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ON THE COMPLETION OF b-METRIC SPACES

NGUYEN VAN DUNG✉ and VO THI LE HANG

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Abstract

Based on the metrisation of b-metric spaces of Paluszyński and Stempak ['On quasi-metric and metric spaces', *Proc. Amer. Math. Soc.* **137**(12) (2009), 4307–4312], we prove that every b-metric space has a completion. Our approach resolves the limitation in using the quotient space of equivalence classes of Cauchy sequences to obtain a completion of a b-metric space.

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1. Introduction

In the 1990s, Czerwik [8, 9] introduced the notion of a b-metric as a generalisation of a metric. Fagin and Stockmeyer [11] also discussed the same relaxation of the triangle inequality, calling the new distance measure nonlinear elastic matching (NEM). They remarked that this measure has been used, for example, for trademark shapes [7] and measuring ice floes [6]. Later Xia [19] used this semimetric distance to study the optimal transport path between probability measures. Xia called these spaces quasimetric spaces, which is the term used in the book by Heinonen [13].

The b-metric spaces were studied by many authors (see [15, Ch. 12] and the references given there). A b-metric space is always understood to be a topological space with respect to the topology induced by its convergence. An *et al.* [3] showed that every b-metric space is a semimetrisable space, proved a Stone-type theorem and obtained a sufficient condition for a b-metric space to be metrisable. A b-metric need not be continuous [3, Examples 3.9 and 3.10]. This fact suggests a strengthening of the notion of a b-metric space called a strong b-metric space by Kirk and Shahzad [15, Definition 12.7]. The fact that every strong b-metric space is dense in a complete strong b-metric space was proved recently in [2, Theorem 2.2].

The technique using the quotient space of equivalence classes of Cauchy sequences, as in [2, Theorem 2.2], is very well known in constructing the completion of a metric space. However, it cannot be applied to obtain the completion of a b-metric space (see [2, Example 2.3]). Thus, the following question was posed.

Question 1.1 [2, Question 2.4]. *Does every b-metric space have a completion?*

Aphane and Moshokoa [4] attempted to adapt the proof of [2, Theorem 2.2] to b-metric spaces, but, in the proof of [4, Theorem 3.1], overlooked the possibility that the inequality

$$d(x_n, y_n) - d(x_m, y_m) \leq |d(x_n, x_n) - s^2 d(x_m, y_m)|$$

may not hold for $s > 1$. So, we need another approach to resolve this issue.

Frink [12, page 133] presented a simple and direct technique for the metrisation of b-metric spaces that was later called the *chain approach*. In 1998 Aimar *et al.* [1] improved Frink's metrisation technique to give a direct proof of a theorem of Macías and Segovia in [16] on the metrisation of a b-metric space (X, D, K). In 2006 Schroeder showed some limits of Frink's construction by providing a counterexample of a b-metric space (X, D, K) for which the function d defined by Frink's metrisation technique is not a metric [18, Example 2]. In 2009 Paluszyński and Stempak [17] also improved Frink's metrisation technique to produce a metric d from a given b-metric space (X, D, K). Recently the authors of [10] gave a simple counterexample to show again the limits of Frink's construction [12, page 134]. They used Frink's metrisation technique to answer two conjectures posed by Berinde and Choban [5] and to compute metrics induced by some examples of b-metrics. They also used the technique to prove a metrisation theorem for 2-generalised metric spaces and to deduce the Banach contraction principle in b-metric spaces and 2-generalised metric spaces from that in metric spaces.

In this paper, by using the metrisation theorem on b-metric spaces from [17], we give an affirmative answer to Question 1.1 by proving that every b-metric space has a completion. Our approach resolves the limitation in using the quotient space of equivalence classes of Cauchy sequences to construct the completion of a metric space.

Now we present some basic notions and results which will be used in what follows.

Definition 1.2 [9]. Let X be a nonempty set, $K \geq 1$ and $D : X \times X \longrightarrow [0, \infty)$ be a function such that for all $x, y, z \in X$:

(1) $D(x, y) = 0$ if and only if $x = y$;
(2) $D(x, y) = D(y, x)$;
(3) $D(x, z) \leq K[D(x, y) + D(y, z)]$.

Then D is called a *b-metric* on X and (X, D, K) is called a *b-metric space*.

Convergence, Cauchy sequences and completeness in b-metric spaces are defined in the same way as in metric spaces.

Definition 1.3 [14, Definition 7]. Let (X, D, K) be a b-metric space.

(1) A sequence $\{x_n\}$ *converges* to x, written $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} D(x_n, x) = 0$.
(2) A sequence $\{x_n\}$ is a *Cauchy sequence* if $\lim_{n,m\to\infty} D(x_n, x_m) = 0$.
(3) (X, D, K) is called *complete* if each Cauchy sequence is a convergent sequence.

THEOREM 1.4 [17, Proposition, page 4308]. *Suppose that* (X, D, K) *is a b-metric space,* $0 < p \leq 1$ *satisfies* $(2K)^p = 2$ *and, for all* $x, y \in X$,

$$d(x,y) = \inf\Big\{ \sum_{i=1}^{n} D^p(x_i, x_{i+1}) : x_1 = x, x_2, \ldots, x_n, x_{n+1} = y \in X, n \in \mathbb{N}\Big\}.$$

Then d is a metric on X satisfying

$$\tfrac{1}{4} D^p(x,y) \leq d(x,y) \leq D^p(x,y) \tag{1.1}$$

for all $x, y \in X$. *In particular, if D is a metric, then* $d = D$.

A map $f : X \longrightarrow Y$ from a b-metric space (X, D, K) into a b-metric space (Y, D', K') is an *isometry* if $D'(f(x), f(y)) = D(x, y)$ for all $x, y \in X$. A b-metric space (X^*, D^*, K^*) is a *completion* of the b-metric space (X, D, K) if (X^*, D^*, K^*) is complete and there exists an isometry $f : X \longrightarrow X^*$ such that $\overline{f(X)} = X^*$. A classical result says that every metric space admits a completion which is unique up to an isometry.

2. Main results

First we prove a technical lemma on the limit of sequences in b-metric spaces.

LEMMA 2.1. *Let* (X, D, K) *be a b-metric space and* $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ *in* (X, D, K). *Then* $\lim_{n\to\infty} D(x_n, y_n) = 0$ *if and only if* $x = y$.

PROOF. *Necessity.* For all n,

$$D(x,y) \leq K[D(x,x_n) + D(x_n,y)] \leq K[D(x,x_n) + K[D(x_n,y_n) + D(y_n,y)]].$$

Letting $n \to \infty$ gives $D(x, y) = 0$. So, $x = y$ and $\lim_{n\to\infty} D(x_n, y_n) = 0 = D(x, y)$.
Sufficiency. For all n,

$$D(x_n,y_n) \leq K[D(x_n,x) + D(x,y_n)] = K[D(x_n,x) + D(y,y_n)].$$

Letting $n \to \infty$ gives $\lim_{n\to\infty} D(x_n, y_n) = 0$. □

The following result gives an affirmative answer to Question 1.1.

THEOREM 2.2. *Let* (X, D, K) *be a b-metric space. Then:*

(1) *the space* (X, D, K) *has a completion* $(X^*, D^*, 4K^3)$*;*
(2) *the completion of* (X, D, K) *is unique in the sense that if* (X_1^*, D_1^*, K_1) *and* (X_2^*, D_2^*, K_2) *are two completions of* (X, D, K)*, then there is a bijective isometry* $\varphi : X_1^* \longrightarrow X_2^*$ *which restricts to the identity on X.*

PROOF. (1). Since $K \geq 1$, the number p with $(2K)^p = 2$ satisfies $0 < p \leq 1$.

Let d be the metric defined in Theorem 1.4. Then the metric space (X, d) has a completion (X^*, d^*). For all $x, y \in X^*$, define

$$D^*(x,y) = \begin{cases} D(x,y) & \text{if } x, y \in X, \\ (d^*)^{1/p}(x,y) & \text{otherwise.} \end{cases}$$

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For all $x, y, z \in X$, we find that $D^*(x, y) \geq 0$, $D^*(x, y) = D^*(y, x)$ and

$$D^*(x, y) = 0 \quad \text{if and only if } x = y.$$

It follows from (1.1) that for all $x, y \in X$,

$$\tfrac{1}{4}(D^*)^p(x, y) = \tfrac{1}{4}D^p(x, y) \leq d(x, y) = d^*(x, y) \leq D^p(x, y) = (D^*)^p(x, y).$$

This implies that for all $x, y \in X^*$,

$$\tfrac{1}{4}(D^*)^p(x, y) \leq d^*(x, y) \leq (D^*)^p(x, y). \tag{2.1}$$

Next we will show that $D^*(x, y) \leq 4K^3[D^*(x, z) + D^*(z, y)]$. From the symmetry of the function D^*, it follows that x and y are interchangeable, so we only need to consider the following six cases.

Case 1: $x, y, z \in X$. Then

$$D^*(x, y) = D(x, y) \leq K[D(x, z) + D(z, y)] = K[D^*(x, z) + D^*(z, y)].$$

Case 2: $x, y \in X$ and $z \notin X$. Then

$$\begin{aligned} D^*(x, z) + D^*(z, y) &= (d^*)^{1/p}(x, z) + (d^*)^{1/p}(z, y) \\ &\geq \frac{1}{2^{(1/p)-1}}(d^*(x, z) + d^*(z, y))^{1/p} \\ &\geq \frac{1}{2^{(1/p)-1}}(d^*(x, y))^{1/p} = \frac{1}{2^{(1/p)-1}}(d(x, y))^{1/p} \\ &\geq \frac{1}{2^{(1/p)-1}}\Big(\frac{D^p(x, y)}{4}\Big)^{1/p} = \frac{1}{K}\frac{D(x, y)}{(2K)^2} = \frac{D^*(x, y)}{4K^3}. \end{aligned}$$

Thus, $D^*(x, y) \leq 4K^3[D^*(x, z) + D^*(z, y)]$.

Case 3: $y, z \in X$ and $x \notin X$. Then

$$\begin{aligned} D^*(x, z) + D^*(z, y) &= (d^*)^{1/p}(x, z) + D(z, y) \\ &\geq (d^*)^{1/p}(x, z) + d^{1/p}(z, y) = (d^*)^{1/p}(x, z) + (d^*)^{1/p}(z, y) \\ &\geq \frac{1}{2^{(1/p)-1}}(d^*(x, y))^{1/p} = \frac{1}{K}D^*(x, y). \end{aligned}$$

Thus, $D^*(x, y) \leq K[D^*(x, z) + D^*(z, y)]$.

Case 4: $z \in X$ and $x, y \notin X$. We find that

$$\begin{aligned} D^*(x, z) + D^*(z, y) &= (d^*)^{1/p}(x, z) + (d^*)^{1/p}(z, y) \\ &\geq \frac{1}{2^{(1/p)-1}}(d^*(x, z) + d^*(z, y))^{1/p} \\ &\geq \frac{1}{2^{(1/p)-1}}(d^*(x, y))^{1/p} = \frac{1}{K}D^*(x, y). \end{aligned}$$

Thus, $D^*(x, y) \leq K[D^*(x, z) + D^*(z, y)]$.

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Case 5: $x \in X$ and $y, z \notin X$. We find that

$$\begin{aligned} D^*(x,z) + D^*(z,y) &= (d^*)^{1/p}(x,z) + (d^*)^{1/p}(z,y) \\ &\geq \frac{1}{2^{(1/p)-1}}(d^*(x,z) + d^*(z,y))^{1/p} \\ &\geq \frac{1}{2^{(1/p)-1}}(d^*(x,y))^{1/p} = \frac{1}{K}D^*(x,y). \end{aligned}$$

Thus, $D^*(x,y) \leq K[D^*(x,z) + D^*(z,y)]$.

Case 6: $x, y, z \notin X$. We find that

$$\begin{aligned} D^*(x,z) + D^*(z,y) &= (d^*)^{1/p}(x,z) + (d^*)^{1/p}(z,y) \\ &\geq \frac{1}{2^{(1/p)-1}}(d^*(x,z) + d^*(z,y))^{1/p} \\ &\geq \frac{1}{2^{(1/p)-1}}(d^*(x,y))^{1/p} = \frac{1}{K}D^*(x,y). \end{aligned}$$

Thus, $D^*(x,y) \leq K[D^*(x,z) + D^*(z,y)]$.

In all cases, $D^*(x,y) \leq 4K^3[D^*(x,z) + D^*(z,y)]$ for all $x, y, z \in X^*$. So, $(X^*, D^*, 4K^3)$ is a b-metric space.

From (2.1) and since (X^*, d^*) is complete, $(X^*, D^*, 4K^3)$ is a complete b-metric space. Further, since $(X, d) = (X, d^*)$ is dense in (X^*, d^*), it follows that $(X, D, 4K^3) = (X, D^*, 4K^3)$ is dense in $(X^*, D^*, 4K^3)$. Convergence in a b-metric space does not depend on the coefficient, so the b-metric space (X, D, K) is also dense in $(X^*, D^*, 4K^3)$. Thus, $(X^*, D^*, 4K^3)$ is a completion of (X, D, K).

(2). Since (X_1^*, D_1^*, K_1^*) and (X_2^*, D_2^*, K_2^*) are completions of the b-metric space (X, D, K), there exist isometries $f_1 : X \longrightarrow X_1^*$ and $f_2 : X \longrightarrow X_2^*$ such that $\overline{f_1(X)} = X_1^*$ and $\overline{f_2(X)} = X_2^*$. For each $x \in X_1^*$, there exists a sequence $\{x_n\} \subset X$ such that $\lim_{n\to\infty} f_1(x_n) = x$ in (X_1^*, D_1^*, K_1^*). Then $\{f_1(x_n)\}$ is a Cauchy sequence in $(f_1(X), D_1^*, K_1^*)$. Therefore, $\{x_n\}$ is a Cauchy sequence in (X, D, K). Thus, $\{f_2(x_n)\}$ is a Cauchy sequence in (X_2^*, D_2^*, K_2^*). Since (X_2^*, D_2^*, K_2^*) is complete, there exists $y \in X_2^*$ such that $\lim_{n\to\infty} f_2(x_n) = y$ in (X_2^*, D_2^*, K_2^*). Put $\varphi(x) = y$. We will prove that $\varphi : X_1^* \longrightarrow X_2^*$ is a bijective isometry and restricts to the identity on X by means of the following four claims.

Claim 1: *the map φ is well defined.* Suppose that the sequences $\{x_n\}$, $\{t_n\} \subset X$ satisfy $\lim_{n\to\infty} f_1(x_n) = \lim_{n\to\infty} f_1(t_n) = x$ in (X_1^*, D_1^*, K_1^*) and $\lim_{n\to\infty} f_2(x_n) = y$, $\lim_{n\to\infty} f_2(t_n) = z$ in (X_2^*, D_2^*, K_2^*). We will prove that $y = z$. Indeed, for all n,

$$D(x_n, t_n) = D_1(f_1(x_n), f_1(t_n)) = D_1^*(f_1(x_n), f_1(t_n)) \leq K_1^*[D_1^*(f_1(x_n), x) + D_1^*(x, f_1(t_n))].$$

So, $\lim_{n\to\infty} D(x_n, t_n) = 0$. Then $\lim_{n\to\infty} D_2(f_2(x_n), f_2(t_n)) = 0$. By Lemma 2.1, we get $y = z$. This proves that the map φ exists.

Claim 2: *φ is injective.* Let $x, t \in X_1^*$ and $\varphi(x) = \varphi(t)$. By the definition of φ, there exist $\{x_n\}, \{t_n\} \subset X$ such that $\lim_{n\to\infty} f_1(x_n) = x$, $\lim_{n\to\infty} f_1(t_n) = t$ and

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$\lim_{n\to\infty} f_2(x_n) = \varphi(x)$, $\lim_{n\to\infty} f_2(t_n) = \varphi(t)$. By Lemma 2.1, since $\varphi(x) = \varphi(t)$, we get $\lim_{n\to\infty} D_2^*(f_2(x_n), f_2(t_n)) = 0$ and so

$$\lim_{n\to\infty} D_1^*(f_1(x_n), f_1(t_n)) = \lim_{n\to\infty} D(x_n, t_n) = \lim_{n\to\infty} D_2^*(f_2(x_n), f_2(t_n)) = 0.$$

Using Lemma 2.1 again, we get $x = t$. So, φ is injective.

Claim 3: *φ is surjective.* Let $y \in X_2^*$. Then there exists $\{x_n\} \subset X$ such that $\lim_{n\to\infty} f_2(x_n) = y$. So, the sequence $\{f_2(x_n)\}$ is Cauchy in (X_2^*, D_2^*, K_2^*). Therefore, the sequence $\{x_n\}$ is Cauchy in (X, D, K) and so the sequence $\{f_1(x_n)\}$ is Cauchy in (X_1^*, D_1^*, K_1^*). So, $\lim_{n\to\infty} f_1(x_n) = x$ exists in (X_1^*, D_1^*, K_1^*). This proves that $y = \varphi(x)$ and that the map φ is surjective.

Claim 4: *φ restricts to the identity on X.* Let $y = \varphi(x)$ and $z = \varphi(t)$, where $x, t \in f_1(X)$. Then $x = f_1(u)$ and $t = f_1(v)$ for some $u, v \in X$. By choosing $x_n = u$ and $t_n = v$ for all n, we get $y = f_2(u)$ and $z = f_2(t)$. So, $D_2^*(y, z) = D(u, v) = D_1^*(x, t)$, that is, φ restricts to the identity on X.

Thus, $\varphi : X_1^* \longrightarrow X_2^*$ is a bijective isometry which restricts to the identity on X. □

Remark 2.3.

(1) If the b-metric D is a metric, then, from Theorem 1.4, $d = D$ and $p = 1$. So, $D^* = d^*$ and the completion of the b-metric space (X, D, K) becomes the usual completion of the metric space (X, D).
(2) D^* is a b-metric with the coefficient $4K^3$ on X^*, but, from the proof of Theorem 2.2(1), we see that D^* is a b-metric with the coefficient K on X.
(3) If the b-metric space (X, D, K) is complete, then $X^* = X$ and (X, D, K) is its completion. So, the completion of the b-metric space presented in [2, Example 2.3] is itself and this resolves the issue mentioned in the Introduction.

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NGUYEN VAN DUNG,
Nonlinear Analysis Research Group,
Ton Duc Thang University, Ho Chi Minh City, Vietnam
and
Faculty of Mathematics and Statistics,
Ton Duc Thang University, Ho Chi Minh City, Vietnam
e-mail: nguyenvandung2@tdt.edu.vn

VO THI LE HANG,
Faculty of Mathematics and Information Technology Teacher Education,
Dong Thap University, Cao Lanh City, Dong Thap Province, Vietnam
e-mail: vtlhang@dthu.edu.vn

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