{-1} \le r^{-\alpha}$ because $\alpha \ge 1$:

$$\chi(s) \le r^{\omega} \left(1 + C'_{d,p,\alpha} \left[A^{b_*} \|\mathbf{v}\|_{\mathrm{BMO}} + A^{\mu_*}\right] \frac{s}{r^{\alpha}}\right). \tag{49}$$

Provided A is large enough, one may amend the previous choice (39) of δ and K to ensure that

$$K \ge \frac{\alpha}{\omega} \left\{\delta + C'_{d,p,\alpha} \left[A^{b_*} \|\mathbf{v}\|_{\mathrm{BMO}} + A^{\mu_*}\right]\right\}, \tag{50}$$

which, in turn, ensures that

$$\forall t \in [0,\gamma], \qquad 1 + C'_{d,p,\alpha} \left[A^{b_*} \|\mathbf{v}\|_{\mathrm{BMO}} + A^{\mu_*}\right] t \le (1-\delta t)(1+Kt)^{\omega/\alpha};$$

that is, with $t = s/r^{\alpha}$:

$$\forall s \in [0,\gamma r^{\alpha}], \qquad \chi(s) \le \left(1 - \frac{\delta s}{r^{\alpha}}\right)(r^{\alpha} + Ks)^{\omega/\alpha}. \tag{51}$$

This concludes the proof of $\psi(s) \in \left(1 - \frac{\delta s}{r^{\alpha}}\right) \mathcal{A}^{p}_{(r^{\alpha}+Ks)^{1/\alpha}}$. □

Remark 21 Our conditions (39) and (50) generalise respectively the conditions 4.6–4.15 of [16]. Both conditions are compatible, provided A is chosen large enough, because

$$\mu > b_* \vee \mu_*.$$

In turn, this condition is satisfied by choosing b_* as small as possible and because $\alpha > \omega \vee (1-\omega)$.

Remark 22 It could be tempting to discard the L^1-property from the atom definition and use Proposition 28 from the Appendix to control this norm a posteriori. In this case, instead of (39) and (50), one is led to choose δ and K such that

$$\frac{\alpha}{\omega}\left\{\delta + C'_{d,p,\alpha}\left[A^{\tilde{b}} \|\mathbf{v}\|_{\mathrm{BMO}} + A^{\tilde{\mu}}\right]\right\} \le K < C_{d,p,\alpha} A^{\tilde{\mu}} - \frac{2\delta}{\frac{d}{\alpha}\left(1-\frac{1}{p}\right)}$$

with $\tilde{b} = \frac{\omega + d(1-\frac{1}{b})}{\omega + d(1-\frac{1}{p})} \in (\tilde{\mu}/\alpha, 1]$ and $\tilde{\mu} = \frac{\alpha}{\omega + d(1-\frac{1}{p})}$. Because both sides are order $A^{\tilde{\mu}}$, it is not clear anymore that the choice can be resolved for some large value of A. This alternate path is thus a subtle deadlock.

4.2. Global propagation

Proposition 23 *In the conditions of Proposition 18, the constants δ, K are such that*

$$\psi_0 \in \mathcal{A}^p_r \qquad \Longrightarrow \qquad \forall s > 0, \qquad \psi(s,\cdot) \in \left(\frac{r^{\alpha}}{r^{\alpha} + Ks}\right)^{\delta/K} \mathcal{A}_{(r^{\alpha}+Ks)^{1/\alpha}}, \tag{52}$$

where ψ denotes the solution of the Cauchy problem (17).

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Proof We keep the assumptions and notations of Proposition 18. Let us split the timeline in consecutive intervals $[\ell\gamma r^\alpha, (\ell+1)\gamma r^\alpha]$ with $\ell \in \mathbb{N}$. For $\ell = 0$, one has

$$\forall s \in [0, \gamma r^\alpha], \qquad 1 - \frac{\delta s}{r^\alpha} \le \left(1 + \frac{Ks}{r^\alpha}\right)^{-\delta/K} = \left(\frac{r^\alpha}{r^\alpha + Ks}\right)^{\delta/K}.$$

Let us assume that, for some integer $\ell \in \mathbb{N}$, one has

$$\psi(\ell\gamma r^\alpha, \cdot) \in (1 + K\ell\gamma)^{-\delta/K} \mathcal{A}_{r(1+K\ell\gamma)^{1/\alpha}}.$$

Then for any $s \in [\ell\gamma r^\alpha, (\ell+1)\gamma r^\alpha]$, Proposition 18 gives

$$\psi(s, \cdot) \in (1 + K\ell\gamma)^{-\delta/K} \left(1 + \frac{KS}{R^\alpha}\right)^{-\delta/K} \mathcal{A}_{(R^\alpha + KS)^{1/\alpha}}$$

with $S = s - \ell\gamma r^\alpha$ and $R = r(1 + K\ell\gamma)^{1/\alpha}$. The new radius is an exact match:

$$(R^\alpha + KS)^{1/\alpha} = (r^\alpha + Ks)^{1/\alpha}. \tag{53}$$

Similarly, the amplitude satisfies

$$(1 + K\ell\gamma)\left(1 + \frac{KS}{R^\alpha}\right) = 1 + \frac{Ks}{r^\alpha}.$$

The proposition thus follows by induction on $\ell \in \mathbb{N}$. □

4.3. Modifications in the case $0 < \alpha < 1$

Throughout Section 4, the assumption $\alpha \ge 1$ has only been used in the third step of the proof of Proposition 18; that is, to ascertain the propagation of the concentration. Let us investigate in this subsection how to deal with the case $0 < \alpha < 1$.

When dealing with the supercritical case, Silvestre [20] assumes a higher regularity for the advection field. We will do the same here and assume respectively that $v \in C^{1-\alpha}(\mathbb{R}^d)$ or $C^{1-\alpha}(\mathbb{T}^d)$. The identity (42) still holds. To deal with the transport term, one uses

$$\left|\mathbf{v}(x) - \bar{\mathbf{v}}_{\mathcal{B}(x(s),r)}\right| \le \frac{1}{|\mathcal{B}(x(s),r)|}\int_{\mathcal{B}(x(s),r)} |\mathbf{v}(x) - \mathbf{v}(y)|\,dy \le \|\mathbf{v}\|_{C^{1-\alpha}} \fint_{\mathcal{B}(x(s),r)} |x-y|^{1-\alpha}\,dy,$$

which gives

$$\left|\mathbf{v}(x) - \bar{\mathbf{v}}_{\mathcal{B}(x(s),r)}\right| \le (|x - x(s)| + r)^{1-\alpha}\|\mathbf{v}\|_{C^{1-\alpha}}. \tag{54}$$

For J_0, one takes $a = \infty$ and the same constraints for the exponents b and c; thus,

$$J_0 \lesssim r^{-(\alpha-\omega)} A^{b_*}\|\mathbf{v}\|_{C^{1-\alpha}}.$$

Similarly, for J_k, one takes $q_1 = \infty$:

$$\int_{E_k(s)} |\mathbf{v} - \bar{\mathbf{v}}_{\mathcal{B}(x(s),r)}||\psi| \lesssim 2^{k(1-\alpha)} r^{1-\alpha}\|\mathbf{v}\|_{C^{1-\alpha}}.$$

Thus, $J_k \lesssim 2^{-k(\alpha-\omega)} r^{-(\alpha-\omega)} \|\mathbf{v}\|_{C^{1-\alpha}}$ and as $\omega < \alpha$, the geometric series in k is convergent. The estimate (46) can therefore be replaced by

$$I_1 \lesssim r^{-(\alpha-\omega)} A^{b_*} \|\mathbf{v}\|_{C^{1-\alpha}} \qquad \text{with} \quad b_* = \frac{1-\omega}{d(1-\frac{1}{p})} + \varepsilon, \quad \varepsilon > 0. \tag{55}$$

For the nonlocal viscous term, the estimate (48) remains valid. The sole difference is that now

$$\frac{d}{d-(1-\omega)} > \frac{d}{d-(\alpha-\omega)}$$

and, consequently, the requirement (45) trumps (47).

Putting together (42) with (55) and (48), one gets

$$|\chi'(s)| \lesssim r^{-(\alpha-\omega)} \left(A^{b_*} \|\mathbf{v}\|_{C^{1-\alpha}} + A^{\mu_*}\right)$$

and

$$\chi(s) \le r^{\omega} \left(1 + C'_{d,p,\alpha} \left[A^{b_*} \|\mathbf{v}\|_{C^{1-\alpha}} + A^{\mu_*}\right] \frac{s}{r^{\alpha}}\right).$$

The conclusion is identical, provided that A is large enough and that the choice of K and δ ensures

$$K \ge \frac{\alpha}{\omega} \left\{\delta + C'_{d,p,\alpha} \left[A^{b_*} \|\mathbf{v}\|_{C^{1-\alpha}} + A^{\mu_*}\right]\right\} \tag{56}$$

instead of (50). Note that to reconcile (56) with (39) for large A, one needs $\alpha > \omega \vee (1-\omega)$, which is always possible if $\alpha > 1/2$. However, when $\alpha \le 1/2$, one needs one final modification, which is to replace the average $\bar{\mathbf{v}}_{\mathcal{B}(x(s),r)}$ by the pointwise value $\mathbf{v}(x(s))$, where

$$\begin{cases} x'(s) = \mathbf{v}(x(s)), \\ x(0) = x_0. \end{cases} \tag{57}$$

In this case, estimate (54) is improved one step further into the following one:

$$|\mathbf{v}(x) - \mathbf{v}(x(s))| \le |x - x(s)|^{1-\alpha} \|\mathbf{v}\|_{C^{1-\alpha}}. \tag{58}$$

This changes J_0 into

$$\widetilde{J}_0 = \int_{\mathcal{B}(x(s),r)} |x - x(s)|^{-(\alpha-\omega)} |\psi|,$$

which is then estimated in a manner identical to I_2. Note that this modification also allows us to drop all requirements concerning b_* and, in particular, (45), which is beneficial for any $\alpha \in (0,1)$. Let us finally point out that, in the other parts of the proof, the requirement $d > \alpha$ now allows for any dimension $d \ge 1$. We have thus established the following statement.

Proposition 24 *Let us assume that* $0 < \alpha < 1$ *and* $d \ge 1$ *and that the velocity field* $\mathbf{v} \in C^{1-\alpha}$ *satisfies*

$$(p-1)\|(\operatorname{div}\mathbf{v})_-\|_{L_t^\infty L_x^{d/\alpha}} \le S_{\alpha/2}(\mathbb{R}^d) \tag{59}$$

for some $p = 2^n$ with $n \in \mathbb{N}$ such that

$$p > \frac{d}{d-(\alpha-\omega)} \qquad \text{with} \qquad 0 < \omega < \alpha.$$

Then there exist constants δ, K and γ (and a lower threshold for A), depending only on d, p, α and $\|\mathbf{v}\|_{C^{1-\alpha}}$, such that for all $r \in (0,1]$, the following implication holds:

$$\psi_0 \in \mathcal{A}_r^p \qquad \Longrightarrow \qquad \forall s > 0, \qquad \psi(s,\cdot) \in \left(1+\frac{Ks}{r^\alpha}\right)^{-\delta/K} \mathcal{A}_{(r^\alpha+Ks)^{1/\alpha}}, \tag{60}$$

where ψ denotes the solution of the Cauchy problem (17).

Remark 25 Note that we can take $p = 2$ in the previous statement (and thus, using Remark 19, claim a similar one in the case of $\mathbb{T}^d$) if

$$\frac{d}{2} > \alpha - \omega.$$

Such a choice is always possible.

5. Proof of Theorem 1

The proof of Theorem 1 is now straightforward.

Given $d \geq 2$ and $1 \leq \alpha \leq 2$ (with $d \geq 3$ when $\alpha = 2$), one chooses $\omega \in (0,1)$ such that $\alpha - 1 < \omega < 1$ if $\alpha < 2$, or $\omega > 1/2$ if $\alpha = 2$. One checks immediately that $\omega < \alpha < d$ and $d > 2(\alpha - \omega)$. Let us now consider an advection vector field $v \in \mathrm{BMO}$ with

$$\|(\operatorname{div} v)_-\|_{L^{d/\alpha}} \leq S_{\alpha/2}.$$

One takes $p = 2$. One chooses the constant A, which is implicit in the definition of atoms, according to the threshold mentioned in Remark 21; this threshold depends solely on d, α and $\|v\|_{\mathrm{BMO}}$. One considers the constants γ, δ and K given by Propositions 18 and 23 and sets

$$\beta = \alpha\delta/K.$$

The value of β depends on d, α and $\|v\|_{\mathrm{BMO}}$.

For $d \geq 1$ and $0 < \alpha < 1$, one chooses ω such that $(\alpha - \frac{d}{2})_+ < \omega < \alpha$ and $p = 2$. In this case, the BMO norm is replaced by the $C^{1-\alpha}$-norm in all computations.

Remark 26 When $\|(\operatorname{div} v)_-\|_{L^{d/\alpha}} < 2S_{\alpha/2}$, one can still run the following proof. However, the choice of A and of all constants then depends not only on $\|v\|_{\mathrm{BMO}}$ but also on $C(v) = 2S_{\alpha/2} - \|(\operatorname{div} v)_-\|_{L^{d/\alpha}} > 0$ and degenerates as $C(v) \to 0$. See Remark 15.

5.1. Propagation of the Hölder regularity

For any solution θ of (1) stemming from $\theta_0 \in C^\beta$ and for $\psi_0 \in \mathcal{A}_r^2$, identity (16) implies that

$$\int_{\mathbb{R}^d} \theta(t,x)\psi_0(x)\,dx = \int_{\mathbb{R}^d} \theta_0(x)\psi(t,x)\,dx,$$

where ψ is the solution of the dual equation (15), which, by Proposition 23, is an atom of calibrated size. Using (14) for θ_0, one gets

$$r^{-\beta}\left|\int_{\mathbb{R}^d}\theta(t,x)\psi_0(x)\,dx\right| \lesssim r^{-\beta}\left(1+\frac{Kt}{r^\alpha}\right)^{-\delta/K}(r^\alpha+Kt)^{\beta/\alpha}\|\theta_0\|_{C^\beta} = \|\theta_0\|_{C^\beta}.$$

A second application of (14) then ensures that $\theta(t)\in C^\beta$ and that

$$\|\theta(t)\|_{C^\beta} \leq C\|\theta_0\|_{C^\beta}.$$

The constant C is the implicit one in (14).

Remark 27 The same argument also gives $\|\theta(t)\|_{C^{\beta'}} \leq C\|\theta_0\|_{C^{\beta'}}$ for any $0\leq\beta'\leq\beta$.

5.2. Gain in Hölder regularity

One can use the Hölder inequality and Proposition 5 with $p=2$ to control

$$\left|\int_{\mathbb{R}^d}\theta_0(x)\psi(t,x)\,dx\right| \leq \|\theta_0\|_{L^q}\|\psi(t)\|_{L^{q'}} \leq \frac{A^{2/q}r^\beta\|\theta_0\|_{L^q}}{(r^\alpha+Kt)^{(\beta+\frac{d}{q})/\alpha}} \lesssim r^\beta t^{-(\beta+\frac{d}{q})/\alpha}\|\theta_0\|_{L^q}$$

for any Lebesgue exponent q such that $2\leq q\leq\infty$. One thus gets a regularisation estimate:

$$\|\theta(t,\cdot)\|_{C^\beta} \simeq \sup_{\substack{0<r\leq 1\\ \psi_0\in\mathcal{A}_r^2}} r^{-\beta}\left|\int_{\mathbb{R}^d}\theta(t,x)\psi_0(x)\,dx\right| \leq Ct^{-(\beta+\frac{d}{q})/\alpha}\|\theta_0\|_{L^q}$$

with a constant C that depends on q and A and thus ultimately on d, α and $\|v\|_{\mathrm{BMO}}$.

Appendix A. On the L^1-control of atoms

Even without the a priori constraint $\|\psi\|_{L^1}\leq 1$, one can control the L^1-norm of atoms (or any L^q norm for $q\leq p$) by a real interpolation estimate.

Proposition 28 *If φ satisfies*

$$\|\varphi\|_{L^p} \leq Ar^{-d(1-\frac{1}{p})}, \qquad \textit{and} \qquad \exists x_0\in\mathbb{R}^d, \quad \int_{\mathbb{R}^d}|\varphi(x)|\Omega(x-x_0)\,dx \leq r^\omega$$

for some $0<r\leq 1$ and $p\in(1,\infty]$, then

$$\|\varphi\|_{L^1} \leq C_{d,p}A^{\omega/(\omega+d(1-\frac{1}{p}))} \tag{61}$$

and, more generally, for any $1\leq q\leq p$, one has

$$\|\varphi\|_{L^q} \leq C_{d,p,q}A^{\frac{\omega+d(1-1/q)}{\omega+d(1-1/p)}}r^{-d(1-\frac{1}{q})}. \tag{62}$$

Remark 29 Compared to Proposition 5, these estimates 'lose' powers of A, which would provoke a critical collision of exponents in the previous proof (see Remark 22).

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Proof For any $\rho \in [0, r]$, one has

$$\|\varphi\|_{L^1} \leq \int_{\mathcal{B}(x_0,\rho)} |\varphi| + \rho^{-\omega} \int_{\mathbb{R}^d \setminus \mathcal{B}(x_0,\rho)} |\varphi(x)| \Omega(x-x_0)\,dx \leq A\left(\frac{\rho}{r}\right)^{d(1-\frac{1}{p})} |\mathcal{B}(0,1)|^{1-\frac{1}{p}} + \left(\frac{r}{\rho}\right)^{\omega},$$

and (61) follows from choosing the optimal value $\rho = r\big(A|\mathcal{B}(0,1)|^{1-\frac{1}{p}}\big)^{-1/(\omega+d(1-\frac{1}{p}))}$. For the second estimate, one proceeds similarly with $\tau \in [0, r]$; for clarity, we do not track the constant related to $|\mathcal{B}(0,1)|$:

$$\begin{aligned}\int_{\mathbb{R}^d} |\varphi|^q &\leq \int_{\mathcal{B}(x_0,\tau)} |\varphi|^q + \int_{\mathbb{R}^d \setminus \mathcal{B}(x_0,\tau)} |\varphi|^q \\ &\lesssim \left(\int_{\mathbb{R}^d} |\varphi|^p\right)^{\frac{q}{p}} \tau^{d(1-\frac{q}{p})} + \left(\tau^{-\omega}\int_{\mathbb{R}^d \setminus \mathcal{B}(x_0,\tau)} |\varphi(x)|\Omega(x-x_0)\,dx\right)^{\frac{p-q}{p-1}} \left(\int_{\mathbb{R}^d} |\varphi|^p\right)^{\frac{q-1}{p-1}},\end{aligned}$$

thanks to the Hölder inequality (with $p/q \geq 1$) for the first term and the interpolation inequality $\|f\|_{L^q} \leq \|f\|_{L^1}^{1-\theta}\|f\|_{L^p}^{\theta}$ with $\theta = (1-\frac{1}{q})/(1-\frac{1}{p}) \in [0,1]$ for the second. We now use the fact that $\varphi \in \mathcal{A}_r^p$ and deduce that

$$\int_{\mathbb{R}^d} |\varphi|^q \lesssim \left(Ar^{-d(1-\frac{1}{p})}\right)^q \tau^{d(1-\frac{q}{p})} + \left(\frac{r}{\tau}\right)^{\omega(\frac{p-q}{p-1})} (A^p r^{-(p-1)d})^{\frac{q-1}{p-1}}.$$

The optimal choice for τ is the one that balances the weight of both terms; it is $\tau = rA^{-p/(d(p-1)+\omega p)}$. The computation then boils down to

$$\int_{\mathbb{R}^d} |\varphi|^q \lesssim r^{-(q-1)d} A^{\frac{dp(q-1)+\omega pq}{d(p-1)+\omega p}} \qquad i.e. \qquad \|\varphi\|_{L^q} \lesssim A^{\frac{\omega+d(1-1/q)}{\omega+d(1-1/p)}} r^{-d(1-\frac{1}{q})}$$

and the lemma is proven. □

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