FINITE DIMENSIONAL APPROXIMATION TO BAND LIMITED WHITE NOISE

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To Professor KIYOSHI NOSHIRO on the occasion of his 60th birthday

1. Introduction. One of the authors discussed finite dimensional approximations to a white noise and a periodic Brownian motion with period 2π on the projective limit space of spheres ([2]). The group of unitary operators derived from the periodic white noise has *a pure point spectrum* which consists of all integers with countably infinite multiplicity. We also have much interest in the investigation of *a band limited white noise* which is another typical example having quite different spectral type. Indeed, the corresponding group of unitary operators has *a continuous spectrum* with countably infinite multiplicity.

A band limited white noise to the band from 0 to W is, as is well known, a Gaussian stationary stochastic process $X_W(t, \omega)$, $-\infty < t < \infty$, $\omega \in \Omega(P)$, which has the following spectral representation:

$$X_W(t) = \int_{-\pi W}^{\pi W} e^{it\lambda} dZ(\lambda), \tag{1}$$

where $dZ(\lambda)$ is a complex Gaussian random measure defined on $\mathscr{B}([-\pi W, \pi W])$, the smallest Borel field generated by all open subsets of $[-\pi W, \pi W]$, satisfying

$$EZ(\Delta) = 0, \quad E|Z(\Delta)|^2 = |\Delta| \quad \text{(the Lebesgue measure of } \Delta) \tag{2}$$

and

$$Z(-\Delta) = \overline{Z(\Delta)}, \qquad \Delta \in \mathscr{B}([-\pi W, \pi W]).$$

The covariance function of $X_W(t)$ is given by the formula

$$\gamma(h) = E(X_W(t+h)\overline{X_W(t)}) = \frac{2}{|h|} \sin \pi |h| W. \tag{3}$$

For simplicity we always assume that $W = 1$ throughout this note.

In order to obtain a finite dimensional approximation to the process $X_W(t)$,

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we shall begin with the construction of a random measure $Z^{(n)}(\lambda)$ which approximates $dZ(\lambda)$ appeared in the expression (1). Our method is quite similar to what was used in the course of approximation to the periodic white noise (cf. [2, §3]).

Having got the Fourier transform of $Z^{(n)}(\lambda)$

$$X^{(n)}(t) = \int_{-\pi}^{\pi} e^{it\lambda} Z^{(n)}(\lambda)\, d\lambda, \qquad -\infty < t \infty,$$

we shall show that the stochastic process $X^{(n)}(t)$ approches to a band limited white noise required to be approximated in the sense to be prescribed as follows: The process $X^{(n)}(t)$ determines a probability measure μ_n on the space of all continuous functions on R^1 with compact uniform topology. Appealing to Prokhorov's theorem [3], we shall prove that there exists a probability measure μ which is the weak limit of μ_n. This measure μ will turn out to be the same measure as the one derived from a band limited white noise to the band from 0 to 1.

2. The complex white noise with circular parameter

We shall first list some results obtained in [1] and [2] which will be needed for our present purpose.

Let S^n be the n-dimensional sphere with radius $\sqrt{n+1}$ and let $x^{(n+1)} = (x_1^{(n+1)}, \ldots, x_{n+1}^{(n+1)})$ be a point of S^n. Then $x^{(n+1)}$ can be expressed in the form

$$x_1^{(n+1)} = \sqrt{n+1} \prod_{i=1}^{n} \sin\theta_i,$$
$$x_k^{(n+1)} = \sqrt{n+1} \cos\theta_{k-1} \prod_{i=k}^{n} \sin\theta_i, \quad 2 \le k \le n,$$
$$x_{n+1}^{(n+1)} = \sqrt{n+1} \cos\theta_n,$$

where $0 \le \theta_1 < 2\pi$, $0 \le \theta_i \le \pi$, $i = 2, 3, \ldots, n$. Let Ω_n be a subset of S^n defined by

$$\Omega_n = \{x^{(n+1)} ; x^{(n+1)} \in S^n, \ 0 < \theta_i < \pi, \ i \ge 2\}$$

and let P_n be the restriction to $\mathscr{B}_n = \mathscr{B}(\Omega_n)$ of the uniform probability measure over S^n. Then we obtain a probability space $(\Omega, \mathscr{B}, P)$ as the projective limit of measure spaces $(\Omega_{2n}, \mathscr{B}_{2n}, P_{2n})$, $n = 1, 2, \ldots$ (see [1]).

Now we can introduce a flow $\{T_\lambda^{(2n)} ; \lambda \text{ real}\}$ on $(\Omega_{2n}, \mathscr{B}_{2n}, P_{2n})$ defined by

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$$(4) \qquad T_\lambda^{(2n)}(x^{(2n+1)}) = \begin{bmatrix} 1 & & & \\ & A_1(\lambda) & & 0 \\ & & \ddots & \\ & 0 & & A_n(\lambda) \end{bmatrix} x^{(2n+1)}, \quad x^{(2n+1)} = \begin{bmatrix} x_1^{(2n+1)} \\ \cdot \\ \cdot \\ \cdot \\ x_{2n+1}^{(2n+1)} \end{bmatrix},$$

where $A_k(\lambda)$'s are given by

$$A_k(\lambda) = \begin{bmatrix} \cos k\lambda & -\sin k\lambda \\ \sin k\lambda & \cos k\lambda \end{bmatrix}, \quad k = 1, 2, \ldots .$$

Since the flows $\{T_\lambda^{(2n)}\}$, $n = 1, 2, \ldots$, form a system of consistent flows, we can uniquely determine a flow $\{T_\lambda\,;\, \lambda \text{ real}\}$ (see [2]). The flow $\{T_\lambda\}$ is obviously a periodic flow with period 2π.

We are now in a position to define a finite dimensional approximation $Z^{(2n)}(\lambda, x^{(2n+1)})$ to the complex white noise $dZ(\lambda, x)$. Let us define unitary groups $\{U_\lambda\,;\, \lambda \text{ real}\}$ and $\{U_\lambda^{(2n)}\,;\, \lambda \text{ real}\}$ by

$$(5) \qquad U_\lambda f(x) = f(T_\lambda x), \text{ for } f \in L^2(\Omega, \mathscr{B}, P), \; -\infty < \lambda < \infty,$$

and

$$(5') \quad U_\lambda^{(2n)} f(x^{(2n+1)}) = f(T_\lambda^{(2n+1)} x^{(2n+1)}), \text{ for } f \in L^2(\Omega_{2n}, \mathscr{B}_{2n}, P_{2n}), \; -\infty < \lambda < \infty,$$

respectively. Then it can be proved that U_λ and $U_\lambda^{(2n)}$ are strongly continuous in λ, λ real, and that both of them are periodic:

$$U_{\lambda+2\pi} = U_\lambda, \quad U_{\lambda+2\pi}^{(2n)} = U_\lambda^{(2n)}.$$

Since $T_\lambda^{(2n)} x^{(2n+1)}$ together with $x^{(2n+1)}$ may be regarded as $(2n+1)$-dimensional vectors, we may consider scalar products such as $(x^{(2n+1)}, a)$, $(T_\lambda^{(2n)} x^{(2n+1)}, b)$, etc., where a and b are $(2n+1)$-demensional vectors. Now let us take a particular $(2n+1)$-dimensional vector a such as

$$a = \left(\frac{1}{2\pi}, \frac{1}{\pi}, 0, \frac{1}{\pi}, 0, \ldots, \frac{1}{\pi}, 0\right).$$

A functional $f_a(x^{(2n+1)})$ defined by

$$f_a(x^{(2n+1)}) = \frac{1+i}{2}(x^{(2n+1)}, a)$$

belongs to $L^2(\Omega_{2n}, \mathscr{B}_{2n}, P_{2n})$. We can therefore apply $U_\lambda^{(2n)}$ to f_a. Define $Z^{(2n)}(\lambda)$ by

(6) $$Z^{(2n)}(\lambda) = U_\lambda f_a(x^{(2n+1)}) + U_{-\lambda}\bar{f}_a(x^{(2n+1)})$$

Then we have the following simple expression

(6′) $$Z^{(2n)}(\lambda) = \frac{1}{2\pi} x_1^{(2n+1)} + \sum_{k=1}^{n} \frac{\cos k\lambda}{\pi} x_{2k}^{(2n+1)} - i\sum_{k=1}^{n} \frac{\sin k\lambda}{\pi} x_{2k+1}^{(2n+1)}$$
$$= Z_1^{(2n)}(\lambda) - iZ_2^{(2n)}(\lambda), \qquad Z_1^{(2n)}(\lambda),\ Z_2^{(2n)}(\lambda) \text{ real.}$$

Note that $Z^{(2n)}(\lambda)$ and $Z_i^{(2n)}(\lambda)$, $i = 1, 2$, can be regarded as random variables not only on $(\Omega_{2n}, \mathscr{B}_{2n}, P_{2n})$ but also on $(\Omega, \mathscr{B}, P)$.

Proposition 1. *i) For any $f \in L^2([-\pi, \pi])$*

$$Z_i^{(2n)}(f) = \int_{-\pi}^{\pi} Z_i^{(2n)}(\lambda) f(\lambda)\, d\lambda, \quad i = 1, 2,$$

belong to real $L^2(\Omega, \mathscr{B}, P)$, and they converge to Gaussian random variables which we denote by $Z_i(f)$, $i = 1, 2$, in $L^2(\Omega, \mathscr{B}, P)$.

ii) For almost all $x \in \Omega$, both $Z_1(\varphi, x)$ and $Z_2(\varphi, x)$, $\varphi \in (\mathscr{D})_{[-\pi, \pi]}$, are continuous linear functionals on $(\mathscr{D})_{[-\pi, \pi]}$.

This proposition can be proved in a similar way to the discussions in [2, § 3] and the proof is omited.

Define $Z^{(2n)}(\Delta) = \int_\Delta Z^{(2n)}(\lambda)\, d\lambda$, then

(7) $$EZ^{(2n)}(\Delta) = 0,\ E(Z^{(2n)}(\Delta_1)\overline{Z^{(2n)}(\Delta_2)}) \to |\Delta_1 \cap \Delta_2| \qquad (n \to \infty)$$

and

$$Z^{(2n)}(-\Delta) = \overline{Z^{(2n)}(\Delta)}.$$

3. Approximation to a band limited white noise

Consider the Fourier transform of $Z^{(2n)}(\lambda)$, $-\pi \le \lambda \le \pi$:

(8) $$X^{(2n)}(t, x^{(2n+1)}) = \int_{-\pi}^{\pi} e^{it\lambda} Z^{(2n)}(\lambda, x^{(2n+1)})\, d\lambda, \quad -\infty < t < \infty.$$

Since the relation (7) holds, $\{X^{(2n)}(t)\ ;\ t \text{ real}\}$ is a real valued second order stochastic process defined on $(\Omega_{2n}, \mathscr{B}_{2n}, P_{2n})$ (hence, on $(\Omega, \mathscr{B}, P)$).

Proposition 2. *For any t, $X^{(2n)}(t)$ approaches to a random variable $\tilde{X}(t)$ of a band limited white noise in the sense of both mean square in $L^2(\Omega, \mathscr{B}, P)$ and almost sure (P) convergence.*

Proof. As was proved in [1], we can show that

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(9) $$\lim_{n\to\infty} y_k^{(2n+1)} = \zeta_k, \qquad y_k^{(2n+1)} = x_{k-n-1}^{(2n+1)}, \qquad k=1, 2, \ldots .$$

exists almost surely. The collection $\{\zeta_k\}$ forms a system of independent Gaussian random variables with mean 0 and variance 1. Since $\sum_{k=-\infty}^{\infty} \left| \frac{\sin(t+k)\pi}{t+k} \right|^2 < \infty$ for every t, we can also prove that

(10) $$\lim_{n\to\infty} X^{(2n)}(t, x^{(2n+1)}) = \frac{2}{\pi} \sum_{k=-\infty}^{\infty} \frac{\sin(t+k)\pi}{t+k} \zeta_k, \text{ a.e. } (P),$$

in a similar manner to [2, § 4].

We denote by $\tilde{X}(t)$ the right hand side of (10). Then $\tilde{X}(t)$, $-\infty<t<\infty$, is obviously a Gaussian process. On the other hand, the band limited white noise $X_1(t)$ ($W=1$) introduced by the formula (1) can be expressed in the form

(11) $$X_1(t) = \frac{2}{\pi} \sum_{k=-\infty}^{\infty} \frac{\sin(t+k)\pi}{t+k} \xi_k,$$

where $\{\xi_k\}$ is a system of independent standard Gaussian random variables. This shows that $\{X_1(t)\}$ and $\{\tilde{X}(t)\}$ are the same process. Consequently, almost sure convergence is proved.

The fact that $X^{(2n)}(t)$ converges to $\tilde{X}(t)$ strongly in $L^2(\Omega, \mathscr{B}, P)$ follows easily from Proposition 1, *i*).

COROLLARY. *Any finite dimensional distribution of the stochastic process* $\{X^{(2n)}(t)\}$ *converges to the finite dimensional distribution of* $\{X_1(t)\}$.

Under these preparations we shall finally show much stronger convergence of $X^{(n)}(t)$ to $X_1(t)$. By the expression (8) we see that $X^{(2n)}(t, x^{(2n+1)})$ is continuous in t for all $x^{(2n+1)} \in \Omega_{2n}$, which means $X^{(2n)}(t)$ determines a probability measure μ_n on the measurable space $(\mathbf{C}, \mathscr{B}_{\mathbf{C}})$, where $\mathbf{C}$ is the space of all continuous functions on R^1 and $\mathscr{B}_{\mathbf{C}}$ is the topological Borel field. The situation is the same for $X_1(t)$ and we denote by μ the derived probability measure from $X_1(t)$. Now we can state

THEOREM. *The measure* μ_n *converges to* μ *weakly.*

Proof. We have already proved that $\mu_n(E)$ tends to $\mu(E)$, as $n\to\infty$, for any cylinder set E of $\mathbf{C}$ (Corollary of Proposition 2). We shall now apply Prokhorov's theorem [3, Chapt. 2] to our discussions. We have

$$E|X^{(2n)}(t) - X^{(2n)}(s)|^2 = \frac{2}{\pi} \sum_{k=-n}^{n} \left| \frac{\sin(t+k)\pi}{t+k} - \frac{\sin(s+k)\pi}{s+k} \right|^2$$

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since the system $\{y_k^{(2n+1)}\ ;\ -n\leq k\leq n\}$ forms an orthonormal basis of $(\Omega_{2n}, \mathscr{B}_{2n}, P_{2n})$. Observing the Fourier coefficients of $e^{it\lambda}-e^{is\lambda}$, we obtain

$$E|X^{(2n)}(t)-X^{(2n)}(s)|^2\leq C\int_{-\pi}^{\pi}|e^{it\lambda}-e^{is\lambda}|^2 d\lambda\leq C'|t-s|^2,$$

where C and C' are constants being independent of n, t, and s. Thus the assumptions of Prokhorov's theorem are satisfied, and hence our theorem is proved.

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