Bull. Aust. Math. Soc. **109** (2024), 464–470
doi:10.1017/S0004972723000631

TRANSCENDENCE OF GENERALISED EULER–KRONECKER CONSTANTS

NEELAM KANDHIL and RASHI LUNIA

(Received 30 April 2023; accepted 26 May 2023; first published online 10 July 2023)

Abstract

We introduce some generalisations of the Euler–Kronecker constant of a number field and study their arithmetic nature.

2020 *Mathematics subject classification*: primary 11J86; secondary 11J81.

Keywords and phrases: linear forms in logarithms, generalised Euler–Kronecker constants.

1. Introduction and preliminaries

In 1740, Euler [2] introduced the Euler–Mascheroni constant, which is defined as

$$\gamma = \lim_{x\to\infty}\Big(\sum_{n\le x}\frac{1}{n} - \log x\Big). \tag{1.1}$$

This constant has been extensively studied (see [4]), but many questions about its behaviour are unanswered. For example, it is not known if γ is rational or irrational. Diamond and Ford [1] introduced a generalisation of Euler's constant as follows. For a nonempty finite set of distinct primes Ω, let P_Ω denote the product of the elements of Ω and $\delta_\Omega = \prod_{p\in\Omega}(1-1/p)$. Then the generalised Euler constant is defined as

$$\gamma(\Omega) = \lim_{x\to\infty}\Big(\sum_{\substack{n\le x\\(n,P_\Omega)=1}}\frac{1}{n} - \delta_\Omega \log x\Big).$$

Note that when $\Omega = \emptyset$, we have $P_\Omega = 1 = \delta_\Omega$ and $\gamma(\Omega) = \gamma$. In this context, Murty and Zaytseva proved the following theorem.

THEOREM 1.1 (Murty and Zaytseva, [8]). *At most one number in the infinite list $\{\gamma(\Omega)\}$, as Ω varies over all finite subsets of distinct primes, is algebraic.*

The second author would like to thank Number Theory plan project, Department of Atomic Energy, for financial support.

We note that γ appears as the constant term in the Laurent series expansion of $\zeta(s)$ around $s = 1$. This observation led Ihara [3] to define the Euler–Kronecker constant associated to a number field as follows.

Let $\mathbf{K}$ be a number field of degree n and let $\mathcal{O}_{\mathbf{K}}$ denote its ring of integers. The Dedekind zeta function of $\mathbf{K}$ is given by

$$\zeta_{\mathbf{K}}(s) = \sum_{(0)\neq\mathfrak{a}\subseteq\mathcal{O}_{\mathbf{K}}} \frac{1}{\mathfrak{N}(\mathfrak{a})^s}, \quad \mathfrak{R}(s) > 1.$$

It has a meromorphic continuation to the entire complex plane with only a simple pole at the point $s = 1$. Its Laurent series expansion around $s = 1$ is given by

$$\zeta_{\mathbf{K}}(s) = \frac{\rho_{\mathbf{K}}}{s-1} + c_{\mathbf{K}} + \mathrm{O}(s-1),$$

where $\rho_{\mathbf{K}} \neq 0$ is the residue of $\zeta_{\mathbf{K}}$ at $s = 1$. Ihara defined the ratio

$$\gamma_{\mathbf{K}} := c_{\mathbf{K}}/\rho_{\mathbf{K}}$$

as the Euler–Kronecker constant of $\mathbf{K}$. In the next section, an expression analogous to (1.1) is given for $\gamma_{\mathbf{K}}$.

The aim of this article is to study the arithmetic nature of generalisations of Euler–Kronecker constants. To do so, we introduce some notation. Let $\mathcal{P}_{\mathbf{K}}$ denote the set of nonzero prime ideals $\mathfrak{p}$ of $\mathcal{O}_{\mathbf{K}}$ and let Ω be a nonempty subset of $\mathcal{P}_{\mathbf{K}}$ (possibly infinite) such that

$$\sum_{\mathfrak{p}\in\Omega} \frac{\log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p}) - 1} < \infty. \tag{1.2}$$

For $\mathbf{K} = \mathbb{Q}$, the set of Pjateckii–Šapiro primes is an example of such an infinite subset. Let $N_\Omega = \{\mathfrak{p} \cap \mathbb{Z} \mid \mathfrak{p} \in \Omega\}$. We set

$$P(\Omega(x)) = \prod_{\mathfrak{p}\in\Omega(x)} \mathfrak{p} \quad \text{and} \quad \delta_{\mathbf{K}}(\Omega(x)) = \prod_{\mathfrak{p}\in\Omega(x)} \Big(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})}\Big),$$

where $\Omega(x) = \{\mathfrak{p} \in \Omega \mid \mathfrak{N}(\mathfrak{p}) \leq x\}$. Then by (1.2), $\lim_{x\to\infty} \delta_{\mathbf{K}}(\Omega(x))$ exists and equals

$$\delta_{\mathbf{K}}(\Omega) = \prod_{\mathfrak{p}\in\Omega} \Big(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})}\Big).$$

Note that $\delta_{\mathbf{K}}(\Omega) = 1$ for $\Omega = \emptyset$. The generalised Euler–Kronecker constant associated to Ω is denoted by $\gamma_{\mathbf{K}}(\Omega)$ and is defined as

$$\lim_{x\to\infty} \Big(\frac{1}{\rho_{\mathbf{K}}} \sum_{\substack{0\neq\mathfrak{a}\subset\mathcal{O}_{\mathbf{K}} \\ \mathfrak{N}(\mathfrak{a})\leq x \\ (\mathfrak{a},P(\Omega(x)))=1}} \frac{1}{\mathfrak{N}(\mathfrak{a})} - \delta_{\mathbf{K}}(\Omega(x)) \log x \Big).$$

In Section 3, we will show that this limit exists. We note that $\gamma_{\mathbf{K}}(\Omega) = \gamma_{\mathbf{K}}$ when $\Omega = \emptyset$. With this set up, we have the following theorem.

https://doi.org/10.1017/S0004972723000631 Published online by Cambridge University Press

THEOREM 1.2. *Let $\{\Omega_i\}_{i\in I}$ be a family of subsets of $\mathcal{P}_{\mathbf{K}}$ satisfying (1.2). Further, suppose that $N_{\Omega_i}\backslash N_{\Omega_j}$ is nonempty and finite for all $i, j \in I$ and $i \neq j$. Then at most one number from the infinite list*

$$\left\{ \frac{\gamma_{\mathbf{K}}(\Omega_i)}{\delta_{\mathbf{K}}(\Omega_i)} \,\middle|\, i \in I \right\}$$

is algebraic.

We digress here a little to make an interesting observation. For $K = \mathbb{Q}$, it is known by Merten's theorem that as $x \to \infty$,

$$\delta_{\mathbb{Q}}(\Omega(x)) \sim \frac{e^{-\gamma}}{\log x}.$$

This makes one wonder if $\gamma_{\mathbf{K}}$ appears as an exponent in the expression for $\mathbf{K} \neq \mathbb{Q}$. A result of Rosen [9] shows that this is not true in general. More precisely, he showed that as $x \to \infty$,

$$\delta_{\mathbf{K}}(\Omega(x)) \sim \frac{e^{-\gamma}}{\rho_{\mathbf{K}} \log x}.$$

2. Preliminaries and lemmas

Let $\mathbf{K}$ be a number field of degree n. Throughout this section, $\mathfrak{p}$ denotes a nonzero prime ideal of $\mathcal{O}_{\mathbf{K}}$. We recall the following result on counting the number of integral ideals of $\mathcal{O}_{\mathbf{K}}$.

LEMMA 2.1 [7, Ch. 11]. *Let a_m be the number of integral ideals of $\mathcal{O}_{\mathbf{K}}$ with norm m. Then, as x tends to infinity,*

$$\sum_{m=1}^{x} a_m = \rho_{\mathbf{K}} x + O(x^{1-1/n}).$$

Using this result, we find the following expression for $\gamma_{\mathbf{K}}$, analogous to (1.1).

LEMMA 2.2. *For any number field $\mathbf{K}$, the limit*

$$\lim_{x\to\infty} \left(\frac{1}{\rho_{\mathbf{K}}} \sum_{\substack{0\neq\mathfrak{a}\subset\mathcal{O}_{\mathbf{K}} \\ \mathfrak{N}(\mathfrak{a})\leq x}} \frac{1}{\mathfrak{N}(\mathfrak{a})} - \log x \right)$$

exists and equals $\gamma_{\mathbf{K}}$.

PROOF. Applying partial summation and Lemma 2.1, the result follows. □

The Möbius function $\mu_{\mathbf{K}}$ and the von Mangoldt function $\Lambda_{\mathbf{K}}$ are defined on $\mathcal{O}_{\mathbf{K}}$ as follows:

https://doi.org/10.1017/S0004972723000631 Published online by Cambridge University Press

$$\mu_{\mathbf{K}}(\mathfrak{a}) = \begin{cases} 1 & \text{if } \mathfrak{a} = \mathcal{O}_{\mathbf{K}}, \\ (-1)^r & \text{if } \mathfrak{a} \text{ is a product of } r \text{ distinct prime ideals}, \\ 0 & \text{otherwise}; \end{cases}$$

$$\Lambda_{\mathbf{K}}(\mathfrak{a}) = \begin{cases} \log \mathfrak{N}(\mathfrak{p}) & \text{if } \mathfrak{a} = \mathfrak{p}^m \text{ for some } \mathfrak{p} \text{ and some integer } m \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

We record the following identities satisfied by these functions which can be derived using techniques similar to [6, Exercises 1.1.2, 1.1.4, 1.1.6].

$$\sum_{J|I} \frac{\mu_{\mathbf{K}}(J)}{\mathfrak{N}(J)} = \prod_{\mathfrak{p}|I} \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})}\right),$$

$$\mu_{\mathbf{K}}(I) \log \mathfrak{N}(I) = -\sum_{J|I} \Lambda_{\mathbf{K}}(J) \mu_{\mathbf{K}}(IJ^{-1}).$$

We end this section by stating the key ingredient in the proof of Theorem 1.2.

LEMMA 2.3 (Lindemann, [5]). *If* $\alpha \neq 0, 1$ *is an algebraic number, then* $\log \alpha$ *is transcendental, where* log *denotes any branch of the logarithm.*

3. Generalised Euler–Kronecker constants

Let $\mathcal{P}_{\mathbf{K}}$ denote the set of nonzero prime ideals of $\mathcal{O}_{\mathbf{K}}$. For any nonempty finite set $\Omega_f \subset \mathcal{P}_{\mathbf{K}}$, we set

$$P(\Omega_f) = \prod_{\mathfrak{p} \in \Omega_f} \mathfrak{p} \quad \text{and} \quad \delta_{\mathbf{K}}(\Omega_f) = \prod_{\mathfrak{p} \in \Omega_f} \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})}\right),$$

with the convention that $P(\Omega_f) = 1 = \delta_{\mathbf{K}}(\Omega_f)$, when $\Omega_f = \emptyset$. Since $\mathcal{O}_{\mathbf{K}}$ is a Dedekind domain, every integral ideal can be uniquely expressed as a product of prime ideals. For ideals

$$\mathfrak{a} = \prod_{\mathfrak{p} \in \mathcal{P}_{\mathbf{K}}} \mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{a})}, \quad \mathfrak{b} = \prod_{\mathfrak{p} \in \mathcal{P}_{\mathbf{K}}} \mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{b})},$$

where all but finitely many $v_{\mathfrak{p}}(\mathfrak{a}), v_{\mathfrak{p}}(\mathfrak{b})$ are zero, we define the greatest common divisor (gcd) of $\mathfrak{a}$ and $\mathfrak{b}$ by

$$(\mathfrak{a}, \mathfrak{b}) = \gcd(\mathfrak{a}, \mathfrak{b}) = \prod_{\mathfrak{p} \in \mathcal{P}_{\mathbf{K}}} \mathfrak{p}^{\min(v_{\mathfrak{p}}(\mathfrak{a}), v_{\mathfrak{p}}(\mathfrak{b}))},$$

where we have denoted $\mathfrak{p}^0$ by $\mathcal{O}_{\mathbf{K}}$. Hence, if the prime factors of $\mathfrak{a}$ and $\mathfrak{b}$ are all distinct, $(\mathfrak{a}, \mathfrak{b}) = \mathcal{O}_{\mathbf{K}}$. We notice that $(\mathfrak{a}, \mathfrak{b}) = \mathfrak{a} + \mathfrak{b}$ as $v_{\mathfrak{p}}(\mathfrak{a} + \mathfrak{b}) = \min(v_{\mathfrak{p}}(\mathfrak{a}), v_{\mathfrak{p}}(\mathfrak{b}))$. From now on, $\mathcal{O}_{\mathbf{K}}$ will be denoted by 1.

https://doi.org/10.1017/S0004972723000631 Published online by Cambridge University Press

LEMMA 3.1. *For a number field $\mathbf{K}$ and a finite set Ω_f, the limit*

$$\lim_{x\to\infty}\Big(\frac{1}{\rho_{\mathbf{K}}}\sum_{\substack{0\neq I\subset\mathcal{O}_{\mathbf{K}}\\ \mathfrak{N}(I)\leq x\\ (I,P(\Omega_f))=1}}\frac{1}{\mathfrak{N}(I)}-\delta_{\mathbf{K}}(\Omega_f)\log x\Big)$$

exists and is denoted by $\gamma_{\mathbf{K}}(\Omega_f)$.

PROOF. Let $\Omega_f\subset\mathcal{P}_{\mathbf{K}}$ and $\mathfrak{p}\in\mathcal{P}_{\mathbf{K}}$ be a prime ideal not in Ω_f. Using

$$\sum_{\substack{0\neq I\subseteq\mathcal{O}_{\mathbf{K}}\\ \mathfrak{N}(I)\leq x\\ (I,\mathfrak{p}P(\Omega_f))=1}}\frac{1}{\mathfrak{N}(I)}=\sum_{\substack{0\neq I\subseteq\mathcal{O}_{\mathbf{K}}\\ \mathfrak{N}(I)\leq x\\ (I,P(\Omega_f))=1}}\frac{1}{\mathfrak{N}(I)}-\frac{1}{\mathfrak{N}(\mathfrak{p})}\sum_{\substack{0\neq I\subseteq\mathcal{O}_{\mathbf{K}}\\ \mathfrak{N}(I)\leq x/\mathfrak{N}(\mathfrak{p})\\ (I,P(\Omega_f))=1}}\frac{1}{\mathfrak{N}(I)},$$

the result follows by induction on the cardinality of Ω_f. □

LEMMA 3.2. *Let Ω_f be a finite set of nonzero prime ideals. Then,*

$$\gamma_{\mathbf{K}}(\Omega_f)=\delta_{\mathbf{K}}(\Omega_f)\Big(\gamma_{\mathbf{K}}+\sum_{\mathfrak{p}\in\Omega_f}\frac{\log\mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})-1}\Big).$$

PROOF. We have

$$\begin{aligned}\sum_{\substack{0\neq I\subset\mathcal{O}_{\mathbf{K}}\\ \mathfrak{N}(I)\leq x\\ (I,P(\Omega_f))=1}}\frac{1}{\mathfrak{N}(I)}&=\sum_{\substack{0\neq I\subset\mathcal{O}_{\mathbf{K}}\\ \mathfrak{N}(I)\leq x}}\frac{1}{\mathfrak{N}(I)}\sum_{J|(I,P(\Omega_f))}\mu(J)\\&=\sum_{J|P(\Omega_f)}\frac{\mu(J)}{\mathfrak{N}(J)}\sum_{\substack{0\neq J_0\subset\mathcal{O}_{\mathbf{K}}\\ \mathfrak{N}(J_0)\leq x/\mathfrak{N}(J)}}\frac{1}{\mathfrak{N}(J_0)}\\&=\sum_{J|P(\Omega_f)}\frac{\mu(J)}{\mathfrak{N}(J)}\Big\{\rho_{\mathbf{K}}\log\frac{x}{\mathfrak{N}(J)}+\rho_{\mathbf{K}}\gamma_{\mathbf{K}}+o(1)\Big\}\\&=\delta_{\mathbf{K}}(\Omega_f)(\rho_{\mathbf{K}}\log x+\rho_{\mathbf{K}}\gamma_{\mathbf{K}}+o(1))-\rho_{\mathbf{K}}\sum_{J|P(\Omega_f)}\frac{\mu(J)}{\mathfrak{N}(J)}\log\mathfrak{N}(J).\end{aligned}$$

We now consider the last term:

$$\begin{aligned}-\sum_{J|P(\Omega_f)}\frac{\mu(J)}{\mathfrak{N}(J)}\log\mathfrak{N}(J)&=\sum_{J|P(\Omega_f)}\frac{1}{\mathfrak{N}(J)}\sum_{J_0|J}\Lambda(J_0)\mu(JJ_0^{-1})\\&=\sum_{J_0|P(\Omega_f)}\frac{\Lambda(J_0)}{\mathfrak{N}(J_0)}\sum_{J_1|P(\Omega_f)J_0^{-1}}\frac{\mu(J_1)}{\mathfrak{N}(J_1)}\\&=\sum_{\mathfrak{p}'\in\Omega_f}\frac{\Lambda(\mathfrak{p}')}{\mathfrak{N}(\mathfrak{p}')}\sum_{J_1|P(\Omega_f)\mathfrak{p}'^{-1}}\frac{\mu(J_1)}{\mathfrak{N}(J_1)}\end{aligned}$$

https://doi.org/10.1017/S0004972723000631 Published online by Cambridge University Press

$$= \sum_{\mathfrak{p}'\in\Omega_f} \frac{\log \mathfrak{N}(\mathfrak{p}')}{\mathfrak{N}(\mathfrak{p}')}\Big(\frac{\delta_{\mathbf{K}}(\Omega_f)}{1-1/\mathfrak{N}(\mathfrak{p}')}\Big)$$
$$= \delta_{\mathbf{K}}(\Omega_f)\sum_{\mathfrak{p}\in\Omega_f}\frac{\log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})-1}.$$

Thus,

$$\lim_{x\to\infty}\Big(\frac{1}{\rho_{\mathbf{K}}}\sum_{\substack{0\neq I\subset \mathcal{O}_{\mathbf{K}}\\ \mathfrak{N}(I)\le x\\ (I,P(\Omega_f))=1}}\frac{1}{\mathfrak{N}(I)} - \delta_{\mathbf{K}}(\Omega_f)\log x\Big) = \delta_{\mathbf{K}}(\Omega_f)\Big(\gamma_{\mathbf{K}} + \sum_{\mathfrak{p}\in\Omega_f}\frac{\log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})-1}\Big). \qquad \square$$

COROLLARY 3.3. *For a number field* $\mathbf{K}$ *and any set* $\Omega\subset\mathcal{P}_{\mathbf{K}}$ *satisfying (1.2), the limit*

$$\lim_{x\to\infty}\Big(\frac{1}{\rho_{\mathbf{K}}}\sum_{\substack{0\neq I\subset \mathcal{O}_{\mathbf{K}}\\ \mathfrak{N}(I)\le x\\ (I,P(\Omega(x)))=1}}\frac{1}{\mathfrak{N}(I)} - \delta_{\mathbf{K}}(\Omega(x))\log x\Big)$$

exists and equals

$$\delta_{\mathbf{K}}(\Omega)\Big(\gamma_{\mathbf{K}} + \sum_{\mathfrak{p}\in\Omega}\frac{\log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})-1}\Big).$$

We denote this limit by $\gamma_{\mathbf{K}}(\Omega)$.

PROOF. Follows from Lemma 3.2 since $\Omega(x)$ is a finite set. $\square$

4. Proof of Theorem 1.2

Suppose there exist $i,j\in I$ such that

$$\frac{\gamma_{\mathbf{K}}(\Omega_i)}{\delta_{\mathbf{K}}(\Omega_i)} \quad\text{and}\quad \frac{\gamma_{\mathbf{K}}(\Omega_j)}{\delta_{\mathbf{K}}(\Omega_j)}$$

are algebraic. Using Corollary 3.3,

$$\frac{\gamma_{\mathbf{K}}(\Omega_i)}{\delta_{\mathbf{K}}(\Omega_i)} - \frac{\gamma_{\mathbf{K}}(\Omega_j)}{\delta_{\mathbf{K}}(\Omega_j)} = \sum_{\mathfrak{p}\in\Omega_i}\frac{\log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})-1} - \sum_{\mathfrak{p}\in\Omega_j}\frac{\log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})-1}, \tag{4.1}$$

which is also an algebraic number. Since the sets $N_{\Omega_i}\backslash N_{\Omega_j}$ and $N_{\Omega_j}\backslash N_{\Omega_i}$ are nonempty and finite, the sets $\Omega_i\backslash\Omega_j$ and $\Omega_j\backslash\Omega_i$ are also finite. Let

$$\Omega_i\backslash\Omega_j = \{\mathfrak{p}_1,\mathfrak{p}_2,\ldots,\mathfrak{p}_n\},\qquad \Omega_j\backslash\Omega_i = \{\mathfrak{q}_1,\mathfrak{q}_2,\ldots,\mathfrak{q}_m\}.$$

Then (4.1) implies

$$\sum_{\mathfrak{p}\in\Omega_i}\frac{\log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})-1} - \sum_{\mathfrak{p}\in\Omega_j}\frac{\log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})-1} = \sum_{s=1}^{n}\frac{\log p_s^{f_s}}{p_s^{f_s}-1} - \sum_{t=1}^{m}\frac{\log q_t^{g_t}}{q_t^{g_t}-1} = \log\Big(\frac{\prod_{s=1}^{n} p_s^{(f_s/p_s^{f_s}-1)}}{\prod_{t=1}^{m} q_t^{(g_t/q_t^{g_t}-1)}}\Big), \tag{4.2}$$

https://doi.org/10.1017/S0004972723000631 Published online by Cambridge University Press

where $\mathfrak{N}(\mathfrak{p}_s) = p_s^{f_s}$ and $\mathfrak{N}(\mathfrak{q}_t) = q_t^{g_t}$. Using Lemma 2.3 and unique prime factorisation of natural numbers, the expression in (4.2) becomes a transcendental number, which gives a contradiction.

Acknowledgements

The authors would like to thank Prof. S. Gun for suggesting the problem and IMSc for providing academic facilities. They would also like to thank Prof. P. Moree and J. Sivaraman for comments on an earlier version of the manuscript, which improved the exposition. The authors would like to thank the anonymous referee for careful reading of the paper and helpful comments. The first author would like to thank the Max-Planck-Institut für Mathematik for providing a friendly atmosphere.

References

[1] H. Diamond and K. Ford, ‘Generalized Euler constants’, *Math. Proc. Cambridge Philos. Soc.* **145**(1) (2008), 27–41.
[2] L. Euler, ‘De Progressionibus harmonicis observationes’, *Comment. Acad. Sci. Petropolitanae* **7** (1740), 150–161.
[3] Y. Ihara, ‘The Euler–Kronecker invariants in various families of global fields’, in: *Arithmetic, Geometry and Coding Theory (AGCT 2005)* (eds. F. Rodier and S. Vladut), Séminaire et Congrès, 21 (Société Mathématique de France, Paris, 2006), 79–102.
[4] J. C. Lagarias, ‘Euler’s constant: Euler’s work and modern developments’, *Bull. Amer. Math. Soc.* **50** (2013), 527–628.
[5] F. Lindemann, ‘Über die Zahl π’, *Math. Ann.* **20**(2) (1882), 213–225.
[6] M. R. Murty, *Problems in Analytic Number Theory*, 2nd edn (Springer, New York, 2008).
[7] M. R. Murty and J. Esmonde, *Problems in Algebraic Number Theory* (Springer, New York, 2005).
[8] M. R. Murty and A. Zaytseva, ‘Transcendence of generalized Euler constants’, *Amer. Math. Monthly* **120**(1) (2013), 48–54.
[9] M. Rosen, ‘A generalization of Mertens’ theorem’, *J. Ramanujan Math. Soc.* **14**(1) (1999), 1–19.

NEELAM KANDHIL, Max-Planck-Institut für Mathematik,
Vivatsgasse 7, D-53111 Bonn, Germany
e-mail: kandhil@mpim-bonn.mpg.de

RASHI LUNIA, The Institute of Mathematical Sciences,
A CI of Homi Bhabha National Institute,
CIT Campus, Taramani, Chennai 600 113, India
e-mail: rashisl@imsc.res.in