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Topological entropy and partially hyperbolic diffeomorphisms

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Abstract. We consider partially hyperbolic diffeomorphisms on compact manifolds. We define the notion of the unstable and stable foliations stably carrying some unique non-trivial homologies. Under this topological assumption, we prove the following two results: if the center foliation is one-dimensional, then the topological entropy is locally a constant; and if the center foliation is two-dimensional, then the topological entropy is continuous on the set of all C^∞ diffeomorphisms. The proof uses a topological invariant we introduced, Yomdin's theorem on upper semi-continuity, Katok's theorem on lower semi-continuity for two-dimensional systems, and a refined Pesin–Ruelle inequality we proved for partially hyperbolic diffeomorphisms.

1. *Introduction and main results*

One of the fundamental invariants in topological dynamics is topological entropy. However, entropy is very hard to compute and its continuity properties are very delicate. For uniformly hyperbolic diffeomorphisms on a compact manifold, the topological entropy is locally a constant. That is, it remains the same under small perturbations. This is due to the structural stability of uniformly hyperbolic systems. In other words, the entropy is stable for hyperbolic systems. The first question we ask in this paper is the following: besides the structurally stable systems, are there any other systems where the topological entropy is stable under perturbations? We will show that the answer to this question is yes and there are classes of partially hyperbolic systems with one-dimensional centers where the topological entropy is locally constant.

Our next question is: when is the topological entropy continuous? This is a difficult problem. In general, entropy is not continuous for C^1 diffeomorphisms. Yomdin [**8**] proved that it is upper semi-continuous for a class of C^∞ diffeomorphisms for any compact manifold M. For $\dim(M) = 2$, Katok [**3**] showed that the entropy is lower semi-continuous for the $C^{1+\alpha}$, $\alpha > 0$, diffeomorphisms on M. Combining these two, we have the continuity of topological entropy for C^∞ diffeomorphisms on compact surfaces.

In this paper we will show that for a large class of partially hyperbolic C^∞ diffeomorphisms with two-dimensional center foliations the topological entropy is continuous. Besides requiring that the center foliation has dimension two or less, we also require that the stable and unstable foliations carry certain homological information of the manifold. We will give a detailed definition later.

Let M be a compact Riemannian manifold and let $f \in \mathrm{PH}^\infty(M)$ be the set of C^∞ partially hyperbolic diffeomorphisms on M. Then f is said to be *partially hyperbolic* if for every $x \in M$ the tangent space at x admits an invariant splitting

$$T_x M = E^s(x) \oplus E^c(x) \oplus E^u(x),$$

into *strongly stable* $E^s(x) = E^s_f(x)$, *central* $E^c(x) = E^c_f(x)$, and *strongly unstable* $E^u(x) = E^u_f(x)$ subspaces and there exist numbers $c_1 > 1$ and

$$0 < \lambda_s < \lambda'_c \le 1 \le \lambda''_c < \lambda_u,$$

such that, for every $x \in M$ and all $i \in \mathbb{N}$,

$$\begin{aligned} &v \in E^s(x) \Rightarrow \|d_x f^i(v)\| \le c_1 \lambda_s^i \|v\|, \\ &v \in E^c(x) \Rightarrow c_1^{-1} (\lambda'_c)^i \|v\| \le \|d_x f^i(v)\| \le c_1 (\lambda''_c)^i \|v\|, \\ &v \in E^u(x) \Rightarrow c_1^{-1} \lambda_u^i \|v\| \le \|d_x f^i(v)\|. \end{aligned} \tag{1}$$

We denote the set of all C^r partially hyperbolic diffeomorphisms by $\mathrm{PH}^r(M)$.

We can state our main results of the paper.

THEOREM 1.1. *Let M be a compact Riemannian manifold and let $f \in \mathrm{PH}^\infty(M)$ be the set of C^∞ partially hyperbolic diffeomorphisms on M. Assume that:*
(1) *the dimension of the center foliation is two or less; and*
(2) *the strong stable and strong unstable foliations stably carry some unique non-trivial homologies.*

Then the topological entropy $h_{\mathrm{top}} : \mathrm{PH}^\infty(M) \to \mathbb{R}$ is continuous at f. Furthermore, if the center foliation has dimension one, then h_{top} is a constant in a small neighborhood of $f \in \mathrm{PH}^1(M)$.

We will define the homologies carried by a foliation. If the stable manifold and unstable manifold are one-dimensional, then the homological condition is a condition on the homotopy class of the map. For the higher-dimensional case, we believe that the same is true, but we are not able to prove that. We do have some open conditions that one can verify. The theorem is not true in general without the assumptions on the homology. In the last section of this paper, we will give some examples where the theorem fails without such assumptions.

Even though our main results are about the topological entropy, the proof, however, relies on smooth ergodic theory, Lyapunov exponents and measure theoretic entropy. In particular, we proved a refined Pesin–Ruelle inequality for partially hyperbolic diffeomorphisms. This is stated in Theorem 3.3, which is of interest in its own right.

2. *Currents, topological and geometric growth*

In this section, we will define some topological invariants for diffeomorphisms with uniformly expanding (or contracting) foliations. The topological invariant was introduced in Saghin and Xia [**5**], where one can find more details and some other applications of the invariant. We will relate the volume growth of an expanding invariant foliation with this topological invariant. If an invariant foliation carries certain non-trivial homological information of the manifold, which we will make precise later, then the volume growth, which is harder to track, is exactly the same as the topological growth. The topological growth can be easily calculated by actions induced by the map on the homology of the manifold.

Let M be an n-dimensional compact Riemannian manifold. Let $f \in \mathrm{Diff}^r(M)$ be a diffeomorphism on M. Let W be a k-dimensional foliation of M, invariant under f, i.e. f maps leaves of W to leaves. This invariant foliation will be, for the purpose of this paper, the strongly stable and strongly unstable foliations. We first define volume growth of f on leaves of W. For any $x \in M$, Let $W(x)$ be the leaf through x and let $W_r(x)$ be the k-dimensional disk on $W(x)$ centered at x, with radius r.

Let

$$\chi_W(x, r) = \limsup_{n\to\infty} \frac{1}{n} \ln(\mathrm{Vol}(f^n(W_r(x)))),$$

where $\chi_W(x, r)$ is the volume growth rate of the foliation at x. Let

$$\chi_W(f) = \sup_{x\in M} \chi_W(x, r).$$

Then, $\chi_W(f)$ is the maximum volume growth rate of W under f. Obviously, the quantity $\chi_W(f)$ is independent of r.

We search for conditions such that $\chi_W(x, r)$ is independent of both x and r. This is not true in general; there need to be certain topological conditions for this to hold. If W is exponentially expanding under the iterates of f, we can formulate this condition in terms of the action f induces on the homology of M. To put it simply, if f is partially hyperbolic with W being part of unstable manifolds, then we require that $f_* : H_*(M, \mathbb{R}) \to H_*(M, \mathbb{R})$ be partially hyperbolic in a compatible way.

More precisely, let W be an f-invariant k-dimensional foliation. We assume that W is orientable and we will fix an orientation for W. Furthermore, we assume that the leaves of W have exponential growth under f. That is, there are constants $\lambda > 1$ and $c_2 > 0$ such that

$$|df_x^n v| \ge c_2\lambda^n|v|,$$

for all $x \in M$, all $v \in T_x W(x)$ and all $n \in \mathbb{N}$, where $W(x)$ is the leaf of W through the point x. Let $W_r(x)$ be the ball of radius r centered at x on the leaf $W(x)$. For any positive integer n, we define the currents C_n by

$$C_n(\omega) = \frac{1}{\mathrm{Vol}(f^n(W_r(x)))} \int_{f^n(W_r(x))} \omega, \tag{2}$$

for any k-form ω on M. These currents depend on x and r. The currents are uniformly bounded so there must be subsequences with weak limits. Let C be such a limit, i.e. we have a sequence $n_i \to \infty$ such that for any k-form ω we have $\lim_{i\to\infty} C_{n_i}(\omega) = C(\omega)$.

A current C is said to be *closed* if, for any exact k-form $\omega = d\alpha$, we have $C(\omega) = C(d\alpha) = 0$. If C is closed, it has a homology class $[C] = h_C \in H_k(M, \mathbb{R})$. This homology class is non-trivial if there exists a closed k-form ω such that $C(\omega) \neq 0$.

We would like to investigate the conditions under which the subsequential limits of the currents C_n are closed. In general, C_n itself is not closed. We believe that it can be approximated by a closed one for large n. From Stokes' theorem, we have

$$\begin{aligned} C_n(\omega) &= \frac{1}{\mathrm{Vol}(f^n(W_r(x)))} \int_{f^n(W_r(x))} d\alpha \\ &= \frac{1}{\mathrm{Vol}(f^n(W_r(x)))} \int_{W_r(x)} (f^*)^n \, d\alpha \\ &= \frac{1}{\mathrm{Vol}(f^n(W_r(x)))} \int_{\partial W_r(x)} (f^*)^n \alpha. \end{aligned}$$

There are reasons to believe that the above sequence always approaches 0 as $n \to \infty$, i.e. every subsequential limit of the currents C_n is closed. Nevertheless, we are not able to show this as of now. However, we can indeed show that this is true in many cases.

The first case is when the dimension of the foliation is one. In this case, α is a real-valued function and hence $\int_{\partial W_r(x)} (f^*)^n \alpha$ is the difference of that function evaluated at the two end points of $f^n(W_r(x))$ and therefore it is uniformly bounded. Thus $C_n(\omega) \to 0$ as $n \to \infty$.

Another case is that when f is close to a linear map on the torus $\mathbb{T}^n$ and W is any of the expanding foliations close to the linear one. We will consider this case in more detail later.

In general, we have the following simple proposition, whose proof is straightforward.

PROPOSITION 2.1. *Let $J_k(x)$ be the Jacobian of f restricted to the unstable subspace of x and let $J_{k-1}(x)$ be the maximal Jacobian on the $(k-1)$-dimensional subspace at x. If $J_k(x) > J_{k-1}(x)$ for all $x \in M$, then all subsequential limits of $\{C_n\}$ in equation (2) are closed.*

The Jacobian condition in the above proposition is an open condition.

Definition 2.2. We say that a k-dimensional invariant foliation W carries a non-trivial homology $h_C \in H_k(M, \mathbb{R})$ if the currents C_n defined above have a closed subsequential limit C and $h_C = [C] \neq 0$.

We say that a k-dimensional invariant foliation W carries a unique non-trivial homology (up to rescale) if all subsequential limits of the currents C_i are closed and the homologies it carries are unique up to scalar multiplication and are uniformly bounded away from zero, for all $x \in M$ and all $r > 0$.

A current is non-trivial if there is a closed k-form ω such that $C(\omega) \neq 0$. The homology class of a non-trivial closed current is non-trivial. One way to show that the closed current C is non-trivial is to show that there is a closed k-form ω such that ω is non-degenerate on $T_x W(x)$ for any $x \in M$. This condition implies that the integral of ω over any oriented segment of W is non-zero, i.e.

$$\int_D \omega \neq 0,$$

for any piece D on a leaf of the foliation W, with its orientation inherited from the leaf. We may assume that the integral is positive by choosing $-\omega$ if necessary. When we have a non-degenerate k-form on the leaves of W, by compactness of the manifold, there exists a constant $c_2 > 1$ such that

$$c_2^{-1}\,\mathrm{Vol}(D) \leq \int_D \omega \leq c_2\,\mathrm{Vol}(D),$$

for any segment D on the leaves of W and therefore

$$c_2^{-1}\,\mathrm{Vol}(f^n(W_r(x))) \leq \int_{f^n(W_r(x))} \omega \leq c_2\,\mathrm{Vol}(f^n(W_r(x))).$$

This implies that $C(\omega) > 0$.

Assume that an invariant foliation W carries a non-trivial homology and let $h_C = [C] \in H_k(M, \mathbb{R})$, where C is the current as defined above. The next proposition shows that h_C is actually an eigenvector of the induced linear map by f on the homology of M.

PROPOSITION 2.3. *Let W be a k-dimensional invariant foliation that carries a unique non-trivial homology h_C. Then h_C is an eigenvector of the induced linear map:*

$$f_* : H_k(M, \mathbb{R}) \to H_k(M, \mathbb{R}).$$

Proof. First we observe that the map f naturally induces an action on the currents, defined by

$$f_*C(\omega) = C(f^*\omega),$$

for any k current C and k-form ω. Obviously, if C is closed, then f_*C is closed too and

$$[f_*C] = f_*h_C \in H_k(M.\mathbb{R}).$$

Let current C be a subsequential limit of $C_n(x, r)$, then

$$C(\omega) = \lim_{i\to\infty} \frac{1}{\mathrm{Vol}(f^{n_i}(W_r(x)))} \int_{f^{n_i}(W_r(x))} \omega,$$

for any k-form on M. Therefore

$$\begin{aligned}
(f_*C)(\omega) &= \lim_{i\to\infty} \frac{1}{\mathrm{Vol}(f^{n_i}(W_r(x)))} \int_{f^{n_i}(W_r(x))} f^*\omega \\
&= \lim_{i\to\infty} \frac{1}{\mathrm{Vol}(f^{n_i}(W_r(x)))} \int_{f^{(n_i+1)}(W_r(x))} \omega \\
&= \lim_{i\to\infty} \frac{\mathrm{Vol}(f^{(n_i+1)}(W_r(x)))}{\mathrm{Vol}(f^{n_i}(W_r(x)))} \frac{1}{\mathrm{Vol}(f^{(n_i+1)}(W_r(x)))} \int_{f^{(n_i+1)}(W_r(x))} \omega.
\end{aligned}$$

Since the ratio $\mathrm{Vol}(f^{(n_i+1)}(W_r(x)))/\mathrm{Vol}(f^{n_i}(W_r(x)))$ is uniformly bounded, both from above and away from zero, there is a convergent subsequence. Without loss of generality, we may assume that the sequence actually converges and there is a constant $\lambda > 0$ such that

$$\lim_{i\to\infty} \frac{\mathrm{Vol}(f^{(n_i+1)}(W_r(x)))}{\mathrm{Vol}(f^{n_i}(W_r(x)))} = \lambda.$$

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This implies that f_*C/λ is also a subsequential limit of the current $C_n(x, r)$. Since W carries a unique non-trivial homology, this limit must be a scalar multiple of C. Therefore, there is a constant c_3 such that we have $f_*C\lambda^{-1} = c_3C$. This implies that

$$f_*h_C = c\lambda h_C,$$

i.e. h_C is an eigenvector of

$$f_* : H_k(M, \mathbb{R}) \to H_k(M, \mathbb{R}),$$

with corresponding eigenvalue $c_3\lambda$.

This proves the proposition. □

Let λ_W be the eigenvalue of f_* corresponding to the eigenvector h_C, as in the above proposition. We call λ_W the *topological growth* of the foliation W. We will see below that the topological growth and the volume growth are the same for a foliation that carries a unique non-trivial homology, except that the volume growth we defined here is an exponent, while the topological growth is a multiplier.

PROPOSITION 2.4. *Let W be a hyperbolic invariant foliation that carries a unique non-trivial homology h_W. Let λ_W be the topological growth of the foliation. Then the volume growth defined before is given by*

$$\chi_W(f) = \ln \lambda_W,$$

for any $x \in M$ and any $r > 0$.

Proof. The volume of a piece of leaf in a foliation depends on the Riemannian metric defined on M. So in general, the volume does not grow uniformly with each iteration. We will need to rescale the volume at each step so that there will be uniform growth. Let $h_W \in H_k(M, \mathbb{R})$ be a homology carried by W. For any $x \in M$ and $r > 0$, we choose a sequence of numbers d_i, $i \in \mathbb{N}$, such that

$$\lim_{i\to\infty} d_iC_i = C \quad \text{and} \quad [C] = h_W/\|h_W\|.$$

This is possible by the uniqueness of homologies carried by the foliation. Moreover, there are numbers $0 < c_4 \le c_5$ such that d_i can be chosen with $c_5^{-1} \le d_i \le c_4^{-1}$. Therefore,

$$\begin{aligned}
(f_*C)(\omega) &= \lim_{i\to\infty} \frac{d_i}{\mathrm{Vol}(f^i(W_r(x)))} \int_{f^i(W_r(x))} f^*\omega \\
&= \lim_{i\to\infty} \frac{d_i}{\mathrm{Vol}(f^i(W_r(x)))} \int_{f^{(i+1)}(W_r(x))} \omega \\
&= \lim_{i\to\infty} \frac{\mathrm{Vol}(f^{(i+1)}(W_r(x)))/d_{i+1}}{\mathrm{Vol}(f^i(W_r(x)))/d_i} \frac{d_{i+1}}{\mathrm{Vol}(f^{(i+1)}(W_r(x)))} \int_{f^{(i+1)}(W_r(x))} \omega \\
&= \lim_{i\to\infty} \frac{\mathrm{Vol}(f^{(i+1)}(W_r(x)))/d_{i+1}}{\mathrm{Vol}(f^i(W_r(x)))/d_i} C(\omega).
\end{aligned}$$

Therefore

$$\lim_{i\to\infty} \frac{\mathrm{Vol}(f^{(i+1)}(W_r(x)))/d_{i+1}}{\mathrm{Vol}(f^i(W_r(x)))/d_i} = f_*C(\omega)/C(\omega) = \lambda_W.$$

This implies that

$$\begin{aligned}
\chi_W(x, r) &= \lim_{i\to\infty}\sup \frac{1}{i} \ln(\mathrm{Vol}(f^i(W_r(x))))\\
&= \lim_{i\to\infty}\sup \frac{1}{i} \ln \mathrm{Vol}(f^i(W_r(x)))\\
&= \lim_{i\to\infty}\sup \frac{1}{i} \ln(d_i^{-1}\mathrm{Vol}(f^i(W_r(x))))\\
&= \lim_{i\to\infty}\sup \frac{1}{i} \ln\bigg(d_0^{-1}\mathrm{Vol}(W_r(x))\bigg(\prod_{j=1}^{i} \frac{d_i^{-1}\mathrm{Vol}(f^i(W_r(x)))}{d_{i-1}^{-1}\mathrm{Vol}(f^{(i-1)}(W_r(x)))}\bigg)\bigg)\\
&= \lim_{i\to\infty}\sup \frac{1}{i} \sum_{j=1}^{i}\bigg(\ln \frac{d_i^{-1}\mathrm{Vol}(f^i(W_r(x)))}{d_{i-1}^{-1}\mathrm{Vol}(f^{(i-1)}(W_r(x)))}\bigg)\\
&= \ln \lambda_W.
\end{aligned}$$

Here we have used the elementary fact that, if $\lim_{i\to\infty} a_i = a$, then

$$\lim_{i\to\infty} \frac{1}{i} \sum_{j=1}^{i} a_i = a.$$

This proves the proposition. □

The next proposition discusses the situation where a foliation carries more than one non-trivial homology.

PROPOSITION 2.5. *Let W be a hyperbolic invariant foliation and let $H \subset H_k(M, \mathbb{R})$ be the set of non-trivial homologies carried by W. Then H spans a linear space, invariant under*

$$f_* : H_k(M, \mathbb{R}) \to H_k(M, \mathbb{R}).$$

Proof. We first observe that $H \subset H_k(M, \mathbb{R})$ is a bounded and closed set. Let $h \in H$ be a homology carried by the foliation W. It follows from the proof of Proposition 2.3 that there exists a constant $c_6 > 0$ such that f_*h/c_6 is also carried by W. The proposition follows. □

Suppose that the foliation W is one-dimensional; then every subsequential limit of the currents is closed, as we have shown. In this case every limit defines a homology class. If we furthermore assume that there is a non-degenerate closed 1-form ω on the leaves of the foliation, then W carries a non-trivial homology. Then this homology class is non-trivial.

Another class of maps that we would like to consider are the ones on the n-torus $\mathbb{T}^n$ close to a linear map. Consider an $n \times n$ matrix A with integer entries and with determinant one. The matrix A induces a linear toral automorphism: $T_A : \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{T}^n$ defined by $T_A x = Ax \mod \mathbb{Z}^n$. If all eigenvalues are away from the unit circle, then T_A is a hyperbolic toral automorphism. If the eigenvalues of A are mixed, with some on the unit circle and some away from the unit circle, then T_A is partially hyperbolic.

In both hyperbolic and partially hyperbolic cases, let E^u be the unstable distribution of the T_A on $\mathbb{T}^n$. At each point $x \in \mathbb{T}^n$, $E^u(x) \subset T_x\mathbb{T}^n$ is the unstable subspace for $dT_A : T_x\mathbb{T}^n \to T_x\mathbb{T}^n$. Let W^u be the unstable foliation generated by E^u; W^u is a hyperplane in $\mathbb{T}^n$. It is easy to see that the currents C_n converge to a unique closed current C and C is non-trivial. Moreover, the eigenvalue corresponding to h_C for the map $f_* : H_*(\mathbb{T}^n, \mathbb{R}) \to H_*(\mathbb{T}^n, \mathbb{R})$ is the product of all eigenvalues outside of the unit circle, i.e. $\lambda_W = \prod_{|\lambda_i|>1} \lambda_i$.

Let f be a map close to T_A. We claim that all the subsequential limits of the currents C_n are closed. This is because the Jacobian for f on the k-dimensional volume is close to the Jacobian for T_A, which is equal to $(\prod_{|\lambda_i|>1} \lambda_i)^n$. Therefore the k-dimensional volume $\mathrm{Vol}(f^n(W_r(x)))$ grows with a factor close to $(\prod_{|\lambda_i|>1} \lambda_i)^n$. However, any $(k-1)$-form $(f^*)^n\alpha$ grows approximately at the rate of the product of $k-1$ eigenvalues. Therefore,

$$\begin{aligned} C_n(\omega) &= \frac{1}{\mathrm{Vol}(f^n(W_r(x)))} \int_{f^n(W_r(x))} d\alpha \\ &= \frac{1}{\mathrm{Vol}(f^n(W_r(x)))} \int_{\partial W_r(x)} (f^*)^n\alpha \to 0, \end{aligned}$$

as $n \to \infty$.

It is also easy to see that every subsequential limit of the currents is non-trivial. For the linear map, there is a coordinate plane $(xn_1, x_{n_2}, \ldots, x_{n_k})$ such that the orthogonal projection of the unstable space E^u to the plane is non-degenerate. Then the k-form $\omega^k = dx_{n_1} \wedge \cdots \wedge dx_{n_k}$ is non-degenerate on the unstable manifolds. Obviously, ω^k is also non-degenerate on the unstable manifolds for all maps close to T_A.

It remains to show that W^u carries a unique homology. We first observe that the map f is homotopic to the linear map T_A and hence the induced maps on the homology are exactly the same. By Proposition 2.5, we have that the set of all homologies carried by W^u span an invariant subspace in $H_k(\mathbb{T}^n, \mathbb{R})$, since the unstable manifold is expanded by approximately a factor of $(\prod_{|\lambda_i|>1} \lambda_i)^n$. Every eigenvector of f_* in this subspace has an eigenvalue close to $(\prod_{|\lambda_i|>1} \lambda_i)^n$. However, there is only one (up to a constant multiple) eigenvector with the eigenvalue $(\prod_{|\lambda_i|>1} \lambda_i)^n$. This implies that all the subsequential limit of the currents is unique up to rescaling and the eigenvalue is exactly $(\prod_{|\lambda_i|>1} \lambda_i)^n$.

3. *Proof of the main results*

In this section, we finish the proof of our main theorems.

First we define topological entropy using (n, ϵ)-separated sets. Let $f : M \to M$ be a homeomorphism on a compact metric space M. For any given positive integer n and positive real number $\epsilon > 0$, a subset $S \subset M$ is said to be (n, ϵ)-separated if, for any two distinct points $x, y \in S$, there is an integer i with $0 \le i \le n$ such that $d_M(f^i(x), f^i(y)) \ge \epsilon$. Let $\#S$ be the cardinality of the set S and let

$$s(n, \epsilon) = \max\{\#S \mid S \subset M \text{ is } (n, \epsilon)\text{-separated}\}.$$

We define

$$h(f, \epsilon) = \lim_{n\to\infty} \sup \frac{1}{n} \ln s(n, \epsilon)$$

and the topological entropy

$$h_{\mathrm{top}}(f) = h(f) = \lim_{\epsilon\to 0^+} h(f, \epsilon).$$

The topological entropy measures the growth of the trajectories. It is certainly related to the geometric growth of the unstable foliations in a partially hyperbolic system. Let $f \in \mathrm{PH}^r(M)$ be a partially hyperbolic diffeomorphism and let W^u be the unstable foliation of f. Let $\chi_u(f) = \chi_{W^u}(f)$ be the geometric growth of f on the unstable foliation W^u. We have the following lemma.

LEMMA 3.1. *With the notation above, $h(f) \geq \chi_u(f)$.*

Proof. For any given $\delta > 0$, choose a point $x \in M$ and a small $r > 0$ such that $\xi_u(x, r) > \xi_u(f) - \delta/2$. By the definition of $\chi_u(x, r)$, there exists $N > 0$ such that for all $n \geq N$ we have the inequality

$$\mathrm{Vol}(f^n(W_r^u(x))) > e^{n(\chi_u(f)-\delta)}.$$

For any $\epsilon > 0$, we consider (n, ϵ)-separated sets on the strongly unstable foliation $W^u(x)$. Let $d_u(y, z)$ be the distance between two points $y, z \in W^u(x)$ measured by the shortest curve in the submanifold $W^u(x)$ between y and z. Clearly $d_M(y, z) \leq d_u(y, z)$, where d_M is the distance between y and z on M. Without loss of generality, we may assume that $\epsilon < r/(\max_{x\in M} \|Df_x\|)$. Let

$$\epsilon' = \inf\{d_M(y, z) \mid y, z \in W^u(x);\ x \in M;\ \epsilon \leq d_u(y, z) \leq r\}.$$

We claim that $\epsilon' > 0$. For otherwise, by the continuity of the leaves, there exist a point $y \in M$ and a sequence of points $z_i \in W^u(y)$ such that $d_M(y, z_i) \to 0$, as $i \to \infty$, and

$$\epsilon \leq d_u(y, z_i) \leq r \quad \text{for all } i \in \mathbb{N}.$$

This is impossible since the set $A = \{z \in W^u(y) \mid \epsilon \leq d_u(y, z) \leq r\}$ is compact and $d_M(y, z) > 0$ for all $z \in A$.

Let $S(n, \epsilon) \subset f^n(W_r^u(x))$ be a finite set such that, for any $x_i, x_j \in S(n, \epsilon)$, $x_i \neq x_j$, we have $d_u(y, z) \geq \epsilon$. There is a constant $c_7 > 0$, depending on the Riemannian metric and k, the dimension of the foliation W^u, such that $\mathrm{Vol}(W_\epsilon^u(x_i)) \leq c_7\epsilon^k$. Let $\#(S(n, \epsilon))$ be the cardinality of the set S. Then the total volume covered by the ϵ balls around the points in $S(n, \epsilon)$ is less than $\#(S(n, \epsilon))c_7\epsilon^k$. Since one can add points to $S(n, \epsilon)$ if the total volume of these ϵ balls is less than the total volume of $W_r^u(x)$, this implies that $S(n, \epsilon)$ can have at least as many points as

$$\mathrm{Vol}(f^n(W_r^u(x)))/(c_7\epsilon^k).$$

The pre-image $f^{-n}S(n, \epsilon) \subset W_r^u(x)$ is an (n, ϵ)-separated set on the unstable foliation W^u. In fact, it is also an (n, ϵ')-separated set on M, where ϵ' is as defined before. For given any two distinct points $y, z \in f^{-n}(S)$, we have $d_u(f^n(y), f^n(z)) \geq \epsilon$ and $d_u(x, y) < r$. Since $\epsilon < r/(\max_{x\in M} \|Df_x\|)$, there exists an integer i, $0 \leq i \leq n$, such that $\epsilon \leq d_u(f^i(y), f^i(z)) \leq r$. Therefore, $d_M(f^i(y), f^i(z)) \geq \epsilon'$.

Finally,

$$\begin{aligned} h(f, \epsilon') &\geq \limsup_{n\to\infty} \frac{1}{n} \ln \#S(n, \epsilon) \\ &\geq \lim_{n\to\infty} \frac{1}{n} \ln(e^{n(\chi_u(f)-\delta)}/(c\epsilon^k)) \\ &= \chi_u(f) - \delta. \end{aligned}$$

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Since $\delta > 0$ is arbitrary,

$$h(f) = \lim_{\epsilon' \to 0^+} h(f, \epsilon') \geq \chi_u(f).$$

This completes the proof. □

Related to the topological entropy is the measure theoretic entropy. Even though our results are about topological entropy, our proof uses results from measure entropy, Lyapunov exponents and smooth ergodic theory. Let ν be an invariant probability measure. Similar to the definition of topological entropy, one can define an entropy, $h_\nu(f) \geq 0$, associated with the invariant measure ν, using the so-called (n, ϵ)-spanning set that covers a ν positive measure set. We refer readers to Pollicott [**3**] and Robinson [**4**] for more details. However, later in the paper, we will use the following equivalent definition.

One can define a *measure entropy* $h_\nu(f)$ as follows. Call $\xi = \{A_1, \ldots, A_r\}$ a (finite) *measurable partition* of X if the A_i are disjoint measurable subsets of X covering X. Now set

$$H(\xi) = \sum_{i=1}^{r} \nu(A_i) \log \nu(A_i).$$

Then the limit

$$\begin{aligned} h_\nu(f, \xi) &= \lim_{n\to\infty} \frac{1}{n} H(\xi \vee f^{-1}\xi \vee \cdots \vee f^{-(n-1)}\xi) \\ &= \lim_{n\to\infty} H\bigg(\bigvee_{i=0}^{n-1} f^{-i}\xi\bigg), \end{aligned}$$

exists and one defines

$$h_\nu(f) = \sup\{h_\nu(f, \xi) : \xi \text{ is a finite measurable partition of } X\}.$$

Let $\xi(A) = \{A_1, \ldots, A_k\}$ and $\zeta(C) = \{C_1, \ldots, C_p\}$ be two finite partitions. We define the *entropy of* ξ *given to* ζ to be

$$\begin{aligned} H(\xi|\zeta) &= -\sum_{j=1}^{p} \nu(C_j) \sum_{i=1}^{k} \frac{\nu(A_i \cap C_j)}{\nu(C_j)} \log \frac{\nu(A_i \cap C_j)}{\nu(C_j)} \\ &= -\sum_{i,j} \nu(A_i \cap C_j) \log \frac{\nu(A_i \cap C_j)}{\nu(C_j)}, \end{aligned}$$

omitting the j-terms when $\nu(C_j) = 0$. Later in this paper we shall use the following fact to compute $h_\nu(f, \xi)$:

$$h_\nu(f, \xi) = \lim_{n\to\infty} H\bigg(\xi \bigg| \bigg(\bigvee_{i=1}^{n} f^{-i}\xi\bigg)\bigg).$$

(See Walters [**6**, pp. 82–83].)

From the definitions, it is easy to show that $h_\nu(f) \leq h(f)$. Moreover, we have the following well-known theorem. We refer the readers to Walters [**7**] for excellent accounts.

THEOREM 3.2. (Variational principle) *Let M_{erg} be the set of all invariant ergodic measures; then $h(f) = \sup_{\nu \in M_{\text{erg}}} h_\nu(f)$. In other words, for all $\varepsilon > 0$, there exists $\nu \in M_{\text{erg}}$ such that $h_\nu(f) > h(f) - \varepsilon$.*

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Another concept we will need to use is the Lyapunov exponents. Let ν be an invariant probability measure for $f \in \mathrm{Diff}^r(M)$. For ν-a.e. $x \in M$, there exist real numbers $\lambda_1(x) > \cdots > \lambda_l(x)$ $(l \leq n)$, positive integers $n_1, \ldots, n_l$ such that $n_1 + \cdots + n_l = n$, and a measurable invariant splitting $T_x M = E_x^1 \oplus \cdots \oplus E_x^l$, with dimension $\dim(E_x^i) = n_i$, such that

$$\lim_{j\to\infty} \frac{1}{j} \log \|D_x f^j(v_i)\| = \lambda_i(x),$$

whenever $v_i \in E_x^i$, $v \neq 0$.

These numbers $\lambda_1(x), \ldots, \lambda_l(x)$ are called the Lyapunov exponents for $x \in M$. If the probability measure ν is ergodic, then these exponents are constants for ν-a.e. $x \in M$. The existence of these Lyapunov exponents is the result of Oseledec's multiplicative ergodic theorem.

An invariant measure ν is called *hyperbolic* on a invariant set Λ if $\nu(\Lambda) > 0$ and ν-a.e. $x \in \Lambda$ has the property that $\lambda_i(x) \neq 0$ for all $i = 1, \ldots, l$.

For a partially hyperbolic diffeomorphism, $f \in \mathrm{PH}(M)$, the Lyapunov exponents can be relabeled in three groups, according to where their corresponding vectors are. We will write λ_i^s for Lyapunov exponents in E^s, λ_i^c for exponents in E^c, and λ_i^u for exponents in E^u.

To study the continuity properties of the topological entropy, we now consider diffeomorphisms close to a given $f \in \mathrm{PH}(M)$. Assume that W_f^u, the unstable foliation for f, carries a unique non-trivial homology. We say that f *stably* carries a unique non-trivial homology if there is a neighborhood V of f in $\mathrm{PH}(M)$ such that, for any $g \in V$, the unstable foliation W_g^u uniquely carries the same homology element (up to rescale). If the unstable foliation is one-dimensional and f carries a unique non-trivial homology, it is easy to show that f *stably* carries a unique homology. We believe that this is true in general, but we are not able to show this. However, if all the subsequential limits of the currents C_n are closed for f and nearby maps, then one can show that f carrying a unique non-trivial homology implies that f stably carries a unique non-trivial homology. This is certainly true for maps on $\mathbb{T}^n$ close to the linear one.

The homologies carried by the stable foliation are defined in the same way by considering f^{-1}.

Under the assumption that f stably carries a unique non-trivial homology, the geometric expansion $\chi_u(f, x)$ is well defined for all $x \in M$ and is constant. Furthermore $\chi_u(f)$ is locally constant on f.

We now proceed with the proof of our main results. We will divide the proof into several cases. Since the upper semi-continuity of topological entropy is known from Yomdin's theorem for C^∞ diffeomorphisms, it suffices to show lower semi-continuity for our results on dimension two.

In the proof we also need to consider f^{-1}. First we recall that $h(f) = h(f^{-1})$. We can also define the volume growth of the stable foliation W^s under f^{-1}. The same as $\chi_u(f)$, we can define $\chi_s(f) = \chi_u(f^{-1})$ and in the same way, we have $h(f) \geq \chi_s(f)$.

Case 1. Either $h(f) = \chi_u(f)$ or $h(f) = \chi_s(f)$. This is a simple case. Assume that $h(f) = \chi_u(f)$. By Proposition 2.4, $\chi_u(f)$ is locally constant, so there exists a neighborhood of V of f in $\mathrm{Diff}^r(M)$ such that, for any $g \in V$, we have $\chi_u(g) = \chi_u(f)$.

Therefore $h(g) \geq \chi_u(g) = \chi_u(f) = h(f)$. That is, $h(f)$ is lower semi-continuous. The case with $h(f) = \chi_s(f)$ is the same.

Case 2. Both $h(f) > \chi_u(f)$ and $h(f) > \chi_s(f)$ hold. This is our main case. Let ν be an ergodic invariant probability measure for $f \in \mathrm{Diff}^r(M)$ and let $\lambda_1, \ldots, \lambda_l$ be the Lyapunov exponents associated with ν. We have the following Pesin–Ruelle inequality:

$$h_\nu(f) \leq \sum_{\lambda_i > 0} \lambda_i.$$

For our purpose, we need a refined version of the Pesin–Ruelle inequality, where we incorporate the geometric expansion $\chi_u(f)$ into the above formula. We have the following theorem.

THEOREM 3.3. *Let $f \in \mathrm{PH}(M)$ be a partially hyperbolic diffeomorphism on a compact manifold M. Let ν be an ergodic measure and let λ_i^c be the Lyapunov exponents corresponding to the center distribution E^c. Then the following estimate holds:*

$$h_\nu(f) \leq \sum_{\lambda_i^c > 0} \lambda_i^c + \chi_u(f).$$

The proof of this theorem is quite involved. We postpone the proof to the next section.

We return to the proof of the main theorem for the case with $h(f) > \chi_u(f)$ and $h(f) > \chi_s(f)$. By the variational principle, for any $\delta > 0$ there is an ergodic measure ν such that

$$h_\nu(f) > h(f) - \delta.$$

Choosing δ such that

$$0 < \delta \leq \min\left\{\frac{h(f) - \chi_u(f)}{3}, \frac{h(f) - \chi_s(f)}{3}\right\},$$

we have

$$h_\nu(f) > h(f) - \delta > \chi_u(f) + \delta.$$

By the above proposition, for such measure ν,

$$h_\nu(f) \leq \chi_u(f) + \sum_{\lambda_i^c > 0} \lambda_i^c.$$

Therefore,

$$\chi_u(f) + \sum_{\lambda_i^c > 0} \lambda_i^c \geq \chi_u(f) + \delta,$$

and so

$$\sum_{\lambda_i^c > 0} \lambda_i^c \geq \delta > 0.$$

We can easily see that at least one of the λ_i^c must be larger than zero.

If $\dim E^c = 1$, then there is only one center exponent and $\lambda^c > 0$. Now consider f^{-1} and recall that $h_\nu(f) = h_\nu(f^{-1})$ and $\chi_s(f) = \chi_u(f^{-1})$; therefore $h_\nu(f^{-1}) > \chi_u(f^{-1})$. Apply the same argument to f^{-1} and we get $-\lambda^c > 0$. But that is a contradiction to $\lambda > 0$. Therefore we cannot have both $h(f) > \chi_u(f)$ and $h(f) > \chi_s(f)$. This implies that we can

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only have Case 1 and $h(f)$ is actually the maximum of $\chi_u(f)$ and $\chi_s(f)$. But both these numbers are locally constant, and therefore $h(f)$ must be locally constant. This proves our theorem for the case where $\dim E^c = 1$.

Assume that $\dim E^c = 2$. For the measure ν, there are two center Lyapunov exponents, λ_1^c and λ_2^c. We may assume that $\lambda_1^c \geq \lambda_2^c$. The above arguments show that $\lambda_1^c > 0$. By considering f^{-1}, we have $-\lambda_2^c > 0$ or $\lambda_2^c < 0$. Since all other Lyapunov exponents are non-zero, this implies that the measure ν is a hyperbolic ergodic measure.

To complete our proof for $\dim E^c = 2$, we need one more result from Katok and Mendoza [**2**].

PROPOSITION 3.4. (Katok and Mendoza) *Assume that ν is an ergodic hyperbolic measure for a $C^{1+\alpha}$ diffeomorphism, $\alpha > 0$. Then, for any $\epsilon > 0$, there exists a uniformly hyperbolic invariant set $\Lambda \subset M$ such that: $h(f|\Lambda) > h_\nu(f) - \epsilon$.*

The proof of this proposition can be found in Pesin [**1**, pp. 122–124].

Hyperbolic invariant sets persist under small perturbations. There is a neighborhood V of f in $\mathrm{Diff}^r(M)$ such that, for any $g \in V$, there is a hyperbolic invariant set Λ_g, close to Λ, such that $g|_{\Lambda_g} : \Lambda_g \to \Lambda_g$ is topologically conjugate to $f|_\Lambda : \Lambda \to \Lambda$. Therefore

$$h(g) \geq h(g|_{\Lambda_g}) = h(f|_\Lambda) > h_\nu(f) - \epsilon > h(f) - \delta - \epsilon.$$

In other words, the entropy of f is lower semi-continuous.

This completes the proof of the main theorem, assuming Theorem 3.3. □

4. *A refined Pesin–Ruelle formula*

In this section, we give a proof of Theorem 3.3, a refined Pesin–Ruelle formula for partially hyperbolic diffeomorphisms.

The usual approach to the proof relies on the fact that, if a partition of the manifold is fine enough, then the diffeomorphism, up to a finite number of iterates, can be approximated by its linearization and, therefore, the growth in the partition can be estimated by Lyapunov exponents. However, the volume growth χ is the opposite of the Lyapunov exponents; it gives estimates of volumes of large surfaces. The difficulty is to incorporate these two, seemingly opposite, concepts into the partitions.

Choose a small $\delta > 0$; for any given $\epsilon > 0$, and any $x \in M$, there is an integer K_x, depending on x, such that

$$\mathrm{Vol}(f^i(W_r(x))) < \delta^k e^{i(\chi(f)+\epsilon)} \tag{3}$$

for all $0 < r \leq 10\delta$ and all $i \geq K_x$. Here k is the dimension of the unstable foliation.

For any positive integer K, let S_K be the set of points such that $K_x \leq K$. Obviously, for any measure ν on M, we have $\nu(M \backslash S_K) \to 0$ as $K \to \infty$. We also observe that there is a constant c_8 such that, for any $x \in M$, any positive integer i and any $r \leq \delta$,

$$\mathrm{Vol}(f^i(W_r(x))) < c_8 \delta^k \left(\sup_{x \in M} \|df\| \right)^{ki}. \tag{4}$$

Fix a positive integer $m = lK$ with a positive integer l and let $B(y, t)$ be a ball centered at y with radius t. Since M is compact, there exists $t_m > 0$ such that, for every $0 < t < t_m$,

$y \in M$ and $x \in B(y, t)$, we have

$$\tfrac{1}{2} d_x f^m(\exp_x^{-1} B(y, t)) \subset \exp_{f^m x}^{-1} f^m(B(y, t)) \subset 2 d_x f^m(\exp_x^{-1} B(y, t)),$$

where $\exp_x$ is the exponential map at $x \in M$.

Now, for any chosen $\epsilon > 0$, there is a positive number $\alpha > 0$, $\alpha < t_m/100$, such that, for any partition ξ with diam $\xi \leq 2\alpha$, we have

$$h_\mu(f^m, \xi) \geq h_\mu(f^m) - \epsilon.$$

Let d_u be the induced metric on $W(x)$, $x \in M$, from the Riemannian structure on M. We introduce a dynamically defined new metric on the manifold. Let J be a positive integer such that the following is true: for any point $x \in M$ and $y \in W(x)$, with $d_u(x, y) \leq \delta$, we have $d_u(f^{-JK}(x), f^{-JK}(y)) \leq \alpha$. Since M is compact and the unstable leaves are uniformly expanding, such an integer J exists. We now define a new metric d^J by

$$d^J(x, y) = d(f^{JK}(x), f^{JK}(y))\alpha/\delta,$$

and this metric also induces a metric, d_u^J, on the unstable leaves, so we have

$$d_u^J(x, y) = d_u(f^{JK}(x), f^{JK}(y))\alpha/\delta.$$

An important property we have for this new metric is that

$$d^J(x, y) \leq \alpha \quad \text{whenever } d(f^{JK}(x), f^{JK}(y)) \leq \delta.$$

The metric d^J depends on the choice of J, which is chosen to be a large integer. A ball with metric d^J is a thin tube-like object. The center direction and the stable direction are very long and the unstable direction is very short. On the unstable manifold, by equation (1), we have

$$d_u^J(x, y) = d_u(f^{JK}(x), f^{JK}(y))\alpha/\delta \geq c_1^{-1}\lambda_u^{JK} d_u(x, y)\alpha/\delta.$$

While on the center or center-stable manifold of any point, if such manifolds do exist, we have

$$d^J(x, y) \leq c_1(\lambda_c'')^{JK} d_u(x, y)\alpha/\delta.$$

By the standard invariant manifold theory, for any point on the manifold M, there exist a center manifold and a center-stable manifold in any sufficiently small neighborhood, even though these manifolds may not be unique and may not form a foliation of M. For a d^J ball on M with a small radius, the ratio of the length in the center-stable direction to that of the unstable direction is at most $2c_1^2(\lambda_c''/\lambda_u)^{JK}$.

For any fixed $\delta > 0$ and small $\alpha > 0$, we will choose J such that

$$c_1^{-1}\lambda_u^{JK}\alpha/\delta > 100 \quad \text{and} \quad c_1(\lambda_c'')^{JK}\alpha/\delta < 1/100. \tag{5}$$

We may increase J by decreasing α. Throughout this section, α can be made arbitrarily small.

Finally we define a new metric ρ on M by

$$\rho(x, y) = d(x, y) + d_J(x, y),$$

for all $x, y \in M$. Clearly ρ is a metric on M. Observe that, by our choice of J in equation (5), on the unstable manifold, $d_u^J(x, y) \geq 100 d_u(x, y)$, and on a center or center-stable manifold, $d^J(x, y) \leq d(x, y)/100$. Therefore, we have the following important property of the metric ρ: it is dominated by the metric d^J in the unstable direction and dominated by d in the center and stable directions. By equation (5), the set, we called it a ρ-ball, given by

$$B_\rho(x, r) = \{y \in M \mid \rho(x, y) \leq r\},$$

contains a d_u^J-ball in the unstable direction with a radius $r/2$ and contains a regular, lower-dimensional ball of radius $r/2$ in the center-stable direction.

We continue our proof of Theorem 3.3. The proof uses proper partitions to estimate the entropy. There is a special partition of the manifold M which is described in the following statement.

LEMMA 4.1. *Given $\epsilon > 0$, there is a partition ξ of M such that:*

(1) $\operatorname{diam} \xi \leq 2\alpha \leq t_m/50$ *and therefore* $h_\mu(f^m, \xi) \geq h_\mu(f^m) - \epsilon$*;*

(2) *for every element $C \in \xi$ there exist ρ-balls $B_\rho(x, r)$ and $B_\rho(x, r')$, such that $\alpha/4 < r' < r < \alpha$ and $B_\rho(x, r') \subset C \subset B_\rho(x, r)$; and*

(3) *there exists $0 < r < t_m/20$ such that if $C \in \xi$ then $C \subset B(y, r)$ for some $y \in M$, and if $x \in C$ then*

$$\tfrac{1}{2} d_x f^m(\exp_x^{-1} B(y, r)) \subset \exp_{f^m x}^{-1} f^m C \subset 2 d_x f^m(\exp_x^{-1} B(y, r)). \tag{6}$$

Proof. To construct such a partition, given $\alpha > 0$, consider a maximal $2\alpha/3$-separated set Γ, with respect to the metric ρ. That is, Γ is a finite set of points for which $\rho(x, y) > 2\alpha/3$ whenever $x, y \in \Gamma$, and, for any point $z \in M$, there is a point $x \in \Gamma$ such that $\rho(x, z) \leq 2\alpha/3$. For $x \in \Gamma$, set

$$D_\Gamma(x) = \{y \in M \mid \rho(y, x) \leq \rho(y, z), z \in \Gamma \setminus \{x\}\}.$$

Obviously, $B_\rho(x, \alpha/3) \subset D_\Gamma(x) \subset B_\rho(x, 2\alpha/3)$. Note that the sets $D_\Gamma(x)$ corresponding to different points $x \in \Gamma$ intersect only along their boundaries, i.e. at a finite number of submanifolds of codimension greater than zero. Since μ is a Borel measure, if necessary, we can move the boundaries slightly so that they have zero measure.

We may choose α arbitrarily small by increasing J. This guarantees the properties in the lemma. □

Continuing with the proof of the theorem, observe that

$$\begin{aligned} h_\nu(f^m, \xi) &= \lim_{k \to \infty} H_\nu(\xi \mid f^m \xi \vee \cdots \vee f^{km} \xi) \\ &\leq H_\nu(\xi \mid f^m \xi) = \sum_{D \in f^m \xi} \nu(D) H(\xi \mid D) \\ &\leq \sum_{D \in f^m \xi} \nu(D) \log \#\{C \in \xi : C \cap D \neq \emptyset\}, \end{aligned} \tag{7}$$

where $H(\xi|D)$ is the entropy of ξ with respect to conditional measure on D induced by ν. To estimate the entropy, we need to know the number of elements $C \in \xi$ that have non-empty intersections with a given element $D \in f^m \xi$. By property (3) of the partition ξ

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(Lemma 4.1), we have a uniform control on the derivatives of f^m for each element $C \in \xi$. If we had used the regular metric in our partition, then the estimate on the growth of the partition could be easily obtained; this would lead to the standard Pesin–Ruelle formula. In our case, we need to estimate the number of intersecting partitions in two different directions: on the unstable direction and on the center and stable directions.

We first estimate the growth of the partition in the unstable direction. Consider the unstable disk $W_r(x)$ with $r \le \delta$. If $W_r(x) \cap S_K \neq \emptyset$, we have an estimate on the k-dimensional volume of $f^K(W_r(x))$,

$$\mathrm{Vol}(f^K(W_r(x))) < \delta^k e^{K(\chi(f)+\epsilon)}.$$

Therefore, there is a constant c_9 such that $f^K(W_r(x))$ contains at most $c_9 e^{K(\chi(f)+\epsilon)}$ non-intersecting disks on the unstable leaf with radius not less than $\delta/20$. Similarly, for any positive integer i,

$$\mathrm{Vol}(f^{iK}(W_r(x))) < \delta^k e^{iK(\chi(f)+\epsilon)}.$$

The set $f^{iK}(W_r(x))$ contains at most $c_1 e^{iK(\chi(f)+\epsilon)}$ non-intersecting disks on the unstable leaf with radius not less than $\delta/20$.

We now translate the above statements in terms of our new metric ρ. Let $D_x \subset W(x)$ be a piece of the unstable manifold contained in a d_u^J-ball of radius α, centered at x. Suppose that $D_x \cap S_K \neq 0$. By the definition of d_u^J, we have $f^{JK}(D_x) \subset W_\delta(f^{JK}(x))$. If $x \in S_K$, then

$$\mathrm{Vol}(f^{(J+i)K}(D_x)) < \delta^k e^{iK(\chi(f)+\epsilon)}.$$

This implies that the d_u^J volume, which we denote by Vol^J, for the set $f^K(D_x)$ is

$$\mathrm{Vol}^J(f^K(D_x)) = \mathrm{Vol}(f^{(J+1)K}(D_x)) < \delta^k e^{K(\chi(f)+\epsilon)},$$

and for any positive integer i,

$$\mathrm{Vol}^J(f^{iK}(D_x)) = \mathrm{Vol}(f^{(J+i)K}(D_x)) < \delta^k e^{iK(\chi(f)+\epsilon)}.$$

Consequently, $f^{iK}(D_x)$ contains at most $c_9 e^{iK(\chi(f)+\epsilon)}$ non-intersecting disks on the unstable leaf with d_u^J radius not less than $\delta/20$.

To summarize, if we partition the unstable leaves with sets which are bounded between d_u^J disks of radius $\alpha/20$ and α, then the f^{iK} image of any element of the partition covers at most $c_1 e^{iK(\chi(f)+\epsilon)}$ number of elements in that partition, provided that the element contains a point in S'_K, where $S'_K = f^{JK}(S_K)$.

Now we consider the partition ξ in Lemma 4.1. By property (2) of the partition ξ therein, we have that, for every element C of ξ, there is $x \in M$ such that $B_\rho(x, \alpha/4) \subset C \subset B_\rho(x, \alpha)$. For any $D \in f^m\xi$, the k-dimensional growth in the unstable direction is controlled by the geometric growth. In the center and stable directions, it is controlled by the derivative of the map f, since each element of the partition is bounded, from both below and above, by balls with the normal metric in the center-stable directions.

Before we proceed, we have the following simple lemma.

LEMMA 4.2. *There exists a constant $K_1 > 0$ such that, for $D \in f^m\xi$,*

$$\#\{C \in \xi \mid C \cap D \neq \emptyset\} \le K_1 \sup\{\|d_x f\|^{mn} \mid x \in M\},$$

where n is the dimension of the manifold.

This can be shown by estimating the ρ volume expansion of each element in C under f^m and using property (2) of Lemma 4.1. Each C is bounded by a product of regular disks in the center-stable manifold and a ρ disk in the unstable direction. On the unstable direction, the ρ volume expansion is bounded by, from inequality (4), $c_8\delta^k(\sup_{x\in M}\|df\|)^{km}$. The expansion of volume in the center and stable directions is bounded by the maximal derivative of the map. The ρ volume in the center-stable direction is close to the real volume. Since the unstable foliation is absolutely continuous, the lemma follows from the Fubini theorem. □

We have a better exponential bound for the number of those sets D such that $D = f(C') \in f^m\xi$ and C' contains regular points for the invariant measure ν. More precisely, given $\epsilon > 0$, let $R_{m,\epsilon}$ be the set of forward regular points $x \in M$ which satisfy the following condition: for $k > m$ and $v \in E^c_x$,

$$e^{k(\lambda(x,v)-\varepsilon)}\|v\| \le \|d_x f^k v\| \le e^{k(\lambda(x,v)+\varepsilon)}\|v\|. \tag{8}$$

Here $\lambda(x, v)$ is the Lyapunov exponent at x corresponding to the vector v,

$$\lambda(x, v) = \lim_{i\to\infty} \frac{1}{i} \ln \|d_x f^i v\|.$$

The limit exists for ν-a.e. $x \in M$.

Finally, we repartition every element of ξ into two sets. For any $C \in \xi$, let

$$C^1 = \{x \in C \mid W_\alpha(x) \cap S'_K \neq \emptyset\},$$

and $C^2 = C\backslash C^1$. Let ξ^1 be the collection of the sets of type C^1 and ξ^2 be the collection of type C^2. Together ξ^1 and ξ^2 form a partition of the manifold; we denote this new partition by ξ'.

The following lemma gives an estimate of the number of intersections of $f^m(\xi^1)$ with ξ'. The total measure for the sets in ξ^2 is small and its contribution to the entropy will be given in another estimate.

LEMMA 4.3. *For any given $\epsilon > 0$, there is an $N > 0$ such that, for any $m > N$, if $C^1 \in \xi^1$ and $D = f^m(C^1) \in f^m\xi'$ such that C^1 has a non-empty intersection with R_m, then there exists a constant $K_2 > 0$ such that*

$$\#\{C \in \xi' \mid C \cap D \neq \emptyset\} \le K_2 e^{\epsilon m} e^{m(\chi_u(f)+\epsilon)} \prod_{i:\lambda^c_i>0} e^{m(\lambda^c_i+\epsilon)}.$$

Proof. To establish the inequality note that

$$\#\{C \in \xi' \mid C \cap D \neq \emptyset\} \le 2\,\mathrm{Vol}_\rho(B)(\mathrm{diam}_\rho\,\xi)^{-n},$$

where $\mathrm{Vol}_\rho(B)$ denotes the ρ volume of

$$B = \{y \in M \mid \rho(y, \exp_{f^m(x)}(d_x f^m(\exp_x^{-1} B'))) < \mathrm{diam}_\rho\,\xi\},$$

where $B' = B_\rho(x, 2\,\mathrm{diam}_\rho\,C') \cap S'_K$, $C' \in \xi^1$, $f^m(C') = D$ and some $x \in C' \cap R_m$. The set B can be thought of as a fattened set D. Let $W^{cs}(x)$ be the center-stable manifold of x. In fact, an approximate one will suffice. Let E be the subset of $W^{cs}(x)$ such that

$$E = \{y \in W^{cs}(x) \mid \rho(y, x) \le 4\alpha\}.$$

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Obviously,

$$B' \subset \{y \in M \mid \rho_u(y,z) \le 3\,\mathrm{diam}_\rho\, C', \text{ for some } z \in E \cap S'_K\},$$

i.e. B' is contained in the product of the set E and unstable disks. Since the unstable foliation is absolutely continuous, by the Fubini theorem, up to a bounded factor, $\mathrm{Vol}_\rho(B')$ is bounded by the product of the volume expansion of unstable disks and the volume expansion of E. By the invariance of the unstable and center foliations, the same is true for $\mathrm{Vol}_\rho(B)$. We have already obtained the ρ volume expansion in the unstable direction. For the set E, it is bounded by a ball with the regular metric, whose tangent space is on the center-unstable direction (or arbitrarily close to the center-stable direction). On both E and $f^m(E)$, the metric ρ is dominated by the regular metric. Also $f^m(E)$ is approximately an $(n-k)$-dimensional ellipsoid, whose total volume is bounded by the product of the lengths of the axes. The length of the axis can be estimated by $d_x f^m|_{E^c \oplus E^s}$, using equations (6) and (8). Those of the axes that correspond to non-positive exponents are at most sub-exponentially larger. The remaining axes are of size at most $e^{m(\lambda_i^c+\epsilon)}$, up to a bounded factor, for all sufficiently large m. Therefore,

$$\begin{aligned}\mathrm{Vol}_\rho(B) &\le K_3 e^{m\epsilon}(\mathrm{diam}_\rho\, B)^n e^{m(\chi_u(f)+\epsilon)} \prod_{i:\lambda_i^c>0} e^{m(\lambda_i^c+\epsilon)}\\ &\le K_3 e^{m\epsilon}(2\,\mathrm{diam}_\rho\, \xi)^n e^{m(\chi_u(f)+\epsilon)} \prod_{i:\lambda_i^c>0} e^{m(\lambda_i^c+\epsilon)},\end{aligned}$$

for some constant $K_3 > 0$. The lemma follows. □

By Lemmas 4.2 and 4.3, we obtain

$$\begin{aligned} mh_\nu(f) - \epsilon &= h_\nu(f^m) - \epsilon \le h_\nu(f^m, \xi)\\ &\le \sum_{f^{-m}(D)=C'\in\xi^1,(C'\cap R_m)\neq\emptyset} \nu(D)\Big(\log K_2 + \epsilon m + m\sum_{i:\lambda_i^c>0}(\lambda_i^c+\epsilon) + m(\chi_u(f)+\epsilon)\Big)\\ &\quad + \sum_{f^{-m}(D)=C'\in\xi^1,(C'\cap R_m)=\emptyset} \nu(D)(\log 2K_1 + nm\log\sup\{\|d_x f\| : x\in M\})\\ &\quad + \sum_{f^{-m}(D)=C'\in\xi^2} \nu(D)(\log 2K_1 + nm\log\sup\{\|d_x f\| : x\in M\})\\ &\le \log K_2 + \epsilon m + m\sum_{i:\lambda_i^c>0}(\lambda_i^c+\epsilon) + m(\chi_u(f)+\epsilon)\\ &\quad + (\log 2K_1 + nm\log\sup\{\|d_x f\| : x\in M\})\nu(M \setminus (R_m \cup S'_K)).\end{aligned}$$

By the multiplicative ergodic theorem, we have

$$\bigcup_{m\ge 0} R_m(\epsilon) = M \pmod 0$$

for every sufficiently small $\epsilon > 0$. Since every point is in S_K for some K and $\nu(S'_K) = \nu(S_K)$, we have $\nu(M\setminus S'_K) \to 0$ as $K \to \infty$. It follows that

$$h_\nu(f) \le \epsilon + \sum_{i:\lambda_i^c>0}(\lambda_i^c+\epsilon) + (\chi_u(f)+\epsilon).$$

Let $\epsilon \to 0$, we obtain the desired upper bound.

This proves the theorem.

5. *Examples*

In this section, we give several examples where the main theorem fails when we drop the assumption on homology.

5.1. *Example 1.*

Our first example is a partially hyperbolic diffeomorphism with a one-dimensional center foliation.

Let M_l be a compact orientable surface of genus $l \geq 2$, with constant negative curvature. Let $g_t : SM_l \to SM_l$ be the geodesic flow on the unit tangent bundle of M_l; g_t is an Anosov flow. For any fixed $t > 0$, g_t is a partially hyperbolic diffeomorphism on SM_l. The stable, unstable and center distributions are all one-dimensional. The topological entropy for g_1 is non-zero and, for any $t \in \mathbb{R}$, it is easy to see that $h_{\text{top}}(g_t) = |t| h_{\text{top}}(g_1)$, i.e. for different values of t, the topological entropy of g_t is different. Therefore in this case the topological entropy is not locally constant. Obviously, the stable and unstable foliations do not carry any non-trivial homology in this case. In fact, geodesic flows are isotopic to identity and any diffeomorphism that is isotopic to identity cannot carry non-trivial homology, since its induced action on homology is trivial.

The map g_t with $t > 0$ satisfies all the conditions of the main theorem except the homology condition. The topological entropy for the maps near g_t fails to be a constant.

An interesting question arises: Without any topological assumptions, is the topological entropy always continuous for partially hyperbolic diffeomorphisms with one-dimensional center?

5.2. *Example 2.*

In this example, the center distribution is two-dimensional and the topological entropy fails to be continuous.

Now consider a map $f : SM_l \times S^1 \to SM_l \times S^1$ defined in the following way. Let $\alpha : S^1 = \mathbb{R}^1/\mathbb{Z}^1 \to S^1$ be a diffeomorphism which is close to identity and has three fixed points $y_i = (i-1)/4$, $i = 1, 2, 3$, which satisfy the following:

$$\alpha'(y_1) > 1, \quad \alpha'(y_3) < 1, \quad \alpha'(y_2) = 1 \quad \text{and} \quad \alpha''(y_2) \neq 0.$$

Define $f(x, y) = (g_{1+\sin(2\pi y)}, \alpha(y))$. Then

$$\begin{aligned} h_{\text{top}}(f) &= \max_i h_{\text{top}}(f | SM_g \times \{y_i\}) \\ &= \max_i (1 + \sin(2\pi y_i)) h_{\text{top}}(g_1) = 2 h_{\text{top}}(g_1). \end{aligned}$$

Now consider a family of diffeomorphisms

$$\alpha_\epsilon = \alpha(y, \epsilon) = \alpha + \epsilon,$$

and

$$f_\epsilon(x, y) = (g_{1+\sin(2\pi y)}, \alpha_\epsilon(y)).$$

For any fixed $\epsilon \geq 0$, f_ϵ is a partially hyperbolic diffeomorphism on $SM_l \times S^1$ with $\dim E^u = \dim E^s = 1$ and $\dim E^c = 2$.

Hyperbolic fixed points persist under small perturbations, and therefore, for every ϵ sufficiently small, α_ϵ have unique fixed points y_1^ϵ and y_3^ϵ close to y_1 and y_3 respectively.

For $i = 2$, since

$$\alpha'(y_2) = 1, \quad \alpha''(y_2) \neq 0 \quad \text{and} \quad \frac{\partial \alpha_\epsilon}{\partial \epsilon}(y_2, 0) = 1 \neq 0,$$

saddle-node bifurcations occur at $(y_2, 0)$. Therefore, when ϵ is small enough, on one side of $\alpha = \alpha_0$ (without loss of generality we can assume that it is on the left side) α_ϵ has no fixed points close to y_2. Therefore, α_ϵ has no other fixed points other than y_1^ϵ and y_3^ϵ, which implies that

$$\begin{aligned} \lim_{\epsilon \to 0^-} h_{\text{top}}(f_\epsilon) &= \lim_{\epsilon \to 0^-} \max_{i=1,3} h_{\text{top}}(f | SM \times y_i^\epsilon) \\ &= h_{\text{top}}(g_1). \end{aligned}$$

This means that the topological entropy of f is not continuous.

In this example, again, the stable and the unstable manifolds fail to carry non-trivial homology. The topological conditions of the main theorem are not satisfied.

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