Proceedings of the Royal Society of Edinburgh, **131A**, 267–277, *2001*

Weighted inequalities for a maximal function on the real line

A. L. Bernardis
PEMA-INTEC, Güemes 3450, (3000) Santa Fe, Argentina
(bernard@alpha.arcride.edu.ar)

F. J. Martín-Reyes
Dpto. de Análisis Matemático, Facultad de Ciencias,
Universidad de Málaga, 29071 Málaga, Spain
(martin@anamat.cie.uma.es)

(MS received 31 August 1999; accepted 11 February 2000)

We consider the maximal operator defined on the real line by

$$M_\alpha f(x) = \sup_{R>0} \frac{1}{(2R)^{1+\alpha}} \int_{R<|x-y|<2R} |f(y)|(|x-y|-R)^\alpha \, \mathrm{d}y, \quad -1<\alpha\leqslant 0,$$

which is related to the Cesàro convergence of the singular integrals. We characterize the weights w for which M_α is of weak type, strong type and restricted weak type (p,p) with respect to the measure $w(x)\,\mathrm{d}x$.

1. Introduction

In this paper we are interested in the study of the boundedness in weighted L^p-spaces of the maximal operator M_α acting on measurable functions on $\mathbb{R}$ and defined by

$$M_\alpha f(x) = \sup_{R>0} \frac{1}{(2R)^{1+\alpha}} \int_{R<|x-y|<2R} |f(y)|(|x-y|-R)^\alpha \, \mathrm{d}y, \quad -1<\alpha\leqslant 0.$$

This operator is interesting by itself and it is useful in the study of the Cesàro-α convergence of singular integrals associated to Calderón–Zygmund kernels (see [1]). Furthermore, M_α is, up to constants, a particular case of the maximal function of positive convolution operators associated with approximations of the identity given by

$$M_\varphi f(x) = \sup_{R>0} \frac{1}{R} \int_{\mathbb{R}} \varphi\left(\frac{x-y}{R}\right) f(y) \, \mathrm{d}y.$$

The operator M_φ was studied in [4], providing access to the study of the Cesàro continuity of order less than one.

On one hand, it follows from [4, theorem 1] that if $\alpha > -1$, then M_α is of restricted weak type $(1/(1+\alpha), 1/(1+\alpha))$ and, consequently, it is of strong type (p,p) for $p > 1/(1+\alpha)$. On the other hand, it was proved in [1] that if w is in the Muckenhoupt $A_{p(1+\alpha)}$ class and $p > 1/(1+\alpha)$ then M_α is of strong type

(p, p) with respect to $w(x)\,\mathrm{d}x$, while if $w \in A_1$, then M_α is of restricted weak type $(1/(1+\alpha), 1/(1+\alpha))$ with respect to $w(x)\,\mathrm{d}x$. The aim of this paper is to characterize the weighted inequalities of restricted weak type, weak type and strong type for M_α. Our results refer only to the case of equal weights.

The study of the boundedness of M_α in weighted L^p-spaces has two main difficulties. The first one is the kernel $(|x-y|-R)^\alpha$. The second one is to find a non-centred maximal operator pointwise equivalent to M_α, as in the case of the Hardy–Littlewood maximal operator, i.e. as in the case $\alpha = 0$.

The paper is organized as follows. We introduce in § 2 a non-centred version of M_α and we prove that it is pointwise equivalent to M_α. Sections 3 and 4 are devoted to characterizing the weighted weak- and strong-type (p, p) inequalities, while the restricted weak-type inequalities with weights are studied in § 5. The main results in the paper are theorems 3.1 and 4.3, where we prove the equivalence for $p > 1$ of the weighted weak-type (p, p) inequality, the weighted strong-type (p, p) inequality for M_α and the fact that w satisfies the following condition: there exists $C > 0$ such that, for any interval I,

$$\left(\int_I w(s)\,\mathrm{d}s\right)^{1/p}\left(\int_I w^{1-p'}(s)|s-x|^{\alpha p'}\,\mathrm{d}s\right)^{1/p'} \leqslant C|I|^{1+\alpha},$$

where x is the centre of I, $|I|$ is the length of I and $1/p+1/p' = 1$. In the final section we observe some relations between the good weights for M_α and the Muckenhoupt A_p-weights.

Throughout the paper, we shall use the following notations. If x and R are real numbers with $R > 0$, the interval $(x-R, x+R)$ is denoted by $I(x, R)$. If $I = I(x, R)$ and λ is a positive number, then $\lambda I = I(x, \lambda R)$, while ∂I is the border of I, i.e. the set $\{x-R, x+R\}$. If $s, t \in \mathbb{R}$ and $A \subset \mathbb{R}$, $d(s, t)$ and $d(s, A)$ are the Euclidean distances from s to t and to A, respectively. By $|A|$ and $w(A)$ we denote the measure of A and the integral $\int_A w(s)\,\mathrm{d}s$, respectively. If $1 < p < \infty$, then p' denotes its conjugate exponent. Finally, the letter C means a positive constant not necessarily the same at each occurrence.

2. The non-centred maximal function

Observe first that, with the notations introduced in § 1, we have that

$$M_\alpha f(x) = \sup_{R>0} \frac{1}{|I(x,R)|^{1+\alpha}} \int_{2I(x,R)\setminus I(x,R)} |f(s)| d(s, I(x,R))^\alpha\,\mathrm{d}s.$$

Notice also that $M_0 f \leqslant M_\alpha f$ (since $\alpha \leqslant 0$) and that $M_0 f$ is pointwise equivalent to the Hardy–Littlewood maximal function

$$Mf(x) = \sup_{R>0} \frac{1}{|I(x,R)|} \int_{I(x,R)} |f(s)|\,\mathrm{d}s.$$

We define the non-centred maximal operator N_α associated with M_α as

$$N_\alpha f(x) = \sup_{I : x \in \frac{1}{2}I} \frac{1}{|I|^{1+\alpha}} \int_{2I\setminus I} |f(s)| d(s, I)^\alpha\,\mathrm{d}s,$$

https://doi.org/10.1017/S0308210500000871 Published online by Cambridge University Press

where the supremum is taken over all the bounded intervals such that $x \in \frac{1}{2}I$. The next proposition shows that M_α and N_α are pointwise equivalent.

PROPOSITION 2.1. *Let $-1 < \alpha \leqslant 0$. There exists a constant C depending only on α such that $M_\alpha f \leqslant N_\alpha f \leqslant C M_\alpha f$, for all measurable functions f.*

Proof. The first inequality is obvious. Let $I = I(z, R)$ be an interval such that $x \in \frac{1}{2}I$. Without loss of generality, we may assume that $x \in (z - \frac{1}{2}R, z]$. Then

$$\begin{aligned}\int_{2I\setminus I} |f(s)| d(s, I)^\alpha \, \mathrm{d}s &= \int_{z-2R}^{z-R} |f(s)|(z - R - s)^\alpha \, \mathrm{d}s + \int_{z+R}^{z+2R} |f(s)|(s - z - R)^\alpha \, \mathrm{d}s \\ &= I + II.\end{aligned}$$

On the one hand, if $L = x - z + R$, we have $\frac{1}{2}R < L \leqslant R$ and $x - 2L \geqslant z - 2R$. Thus

$$\begin{aligned}I &\leqslant \int_{z-2R}^{x-2L} |f(s)|(z - R - s)^\alpha \, \mathrm{d}s + \int_{x-2L}^{x-L} |f(s)|(x - L - s)^\alpha \, \mathrm{d}s \\ &\leqslant (\tfrac{1}{2}R)^\alpha \int_{x-L-R}^{x+L+R} |f(s)| \, \mathrm{d}s + \int_{2I(x,L)\setminus I(x,L)} |f(s)| d(s, I(x, L))^\alpha \, \mathrm{d}s.\end{aligned}$$

On the other hand, if $T = z + R - x$, then $R \leqslant T < \frac{3}{2}R$ and $x + 2T \geqslant z + 2R$. Therefore,

$$II \leqslant \int_{x+T}^{x+2T} |f(s)|(s - x - T)^\alpha \, \mathrm{d}s \leqslant \int_{2I(x,T)\setminus I(x,T)} |f(s)| d(s, I(x, T))^\alpha \, \mathrm{d}s.$$

Putting together the inequalities, we get

$$\begin{aligned}\frac{1}{|I|^{1+\alpha}} \int_{2I\setminus I} |f(s)| d(s, I)^\alpha \, \mathrm{d}s \leqslant C \frac{1}{|I|} \int_{I(x,L+R)} |f(s)| \, \mathrm{d}s \\ + \frac{1}{|I|^{1+\alpha}} \int_{2I(x,L)\setminus I(x,L)} |f(s)| d(s, I(x, L))^\alpha \, \mathrm{d}s \\ + \frac{1}{|I|^{1+\alpha}} \int_{2I(x,T)\setminus I(x,T)} |f(s)| d(s, I(x, T))^\alpha \, \mathrm{d}s.\end{aligned}$$

Since the lengths of the intervals I, $I(x, L + R)$, $I(x, L)$ and $I(x, T)$ are essentially the same, the right-hand side is dominated by $C[Mf(x) + M_\alpha f(x)] \leqslant C M_\alpha f(x)$ and we are done. □

3. Weighted weak-type inequalities

The first main result of the paper characterizes the weighted weak-type inequalities for the maximal operator M_α by means of a Muckenhoupt-type condition.

THEOREM 3.1. *Let w be a non-negative measurable function on $\mathbb{R}$ and let $-1 < \alpha \leqslant 0$. If $1 < p < \infty$, then the following are equivalent.*

https://doi.org/10.1017/S0308210500000871 Published online by Cambridge University Press

(i) *M_α is of weak type (p,p) with respect to $w(x)\,\mathrm{d}x$, i.e. there exists C such that*

$$w(\{M_\alpha f > \lambda\}) \leqslant C\lambda^{-p}\int |f|^p w,$$

for all $\lambda > 0$ and all $f \in L^p(w)$.

(ii) *w satisfies $A_{p,\alpha}$, i.e. there exists C such that, for any interval I,*

$$\biggl(\int_{\frac{1}{2}I} w\biggr)^{1/p}\biggl(\int_{2I\setminus I} w^{1-p'}(s)d(s,I)^{\alpha p'}\,\mathrm{d}s\biggr)^{1/p'} \leqslant C|I|^{1+\alpha}.$$

Remark 3.2. Observe that for $\alpha < 0$ the weighted weak-type (p,p) inequality is not possible for $1 < p \leqslant 1/(1+\alpha)$ unless $w = 0$ almost everywhere, since if $1 < p \leqslant 1/(1+\alpha)$, then (ii) does not hold. As a consequence, we have that the weighted weak-type $(1,1)$ inequality for M_α with $\alpha < 0$ never holds. In other words, the weak-type $(1,1)$ inequality makes sense only for $\alpha = 0$. In this case (M_0 is pointwise equivalent to the Hardy–Littlewood maximal operator), the weighted weak-type inequalities are characterized by the well-known Muckenhoupt A_p-conditions. This is the reason why we do not include the case $p = 1$ in the statement of the theorem.

Proof of theorem 3.1. By 2.1, statement (i) is equivalent to the weighted weak-type (p,p) inequality for N_α. Then (ii) follows from (i) by standard arguments, i.e. roughly speaking, applying (i) (with N_α) to the functions

$$w^{1-p'}(s)d(s,I)^{\alpha(p'-1)}\chi_{2I\setminus I}(s).$$

In order to prove (ii) $\Rightarrow$ (i), we need to know that w is a doubling weight, i.e. that $w(2I) \leqslant Cw(I)$ for all intervals I.

Lemma 3.3. *If $1 < p < \infty$, $-1 < \alpha \leqslant 0$ and w satisfies $A_{p,\alpha}$, then w is a doubling weight.*

We postpone the proof of lemma 3.3 and continue with the proof of the theorem. Assume that (ii) holds. Let $x \in \mathbb{R}$ and let I be any interval with centre x. By the Hölder inequality and the $A_{p,\alpha}$-condition, we obtain

$$\begin{aligned}\int_{2I\setminus I} |f(s)|d(s,I)^\alpha\,\mathrm{d}s &\leqslant \biggl(\int_{2I\setminus I} |f|^p w\biggr)^{1/p}\biggl(\int_{2I\setminus I} w^{1-p'}(s)d(s,I)^{\alpha p'}\,\mathrm{d}s\biggr)^{1/p'}\\ &\leqslant C\biggl(\int_{2I\setminus I} |f|^p w\biggr)^{1/p}\biggl(\int_{\frac{1}{2}I} w\biggr)^{-1/p}|I|^{1+\alpha}.\end{aligned}$$

Since w is a doubling weight (lemma 3.3), we get

$$\frac{1}{|I|^{1+\alpha}}\int_{2I\setminus I} |f(s)|d(s,I)^\alpha\,\mathrm{d}s \leqslant C\biggl(\int_{2I} |f|^p w\bigg/\int_{2I} w\biggr)^{1/p}.$$

Therefore,

$$M_\alpha f(x) \leqslant C[\mathcal{M}_w(|f|^p)]^{1/p}(x),$$

https://doi.org/10.1017/S0308210500000871 Published online by Cambridge University Press

where

$$\mathcal{M}_w g(x) = \sup_{R>0}\left[\frac{1}{w(I(x,R))}\int_{I(x,R)} |g|w\right].$$

Now (i) follows from the above inequality and the well-known fact that $\mathcal{M}_w$ is of weak type $(1,1)$ with respect to $w(x)\,\mathrm{d}x$. □

Proof of lemma 3.3. If $I = I(x,R)$, we obtain, by $A_{p,\alpha}$ and the Hölder inequality, that

$$\begin{aligned}\biggl(\int_{\frac{1}{2}I} w\biggr)^{1/p}&\biggl(\int_{2I\setminus I} w^{1-p'}(s)d(s,I)^{\alpha p'}\,\mathrm{d}s\biggr)^{1/p'}\\ &\leqslant C\int_{x+R}^{x+2R} d(s,I)^{\alpha}\,\mathrm{d}s\\ &\leqslant C\biggl(\int_{x+R}^{x+2R} w\biggr)^{1/p}\biggl(\int_{x+R}^{x+2R} w^{1-p'}(s)d(s,I)^{\alpha p'}\,\mathrm{d}s\biggr)^{1/p'}.\end{aligned}$$

Since $(x+R, x+2R)$ is contained in $2I\setminus I$, we have

$$w((x-\tfrac{1}{2}R, x+\tfrac{1}{2}R)) \leqslant Cw((x+R, x+2R)),$$

for every $x \in \mathbb{R}$ and all positive R. Applying this property to the intervals $(x-2R, x-R)$ and $(x-R, x-\frac{1}{2}R)$ instead of $(x-\frac{1}{2}R, x+\frac{1}{2}R)$, we obtain that

$$w((x-2R, x-\tfrac{1}{2}R)) \leqslant C[w(\tfrac{1}{2}I) + w(\tfrac{1}{4}I)] \leqslant Cw(\tfrac{1}{2}I).$$

Analogously, we have

$$w((x+\tfrac{1}{2}R, x+2R)) \leqslant Cw(\tfrac{1}{2}I).$$

Summing the inequalities, we get that $w(2I\setminus\frac{1}{2}I) \leqslant Cw(\frac{1}{2}I)$. Thus

$$w(2I) \leqslant Cw(\tfrac{1}{2}I) \leqslant Cw(I).$$

□

4. Weighted strong-type inequalities

We start by establishing different characterizations of the $A_{p,\alpha}$-condition, which are a key step for the study of the strong-type inequalities. In order to state the result, we recall that if μ is a Borel measure, then it is said that a non-negative measurable function w satisfies $A_p(\mu)$, $1 < p < \infty$, if there exists a positive constant C such that

$$\biggl(\int_I w\,\mathrm{d}\mu\biggr)^{1/p}\biggl(\int_I w^{1-p'}\,\mathrm{d}\mu\biggr)^{1/p'} \leqslant C\mu(I),$$

for all bounded intervals I (see, for instance, [5]). (If μ is the Lebesgue measure, then $A_p(\mu)$ is the Muckenhoupt A_p-condition.)

Proposition 4.1. *Let $-1 < \alpha \leqslant 0$ and $p > 1$. Let w be a non-negative measurable function. The following statements are equivalent.*

https://doi.org/10.1017/S0308210500000871 Published online by Cambridge University Press

(a) *w satisfies* $A_{p,\alpha}$.

(b) *There exists* C *such that, for any interval* I,

$$\Big(\int_I w\Big)^{1/p}\Big(\int_I w^{1-p'}(s)d(s,x)^{\alpha p'}\,\mathrm{d}s\Big)^{1/p'} \leqslant C|I|^{1+\alpha},$$

where x *is the centre of* I.

(c) *The functions* $s \to w(s)d(s,z)^{-\alpha}$ *satisfy* $A_p(\mu_z)$ *with a constant independent of* $z \in \mathbb{R}$, *where* $\mathrm{d}\mu_z = d(s,z)^{\alpha}\,\mathrm{d}s$, *i.e. there exists* C *such that, for any interval* I *and all* $z \in \mathbb{R}$,

$$\Big(\int_I w\Big)^{1/p}\Big(\int_I w^{1-p'}(s)d(s,z)^{\alpha p'}\,\mathrm{d}s\Big)^{1/p'} \leqslant C\int_I d(s,z)^{\alpha}\,\mathrm{d}s.$$

Proof. It is clear that (c) $\Rightarrow$ (b). Therefore, we only have to prove (a) $\Rightarrow$ (b) $\Rightarrow$ (c) and (b) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b). Let $I=(a,b)$, $I^-=(a,x)$ and $I^+=(x,b)$, where x is the centre of I. It suffices to establish that

$$\Big(\int_I w\Big)^{1/p}\Big(\int_{I^*} w^{1-p'}(s)d(s,x)^{\alpha p'}\,\mathrm{d}s\Big)^{1/p'} \leqslant C|I|^{1+\alpha}$$

for $I^*=I^-$ and $I^*=I^+$. We shall only prove the inequality for I^-, since the other one is proved in a similar way. Let J be the interval with left end point equal to x and the same length as I. It is clear that $I^- \subset 2J\setminus J$ and $d(s,x)=d(s,J)$ for all $s \in I^-$. These properties, together with the fact that w satisfies $A_{p,\alpha}$ (and therefore is a doubling weight), give

$$\begin{aligned}\Big(\int_I w\Big)^{1/p}&\Big(\int_{I^-} w^{1-p'}(s)d(s,x)^{\alpha p'}\,\mathrm{d}s\Big)^{1/p'}\\ &\leqslant C\Big(\int_{\frac{1}{2}J} w\Big)^{1/p}\Big(\int_{2J\setminus J} w^{1-p'}(s)d(s,J)^{\alpha p'}\,\mathrm{d}s\Big)^{1/p'} \leqslant C|I|^{1+\alpha}.\end{aligned}$$

(b) $\Rightarrow$ (c). Let $I=(a,b)$. We shall consider the following two cases: (1) $z \in [a,b]$ and (2) $z \notin [a,b]$.

Case 1. Let $J \supset I$ be an interval centred at z such that $|I| \leqslant |J| \leqslant 2|I|$. Enlarging the interval I to J and applying (b), we obtain

$$\Big(\int_I w\Big)^{1/p}\Big(\int_I w^{1-p'}(s)d(s,z)^{\alpha p'}\,\mathrm{d}s\Big)^{1/p'} \leqslant C|J|^{1+\alpha} = C|I|^{1+\alpha}.$$

https://doi.org/10.1017/S0308210500000871 Published online by Cambridge University Press

If $z = a$ or $z = b$, we are done. If $z \in (a, b)$, we have

$$\begin{aligned} |I|^{1+\alpha} &\leqslant C[(b-z)^{1+\alpha} + (z-a)^{1+\alpha}] \\ &= C\biggl[(b-z)^{\alpha}\int_z^b \mathrm{d}s + (z-a)^{\alpha}\int_a^z \mathrm{d}s\biggr] \\ &\leqslant C\biggl[\int_z^b (s-z)^{\alpha}\,\mathrm{d}s + \int_a^z (z-s)^{\alpha}\,\mathrm{d}s\biggr] \\ &= C\int_a^b d(s,z)^{\alpha}\,\mathrm{d}s. \end{aligned}$$

Putting together the last inequalities, we obtain (c) for $z \in (a, b)$.

CASE 2. We shall prove it only for $z > b$. First observe that the function

$$g(s) = \biggl(\frac{d(s,z)}{d(s,b)}\biggr)^{\alpha p'}$$

is decreasing in the interval (a, b). Therefore,

$$\biggl(\int_I w^{1-p'}(s)d(s,z)^{\alpha p'}\,\mathrm{d}s\biggr)^{1/p'} \leqslant \biggl(\frac{z-a}{b-a}\biggr)^{\alpha}\biggl(\int_I w^{1-p'}(s)d(s,b)^{\alpha p'}\,\mathrm{d}s\biggr)^{1/p'}.$$

Using case 1 ($z = b$) and the fact that $\alpha \leqslant 0$, we obtain

$$\biggl(\int_I w\biggr)^{1/p}\biggl(\int_I w^{1-p'}(s)d(s,z)^{\alpha p'}\,\mathrm{d}s\biggr)^{1/p'} \leqslant C(z-a)^{\alpha}|I| \leqslant C\int_I d(z,s)^{\alpha}\,\mathrm{d}s.$$

(b) $\Rightarrow$ (a). First we observe that (b) implies that w is doubling. Now, let $I = I(x, R)$ be any interval. Applying (b), we have

$$\biggl(\int_{x-2R}^{x} w\biggr)^{1/p}\biggl(\int_{x-2R}^{x} w^{1-p'}(s)d(s,x-R)^{\alpha p'}\,\mathrm{d}s\biggr)^{1/p'} \leqslant CR^{1+\alpha}.$$

Restricting the interval $(x - 2R, x)$ in the second integral and using the fact that w is doubling, we obtain

$$\biggl(\int_{x-R/2}^{x+R/2} w\biggr)^{1/p}\biggl(\int_{x-2R}^{x-R} w^{1-p'}(s)d(s,I)^{\alpha p'}\,\mathrm{d}s\biggr)^{1/p'} \leqslant CR^{1+\alpha}.$$

Analogously, we get the same inequality changing the interval $(x - 2R, x - R)$ to $(x + R, x + 2R)$. Finally, (a) follows adding both inequalities. □

As a consequence of the characterizations obtained in proposition 4.1, we have the following proposition.

PROPOSITION 4.2. *Let* $-1 < \alpha \leqslant 0$, $1 < p < \infty$ *and let* w *be a non-negative measurable function on the real line. If* w *satisfies* $A_{p,\alpha}$, *then there exists* $\epsilon > 0$, $0 < \epsilon < p - 1$, *such that* w *satisfies* $A_{p-\epsilon,\alpha}$.

https://doi.org/10.1017/S0308210500000871 Published online by Cambridge University Press

Proof. We know by proposition 4.1 that $w(s)d(s,z)^{-\alpha}$ satisfies $A_p(\mu_z)$ with an $A_p(\mu_z)$-constant independent of z. Then (see [5]) there exists $\epsilon > 0$, depending only on the $A_p(\mu_z)$-constant, such that $w(s)d(s,z)^{-\alpha}$ satisfies $A_{p-\epsilon}(\mu_z)$, where the $A_{p-\epsilon}(\mu_z)$-constant depends only on the $A_p(\mu_z)$-constant and ϵ. Applying again proposition 4.1, we obtain that w satisfies $A_{p-\epsilon,\alpha}$. □

It is clear that Marcinkiewicz's interpolation theorem and results 3.1, 4.1 and 4.2 give immediately the characterization of the weighted strong-type inequality.

THEOREM 4.3. *Let $-1 < \alpha \leqslant 0$ and $p > 1$. Let w be a non-negative measurable function on $\mathbb{R}$. The following statements are equivalent.*

(a) *M_α is of strong type (p,p) with respect to $w(x)\,\mathrm{d}x$, i.e. there exists C such that*

$$\int |M_\alpha f|^p w \leqslant C \int |f|^p w,$$

for all $f \in L^p(w)$.

(b) *w satisfies $A_{p,\alpha}$ or, equivalently, there exists C such that, for any interval I,*

$$\left(\int_I w\right)^{1/p} \left(\int_I w^{1-p'}(s) d(s,x)^{\alpha p'}\,\mathrm{d}s\right)^{1/p'} \leqslant C|I|^{1+\alpha},$$

where x is the centre of I.

5. Restricted weak-type inequalities

As we said above, the operator M_α is not of weak type $(1/(1+\alpha), 1/(1+\alpha))$ with respect to Lebesgue measure if $\alpha < 0$, but it is of restricted weak type $(1/(1+\alpha), 1/(1+\alpha))$; in other words, M_α satisfies the weak-type inequality for characteristic functions or, equivalently, M_α maps the Lorentz space $L_{1/(1+\alpha),1}(\mathrm{d}x)$ into the Lorentz space $L_{1/(1+\alpha),\infty}(\mathrm{d}x)$. Therefore, it is interesting to study the weights w such that $w(\{x : M_\alpha \chi_E(x) > \lambda\}) \leqslant C\lambda^{-p} w(E)$ for all $\lambda > 0$ and all measurable sets $E \subset \mathbb{R}$.

THEOREM 5.1. *Let w be a non-negative measurable function on $\mathbb{R}$ and let $-1 < \alpha \leqslant 0$. If $1 \leqslant p < \infty$, then the following are equivalent.*

(a) *M_α is of restricted weak type (p,p) with respect to $w(x)\,\mathrm{d}x$, i.e. there exists C such that $w(\{x : M_\alpha \chi_E(x) > \lambda\}) \leqslant C\lambda^{-p} w(E)$ for all $\lambda > 0$ and all measurable $E \subset \mathbb{R}$.*

(b) *w satisfies $RA_{p,\alpha}$, i.e. there exists C such that, for every interval I and all measurable $E \subset \mathbb{R}$,*

$$\left(\int_{\frac{1}{2}I} w\right)\left(\int_{E\cap(2I\setminus I)} d(s,I)^\alpha\,\mathrm{d}s\right)^p \leqslant C|I|^{(1+\alpha)p} \int_{E\cap(2I\setminus I)} w.$$

Proof. Using proposition 2.1, we see that (b) follows from (a), since

$$N_\alpha \chi_{E\cap(2I\setminus I)}(x) \geqslant \frac{1}{|I|^{1+\alpha}} \int_{E\cap(2I\setminus I)} d(s,I)^\alpha\,\mathrm{d}s,$$

https://doi.org/10.1017/S0308210500000871 Published online by Cambridge University Press

for all $x \in \frac{1}{2}I$. Assume now that (b) holds. We shall need the following lemma.

LEMMA 5.2. *If $-1 < \alpha \leqslant 0$ and w satisfies $RA_{p,\alpha}$, then w is a doubling weight.*

We postpone the proof of the lemma and continue with the proof of the theorem. By (b) and the fact that w is a doubling weight, we have

$$\frac{\int_{E\cap(2I\setminus I)} d(s,I)^{\alpha}\,\mathrm{d}s}{|I|^{1+\alpha}} \leqslant C\biggl(\frac{w(E\cap(2I\setminus I))}{w(2I)}\biggr)^{1/p}.$$

Therefore, $M_{\alpha}\chi_E \leqslant C(\mathcal{M}_w\chi_E)^{1/p}$. As in the proof of theorem 3.1, we obtain (a) using the fact that $\mathcal{M}_w$ is of weak type (1,1) with respect to $w(x)\,\mathrm{d}x$. □

Proof of lemma 5.2. Let $I = I(x,R)$ be any interval. Since w satisfies $RA_{p,\alpha}$, taking $E = (x+R, x+2R)$, we obtain

$$\biggl(\int_{x-R/2}^{x+R/2} w\biggr)\biggl(\int_{x+R}^{x+2R} d(s,I)^{\alpha}\,\mathrm{d}s\biggr)^{p} \leqslant C|I|^{(1+\alpha)p}\int_{x+R}^{x+2R} w,$$

and therefore $w((x-\frac{1}{2}R, x+\frac{1}{2}R)) \leqslant Cw((x+R,x+2R))$. Now we continue as in the proof of lemma 3.3. □

We can give equivalent formulations of the $RA_{p,\alpha}$-condition in the same way as we did with the $A_{p,\alpha}$-condition in proposition 4.1. We collect them in the next proposition. We omit the proof, since it is similar to the proof of proposition 4.1.

PROPOSITION 5.3. *Let $-1 < \alpha \leqslant 0$ and $p \geqslant 1$. Let w be a non-negative measurable function. The following statements are equivalent.*

(a) *w satisfies $RA_{p,\alpha}$.*

(b) *There exists C such that, for any interval I and all measurable $E \subset I$,*

$$\biggl(\int_I w\biggr)\biggl(\int_E d(s,x)^{\alpha}\,\mathrm{d}s\biggr)^{p} \leqslant C|I|^{(1+\alpha)p}\int_E w,$$

where x is the centre of I.

(c) *There exists C such that, for any interval I, all measurable $E \subset I$ and all $z \in \mathbb{R}$,*

$$\biggl(\int_I w\biggr)\biggl(\int_E d(s,z)^{\alpha}\,\mathrm{d}s\biggr)^{p} \leqslant C\biggl(\int_I d(s,z)^{\alpha}\biggr)^{p}\int_E w.$$

6. Further results

This section is devoted to establishing several relations among the classes of weights considered in the previous sections. Some of them are proven easily; for instance:

(a) if $1 < p < \infty$ and $\alpha \leqslant \beta$, then $A_{p,\alpha} \subset A_{p,\beta}$; and

(b) if $1 < p \leqslant q < \infty$, then $A_{p,\alpha} \subset A_{q,\alpha}$.

Others relations appear in the next proposition.

https://doi.org/10.1017/S0308210500000871 Published online by Cambridge University Press

Proposition 6.1. *If* $-1 < \alpha \leqslant 0$ *and* $p(1+\alpha) > 1$, *then* $A_{p(1+\alpha)} \subset A_{p,\alpha} \subset A_p$ *and* $A_{p,\alpha} \neq A_r$ *for all* $r > p(1+\alpha)$.

Proof. Taking $\beta = 0$ in (a), we obtain $A_{p,\alpha} \subset A_p$ and, applying the Hölder inequality, we get that $A_r \subset A_{p,\alpha}$ for all r with $1 < r < p(1+\alpha)$. Keeping in mind that $w \in A_{p(1+\alpha)}$ implies $w \in A_r$ for some $r < p(1+\alpha)$ (see [3,5]), we have that $A_{p(1+\alpha)} \subset A_{p,\alpha}$. In order to see that $A_{p,\alpha} \neq A_r$ for all $r > p(1+\alpha)$, we consider the functions $w(x) = |x|^\gamma$. It is well known (see [3]) that $w \in A_r$ if and only if $-1 < \gamma < r-1$. On the other hand, if $w \in A_{p,\alpha}$, then (proposition 4.1(b))

$$\int_{-a}^{a} w^{1-p'}(s)d(s,0)^{\alpha p'}\,\mathrm{d}s = \int_{-a}^{a} |s|^{\gamma(1-p')+\alpha p'}\,\mathrm{d}s < \infty.$$

This implies that $\gamma < p(1+\alpha) - 1$. Therefore, if $p(1+\alpha) - 1 < \gamma < r-1$, then $w \in A_r$ and $w \notin A_{p,\alpha}$. □

Remark 6.2. The same argument in the proof of the above proposition shows that if $p(1+\alpha) > 1$, then $w(x) = |x|^\gamma \in A_{p,\alpha}$ if and only if $-1 < \gamma < p(1+\alpha) - 1$. However, we do not know if $A_{p(1+\alpha)}$ is equal to $A_{p,\alpha}$ for $\alpha < 0$.

Now we check the same kind of relations among the classes $RA_{p,\alpha}$. Clearly, properties (a) and (b) also hold for the classes $RA_{p,\alpha}$. If we denote the classes $RA_{p,0}$ by RA_p, we can prove the following proposition.

Proposition 6.3. *If* $-1 < \alpha \leqslant 0$ *and* $p(1+\alpha) \geqslant 1$, *then* $RA_{p(1+\alpha)} \subset RA_{p,\alpha} \subset RA_p$ *and* $RA_{p,\alpha} \neq RA_r$ *for all* $r > p(1+\alpha)$.

Proof. The relation $RA_{p,\alpha} \subset RA_p$ is obvious. Now let $w \in RA_{p(1+\alpha)}$. Then (proposition 5.3(b) for $\alpha = 0$)

$$\left(\int_I w\right)|E|^{p(1+\alpha)} \leqslant C|I|^{p(1+\alpha)} \int_E w$$

for all intervals I and all measurable $E \subset I = I(x,R)$. Since

$$\int_E d(s,x)^\alpha\,\mathrm{d}s \leqslant |E|^{1+\alpha},$$

we obtain that $w \in RA_{p,\alpha}$, by proposition 5.3. In order to prove that $RA_r \neq RA_{p,\alpha}$ for all $r > p(1+\alpha)$, let us consider $w(x) = |x|^{r-1}$. It was noticed in [2] that $w \in RA_r$. However, w does not belong to $RA_{p,\alpha}$ because if $w \in RA_{p,\alpha}$, then (proposition 5.3(b))

$$\left(\int_{-a}^{a} w\right)\left(\int_0^\epsilon d(s,0)^\alpha\,\mathrm{d}s\right)^p \leqslant C(2a)^{p(1+\alpha)} \int_0^\epsilon w,$$

for all a and ϵ with $0 < \epsilon < a$ or, equivalently, $(a/\epsilon)^r \leqslant C(a/\epsilon)^{p(1+\alpha)}$, $0 < \epsilon < a$, which is a contradiction since $r > p(1+\alpha)$. □

Remark 6.4. With the same arguments as those above, we can easily see that $w(x) = |x|^\gamma \in RA_{p,\alpha}$ if and only if $-1 < \gamma \leqslant p(1+\alpha) - 1$. On the other hand, as in the case of the $A_{p,\alpha}$-classes, we do not know if $RA_{p(1+\alpha)}$ is equal to $RA_{p,\alpha}$ when $\alpha < 0$, but the equality holds when p is the endpoint, i.e. if $p = 1/(1+\alpha)$.

https://doi.org/10.1017/S0308210500000871 Published online by Cambridge University Press

PROPOSITION 6.5. $RA_{1/(1+\alpha),\alpha} = RA_1 = A_1$.

Proof. First, notice that by proposition 6.3 (with $p = 1/(1+\alpha)$), we have that $RA_1 \subset RA_{1/(1+\alpha),\alpha}$. Second, RA_1 is clearly equivalent to A_1 since the restricted weak type $(1,1)$ is equivalent to the weak type $(1,1)$. It only remains to show that $RA_{1/(1+\alpha),\alpha} \subset A_1$. Let $w \in RA_{1/(1+\alpha),\alpha}$ and let I be any bounded interval. Applying proposition 5.3(c) to $E = (z-\epsilon, z+\epsilon) \subset I$, we get

$$w(I)\biggl(\int_{z-\epsilon}^{z+\epsilon} d(s,z)^{\alpha}\,\mathrm{d}s\biggr)^{1/(1+\alpha)} \leqslant C|I| \int_{z-\epsilon}^{z+\epsilon} w.$$

Thus

$$\frac{1}{|I|}\int_I w \leqslant C\frac{1}{2\epsilon}\int_{z-\epsilon}^{z+\epsilon} w.$$

If we let ϵ go to 0, we obtain

$$\frac{1}{|I|}\int_I w \leqslant Cw(z)$$

for almost every $z \in I$, i.e. w is in the class A_1 of Muckenhoupt. □

Acknowledgments

A.L.B. was supported by CONICET, Prog. CAI+D-UNL and PICT1998 (Codigo: 03-04186). F.J.M.-R. was partly supported by FOMEC, DGES Grant (PB97-1097) and Junta de Andalucía.

References

1 A. L. Bernardis and F. J. Martín-Reyes. Singular integrals in the Cesáro sense. *J. Fourier Analysis Appl.* **6**(2) (2000), 143–152.

2 H.-M. Chung, R. A. Hunt and D. S. Kurtz. The Hardy–Littlewood maximal function on $L(p,q)$ spaces with weights. *Indiana Univ. Math. J.* **31** (1982), 109–120.

3 J. García-Cuerva and J. L. Rubio de Francia. *Weighted norm inequalities and related topics* (Amsterdam: North-Holland, 1985).

4 W. Jourkat and J. Troutman. Maximal inequalities related to generalized a.e. *Trans. Am. Math. Soc.* **252**, 1979, 49–64.

5 J.-O. Stromberg and A. Torchinsky. *Weighted Hardy inequalities*. Lecture Notes in Mathematics, vol. 1381 (Springer, 1989).

(*Issued 27 April 2001*)