Proceedings of the Edinburgh Mathematical Society (2020) **63**, 1092–1099
doi:10.1017/S0013091520000358

ON THE ADJOINT REPRESENTATION OF A HOPF ALGEBRA

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(Received 21 April 2020; first published online 11 November 2020)

Abstract We consider the adjoint representation of a Hopf algebra H focusing on the locally finite part, $H_{\mathrm{ad}\,fin}$, defined as the sum of all finite-dimensional subrepresentations. For virtually cocommutative H (i.e., H is finitely generated as module over a cocommutative Hopf subalgebra), we show that $H_{\mathrm{ad}\,fin}$ is a Hopf subalgebra of H. This is a consequence of the fact, proved here, that locally finite parts yield a tensor functor on the module category of any virtually pointed Hopf algebra. For general Hopf algebras, $H_{\mathrm{ad}\,fin}$ is shown to be a left coideal subalgebra. We also prove a version of Dietzmann's Lemma from group theory for Hopf algebras.

Keywords: Infinite-dimensional Hopf algebra; cocommutative Hopf algebra; pointed Hopf algebra; Hopf subalgebra; adjoint representation; locally finite part; tensor functor; coideal subalgebra; Δ-methods; Dietzmann's Lemma

2020 Mathematics subject classification: Primary 16T05; 16T2

1. Introduction

Originating in the work of Heinz Hopf in algebraic topology [8], the subject of Hopf algebras nowadays belongs very much to the mainstream of research in non-commutative algebra. Indeed, Hopf algebras provide a common approach to groups and Lie algebras, inasmuch as they encompass both group algebras and enveloping algebras; they are essential in the foundational aspects of algebraic groups and group schemes; and they are at the core of more recent developments in representation theory and in the theory of knot invariants. This note concerns an aspect of Hopf algebras that, in the special context of group algebras, has long been explored under the name 'Δ-methods' [15]; they were also investigated for enveloping algebras of Lie algebras [2–4] and for quantum enveloping algebras [9, 11]. In contrast with these earlier publications, our approach is independent of the specifics of group algebras and (quantum) enveloping algebras and uses general Hopf-theoretic methods. We assume that the reader has at least a passing familiarity with Hopf algebras; see [16] for background.

Let H be a Hopf algebra over a field $\Bbbk$, with comultiplication $\Delta h = h_{(1)} \otimes h_{(2)}$, antipode S, and counit ε. The left *adjoint action* of H on itself is defined by

$$k.h = k_{(1)} h \, \mathsf{S}(k_{(2)}) \quad (h, k \in H). \tag{1}$$

This action makes H a left H-module algebra that will be denoted by H_{ad}. Our main interest is in the locally finite part,

$$H_{\mathrm{ad}\,fin} = \{h \in H \mid \dim_{\Bbbk} H.h < \infty\}.$$

Of course, if H is finite dimensional, then $H_{\mathrm{ad}\,fin} = H$; so we are primarily concerned with infinite-dimensional Hopf algebras. In the case of a group algebra $H = \Bbbk G$, it is well known and easy to see that $H_{\mathrm{ad}\,fin} = \Bbbk\Delta$, the subgroup algebra of the so-called FC-centre $\Delta = \Delta(G) = \{g \in G \mid g \text{ has finitely many} G - \text{conjugates}\}$. The Δ-notation for FC-centers, introduced by Passman and not to be confused with the traditional comultiplication notation, has led to the nomenclature 'Δ-methods' in the study of group algebras: a significant number of ring-theoretic properties of group algebras are controlled by the FC-centre; see [15].

Locally finite parts may of course be defined for arbitrary representations as the sum of all finite-dimensional subrepresentations: for any left module V over a $\Bbbk$-algebra R,

$$V_{\mathrm{fin}} \overset{\mathrm{de}}{=} \{v \in V \mid \dim_{\Bbbk} R.v < \infty\}.$$

This gives a functor $\,\cdot_{fin}$ on the category ${}_R\mathsf{Mod}$ of left R-modules. If R is finitely generated as a right module over some subalgebra T, then $V_{\mathrm{fin}} = \{v \in V \mid \dim_{\Bbbk} T.v < \infty\}$ for any $V \in {}_R\mathsf{Mod}$. Adopting group-theoretical terminology, we will call a Hopf algebra H *virtually* of type $\mathcal{C}$, where $\mathcal{C}$ is a given class of Hopf algebras, if H is finitely generated as right module over some Hopf subalgebra $K \in \mathcal{C}$.

The locally finite part A_{fin} of any left H-module algebra A is a subalgebra that contains the algebra of H-invariants, $A^H = \{a \in A \mid h.a = \langle \varepsilon, h \rangle a \text{ for all } h \in H\}$. For $A = H_{\mathrm{ad}}$, the invariant algebra coincides with the centre of H [12, Lemma 10.1]. The centre is rarely a Hopf subalgebra of H, even if H is a group algebra, and $H_{\mathrm{ad}\,fin}$ need not be a Hopf subalgebra either in general, for example, when H is a quantized enveloping algebra; see [1, Example 2.8] or [11]. However, we have the following result. Recall that a left coideal subalgebra of H is a subalgebra C that is also a left coideal of H, i.e., $\Delta(C) \subseteq H \otimes C$.

Theorem 1. *(a) $H_{\mathrm{ad}\,fin}$ is always a left coideal subalgebra of H.*

(b) If H is virtually cocommutative, then $H_{\mathrm{ad}\,fin}$ is a Hopf subalgebra of H.

For a quantized enveloping algebra of a complex semisimple Lie algebra, $H = U_q(\mathfrak{g})$, part (a) is due to Joseph and Letzter: it follows from [9, Theorem 4.10] that $U_q(\mathfrak{g})_{\mathrm{ad}\,fin}$ is a left coideal subalgebra and this fact is also explicitly stated as [11, Theorem 5.1]. Part (b) extends an earlier result of Bergen [1, Theorem 2.18] to arbitrary characteristics. Bergen's

https://doi.org/10.1017/S0013091520000358 Published online by Cambridge University Press

proof is based on his joint work with Passman [4], which determines $H_{\mathrm{ad}\,fin}$ explicitly for group algebras and for enveloping algebras of Lie algebras in characteristic 0.

Some of our work is in the context of (virtually) pointed Hopf algebras. Over an algebraically closed field, all cocommutative Hopf algebras are pointed [17, Lemma 8.0.1]. However, many pointed Hopf algebras of interest are not necessarily cocommutative. Examples include the algebras of polynomial functions of solvable connected affine algebraic groups (over an algebraically closed base field) and quantized enveloping algebras of semisimple Lie algebras. It turns out that $\,\mathbf{.}_{fin}$ is a tensor functor for virtually pointed Hopf algebras.

Theorem 2. *If H is virtually pointed, then $(V \otimes W)_{fin} = V_{fin} \otimes W_{fin}$ for any $V, W \in {}_H\mathsf{Mod}$.*

A standard group-theoretic fact, known as Dietzmann's Lemma, states that any finite subset of a group that is stable under conjugation and consists of torsion elements generates a finite subgroup ([7] or [10, §53]). Our final result is the following version of Dietzmann's Lemma for arbitrary Hopf algebras.

Proposition 3. *Let $C_1, \ldots, C_k$ be finite-dimensional left coideal subalgebras of H and assume that $C = \sum_{i=1}^{k} C_i$ is stable under the adjoint action of H. Then C generates a finite-dimensional subalgebra of H.*

Of course, the subalgebra that is generated by C is also a left coideal subalgebra, stable under the adjoint H-action, and it is contained in $H_{\mathrm{ad}\,fin}$. Our proof of Proposition 3 will show that if all C_i are in fact sub-bialgebras of H, then it suffices to assume that C is stable under the adjoint actions of all C_i.

Notations and conventions. We work over an arbitrary base field $\Bbbk$ and continue to write $\otimes = \otimes_{\Bbbk}$. As usual, $\,\mathbf{.}^{*} = \mathrm{Hom}_{\Bbbk}(\,\mathbf{.}\,, \Bbbk)$ denotes linear duals of $\Bbbk$-vector spaces. Throughout, H is a Hopf $\Bbbk$-algebra with counit ε, antipode S, and comultiplication $\Delta h = h_{(1)} \otimes h_{(2)}$.

2. Proofs

2.1. Proof of Theorem 1(a)

It follows from (1) that the comultiplication satisfies

$$\Delta(k.h) = k_{(1)}h_{(1)}\mathsf{S}(k_{(3)}) \otimes k_{(2)}.h_{(2)} \quad (h, k \in H). \tag{2}$$

For a given $a \in H_{\mathrm{ad}\,fin}$, consider the subspace $V = H.a$ and let L denote the left coideal of H that is generated by V. Since $\dim_{\Bbbk} V < \infty$, we also have $\dim_{\Bbbk} L < \infty$ by the Finiteness Theorem for comodules [12, 9.2.2]. For any $h \in H$, we have $h_{(1)} \otimes h_{(2)}.a \otimes h_{(3)} \in H \otimes V \otimes H$. Applying $\mathrm{Id} \otimes \Delta \otimes \mathrm{Id}$ and using (2), we obtain

$$h_{(1)} \otimes h_{(2)}a_{(1)}\mathsf{S}(h_{(4)}) \otimes h_{(3)}.a_{(2)} \otimes h_{(5)} \in H \otimes H \otimes L \otimes H,$$

which in turn implies

$$\mathsf{S}(h_{(1)})h_{(2)}a_{(1)}\mathsf{S}(h_{(4)})h_{(5)} \otimes h_{(3)}.a_{(2)} = a_{(1)} \otimes h.a_{(2)} \in H \otimes L.$$

Consequently, $\Delta a = a_{(1)} \otimes a_{(2)} \in H \otimes H_{\mathrm{ad}\,fin}$, proving Theorem 1(a).

2.2. Tensors

Let V and W be left H-modules, with H-operations indicated by a dot, and view $V \otimes W$ as H-module with the usual H-operation on tensors: $h.(v \otimes w) = h_{(1)}.v \otimes h_{(2)}.w$. For any $\Bbbk$-subspace $U \subseteq V \otimes W$, put

$$U' = \sum_{f \in W^*} (\mathrm{Id}_V \otimes f)(U) \subseteq V \otimes \Bbbk = V$$

and

$$U' = \sum_{f \in V^*} (f \otimes \mathrm{Id}_W)(U) \subseteq \Bbbk \otimes W = W.$$

Parts (a), (b) of the following lemma are standard (e.g., [5, chap. II §7.8]), but (c) may be new.

Lemma 4 (notation as above). *(a) U' is a $\Bbbk$-subspace of V such that $U \subseteq U' \otimes W$; in fact, U' is the smallest such subspace (contained in all others). Similarly for U''. Moreover, $U \subseteq U' \otimes U''$.*

(b) U is finite dimensional if and only if U' and U'' are both finite dimensional.

(c) Assume that H is pointed. If U is an H-submodule of $V \otimes W$, then U' and U'' are H-submodules of V and W, respectively.

Proof. (a) If $U \subseteq V' \otimes W$ for a subspace $V' \subseteq V$, then $(\mathrm{Id}_V \otimes f)(U) \subseteq (\mathrm{Id}_V \otimes f)(V' \otimes W) \subseteq V' \otimes \Bbbk = V'$ for all $f \in W^*$. Thus, $U' \subseteq V'$, proving the minimality statement. On the other hand, any $u \in U$ can be written as a finite sum $u = \sum_i v_i \otimes w_i$ with $v_i \in V$, $w_i \in W$ and the w_i may be chosen to be $\Bbbk$-linearly independent. Fixing $f_j \in W^*$ such that $\langle f_j, w_i \rangle = \delta_{i,j}$, we obtain $(\mathrm{Id}_V \otimes f_i)(u) = v_i \in U'$. This shows that $U \subseteq U' \otimes W$. Similarly, $U \subseteq V \otimes U''$ and so $U \subseteq (U' \otimes W) \cap (V \otimes U') = U' \otimes U''$.

(b) One direction is clear from the inclusion $U \subseteq U' \otimes U''$. Now assume that $\dim_{\Bbbk} U = 1$, say $U = \Bbbk u$ with u as in the proof of (a). Then U' is generated by the vectors $(\mathrm{Id}_V \otimes f)(u) = \sum_i v_i \langle f, w_i \rangle$ belonging to the subspace generated by the (finitely many) v_i. Thus U' is finite dimensional in this case. Since $\boldsymbol{.}'$ evidently commutes with summation of subspaces, it follows that U' is finite dimensional whenever U is so. The argument for U'' is analogous.

(c) We will prove the following more general claim, for H pointed.

Claim. *Let $U \subseteq V \otimes W$ be a $\Bbbk$-subspace. Then $H.U' \subseteq U''$ if and only if $H.U \subseteq V \otimes U''$.*

One direction is clear: $H.U \subseteq H.(U' \otimes U'') \subseteq H.U' \otimes H.U' \subseteq V \otimes U''$ if $H.U' \subseteq U''$. The reverse implication will be proved below. Granting it for now, we may also conclude by symmetry that $H.U \subseteq U' \otimes W$ implies $H.U' \subseteq U'$. If U is an H-submodule of $V \otimes W$, then $H.U \subseteq U \subseteq U' \otimes U''$ and hence $H.U' \subseteq U'$ and $H.U' \subseteq U''$ by the Claim, proving (c).

To prove the remaining direction of the Claim, assume that $H.U \subseteq V \otimes U''$. Fix a $\Bbbk$-basis $(v_i)_i$ of V and write an arbitrary given $u \in U$ as $u = \sum_i v_i \otimes w_i$ with $w_i = w_i(u) \in$

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W. As in the proof of (a), one sees that the vectors w_i for the various $u \in U$ generate the vector space U''. We need to show that $h.w_i \in U''$ for all i and all $h \in H$. Let $(H_n)_{n \geq 0}$ denote the coradical filtration of H. Then $h \in H_n$ for some n. If $n = 0$, then we may assume that h is grouplike. Thus, $h.u = \sum_i h.v_i \otimes h.w_i \in H.U \subseteq V \otimes U''$. Since h is invertible, $(h.v_i)_i$ is a $\Bbbk$-basis of V and it follows as in the proof of (a) that $h.w_i \in U''$, as desired. Now let $n > 0$ and assume that $H_{n-1}.w_i \subseteq U''$ for all i. By the Taft–Wilson Theorem (see [14, Theorem 5.4.1] or [16, Section 4.3]), we may assume that $\Delta h = x \otimes h + h \otimes y + \sum_j h'_j \otimes h''_j$ with $x, y \in H_0$ grouplike and $h'_j, h''_j \in H_{n-1}$. Thus,

$$\sum_i x.v_i \otimes h.w_i + \sum_i h.v_i \otimes y.w_i + \sum_{i,j} h'_j.v_i \otimes h''_j.w_i = h.u \in H.U \subseteq V \otimes U''.$$

By induction, all $y.w_i, h''_j.w_i \in U''$. Therefore, $\sum_i x.v_i \otimes h.w_i \in V \otimes U''$. Since $(x.v_i)_i$ is a $\Bbbk$-basis of V, we deduce once more that $h.w_i \in U''$ for all i, completing the proof. □

2.3. Proof of Theorem 2

We may assume that H is pointed (§1.3). For given $V, W \in {}_H\mathsf{Mod}$, the inclusion $(V \otimes W)_{\text{fin}} \supseteq V_{\text{fin}} \otimes W_{\text{fin}}$ follows directly from the fact that $H.(v \otimes w) \subseteq H.v \otimes H.w$ for all $v \in V$ and $w \in W$. For the reverse inclusion, let $u \in (V \otimes W)_{\text{fin}}$ and put $U = H.u$. Then U is a finite-dimensional H-submodule of $V \otimes W$, and so Lemma 4 gives that $U \subseteq U' \otimes U''$ with $U' \subseteq V$ and $U' \subseteq W$ being finite-dimensional H-submodules. Therefore, $U' \subseteq V_{\text{fin}}$ and $U' \subseteq W_{\text{fin}}$, proving the desired inclusion.

2.4. Field extensions

Let R be a $\Bbbk$-algebra and let $F/\Bbbk$ be a field extension. Consider the F-algebra $R_F = R \otimes F$ and, for any $V \in {}_R\mathsf{Mod}$, put $V_F = V \otimes F \in {}_{R_F}\mathsf{Mod}$.

Lemma 5 (notation as above). $(V_F)_{\mathit{fin}} = (V_{\mathit{fin}})_F$.

Proof. The inclusion $(V_F)_{\text{fin}} \supseteq (V_{\text{fin}})_F$ is evident. For the reverse inclusion, let $v \in (V_F)_{\text{fin}}$ be given and write $v = \sum_{i=1}^r v_i \otimes \lambda_i$, where $v_i \in V$ and the λ_i are $\Bbbk$-linearly independent elements of F. We need to show that all $v_i \in V_{\text{fin}}$ for all i. Suppose this fails for $i = 1$, say. Replacing v by $v\lambda_1^{-1}$, we may assume that $\lambda_1 = 1$. Since $\dim_\Bbbk R.v_1 = \infty$, we may recursively construct a sequence in R by putting $r_0 = 1$ and choosing $r_n \in R$ so that $r_n.v_1 \notin \langle r_j.v_i \mid i \leq r, j < n\rangle_\Bbbk$. Since $v \in (V_F)_{\text{fin}}$, we can let m denote the first index so that the family $(r_j.v)_{j=0}^m$ is F-linearly dependent, say $\sum_{j=0}^m r_j.v\,\xi_j = 0$ with $\xi_j \in F$ not all 0. Then $\xi_m \neq 0$ by minimality of m; so we may assume that $\xi_m = 1$. Thus,

$$0 = r_m.v + \sum_{j=0}^{m-1} r_j.v\,\xi_j = \sum_{i=1}^r r_m.v_i \otimes \lambda_i + \sum_{i=1}^r \sum_{j=0}^{m-1} r_j.v_i \otimes \lambda_i\xi_j.$$

Choose a $\Bbbk$-linear projection $\pi\colon F \twoheadrightarrow \Bbbk$ with $\pi(1) = 1$ but $\pi(\lambda_i) = 0$ for $i = 2, \ldots, r$ and apply the projection $\mathrm{Id}_V \otimes \pi\colon V_F \twoheadrightarrow V$ to the above relation to obtain

$$0 = r_m.v_1 + \sum_{i=1}^r \sum_{j=0}^{m-1} r_j.v_i\pi(\lambda_i\xi_j).$$

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Thus, $r_m.v_1$ is a $\Bbbk$-linear combination of the vectors $r_j.v_i$ $(i \leq r, j < m)$, contrary to our construction of the sequence $(r_n)_{n\geq 0}$. □

We can now note the following consequence of Theorem 2.

Corollary 6. *If H is virtually cocommutative, then $(V \otimes W)_{fin} = V_{fin} \otimes W_{fin}$ for all $V, W \in {}_H\mathsf{Mod}$.*

Proof. Let F denote an algebraic closure of $\Bbbk$. Then H_F is virtually pointed and we may apply Theorem 2 and Lemma 5 with $R = H$ to obtain $(V \otimes W)_{\text{fin}} = (V_F \otimes_F W_F)_{\text{fin}} \cap (V \otimes W) = ((V_{\text{fin}})_F \otimes_F (W_{\text{fin}})_F) \cap (V \otimes W) = V_{\text{fin}} \otimes W_{\text{fin}}$. □

2.5. Proof of Theorem 1(b)

If $k \in K$ for some cocommutative Hopf subalgebra $K \subseteq H$, then Equation (2) becomes

$$\Delta(k.h) = k_{(1)}h_{(1)}\mathsf{S}(k_{(2)}) \otimes k_{(3)}.h_{(2)} = k_{(1)}.h_{(1)} \otimes k_{(2)}.h_{(2)} = k.\Delta h.$$

So Δ is a map in ${}_K\mathsf{Mod}$. This also holds for the antipode $\mathsf{S}\colon H_{\text{ad}} \to H_{\text{ad}}$ as is easily checked. Assuming H to be finitely generated as right K-module, locally finite parts can be calculated for K (§1.3) and we obtain $\Delta(H_{\text{ad}\,fin}) \subseteq (H_{\text{ad}} \otimes H_{\text{ad}})_{\text{fin}} = H_{\text{ad}\,fin} \otimes H_{\text{ad}\,fin}$, where the last equality holds by Corollary 6, and $\mathsf{S}(H_{\text{ad}\,fin}) \subseteq H_{\text{ad}\,fin}$. Thus, $H_{\text{ad}\,fin}$ is a Hopf subalgebra if H is virtually cocommutative. □

2.6. Proof of Proposition 3

For any $\Bbbk$-subspace $V \subseteq H$ and any $n \in \mathbb{Z}_+$, let $V^{(n)} \subseteq H$ denote the subspace that is generated by the products $v_1 v_2 \ldots v_m$ with $v_i \in V$ and $m \leq n$. Thus, the subalgebra that is generated by $C = \sum_{i=1}^{k} C_i$ is equal to $\bigcup_{n\geq 0} C^{(n)}$. We must show that this union is finite dimensional. Since C is finite dimensional, so are all $C^{(n)}$. Therefore, it suffices to show that $C^{(k)} = C^{(k+i)}$ for all $i \geq 0$. To prove this, let $s > k$ and consider a monomial of length s,

$$x = c_{i_1} c_{i_2} \ldots c_{i_s} \quad (c_i \in C_i).$$

We will show that $x \in C^{(s-1)}$, which will prove the desired equality $C^{(s)} = C^{(s-1)}$.

Observe that not all indices i_l in the above expression for x can be distinct. Let β denote the shortest gap between any two factors in x having the same index. If $\beta = 0$, then two adjacent factors belong to the same C_i and we may replace the pair by their product in C_i, thereby representing x as an element of $C^{(s-1)}$. Now assume that $\beta > 0$ and, without loss, assume that a pair of factors with index $i = 1$ has gap β. Thus, x contains a length-2 submonomial of the form cd with $c \in C_1$ being the first factor of the pair and $d \in C_i$ $(i \neq 1)$. Using the general formula $hk = (h_{(1)}.k)h_{(2)}$ for $h, k \in H$ we may write $cd = (c_{(1)}.d)c_{(2)}$, a finite sum with all $c_{(1)}.d \in C$ and all $c_{(2)} \in C_1$, because $\Delta C_1 \subseteq H \otimes C_1$ and $H.C_i \subseteq C$. (If C_1 is a sub-bialgebra, then it suffices to assume that $C_1.C \subseteq C$.) We may further expand all $c_{(1)}.d \in C$ into sums with terms from the various C_i. Thus, with y and z denoting the initial and final segments of x (possibly empty) before and after cd, respectively, the resulting sum for $x = ycdz = y(c_{(1)}.d)c_{(2)}z$ consists of length-s monomials of the same form as x above, but having a lower β-value. We may therefore argue by induction to finish the proof.

3. Remarks

For the special case of cocommutative Hopf algebras, part (b) of Theorem 1 is a consequence of part (a), because left coideal subalgebras then coincide with Hopf subalgebras. However, virtually cocommutative Hopf algebras form a much wider class, which includes all finite-dimensional Hopf algebras. Since Theorem 1 is trivial in the finite-dimensional case, we mention the following infinite-dimensional example.

Let n be odd and let $q \in \Bbbk$ be a primitive nth root of unity. Let $U = U_q(\mathfrak{sl}_2)$ be the quantum enveloping algebra with standard generators $E, F, K^{\pm 1}$ as in [6, I.3 and III.2]. The elements E^n and F^n belong to the centre of U and $I = E^nU + F^nU$ is a Hopf ideal of U, because $\mathsf{S}I \subseteq I$ and $\Delta E^n = E^n \otimes 1 + K^n \otimes E^n$, $\Delta F^n = F^n \otimes K^{-n} + 1 \otimes F^n$ by the q-binomial formula; see [12, Exercise 9.3.14]. So $H = U/I$ is a Hopf algebra; it is not cocommutative but virtually cocommutative, being finitely generated as (right and left) module over the cocommutative Hopf subalgebra $\Bbbk[K^{\pm 1}]$. In this example, $H_{\mathrm{ad}\,fin} = H$, because the adjoint (conjugation) action of K on H is locally finite.

In general, H need not be free over $H_{\mathrm{ad}\,fin}$. To see this, we recall the following result of Masuoka [13]. Assume that H is pointed and let B be a left coideal subalgebra of H. Then H is free as a right or left B-module if and only if $\mathsf{S}(B \cap \Bbbk G) = B \cap \Bbbk G$, where $G = G(H)$ is the group of grouplike elements. This condition is often easy to check. For example, taking $H = U_q(\mathfrak{g})$, $B = H_{\mathrm{ad}\,fin}$ and using the notation of [9, Theorem 4.10], we have $\tau(\lambda) \in H_{\mathrm{ad}\,fin} \cap G(H)$ but $\mathsf{S}(\tau(\lambda)) = \tau(-\lambda) \notin H_{\mathrm{ad}\,fin}$ for $\lambda \in -R^+(\pi)$. So H is not free over $H_{\mathrm{ad}\,fin}$ in this case.

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