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RESEARCH ARTICLE

L^p harmonic 1-forms on hypersurfaces with finite index

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Abstract
In the present note, we establish a finiteness theorem for L^p harmonic 1-forms on hypersurfaces with finite index, which is an extension of the result of Choi and Seo (*J. Geom. Phys.* **129** (2018), 125–132).

1. Introduction

It is an interesting problem in geometry and topology to find sufficient conditions on the manifold for the space of harmonic k-forms to be trivial. The nonexistence of nontrivial L^2 harmonic 1-forms on a complete noncompact submanifold has been studied by many geometers.

Palmer [23] proved that a complete minimal hypersurface in the Euclidean space $\mathbb{R}^{n+1}$ has no nontrivial L^2 harmonic 1-forms. Thereafter, using the Bochner's vanishing technique, Miyaoka [22] obtained the nonexistence of nontrivial L^2 harmonic 1-forms on complete orientable noncompact stable minimal hypersurface in a Riemannnian manifold with nonnegative sectional curvature. Later, this result was extended to more general ambient spaces [16, 20, 21]. When the curvature of the ambient manifold is negative, Seo [28] proved that such a vanishing theorem holds for a complete stable minimal hypersurface in $\mathbb{H}^{n+1}$ with a further assumption about the first eigenvalue of Laplacian ($\lambda_1 > (2n-1)(n-1)$). Dung and Seo [8] dealed with case of the curvature of the ambient manifold is pinched and obtained the corresponding vanishing result for a complete noncompact stable non-totally geodesic minimal hypersurface in Riemannian manifold N with $K \le K_N (K \le 0)$ and $\lambda_1(M) > -K(2n-1)(n-1)$.

A natural question is that how about the nonexistence results of nontrivial $L^p (p \ne 2)$ harmonic 1-forms? Yau [33] proved that there is no nonconstant $L^p (1 < p < \infty)$ harmonic function on a complete Riemannian manifold. Later, Li and Schoen [19] proved that Yau's result is valid for $L^p (0 < p < \infty)$ harmonic functions on a complete manifold with nonnegative Ricci curvature. For L^p harmonic forms, Greene and Wu [12, 13] presented a vanishing theorem for the complete Riemannian manifolds or Kähler manifolds of nonnegative curvature. Recently, under the stability assumption, Seo [26] obtained that there is no nontrivial L^{2p} harmonic 1-form on a stable minimal hypersurface M^n of Riemannian manifold N with $K_N \ge K (K \le 0)$, provided $\lambda_1(M) > \frac{-2n(n-1)^2p^2K}{2n-[(n-1)p-n]^2}$ for $0 < p < \frac{n}{n-1} + \sqrt{2n}$. Moreover, Dung and Seo [9] studied the same topic on a complete δ-stability hypersurface in a Riemannian manifold with nonnegative sectional curvature. The first author and Lv [6] also investigated the nonexistence of nontrivial L^p harmonic 1-form of a complete δ-stable hypersurface with weighted Poincaré inequality in a Riemannian manifold with sectional curvature bounded below by a nonpositive function. Most recently, without the stability assumption, Choi and Seo [7] proved the following finiteness theorem.

Theorem 1.1 ([7]). *Let N be an $(n+1)$-dimensional complete simply connected Riemannian manifold with sectional curvature K_N satisfying $-k^2 \leq K_N \leq 0$ for a nonzero constant k. Let M be an $n(n \geq 3)$-dimensional complete noncompact minimal hypersurface with finite index in N. For $\frac{n-2}{n-1} < p < \frac{n}{n-1}$, assume that*

$$\lambda_1(M) > \max\left\{\frac{(n-1)^2k^2p^2}{(n-1)p-n+2}, \frac{n(n-1)k^2p}{n-(n-1)p}\right\}.$$

Then, $\dim H^1(L^{2p}(M)) < \infty$.

In this paper, removing the minimality assumption of M in Theorem 1.1, we can obtain the following finiteness result.

Theorem 1.2. *Let N be an $(n+1)$-dimensional complete simply connected Riemannian manifold with sectional curvature K_N satisfying $-k^2 \leq K_N \leq 0$ for a nonzero constant k. Let M be an n-dimensional $(3 \leq n \leq 6)$ complete noncompact hypersurface with $(\int_M H^n)^{\frac{2}{n}} < \frac{1}{S(n)}$ and finite index in N, where S(n) is the Sobolev constant. For $\frac{n-2}{n-1} < p < \frac{2}{\sqrt{n-1}}$, assume that $|A|$ is bounded and*

$$\lambda_1(M) > \max\left\{\frac{(n-1)^2k^2p^2}{(n-1)p-n+2}, \frac{n\sqrt{n-1}k^2p}{2-p\sqrt{n-1}}\right\}.$$

Then, $\dim H^1(L^{2p}(M)) < \infty$.

Corollary 1.3. *Let M be an n-dimensional $(3 \leq n \leq 4)$ complete noncompact hypersurface with finite index in hyperbolic space $\mathbb{H}^{n+1}$. If $\lambda_1(M) > \frac{n\sqrt{n-1}}{2-\sqrt{n-1}}$ and $(\int_M H^n)^{\frac{2}{n}} < \frac{1}{S(n)}$, where S(n) is the Sobolev constant, then* $\dim H^1(L^2(M)) < \infty$. *Moreover, M has finitely many ends.*

We say that an n-dimensional complete Riemannian manifold M has property $(\mathcal{P}_\rho)$, if a weighted Poincaré inequality is valid on M with some nonnegative weight function $\rho(x)$, namely

$$\int_M \rho(x)\eta^2 \leq \int_M |\nabla\eta|^2, \quad \forall \eta \in C_0^\infty(M). \tag{1.1}$$

Moreover, the ρ-metric, defined by $ds_\rho^2 = \rho ds_M^2$ is complete. In particular, if $\lambda_1(M)$ is assumed to be positive, then obviously M possesses property $(\mathcal{P}_\rho)$ with $\rho = \lambda_1(M)$. So, the notion of property $(\mathcal{P}_\rho)$ may be viewed as a generalization of the assumption $\lambda_1(M) > 0$. Recently, Sang and Thanh [25] proved that a complete noncompact stable minimal hypersurface with property $(\mathcal{P}_\rho)$ in Riemannian manifold N has no nontrivial L^2 harmonic 1-form if the sectional curvature of N satisfies $K_N(x) \geq -\frac{(1-\tau)\rho(x)}{(2n-1)(n-1)}$, $0 < \tau \leq 1$ and $\rho(x)$ satisfies certain growth condition. Motivated by [4, 5, 6, 9, 25], we can obtain an another improvement of Theorem 1.1. More precisely, we have the following theorem.

Theorem 1.4. *Let $M^n (3 \leq n \leq 6)$ be a complete noncompact hypersurface with property $(\mathcal{P}_\rho)$ in an $(n+1)$-dimensional Riemannian manifold N. Assume that ρ is bounded and*

$$0 \geq K_N(x) \geq -\frac{(1-\tau)\rho(x)}{(2n-1)(n-1)}, \quad (\forall x \in M)$$

for some $\tau : \frac{122-51\sqrt{5}}{12+4\sqrt{5}} < \tau \leq 1$. If M has finite index, then $\dim H^1(L^{2p}(M)) < \infty$ *for any constant p satisfying $C_1(n,\tau) < p < C_2(n,\tau)$, where*

$$C_0 = \frac{(2\sqrt{n-1}+n)(1-\tau)}{(2n-1)(n-1)},$$

$$C_1(n,\tau) = \frac{2\left(1-\sqrt{1-\frac{n-2}{2\sqrt{n-1}}(1+C_0)}\right)}{\sqrt{n-1}(1+C_0)},$$

$$C_2(n,\tau) = \frac{2\left(1+\sqrt{1-\frac{n-2}{2\sqrt{n-1}}(1+C_0)}\right)}{\sqrt{n-1}(1+C_0)}.$$

when $\tau = 1$*, we have*

Corollary 1.5. *Let* $M^n(3 \leq n \leq 6)$ *be a complete noncompact hypersurface with property* $(\mathcal{P}_\rho)$ *in* $\mathbb{R}^{n+1}$*. If M has finite index and* ρ *is bounded, then* $\dim H^1(L^{2p}(M)) < \infty$ *for any constant p satisfying* $C_1(n) < p < C_2(n)$*, where*

$$C_1(n) = \frac{2}{\sqrt{n-1}}\left(1-\sqrt{1-\frac{n-2}{2\sqrt{n-1}}}\right),$$

$$C_2(n) = \frac{2}{\sqrt{n-1}}\left(1+\sqrt{1-\frac{n-2}{2\sqrt{n-1}}}\right).$$

Moreover, we can prove a similar finiteness theorem for L^p harmonic 1-forms on complete noncompact hypersurfaces with property $(\mathcal{P}_\rho)$ as Theorem 1.4 except the condition that the lower bound of K_N depends on n, p, ρ. More precisely, we have

Theorem 1.6. *Let* N^{n+1} *be an* $(n+1)$*-dimensional Riemannian manifold, and* $M^n(3 \leq n \leq 6)$ *be a complete noncompact hypersurface satisfying weighted Poincaré inequality* $(\mathcal{P}_\rho)$ *for some nonnegative bounded function* ρ *in N. If M has finite index and*

$$0 \geq K_N > -\frac{4p(n-1)-2(n-2)-(n-1)\sqrt{n-1}p^2}{p^2(n-1)(2n-2+n\sqrt{n-1})}\rho,$$

where p satisfies

$$\frac{2}{\sqrt{n-1}}\left(1-\sqrt{1-\frac{n-2}{2\sqrt{n-1}}}\right) < p < \frac{2}{\sqrt{n-1}}\left(1+\sqrt{1-\frac{n-2}{2\sqrt{n-1}}}\right).$$

Then, $\dim H^1(L^{2p}(M)) < \infty$.

2. Some lemmas

In this section, we will recall some useful results which will be adopted in the proof of main theorems. The most basic one is the following Weitzenböck formula.

Lemma 2.1 ([18]). *Given a Riemannian manifold* M^n*, for any* 1*-form* ω *on* M^n*, we have*

$$\triangle|\omega|^2 = 2|\nabla\omega|^2 + 2\langle\Delta\omega,\omega\rangle + 2\mathrm{Ric}(\omega^\sharp,\omega^\sharp),$$

where $\omega^\sharp$ *is the dual vector field of* ω*.*

Besides, the Kato inequality is also a fundamental technique.

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Lemma 2.2 ([1]). *Given a Riemannian manifold M^n, for any closed and coclosed k-form ω on M^n, we have*

$$|\nabla\omega|^2 \geq (1+C_{n,k})|\nabla|\omega||^2, \quad \text{where} \quad C_{n,k} = \begin{cases} \frac{1}{n-k}, & 1 \leq k \leq \frac{n}{2}. \\ \frac{1}{k}, & \frac{n}{2} \leq k \leq n-1. \end{cases}$$

What's more, Shiohama and Xu [29] proved the following estimation on the Ricci curvature of submanifold.

Lemma 2.3 ([29]). *Let M be an n-dimensional complete immersed hypersurface in a Riemannian manifold N. If all the sectional curvatures of N are bounded pointwise from below by a function k, then*

$$\text{Ric} \geq (n-1)k - \frac{n-1}{n}|A|^2 + 2(n-1)H^2 - \frac{(n-2)\sqrt{n(n-1)}}{n}|H|\sqrt{|A|^2 - nH^2}, \tag{2.1}$$

where H is the mean curvature and A is the second fundamental form of M.

We should note in [29], the author assumed that all the sectional curvatures of N are bounded below by a constant k. But according to his argument, this assumption was only used in the end of the proof; hence, this method can be used to prove the above lemma without any change. Under the same assumption, the following lemma estimates the right hand side of (2.1).

Lemma 2.4 ([6]). *Let M^n be an n-dimensional orientable submanifold in Riemannian manifold N. We have*

$$2(n-1)H^2 - \frac{(n-2)\sqrt{n(n-1)}}{n}|H|\sqrt{|A|^2 - nH^2} \geq \frac{2(n-1) - n\sqrt{n-1}}{2n}|A|^2. \tag{2.2}$$

Definition 2.5. *Let M^n be an n-dimensional orientable hypersurface in a Riemannian manifold N. We say M is stable if the following inequality*

$$\int_M |\nabla\eta|^2 \geq \int_M \left(|A|^2 + \overline{\text{Ric}}(\nu,\nu)\right)\eta^2 \tag{2.3}$$

holds for any $\eta \in C_0^\infty(M)$, where ν is a unit normal vector field on M, $\overline{\text{Ric}}$ is the Ricci curvature of N, and A is the second fundamental form of M.

Now, we will give a condition to ensure that the volume of Riemannian manifold to be infinite.

Lemma 2.6 ([9]). *Let M be a complete oriented noncompact immersed hypersurface in a complete Riemannian manifold N^{n+1} with nonnegative sectional curvature. If the stability inequality (2.3) holds on M, then the volume of M is infinite.*

In addition, the following Hoffman-Spruck inequality generalizes the Poincaré inequality and relates it to the Sobolev inequality.

Lemma 2.7 ([14]). *Let $x : M^n \hookrightarrow N$ be an isometric immersion of a complete manifold M in a complete simply connected manifold N with nonpositive sectional curvature. Then, the following inequality holds:*

$$\left(\int_M h^{2n} dV\right)^{\frac{n-2}{n}} \leq S(n) \int_M (|\nabla h|^2 + (h|H|)^2) dV,$$

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for all nonnegative C^1-functions $h: M^n \to \mathbb{R}$ with compact support, where S(n) is the Sobolev constant, which is positive and only depends on n.

The following Cauchy inequality gives the L^2 upper bound of a nonnegative sub-eigenfunction.

Lemma 2.8 ([18]). *Let M be an n-dimensional complete noncompact Riemannian manifold. For $x \in M$ and a constant $\kappa \geq 0$, we assume that the Ricci curvature of M satisfies*

$$\mathrm{Ric} \geq -(n-1)\kappa$$

on the geodesic ball $B_x(4r)$ centered at p with radius $4r$. Let $0 < \delta < \frac{1}{2}$ and $\lambda > 0$ be two fixed constants. Then there exists a positive constant $C = C(r, \delta, \lambda, \kappa)$ so that if any nonnegative function $\eta \in C^\infty(B_x(2r))$ satisfying the differential inequality $\triangle\eta \geq -\lambda\eta$, then

$$\sup_{B_x((1-\delta)r)} \eta^2 \leq \frac{C}{\mathrm{Vol}(B_x(r))} \int_{B_x(r)} \eta^2.$$

The last lemma associates the L^2 and L^∞ norms of harmonic forms with the dimension of the space of harmonic forms.

Lemma 2.9 ([17, 24]). *Let K be a finite dimensional subspace of L^{2p} harmonic q-forms on an m-dimensional complete noncompact Riemannian manifold M for any $p > 0$. Then, there exists $\eta \in K$ such that*

$$(\dim K)^{\min\{1,p\}} \int_{B_x(r)} |\eta|^{2p} \leq \mathrm{Vol}(B_x(r)) \cdot \min\left\{\binom{m}{q}, \dim K\right\}^{\min\{1,p\}} \cdot \sup_{B_x(r)} |\eta|^{2p},$$

for any $x \in M$ and $r > 0$.

3. Proofs of the theorems

Proof of Theorem 1.2. Let ω be a L^{2p} harmonic 1-form. Using the Weitzenböck formula and the Kato inequality, we can get that

$$|\omega|\triangle|\omega| \geq \frac{1}{n-1}|\nabla|\omega||^2 + \mathrm{Ric}(\omega^\sharp, \omega^\sharp). \tag{3.1}$$

Under our hypothesis on the sectional curvature of N, we can estimate the Ricci curvature of M by using Lemmas 2.3 and 2.4:

$$\begin{aligned}\mathrm{Ric}_M &\geq -(n-1)k^2 + \frac{2(n-1) - n\sqrt{n-1}}{2n}|A|^2 - \frac{n-1}{n}|A|^2\\ &= -(n-1)k^2 - \frac{\sqrt{n-1}}{2}|A|^2.\end{aligned}$$

Thus equation (3.1) becomes

$$|\omega|\triangle|\omega| \geq \frac{1}{n-1}|\nabla|\omega||^2 - (n-1)k^2|\omega|^2 - \frac{\sqrt{n-1}}{2}|A|^2|\omega|^2. \tag{3.2}$$

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Furthermore, using (3.2) we have that

$$\begin{aligned}|\omega|^p\triangle|\omega|^p &= |\omega|^p\Big(p(p-1)|\omega|^{p-2}|\nabla|\omega||^2 + p|\omega|^{p-1}\triangle|\omega|\Big)\\ &= \frac{p-1}{p}|\nabla|\omega|^p|^2 + p|\omega|^{2p-2}|\omega|\triangle|\omega|\\ &\geq \Big(1-\frac{n-2}{(n-1)p}\Big)|\nabla|\omega|^p|^2 - \frac{p\sqrt{n-1}}{2}|A|^2|\omega|^{2p} - (n-1)k^2p|\omega|^{2p}. \end{aligned} \tag{3.3}$$

Since M has finite index, there exists a compact subset $\Omega \subset M$ such that $M\backslash\Omega$ is stable ([10, 31]). Without loss of generality, we assume that $\Omega = B_x(R_0)$. Then, according to Definition 2.5, for any compactly supported Lipschitz function η on $M\backslash B_x(R_0)$,

$$\int_{M\setminus B_x(R_0)} |\nabla\eta|^2 \geq \int_{M\setminus B_x(R_0)} \Big(|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu)\Big)\eta^2.$$

The assumption on the sectional curvature of N implies that $\overline{\mathrm{Ric}}(\nu,\nu) \geq -nk^2$ and

$$\int_{M\backslash B_x(R_0)} |\nabla\eta|^2 \geq \int_{M\backslash B_x(R_0)} \Big(|A|^2 - nk^2\Big)\eta^2. \tag{3.4}$$

for all compactly supported Lipschitz function η on $M\backslash B_x(R_0)$. Replacing η by $\eta|\omega|^p$ in (3.4), we get

$$\int_{M\backslash B_x(R_0)} |A|^2\eta^2|\omega|^{2p} - nk^2\int_{M\backslash B_x(R_0)} \eta^2|\omega|^{2p} \leq \int_{M\backslash B_x(R_0)} |\nabla(\eta|\omega|^p)|^2. \tag{3.5}$$

Moreover, the domain monotonicity of eigenvalues implies that

$$\lambda_1(M) \leq \lambda_1(M\backslash B_x(R_0)) \leq \frac{\int_{M\backslash B_x(R_0)} |\nabla\eta|^2}{\int_{M\backslash B_x(R_0)} \eta^2}$$

for any compactly supported Lipschitz function η on $M\backslash B_x(R_0)$. Replacing η by $\eta|\omega|^p$ in this inequality and using (3.5), we have

$$\int_{M\backslash B_x(R_0)} \eta^2|\omega|^{2p} \leq \frac{1}{\lambda_1(M)}\int_{M\backslash B_x(R_0)} |\nabla(\eta|\omega|^p)|^2. \tag{3.6}$$

and

$$\begin{aligned}\int_{M\backslash B_x(R_0)} |A|^2\eta^2|\omega|^{2p} &\leq \Big(1+\frac{nk^2}{\lambda_1(M)}\Big)\int_{M\backslash B_x(R_0)} |\nabla(\eta|\omega|^p)|^2\\ &= \Big(1+\frac{nk^2}{\lambda_1(M)}\Big)\int_{M\backslash B_x(R_0)} \Big(\eta^2|\nabla|\omega|^p|^2 + |\nabla\eta|^2|\omega|^{2p} + 2\eta|\omega|^p\langle\nabla\eta,\nabla|\omega|^p\rangle\Big). \end{aligned} \tag{3.7}$$

Applying the divergence theorem, we get

$$2\int_{M\backslash B_x(R_0)} \eta|\omega|^p\langle\nabla\eta,\nabla|\omega|^p\rangle = -\int_{M\backslash B_x(R_0)} \Big(\eta^2|\nabla|\omega|^p|^2 + \eta^2|\omega|^p\triangle|\omega|^p\Big). \tag{3.8}$$

Therefore,

$$\int_{M\backslash B_x(R_0)} |A|^2\eta^2|\omega|^{2p} \leq \Big(1+\frac{nk^2}{\lambda_1(M)}\Big)\int_{M\backslash B_x(R_0)} \Big(|\nabla\eta|^2|\omega|^{2p} - \eta^2|\omega|^p\triangle|\omega|^p\Big). \tag{3.9}$$

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From (3.3) and (3.9), we have

$$\begin{aligned}\int_{M\setminus B_x(R_0)} |A|^2\eta^2|\omega|^{2p} \leq \Big(1+\frac{nk^2}{\lambda_1(M)}\Big)\int_{M\setminus B_x(R_0)} |\nabla\eta|^2|\omega|^{2p}\\ -\Big(1+\frac{nk^2}{\lambda_1(M)}\Big)\Big(1-\frac{n-2}{(n-1)p}\Big)\int_{M\setminus B_x(R_0)} \eta^2|\nabla|\omega|^p|^2\\ +\Big(1+\frac{nk^2}{\lambda_1(M)}\Big)\int_{M\setminus B_x(R_0)} \Big(\frac{p\sqrt{n-1}}{2}|A|^2+(n-1)k^2p\Big)\eta^2|\omega|^{2p}.\end{aligned}$$

Therefore, the assumption on $\lambda_1(M)$ implies

$$\begin{aligned}\Big(1-\frac{n-2}{(n-1)p}\Big)\int_{M\setminus B_x(R_0)} \eta^2|\nabla|\omega|^p|^2\\ \leq \int_{M\setminus B_x(R_0)} |\nabla\eta|^2|\omega|^{2p}+(n-1)k^2p\int_{M\setminus B_x(R_0)} \eta^2|\omega|^{2p}\\ +\left(\frac{p\sqrt{n-1}}{2}-\frac{1}{1+\frac{nk^2}{\lambda_1(M)}}\right)\int_{M\setminus B_x(R_0)} |A|^2\eta^2|\omega|^{2p}\\ \leq \int_{M\setminus B_x(R_0)} |\nabla\eta|^2|\omega|^{2p}+(n-1)k^2p\int_{M\setminus B_x(R_0)} \eta^2|\omega|^{2p}.\end{aligned} \tag{3.10}$$

On the other hand, applying Young's inequality in (3.6), we obtain

$$\int_{M\setminus B_x(R_0)} \eta^2|\omega|^{2p} \leq \frac{1}{\lambda_1(M)}\cdot\frac{1+\varepsilon}{\varepsilon}\int_{M\setminus B_x(R_0)} |\nabla\eta|^2|\omega|^{2p}+\frac{1+\varepsilon}{\lambda_1(M)}\int_{M\setminus B_x(R_0)} \eta^2|\nabla|\omega|^p|^2$$

for any $\varepsilon > 0$. Combining this with (3.10), we get

$$\begin{aligned}\Big(1-\frac{n-2}{(n-1)p}-\frac{(n-1)k^2p(1+\varepsilon)}{\lambda_1(M)}\Big)\int_{M\setminus B_x(R_0)} \eta^2|\nabla|\omega|^p|^2\\ \leq \Big(1+\frac{(n-1)k^2p}{\lambda_1(M)}\cdot\frac{1+\varepsilon}{\varepsilon}\Big)\int_{M\setminus B_x(R_0)} |\nabla\eta|^2|\omega|^{2p}.\end{aligned}$$

Using the assumption on $\lambda_1(M)$, we choose a sufficiently small $\varepsilon > 0$ such that

$$1-\frac{n-2}{(n-1)p}-\frac{(n-1)k^2p(1+\varepsilon)}{\lambda_1(M)} > 0.$$

Then we have

$$\int_{M\setminus B_x(R_0)} \eta^2|\nabla|\omega|^p|^2 \leq C_1\int_{M\setminus B_x(R_0)} |\nabla\eta|^2|\omega|^{2p}, \tag{3.11}$$

for some positive constant C_1 which depends only on p, n, k and $\lambda_1(M)$. Moreover, from Lemma 2.7 and Hölder inequality, we obtain

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$$\Big(\int_{M\backslash B_x(R_0)}(\eta|\omega|^p)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}\le S(n)\int_{M\backslash B_x(R_0)}\big(|\nabla(\eta|\omega|^p)|^2+\eta^2|\omega|^{2p}|H|^2\big)$$
$$\le 2S(n)\int_{M\backslash B_x(R_0)}\eta^2|\nabla|\omega|^p|^2$$
$$+2S(n)\int_{M\backslash B_x(R_0)}|\nabla\eta|^2|\omega|^{2p}$$
$$+S(n)\Big(\int_{M\backslash B_x(R_0)}|H|^n\Big)^{\frac{2}{n}}\Big(\int_{M\backslash B_x(R_0)}(\eta|\omega|^p)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}},\qquad(3.12)$$

i.e.

$$\Big(1-S(n)\|H\|^2_{L^n(M)}\Big)\Big(\int_{M\backslash B_x(R_0)}(\eta|\omega|^p)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}$$
$$\le 2S(n)\int_{M\backslash B_x(R_0)}\eta^2|\nabla|\omega|^p|^2+2S(n)\int_{M\backslash B_x(R_0)}|\nabla\eta|^2|\omega|^{2p}.\qquad(3.13)$$

Combining (3.11) and (3.13) and the assumption $1-S(n)\|H\|^2_{L^n(M)}>0$, we get

$$\Big(\int_{M\backslash B_x(R_0)}(\eta|\omega|^p)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}\le C_2\int_{M\backslash B_x(R_0)}|\nabla\eta|^2|\omega|^{2p},\qquad(3.14)$$

for some positive constant C_2. Now we choose our test function $0\le\eta\le 1$ as in [7]: given $R>R_0+1$,

$$\eta=\begin{cases}1,\ on\ B_x(R)\backslash B_x(R_0+1)\\ 0,\ on\ B_x(R_0)\cup(M\backslash B_x(2R)),\end{cases}$$

$|\nabla\eta|\le C_3$ on $B_x(R_0+1)\backslash B_x(R_0)$ and $|\nabla\eta|\le\frac{C_3}{R}$ on $B_x(2R)\backslash B_x(R)$. Applying this test function η to (3.14), we get

$$\Big(\int_{B_x(R)\backslash B_x(R_0+1)}|\omega|^{\frac{2np}{n-2}}\Big)^{\frac{n-2}{n}}\le C_4\int_{B_x(R_0+1)\backslash B_x(R_0)}|\omega|^{2p}+\frac{C_4}{R^2}\int_{B_x(2R)\backslash B_x(R)}|\omega|^{2p}.$$

Letting $R\to\infty$ and using the assumption that $\int_M|\omega|^{2p}<\infty$, we obtain

$$\Big(\int_{M\backslash B_x(R_0+1)}|\omega|^{\frac{2np}{n-2}}\Big)^{\frac{n-2}{n}}\le C_4\int_{B_x(R_0+1)\backslash B_x(R_0)}|\omega|^{2p}.\qquad(3.15)$$

Then, using Hölder inequality and (3.15), we conclude that

$$\int_{B_x(R_0+2)\backslash B_x(R_0+1)}|\omega|^{2p}\le\big(\mathrm{Vol}(B_x(R_0+2))\big)^{\frac{2}{n}}\Big(\int_{B_x(R_0+2)\backslash B_x(R_0+1)}|\omega|^{\frac{2np}{n-2}}\Big)^{\frac{n-2}{n}}$$
$$\le C_4\cdot\big(\mathrm{Vol}(B_x(R_0+2))\big)^{\frac{2}{n}}\int_{B_x(R_0+1)\backslash B_x(R_0)}|\omega|^{2p}.\qquad(3.16)$$

Adding $\int_{B_x(R_0+1)\backslash B_x(R_0)}|\omega|^{2p}$ to both sides of (3.16), we get

$$\int_{B_x(R_0+2)\backslash B_x(R_0)}|\omega|^{2p}\le\Big(1+C_4\cdot\big(\mathrm{Vol}(B_x(R_0+2))\big)^{\frac{2}{n}}\Big)\int_{B_x(R_0+1)\backslash B_x(R_0)}|\omega|^{2p}.$$

Again adding $\int_{B_x(R_0)}|\omega|^{2p}$ to both sides infers

$$\int_{B_x(R_0+2)}|\omega|^{2p}\le C_5\int_{B_x(R_0+1)}|\omega|^{2p}.\qquad(3.17)$$

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On the other hand, since $|\omega|$ satisfies the differential inequality (3.3), Lemma 2.8 implies that

$$\sup_{B_x((1-\delta)(R_0+2))} |\omega|^{2p} \leq C_6 \int_{B_x(R_0+2)} |\omega|^{2p}$$

for some positive constant $C_6 = C_6(\delta, n, k, \mathrm{Vol}(B_x(R_0+2)), \sup_{B_x(R_0+2)} |A|^2)$. For a sufficiently small $\delta > 0$ such that $(1-\delta)(R_0+2) > R_0+1$,

$$\sup_{B_x(R_0+1)} |\omega|^{2p} \leq C_6 \int_{B_x(R_0+2)} |\omega|^{2p}.$$

Together with (3.17), we have

$$\sup_{B_x(R_0+1)} |\omega|^{2p} \leq C_7 \int_{B_x(R_0+1)} |\omega|^{2p}. \tag{3.18}$$

for some positive constant $C_7 = C_7(n, p, k, R_0, \lambda_1(M), S(n), \mathrm{Vol}(B_x(R_0+2)))$. In what follows, as in [7], we consider any finite dimensional subspace $K \subset H^1(L^{2p}(M))$. According to Lemma 2.9, we see that there exists an L^{2p} harmonic 1-form $\omega \subset K$ such that

$$(\dim K)^{\min\{1,p\}} \int_{B_x(R_0+1)} |\omega|^{2p} \leq \mathrm{Vol}(B_x(R_0+1)) \cdot \min\{n, \dim K\}^{\min\{1,p\}} \cdot \sup_{B_x(R_0+1)} |\omega|^{2p}.$$

From (3.18), we have

$$\dim K \leq \Big(C_7 \cdot \mathrm{Vol}(B_x(R_0+1))\Big)^{\frac{1}{\min\{1,p\}}} \cdot \min\{n, \dim K\},$$

which implies that $\dim K$ is bounded by a fixed constant. Since K is an arbitrary subspace of finite dimension, we get that $\dim H^1(L^{2p}(M)) < \infty$. □

Proof of Theorem 1.4. Let ω be a L^{2p} harmonic 1-form. Using Weitzenböck formula and Kato inequality, we can get that

$$|\omega|\triangle|\omega| \geq \frac{1}{n-1}|\nabla|\omega||^2 + \mathrm{Ric}(\omega^\sharp, \omega^\sharp). \tag{3.19}$$

Under our hypothesis on the sectional curvature of N, we can estimate the Ricci curvature of M by using Lemmas 2.3 and 2.4:

$$\begin{aligned}
\mathrm{Ric}_M &\geq -(n-1)\frac{(1-\tau)\rho}{(2n-1)(n-1)} + (n-1)H^2 - \frac{n-1}{n}|\Phi|^2 - \frac{(n-2)\sqrt{n(n-1)}}{n}|H||\Phi| \\
&= -\frac{(1-\tau)\rho}{2n-1} + 2(n-1)H^2 - \frac{(n-2)\sqrt{n(n-1)}}{n}|H|\sqrt{|A|^2 - nH^2} - \frac{n-1}{n}|A|^2 \\
&\geq -\frac{(1-\tau)\rho}{2n-1} + \frac{2(n-1) - n\sqrt{n-1}}{2n}|A|^2 - \frac{n-1}{n}|A|^2 \\
&= -\frac{(1-\tau)\rho}{2n-1} - \frac{\sqrt{n-1}}{2}|A|^2.
\end{aligned}$$

Thus, equation (3.19) becomes

$$|\omega|\triangle|\omega| \geq \frac{1}{n-1}|\nabla|\omega||^2 - \frac{(1-\tau)\rho}{2n-1}|\omega|^2 - \frac{\sqrt{n-1}}{2}|A|^2|\omega|^2. \tag{3.20}$$

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Given any $\alpha > 0$, using (3.20) we have that

$$\begin{aligned}|\omega|^\alpha \triangle|\omega|^\alpha &= |\omega|^\alpha\Big(\alpha(\alpha-1)|\omega|^{\alpha-2}|\nabla|\omega||^2 + \alpha|\omega|^{\alpha-1}\triangle|\omega|\Big)\\ &= \frac{\alpha-1}{\alpha}|\nabla|\omega|^\alpha|^2 + \alpha|\omega|^{2\alpha-2}|\omega|\triangle|\omega|\\ &\geq \Big(1-\frac{n-2}{(n-1)\alpha}\Big)|\nabla|\omega|^\alpha|^2 - \frac{\alpha\sqrt{n-1}}{2}|A|^2|\omega|^{2\alpha} - \frac{(1-\tau)\rho\alpha}{2n-1}|\omega|^{2\alpha}. \end{aligned} \tag{3.21}$$

Since M has finite index, as in the proof of Theorem 1.2, we assume that $M\backslash B_x(R_0)$ is stable. In other words,

$$\int_{M\backslash B_x(R_0)} |\nabla\eta|^2 \geq \int_{M\backslash B_x(R_0)} \Big(|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu)\Big)\eta^2. \tag{3.22}$$

for all compactly supported Lipschitz function η on $M\backslash B_x(R_0)$. Replacing η by $|\omega|^{(s+1)\alpha}\eta$ in (3.22) and applying the lower bound of sectional curvature of N allow us to conclude that

$$\begin{aligned}&\int_{M\backslash B_x(R_0)} |A|^2|\omega|^{2(s+1)\alpha}\eta^2\\ &\quad\leq \int_{M\backslash B_x(R_0)} |\nabla(|\omega|^{(s+1)\alpha}\eta)|^2 + \frac{n(1-\tau)}{(2n-1)(n-1)}\int_{M\backslash B_x(R_0)} \rho|\omega|^{2(s+1)\alpha}\eta^2\\ &\quad= (s+1)^2\int_{M\backslash B_x(R_0)} |\omega|^{2s\alpha}|\nabla|\omega|^\alpha|^2\eta^2 + \int_{M\backslash B_x(R_0)} |\omega|^{2(s+1)\alpha}|\nabla\eta|^2\\ &\qquad+ 2(s+1)\int_{M\backslash B_x(R_0)} |\omega|^{(2s+1)\alpha}\eta\langle\nabla\eta,\nabla|\omega|^\alpha\rangle\\ &\qquad+ \frac{n(1-\tau)}{(2n-1)(n-1)}\int_{M\backslash B_x(R_0)} \rho|\omega|^{2(s+1)\alpha}\eta^2. \end{aligned} \tag{3.23}$$

On the other hand, for $s > 0$ and a smooth function η with compactly support in M, multiplying both sides of the inequality (3.21) by $|\omega|^{2s\alpha}\eta^2$ and integrating over M, we obtain that

$$\begin{aligned}&\Big(1-\frac{n-2}{(n-1)\alpha}\Big)\int_{M\backslash B_x(R_0)} |\omega|^{2s\alpha}|\nabla|\omega|^\alpha|^2\eta^2\\ &\leq \int_{M\backslash B_x(R_0)} |\omega|^{(2s+1)\alpha}\eta^2\triangle|\omega|^\alpha + \frac{\alpha\sqrt{n-1}}{2}\int_{M\backslash B_x(R_0)} |A|^2|\omega|^{2(s+1)\alpha}\eta^2\\ &\quad+ \frac{\alpha(1-\tau)}{2n-1}\int_{M\backslash B_x(R_0)} \rho|\omega|^{2(s+1)\alpha}\eta^2\\ &= -(2s+1)\int_{M\backslash B_x(R_0)} |\omega|^{2s\alpha}|\nabla|\omega|^\alpha|^2\eta^2 - 2\int_{M\backslash B_x(R_0)} \eta|\omega|^{(2s+1)\alpha}\langle\nabla\eta,\nabla|\omega|^\alpha\rangle\\ &\quad+ \frac{\alpha\sqrt{n-1}}{2}\int_{M\backslash B_x(R_0)} |A|^2|\omega|^{2(s+1)\alpha}\eta^2 + \frac{\alpha(1-\tau)}{2n-1}\int_{M\backslash B_x(R_0)} \rho|\omega|^{2(s+1)\alpha}\eta^2, \end{aligned}$$

i.e.

$$\left(2(s+1)-\frac{n-2}{(n-1)\alpha}\right)\int_{M\setminus B_x(R_0)}|\omega|^{2s\alpha}|\nabla|\omega|^{\alpha}|^2\eta^2$$
$$\leq -2\int_{M\setminus B_x(R_0)}\eta|\omega|^{(2s+1)\alpha}\langle\nabla\eta,\nabla|\omega|^{\alpha}\rangle+\frac{\alpha\sqrt{n-1}}{2}\int_{M\setminus B_x(R_0)}|A|^2|\omega|^{2(s+1)\alpha}\eta^2$$
$$+\frac{\alpha(1-\tau)}{2n-1}\int_{M\setminus B_x(R_0)}\rho|\omega|^{2(s+1)\alpha}\eta^2. \tag{3.24}$$

Combining (3.24) with (3.23), we obtain that

$$\left(2(s+1)-\frac{n-2}{(n-1)\alpha}-\frac{\alpha\sqrt{n-1}}{2}\cdot(s+1)^2\right)\int_{M\setminus B_x(R_0)}|\omega|^{2s\alpha}|\nabla|\omega|^{\alpha}|^2\eta^2$$
$$\leq\frac{\alpha\sqrt{n-1}}{2}\int_{M\setminus B_x(R_0)}|\omega|^{2(s+1)\alpha}|\nabla\eta|^2+E\int_{M\setminus B_x(R_0)}\rho|\omega|^{2(s+1)\alpha}\eta^2$$
$$+\left(\alpha\sqrt{n-1}\cdot(s+1)-2\right)\int_{M\setminus B_x(R_0)}|\omega|^{(2s+1)\alpha}\eta\langle\nabla\eta,\nabla|\omega|^{\alpha}\rangle, \tag{3.25}$$

where

$$E=\left(\frac{n\sqrt{n-1}}{2}+n-1\right)\cdot\frac{\alpha(1-\tau)}{(2n-1)(n-1)}.$$

From the assumption of weighted Poincaré inequality (1.1), we obtain that

$$\int_{M\setminus B_x(R_0)}\rho(|\omega|^{2(s+1)\alpha}\eta^2)\leq\int_{M\setminus B_x(R_0)}|\nabla(|\omega|^{(s+1)\alpha}\eta)|^2$$
$$=(s+1)^2\int_{M\setminus B_x(R_0)}|\omega|^{2s\alpha}|\nabla|\omega|^{\alpha}|^2\eta^2+\int_{M\setminus B_x(R_0)}|\omega|^{2(s+1)\alpha}|\nabla\eta|^2$$
$$+2(s+1)\int_{M\setminus B_x(R_0)}|\omega|^{(2s+1)\alpha}\eta\langle\nabla\eta,\nabla|\omega|^{\alpha}\rangle. \tag{3.26}$$

Plugging (3.26) into (3.25) implies that

$$B\int_{M\setminus B_x(R_0)}|\omega|^{2s\alpha}|\nabla|\omega|^{\alpha}|^2\eta^2\leq C\int_{M\setminus B_x(R_0)}|\omega|^{2(s+1)\alpha}|\nabla\eta|^2+2D\int_{M\setminus B_x(R_0)}|\omega|^{(2s+1)\alpha}\eta\langle\nabla\eta,\nabla|\omega|^{\alpha}\rangle, \tag{3.27}$$

where

$$B=2(s+1)-\frac{n-2}{(n-1)\alpha}-\frac{\alpha\sqrt{n-1}}{2}\cdot(s+1)^2-E(s+1)^2,$$
$$C=\frac{\alpha\sqrt{n-1}}{2}+E,$$
$$D=\frac{\alpha\sqrt{n-1}}{2}\cdot(1+s)-1+E(s+1).$$

For any $\varepsilon>0$, using Cauchy-Schwarz inequality, we can rewrite equation (3.27) as

$$\left(B-|D|\varepsilon\right)\int_{M\setminus B_x(R_0)}|\omega|^{2s\alpha}|\nabla|\omega|^{\alpha}|^2\eta^2\leq\left(C+|D|\frac{1}{\varepsilon}\right)\int_{M\setminus B_x(R_0)}|\omega|^{2(s+1)\alpha}|\nabla\eta|^2. \tag{3.28}$$

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Now let $p=(s+1)\alpha$, we see that

$$\begin{aligned} B &= 2(s+1) - \frac{n-2}{(n-1)\alpha} - \frac{\alpha\sqrt{n-1}}{2}\cdot(s+1)^2 - E(s+1)^2 \\ &= \frac{1}{\alpha}\Big\{2p - \frac{n-2}{n-1} - \frac{p^2\sqrt{n-1}}{2} - \Big(\frac{n\sqrt{n-1}}{2}+n-1\Big)\frac{(1-\tau)p^2}{(2n-1)(n-1)}\Big\} \\ &= \frac{1}{\alpha}\Big\{2p - \frac{n-2}{n-1} - \frac{\sqrt{n-1}}{2}\Big[1+\Big(n+2\sqrt{n-1}\Big)\frac{(1-\tau)}{(2n-1)(n-1)}\Big]p^2\Big\}. \end{aligned} \tag{3.29}$$

Let

$$f(p) = -\frac{\sqrt{n-1}}{2}\Big[1+\Big(n+2\sqrt{n-1}\Big)\frac{(1-\tau)}{(2n-1)(n-1)}\Big]p^2 + 2p - \frac{n-2}{n-1},$$

then the discriminant of $f(p)$ is

$$\Delta = 4\Big(1 - \frac{n-2}{2\sqrt{n-1}}\Big[1+\frac{(n+2\sqrt{n-1})(1-\tau)}{(2n-1)(n-1)}\Big]\Big) > 0 \tag{3.30}$$

when $2\le n\le 6$ and $\frac{122-51\sqrt{5}}{12+4\sqrt{5}} < \tau \le 1$. Consequently, the condition $C_1 < p < C_2$ allows us to conclude that $f(p)>0$, or equivalently $B>0$. Therefore, for a sufficiently small $\varepsilon>0$, we have $B-|D|\varepsilon>0$. Then, the inequality (3.28) becomes

$$\int_{M\backslash B_x(R_0)} |\nabla|\omega|^p|^2 \le C_0\int_{M\backslash B_x(R_0)} |\omega|^{2p} \tag{3.31}$$

for some positive constant $C_0 = C_0(n,p,\tau)$. Then following the same method as in Theorem 1.2, we can obtain $\dim H^1(L^{2p}(M)) < \infty$. □

Remark 3.1. *If we assume further that index$(M)=0$ (i.e. M is stable) in Theorem 1.4, then $H^1(L^{2p}(M))$ is trivial* [6].

Proof of Theorem 1.6. Let $K_N \ge -k\rho$, where $k < \frac{4p(n-1)-2(n-2)-(n-1)\sqrt{n-1}p^2}{p^2(n-1)(2n-2+n\sqrt{n-1})}$. Similarly as in the proof of Theorem 1.4, we can obtain that

$$\begin{aligned} \widetilde{B}\int_{M\backslash B_x(R_0)} |\omega|^{2s\alpha}|\nabla|\omega|^\alpha|^2\eta^2 &\le \widetilde{C}\int_{M\backslash B_x(R_0)} |\omega|^{2(s+1)\alpha}|\nabla\eta|^2 \\ &\quad + 2\widetilde{D}\int_{M\backslash B_x(R_0)} |\omega|^{(2s+1)\alpha}\eta\langle\nabla\eta, \nabla|\omega|^\alpha\rangle \end{aligned}$$

for any compactly supported nonconstant Lipschitz function η on $M\backslash B_x(R_0)$, where

$$\begin{aligned} \widetilde{B} &= 2(s+1) - \frac{n-2}{(n-1)\alpha} - \frac{\alpha\sqrt{n-1}}{2}\cdot(s+1)^2 - \widetilde{E}(s+1)^2, \\ \widetilde{C} &= \frac{\alpha\sqrt{n-1}}{2} + \widetilde{E}, \\ \widetilde{D} &= \frac{\alpha\sqrt{n-1}}{2}\cdot(s+1) - 1 + \widetilde{E}(s+1), \\ \widetilde{E} &= \Big(\frac{n\sqrt{n-1}}{2}+n-1\Big)k\alpha. \end{aligned}$$

For any $\varepsilon>0$, applying Cauchy-Schwarz inequality, we have that

$$\Big(\widetilde{B}-|\widetilde{D}|\varepsilon\Big)\int_{M\backslash B_x(R_0)} |\omega|^{2s\alpha}|\nabla|\omega|^\alpha|^2\eta^2 \le \Big(\widetilde{C}+|\widetilde{D}|\frac{1}{\varepsilon}\Big)\int_{M\backslash B_x(R_0)} |\omega|^{2(s+1)\alpha}|\nabla\eta|^2.$$

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Let $p = (s+1)\alpha$, then we have

$$\widetilde{B} = \frac{1}{\alpha}\Big\{2p - \frac{n-2}{n-1} - \frac{\sqrt{n-1}}{2}p^2 - \Big(\frac{n\sqrt{n-1}}{2} + n - 1\Big)kp^2\Big\}.$$

Let $\widetilde{f}(p) = -(n-1)\sqrt{n-1}p^2 + 4(n-1)p - 2(n-2)$, then the discriminant of $\widetilde{f}(p)$ is

$$\Delta = 16(n-1)^2\Big(1 - \frac{n-2}{2\sqrt{n-1}}\Big) > 0,$$

which is satisfied when $3 \le n \le 6$. Thus from the assumption on p, we see that $\widetilde{f}(p) > 0$. Moreover, the condition $k < \frac{4p(n-1)-2(n-2)-(n-1)\sqrt{n-1}p^2}{p^2(n-1)(2n-2+n\sqrt{n-1})}$ allow us to conclude that

$$\begin{aligned}\widetilde{B} &= \frac{1}{\alpha}\Big\{2p - \frac{n-2}{n-1} - \frac{p^2\sqrt{n-1}}{2} - \Big(\frac{n\sqrt{n-1}}{2} + n - 1\Big)kp^2\Big\}\\ &= \frac{1}{\alpha}\Big\{\frac{4(n-1)p - 2(n-2) - (n-1)\sqrt{n-1}p^2}{2(n-1)} - \Big(\frac{n\sqrt{n-1}}{2} + n - 1\Big)kp^2\Big\}\\ &= \frac{1}{\alpha}\Big\{\frac{\widetilde{f}(p)}{2(n-1)} - \Big(\frac{n\sqrt{n-1}}{2} + n - 1\Big)kp^2\Big\} > 0.\end{aligned}$$

Therefore, for a sufficiently small $\varepsilon > 0$, we have $\widetilde{B} - |\widetilde{D}|\varepsilon > 0$. Using same argument as before, we can complete the proof of Theorem 1.6. □

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