The Bulletin of Symbolic Logic
Volume 20, Number 3, September 2014

COMPUTABLE ABELIAN GROUPS

ALEXANDER G. MELNIKOV

Abstract. We provide an introduction to methods and recent results on infinitely generated abelian groups with decidable word problem.

§1. Introduction. In this article we review results on algorithmic presentations of infinitely generated abelian groups. A presentation $\langle X|R\rangle$ of a group is *computable* if both the generators X and the relations R can be algorithmically enumerated and the word problem in $\langle X|R\rangle$ is decidable. This is equivalent to saying that there exists a naming of the elements of the group by natural numbers such that the operation becomes a computable function on the respective numbers (Rabin [93], Mal'cev [79]). All results discussed in the paper are related to the following general research program that goes back to Mal'cev [80]:

Study countable abelian groups that admit computable presentations.

We will shortly clarify the terminology.

1.1. Groups and their presentations. Recall that a *presentation* of a group G is a pair $\langle X|R\rangle$ such that R is a set of elements of the free group $\mathbb{F}(X)$ generated by X, and $G \cong \mathbb{F}(X)/N(R)$, where $N(R)$ is the least normal subgroup of $\mathbb{F}(X)$ containing R. We also say that X is the set of generators and R are the relations on these generators.

There are two conflicting interpretations of the term *recursive presentation* in the literature. In combinatorial group theory, a presentation $\langle X|R\rangle$ of a group G is called *recursive* if both X and R can be algorithmically ("recursively") listed. If a group has a recursive presentation then it is called *recursively presented* (see, e.g., Higman [55]). In effective algebra, a *recursive presentation* is a synonym of a *computable presentation* which will be defined shortly. Since these two approaches are not equivalent in general, to avoid confusion we say that a presentation $\langle X|R\rangle$ of a group G is *computably enumerable* (c.e.) if both the generators X and the relations R can be algorithmically listed. If a group has a c.e. presentation we will say that it is c.e. presented.

Received May 1, 2014.
Key words and phrases. Abelian groups, logic, computability theory.

1079-8986/14/2003-0003/$3.40
DOI:10.1017/bsl.2014.32

We are concerned with "effective" presentations of groups, but are c.e. presentations "effective" enough? Recall that the word problem in $\langle X|R\rangle$ is *decidable* if there exists an algorithm for checking if $x = y \bmod N(R)$. It is easy to construct a c.e. presented group with undecidable word problem (see, e.g., Higman [55] for an elementary example). It is not difficult to see that an infinite group has a c.e. presentation with decidable word problem exactly if it has a *computable presentation* as defined below:

Definition 1.1 (Mal'cev [79], Rabin [93]). A *computable presentation* or a *constructivization* of a countably infinite algebraic structure $\mathcal{A}$ (e.g., a group) is an algebraic structure $\mathcal{C} \cong \mathcal{A}$ upon the domain of natural numbers $\mathbb{N}$ such that the operations of $\mathcal{C}$ are Turing computable functions.

Definition 1.1 also generalizes the earlier notion of an explicitly presented field due to van der Waerden [108] that was later formalized by Fröhlich and Shepherdson [41]. Definition 1.1 is typically excepted as the standard approach to "effectively presented" countable structures (see books Ash and Knight [4] and Ershov and Goncharov [37]).

1.2. Computable groups. Which countable groups admit computable presentations? Groups with undecidable word problem famously exist already among finitely presented ones, see Novikov [89] and Boone [9]. In fact, it is not known which finitely generated or finitely presented groups admit computable presentations, but there are many partial results towards this problem. See Miller [87] for a survey of results on finitely generated computably presented groups.

The infinitely generated case is even harder. But what if we restrict ourselves to countable *abelian* groups? The class of countable abelian groups is relatively well-understood (Fuchs [44, 45], Loth [76]), and the word problem is decidable in every finitely generated abelian group. Also, there is a long tradition of mixing logic and abelian group theory (e.g., Szmielew [105]). So it seems reasonable to first study countable abelian groups before approaching any other class of infinitely generated groups. Despite of all these promising simplifications, we will see that classifying computable abelian groups up to isomorphism is an enormous task. Nonetheless, we can make progress within some sufficiently broad natural classes of abelian groups.

1.3. Computable abelian groups. In the early 1960's, Mal'cev [80] initiated the systematic study of computable abelian groups. Among other results, Mal'cev gave a full classification of computable subgroups of $(\mathbb{Q}, +)$ up to isomorphism in terms of computably enumerable *types* (to be discussed in Subsection 4.2.1). Mal'cev [80] also made the following fundamental observation:

An infinitely generated abelian group may have two computable presentations that are not computably isomorphic.

This is not true for finitely generated groups. This observation led Mal'cev to the problems of uniqueness and determining the number of algorithmic presentations of a group up to computable isomorphism.

https://doi.org/10.1017/bsl.2014.32 Published online by Cambridge University Press

Mal'cev challenged the large Soviet algebra and logic group asking various questions related to computable abelian groups and isomorphisms between their computable presentations. His interest led to a systematic development of the subject within the USSR, see a survey paper of Khisamiev [66]. At the same time similar questions were independently raised by mathematicians in the USA and Australia, see e.g. Lin [12], Smith [101], and Crossley (ed.) [20]. As we will see, some of the fundamental results of the theory were independently discovered by representatives of these two isolated mathematical traditions.

1.4. The selection of topics. Some topics that were covered in the earlier surveys of Khisamiev [66] and Downey [26] will be omitted here. Very little progress on decidable abelian groups (to be defined) has been done since [66] appeared in print, and thus we will not concentrate on decidable presentations. On the other hand, we will cover some topics that were not touched in surveys [26, 66] such as computable completely decomposable groups and ordered abelian groups. For the sake of exposition we will also sketch proofs of many results. The main reason we provide sketches is that many important results are scattered throughout the literature and some of these results were only published in Russian. Since we are limited in space, we will often give only a proof idea.

To piece together the various results in a logically structured way, we will mostly concentrate on the following more specific problems and their variations:

The Main Problems. Let K be a class of countable abelian groups.

I. Describe the isomorphism types of c.e. and computably presented members of K.
II. Measure the algorithmic complexity of isomorphisms between different computable presentations of a group in K.
III. Investigate direct decompositions of computable members of K.
IV. Study various notions of independence and different kinds of bases for computable members of K.

Problems I–IV above are closely related. In the next few lines we give one of the many examples illustrating this correlation.

In the context of computable completely decomposable groups, investigations towards Problem II lead to an index set result, and the latter can be viewed as a first step towards classification of such groups up to isomorphism, i.e., towards Problem I. The proof of this result relies on an analysis of effective full decomposability; the latter is directly related to Problem III. Furthermore, the main algebraic tool for the proof is a special notion of independence which is within the scope of Problem IV. (See Subsection 4.2 for more details.)

§2. Preliminaries.

2.1. Abelian groups. All groups in this paper are countable, additive and abelian. We assume that the reader is familiar with the notions of a factor-group, order of an element, direct sum, free abelian group, and with the elementary classification of finitely generated abelian groups. The standard references for pure abelian group theory are Fuchs [44, 45] and Kaplansky [61]; we also recommend Kurosh [73] for a smooth and gentle introduction. We briefly go through some basic notions special to the field of abelian groups. Further notions will be introduced when needed.

2.1.1. Abelian groups as $\mathbb{Z}$-modules. Let A be an abelian group. We can make A a $\mathbb{Z}$-module as follows. Given a positive $n \in \mathbb{Z}$ and $a \in A$, define

$$na = a + \cdots (n \text{ times}) \cdots + a,$$

and also define $(-n)a = -(na)$. We do not adjoin the module operation to the signature of groups and use it as an abbreviation. In a torsion-free group A, $na \neq 0$ for any $n \neq 0$ and each nonzero $a \in A$. In a torsion group, the least nonzero n such that $na = 0$ is the order of $a \neq 0$. For every abelian group A, the collection of all its elements of finite order forms a subgroup $T(A)$. Then $A/T(A)$ is torsion-free. The torsion subgroup $T(A)$ further splits into a direct sum of its maximal p-subgroups $T_p(A)$.

For $a \in A$ and a nonzero $n \in \mathbb{Z}$, the equation $nx = a$ does not have to be solvable in A. If there is such a solution, then we write $n|a$ and say that n divides a. If for every $k \in \mathbb{N}$ we have $n^k|a$, then we write $n^\infty|a$ and say that n infinitely divides a.

A subgroup B of A is *pure* or *serving* if for each $b \in B$ and $n \in \mathbb{Z}$, if $n|b$ in A then $n|b$ already in B. A group D is *divisible* if $n|d$ for every nonzero $n \in \mathbb{N}$ and every $d \in D$. Every abelian group is contained in its *divisible hull* (also called *divisible closure*). In a torsion-free group, there may exist at most one solution for $nx = a$. Thus, for a torsion-free group A and a subset X of A, we can define the *pure closure* $(X)^*_A$ of X in A to be the least pure subgroup of A containing X.

2.1.2. Linear dependence. Let A be an abelian group. Then $a_1, \dots, a_k \in A$ are *linearly independent* if for each $m_0, \dots, m_i \in \mathbb{Z}$, the equality

$$m_1 a_1 + \cdots + m_k a_k = 0,$$

implies $m_i a_i = 0$ for all $i \leq k$. We say that $a_1, \dots, a_k$ are linearly dependent, otherwise. A *basis* of A is its maximal linearly independent subset. Every generating set of a free abelian group is a basis, but not every basis is necessarily a generating set.

We will be using linear dependence mostly in the context of countable torsion-free abelian groups. Countable torsion-free abelian groups are exactly the additive subgroups of $V_\infty \cong \bigoplus_{i \in \omega} \mathbb{Q}$, the $\mathbb{Q}$-vector space of dimension ω. The rank of a (countable) torsion-free abelian group A is the smallest $\alpha \leq \omega$ such that $A \leqq \bigoplus_{i \in \alpha} \mathbb{Q}$. One can show that the cardinality of any basis of a torsion-free abelian A is exactly the rank of A.

Recall that an abelian group is *free* if it has a group-presentations with no relations except for the ones saying that its generators commute. Such groups are isomorphic to direct powers of $\mathbb{Z}$. Every subgroup of a free abelian group is free itself [44]. We will use the following well-known fact about generating sets of subgroups of free abelian groups.

LEMMA 2.1 (Rado). *Let $G \leqq F$ be free abelian groups. There exist generating sets $g_1, \dots, g_k$ and $f_1, \dots f_m$ $(k \le m)$ of G and F, respectively, and integers $n_1, \dots, n_k$ such that for each $i \le k$, we have $g_i = n_i f_i$.*

See Fuchs [44] for a proof. We will also be using the fact that a pure cyclic subgroup of an abelian group detaches as its direct summand.

2.2. Computability theory and computable structures. In contrast to combinatorial group theory, the study of computable abelian groups heavily relies on methods of computability theory and computable model theory. In fact, it is traditionally viewed as a topic in effective algebra.

We follow the standard terminology of computability theory. We assume that the reader is familiar with the notions of computable (recursive) and computably enumerable (c.e.) set, an oracle computation, Turing and many-one reducibility, and Turing degree. We also assume that the reader is familiar with the Arithmetical hierarchy (i.e., lightface Σ^0_n, Π^0_n, Δ^0_n) and its relation to c.e. sets and sets c.e. relative to $0^{(n)}$. The latter stands for the Halting problem iterated n times, i.e., the Halting problem for Turing machines with an oracle for $0^{(n-1)}$). These notions can be found in Soare [102] and Rogers [96]. Some of our sketches will refer to the priority method. A detailed exposition of the priority method can be found in Soare [102]. For a very readable introduction to priority methods, see the first few chapters of Downey and Hirschfeldt [29]. We discuss below some computability–theoretic and syntactical notions specific to computable structure theory. These can be found in Ash and Knight [4].

2.2.1. The Hyperarithmetical hierarchy. The (lightface) Σ^0_{n+1}-sets are exactly the sets computably enumerable (c.e.) relative to $0^{(n)}$. Using a transfinite recursion scheme, we can iterate the Turing jump over computable ordinals. This way we obtain a transfinite extension of the Arithmetical hierarchy called the Hyperarithmetical hierarchy. There are two conflicting notations in the literature. In this paper we follow Soare [102] and take $0^{(\omega+1)}$ as the $\Sigma^0_{\omega+1}$-complete degree.

The class Σ^1_1 is defined as the collection of sets of the form $\{y : \exists f \forall n R(y, f(n))\}$, where R is a computable predicate and $f : \mathbb{N} \to \mathbb{N}$ is a function. The Π^1_1-sets are the complements of Σ^1_1-sets, and the class $\Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1$ is equal to the union of Σ^0_α where α ranges over all computable ordinals. The definition can be iterated to obtain the Analytical hierarchy, but we will not go beyond Σ^1_1- and Π^1_1-sets in our paper.

2.2.2. Index sets. We will measure the complexity of various properties of computable groups using the following common approach (see, e.g., Goncharov and Knight [50] and Ash and Knight [4]). The standard effective listing of all partial computable functions leads to an effective enumeration

$$M_0, M_1, M_2, \dots, M_k, \dots$$

of all partial computable structures upon the domain $\mathbb{N}$ that includes all total computable algebraic structures. (A structure is partial if its operations are not necessarily defined on all elements. A structure is total, otherwise. There is no effective listing of total computable structures because there is no effective enumeration of total computable functions.)

Informally, given a property $\mathbb{P}$, we can ask which computable structures in this list satisfy $\mathbb{P}$. For example, $\mathbb{P}$ could be "M_i is the free abelian group of rank ω". Then the complexity of the set $\{i : M_i \text{ satisfies } \mathbb{P}\}$ reflects the complexity of $\mathbb{P}$. The property may mention two or more structures (e.g. "$M_i \cong M_j$"), in this case we ask for the complexity of the set of tuples satisfying $\mathbb{P}$.

Given a class K of structures, we form the *index set* $I(K) = \{i : M_i \in K\}$ and the *isomorphism problem* $E(K) = \{(i, j) : M_i, M_j \in K \text{ and } M_i \cong M_j\}$ for K. It is rather typical that algebraically well-understood classes have the respective index sets and isomorphism problems of low complexities. In contrast, "wild" classes such as computable linear orderings have at least one of these sets not even hyperarithmetical. See Goncharov and Knight [50] for a discussion and examples. If a class of computable structures K has a relatively low complexity of the isomorphism problem and index set (e.g., arithmetical), then we say that K has a classification. Such a classification may be "weak" in the sense that we do not necessarily have an explicit and convenient list of isomorphism invariants. Nonetheless, we get *some* listing of the isomorphism types and give a bound on the complexity of this listing. It is also rather typical that the index set and isomorphism problem are complete in the respective classes, and thus no "stronger" classification is possible. We cite Lempp [74] for an application of index sets to the classification problem for torsion-free (nonabelian) finitely presented groups.

2.2.3. Relativized computability and degree spectra. Although the main objects of our study are computable groups and isomorphisms between them, noncomputable invariants and processes arise naturally in effective abelian group theory.

DEFINITION 2.2. Let $\mathbf{a}$ be a Turing degree. A countable algebraic structure A is $\mathbf{a}$-presentable, or computable relative to $\mathbf{a}$, if there exists a 1-1 numbering of its universe under which the operations on A become $\mathbf{a}$-computable.

The notion defined below has recently become central in the study of noncomputable algebraic structures.

DEFINITION 2.3 (Jockusch). The *degree spectrum* of a countable algebraic structure A is the collection $DSp(A) = \{\mathbf{a} : A \text{ is } \mathbf{a}\text{-presentable}\}$.

For example, every c.e. presented group has a $0'$-computable presentation. (As we will observe in Corollary 5.5, the converse fails already for direct sums of cyclic abelian groups.)

2.2.4. Computable categoricity and beyond. The central notion of computable algebra related to the complexity of isomorphisms (Problem II) is:

DEFINITION 2.4. A computable structure A is *computably categorical* or *autostable* if for any two computable presentations B and C of A there exists a computable isomorphism $f : B \cong C$.

Mal'cev [79] was the first to state Definition 2.4 in full generality. We believe that Rabin, Fröhlich, and Shepherdson, and perhaps van der Waerden were aware of this notion before Mal'cev, but they restricted themselves to groups and/or fields.

Computably categorical algebraic structures in common classes tend to be algebraically trivial, see Goncharov [49], Ash and Knight [4], and Ershov and Goncharov [37] for examples and see Chapter 10.4 of Ash and Knight [4] for a formal clarification of this phenomenon. As we will see, computably categorical abelian groups fall into this general pattern. Thus, we typically have to deal with computable abelian groups that are *not* computably categorical. If a structure is not computably categorical, then perhaps a stronger oracle can compute an isomorphism. For example, the additive group of the $\mathbb{Q}$-vector space of infinite dimension is *not* computably categorical (to be discussed), but if we had access to the halting problem, we would be able to compute an isomorphism between any two computable presentations of the group. Furthermore, $0'$ is sharp in this case, i.e., we would not be able to do that with a weaker oracle. The following approach to structures that are not computably categorical generalizing the example above goes back to Ash:

Definition 2.5 (Ash). A computable algebraic structure (e.g., a group) A is Δ^0_α-categorical if for any two computable presentations B and C of A there exists a Δ^0_α-isomorphism $f : B \cong C$.

In the definition above, α is a computable ordinal. Definition 2.5 and its variations play a central role in modern computable structure theory.

§3. Basic properties of computable abelian groups. This section contains several important observations and notions that will be used throughout the paper.

3.0.5. Definition 1.1 revisited. We say that $H \leqq A$ is a computable subgroup of a (computable) group A if the domain of H is a computable subset of the domain of A. We define Σ^0_n- and Π^0_n-subgroups in a similar fashion. The free abelian group on countably many generators clearly has a computable presentation with a computable generating set. We denote this presentation by $\mathbb{Z}_\omega$. An infinite countable abelian group A has a computable (c.e.) presentation if, and only if, A is isomorphic to $\mathbb{Z}_\omega/H$, where $H \leqq \mathbb{Z}_\omega$ is a computable (respectively, c.e.) subgroup of $\mathbb{Z}_\omega$.

We arrive at a natural generalization of Definition 1.1.

Definition 3.1. Say that an abelian group A is Σ^0_n-presentable if $A \cong \mathbb{Z}_\omega/H$ for some Σ^0_n-subgroup H of $\mathbb{Z}_\omega$, and say that A is Π^0_n-presentable if $A \cong \mathbb{Z}_\omega/H$ for a Π^0_n-subgroup H of $\mathbb{Z}_\omega$. Define Δ^0_n-presentations similarly.

Note that a group admits a Δ^0_n-presentation iff it has a $0^{(n-1)}$-computable copy, and Σ^0_{n+1}-presentations are exactly the ones that are c.e. relative to $0^{(n)}$. A more detailed analysis of Σ^0_n- and Π^0_n-presentable abelian groups can be found in Downey [26] and Khisamiev [66].

We will be frequently using that a c.e. subgroup of a computable group has a computable presentation. The presentation can be obtained by a process sometimes called "padding". That is, we use the operation of the larger group but we assign new names to elements of the subgroup. Note that it does not imply that a c.e. subgroup is always a computable subgroup (viewed as a subset), according to the terminology above. On the other hand, it explains why Higman [55] called c.e.-presented groups "recursive". Indeed, for each c.e.-presentation $\langle \mathbb{F}|N\rangle$ the normal subgroup N generated by the relations has a computable presentation C, and furthermore there exists a computable isomorphism $f : C \to N \leqq \mathbb{F}$. Nonetheless, the c.e. range of f does not have to be computable in general.

Another stronger form of Definition 1.1 uses the first-order diagram of a structure. Recall that the *atomic diagram* of an algebraic structure A in language $\mathcal{L}$ is the collection of quantifier-free first order $\mathcal{L}_{a\in A}$-formulae that hold on A. Here $\mathcal{L}_{a\in A}$ is the language $\mathcal{L}$ augmented by constants for each element of A (the latter can be identified with the natural numbers). The a group has a computable copy if, and only if, its atomic diagram is a computable set under the standard Gödel numbering. The *full diagram* of a structure is the collection of *all* first-order $\mathcal{L}_{a\in A}$-formulae that hold on A. The definition below goes back to Ershov:

Definition 3.2. A countable structure is *decidable* or *strongly constructive* if its full diagram is a computable set, under the standard Gödel numbering.

See Ershov [38] for more on decidable models. In the context of abelian groups, the standard reference is Khisamiev [66]. The following result can be derived from ([38], Proposition 5 on p. 316):

Theorem 3.3 (Ershov). *A computable abelian group A is decidable if, and only if, $Th(A)$ is decidable and the unary predicates $p^k|\cdot$ are computable, uniformly in k and p.*

3.0.6. Divisible hull. Recall that every abelian group A is contained in its divisible hull $D(A)$ which is also called the divisible closure of A.

Fact 3.4 (Smith [101]). *Every computable abelian group has a computable presentation that is computably embedded into its divisible hull.*

The proof is a straightforward effectvization of the classical argument. In the context of reverse mathematics, a detailed discussion of the result can be found in Simpson [99]. An analysis of the uniqueness of $D(A)$ up to a computable isomorphism that agrees on A, in the spirit of Rabin Embedding Theorem for algebraically closed fields (Rabin [93]), can be found in Smith [101].

3.0.7. Splitting a torsion group. Recall that every torsion group T splits into a direct sum of p-groups T_p. The effective content of this result is:

Fact 3.5. *A countable torsion abelian group T has a computable presentation if, and only if, T_p are computable uniformly in p.*

The proof is again elementary. One might expect that Fact 3.5 reduces the study of effective procedures in torsion groups to p-groups, but we will see this is not quite the case.

§4. Torsion-free groups. Recall that a group is torsion-free if every nonzero element of the group is of infinite order. In pure algebra, the class of countable torsion-free abelian groups has been serving as a source of striking counterexamples, see Fuchs [45]. The class of countable torsion-free abelian groups is believed to be very "complicated" since only a few special classes of countable torsion-free abelian groups are classically well-understood.

What can be said about c.e. and computable torsion-free abelian groups? Can we classify such groups up to isomorphism? We will explain why the answer to the latter question is believed to be negative in general, but we will accumulate a good deal of information about the special class of *completely decomposable* abelian groups.

The section is subdivided into subsections and paragraphs.

In Subsection 4.1 we give an effective analysis of linear dependence and generating sets. In Subsection 4.1.1 we use linear independence to characterize computable categoricity, and in Subsection 4.1.2 we manipulate with finitely generated subgroups to show that every c.e. presented torsion-free abelian group admits a computable presentation.

In Subsection 4.2 we discuss computable completely decomposable groups. In Subsection 4.2.1 give a detailed analysis of the most elementary case of rank 1 groups, and in Subsection 4.2.2 we briefly discuss the case of any finite rank. The homogeneous case is studied in Subsection 4.2.3, and Subsection 4.2.4 contains results on arbitrary computable completely decomposable groups.

In Subsection 4.3 we finally approach computable torsion-free abelian groups that are not necessarily completely decomposable. In Section 4.3.1 we introduce a general method of constructing computable torsion-free abelian groups. In Section 4.3.2 and Subsection 4.3.3 we apply the method to show that computable torsion-free abelian groups are "unclassifiable". In Subsection 4.3.4 we discuss further applications of this method to degree spectra of torsion-free abelian groups, and in Subsection 4.3.5 we apply the method to study indecomposable computable abelian groups.

4.1. Linear independence and generating sets.

4.1.1. Linear dependence algorithm. We say that a computable torsion-free abelian group A has a *linear dependence algorithm* if given any $a_1, \dots, a_k \in A$ we can uniformly decide if $a_1, \dots, a_k$ are linearly dependent or not. The fact below is well-known.

FACT 4.1 (Mal'cev [80]). *For a computable torsion-free abelian group A, the following are equivalent*:

1. *A has a linear dependence algorithm*;
2. *A has a computable base*;
3. *A has a c.e. base.*

https://doi.org/10.1017/bsl.2014.32 Published online by Cambridge University Press

SKETCH. The implications (1) $\rightarrow$ (2) and (2) $\rightarrow$ (3) are straightforward. For (3) $\rightarrow$ (1), note that for every nonzero element a_i there exists an $n_i \in \mathbb{Z}$ such that na belongs to the free group $\bigoplus_{b\in B} \mathbb{Z}b$ generated by a c.e. basis B. Given $a_1, \dots, a_k$, find $n_1, \dots, n_k \in \mathbb{Z}$ with that properly. Now $a_1, \dots, a_k$ are linearly independent if, and only if, $n_1 a_1, \dots, n_k a_k$ are so *within* $\bigoplus_{b\in B} \mathbb{Z}b$. The latter can be decided using the standard matrix analysis. ⊣

Mal'cev [80] was possibly the first to discover that the divisible group $\bigoplus_{i\in\omega} \mathbb{Q}$ has a presentation with no linear dependence algorithm. The group $\bigoplus_{i\in\omega} \mathbb{Q}$ is naturally a $\mathbb{Q}$-vector space. There has been a line of study into the *Turing degrees* of linear dependence of computable and c.e. presented and computable vector spaces, see Metakides and Nerode [85] and Shore [98]. For instance, one can show that there exists a computable copy of $\bigoplus_{i\in\omega} \mathbb{Q}$ in which any basis computes the Halting problem. See Simpson [99] for a discussion. But rather than analyzing degrees of linear dependence, we present two results on bases of arbitrary computable torsion-free abelian groups that contrast each other.

PROPOSITION 4.2 (Dobritsa [25]). *Every computable torsion-free abelian group has a presentation with a computable basis.*

PROOF SKETCH. We use the notion of a finite partial group which is self-explanatory. Also, for an integer $t \geq 0$, say that $\{a_1, \dots a_k\} \subseteq A$ is *t-dependent* in A if $m_1 a_1 + \dots + m_k a_k = 0$ for some $m_1, \dots, m_k \in Z$ with $0 < |m_i| \leq t$. The desired presentation is constructed by stages. At the end of stage t we have a finite partial group C_t, its t-basis B_t, and an embedding of C_t into A. At stage t, we extend C_t and the embedding using another element of A that is t-independent of the image $\{a_0, \dots, a_s\}$ of $B_t = \{b_0, \dots, b_s\}$. There is one crucial subtlety in the proof: At a later stage s we may discover a_i is s-dependent on $a_0, \dots, a_{i-1}$. We then do the following. For each integer $j \in [i, s]$, we re-define b_j by setting it equal to $a_j + s!\, c_j$, where $c_j \in A$ are first found with the property:

For any choice of nonzero integers k_j with $|k_j| \leq s!$, the set $\{a_0, \dots, a_{i-1}, a_i + k_i c_i, \dots, a_s + k_s c_s\}$ is s-independent.

The coefficients $s!$ allow to preserve the already declared divisibility conditions. We leave the verification to the reader. ⊣

In contrast, based on Nurtazin [90], we prove:

PROPOSITION 4.3. *Every computable torsion-free abelian group A of infinite rank has a presentation in which linear independence is undecidable.*

If A is divisible then the usual trick of making elements equal to large linear combinations of the previously enumerated ones would do the job. However, if A is not divisible, then we need more.

The proof below relies on Lemma 2.1 and is different from the purely combinatorial techniques of Nurtazin and other similar proofs in the literature.

PROOF SKETCH. For simplicity, suppose A has a computable basis $B = \{b_i : i \in \omega\}$ (Proposition 4.2). At a stage s, we have a finitely generated partial group D_s isomorphic to A_s, and an embedding $\psi_s : D_s \rightarrow A_s$.

We will build $\psi = \lim_s \psi_s$ a Δ^0_2 isomorphism of $D = \cup_s D_s$ onto $A = \cup_s A_s$. This *global strategy* is split into sub-strategies R_j. Each R_j attempts to define and keep ψ unchanged on the least pure subgroup $(B_j)^*$ of A containing $B_j = \{b_i : i \leq j\}$. At stage s, the *diagonalization strategy* N_e searches for an element $x \in D$ outside the pre-image U_s of $(B_e)^*_s$ so that the e-th potential dependence algorithm declares x independent of $\psi_s^{-1} B_e$. Suppose such an element x is found. The purity of $(B_e)^*$ guarantees that

$$D_s = U_s \oplus C,$$

where C is a finite partial free group, see Lemma 2.1. (Note that we are dealing with partial groups here. The reader may verify that it is not a problem though.)

Assuming the above, we have $A_s = (B_e)^*_s \oplus \psi_s C$. We then declare the generators of C equal to linear combinations of the generators of $U_s = \psi_s^{-1}(B_e)^*_s$ that use very large coefficients never seen so far in the construction. Note that we necessarily make x dependent on B_e. After this is done, we introduce new elements, put them into D_s, and correct the embedding ψ_s extending its range onto C. The construction can be organized using a finite injury priority method. $\dashv$

In contrast, if the rank of a torsion-free abelian group is finite, then we can fix its finite basis and nonuniformly find its image in every computable presentation of the group. Recall that a group is *autostable* or *computably categorical* if any two computable presentations of the group are computably isomorphic. An immediate consequence of Propositions 4.2 and 4.3 is the following:

Corollary 4.4 (Nurtazin, Dobrica). *A computable torsion-free abelian group is computably categorical if, and only if, its rank is finite.*

More can be said.

Corollary 4.5 (Goncharov). *Every computably presentable torsion-free abelian group has either one or infinitely many computable presentations, up to computable isomorphism.*

Proof. Goncharov [37, 47] showed that if an algebraic structure $\mathcal{M}$ has two computable presentations, A and B, such that $A \cong_{\Delta^0_2} B$ but[1] $A \ncong_{comp} B$, then $\mathcal{M}$ has infinitely many different computable presentations up to computable isomorphism. Propositions 4.2 and 4.3 produce two presentations of a torsion-free abelian group of infinite rank that are not computably isomorphic, but that are Δ^0_2-isomorphic. $\dashv$

4.1.2. Computable presentations vs. c.e. presentations. We use the technique of generating sets and bases to establish:

Theorem 4.6 (Khisamiev [64]). *There exists a uniform procedure that on input a c.e. presentation of a torsion-free abelian group outputs its computable presentation.*

[1] We believe that our notations are self-explanatory.

https://doi.org/10.1017/bsl.2014.32 Published online by Cambridge University Press

PROOF IDEA. The theorem fails for groups that are not torsion-free (to be stated in Corollary 5.4). We briefly explain why being torsion-free matters. At every stage, we have a finitely generated partial group C_s and an embedding of C_s into a c.e. presentation U of A. Suppose C_s is generated by $b_1, b_2, \dots, b_{k(s)}$. At a later stage we may discover that the image of $h = \sum_i m_i b_i$ in U is declared equal to 0. By Lemma 2.1, we can pick a new collection of generators $g_1, \dots, g_{k(s)}$ of C_s such that $n g_{k(s)} = h$ for some n. (Notice that here we are dealing with partial groups.) *It is crucial that the image of* $g_{k(s)}$ *in* U *must be* 0 *as well, because* U *is torsion-free.* We thus can safely dispose $g_{k(s)}$ by declaring it equal to a linear combination of the rest of the generators using sufficiently large coefficients. We then introduce a new generator $g'_{k(s)}$ which will replace $g_{k(s)}$, and re-define the embedding accordingly. Now one needs to show that the map is eventually stable, i.e., is Δ^0_2. This can be done by a careful dynamic analysis of ranks of finitely generated subgroups. We omit technical details. $\dashv$

Theorem 4.6 solves a problem left open in Baumslag, Dyer, and Miller [8]. They were interested in homologies of finitely presented groups that turned to be c.e.-presented. We mention that Khisamiev originally proved a more general fact that he implicitly stated in terms of linear dependence, see his survey paper [66]. Khisamiev's original proof was a relatively complicated combinatorial argument. Our simple proof idea based on Lemma 2.1 and its modifications (see Fuchs [44]) can be extended to this more general setting as well, but we leave it to the reader. Khisamiev [69] also strengthened the result by showing that every Σ^0_{n+1}-presented torsion-free abelian group has a Π^0_n-presentation. A further discussion of related results can be found in Downey [26] and Khismaiev [66].

4.2. Completely decomposable groups. A group G is *completely decomposable* (c.d.) if $G \cong \bigoplus_{i \in I} H_i$, where $H_i \leqq \mathbb{Q}$ for all $i \in I$. Any decomposition of G into subgroups of $\mathbb{Q}$ is called *complete*, *full*, or *elementary*, and the direct components in a complete decomposition of G are its *elementary summands*. The isomorphism types of the elementary summands of G full determine the isomorphism type of completely decomposable group G.

Baer [6] was the first to systematically study completely decomposable groups. Algebraic properties of completely decomposable groups and their pure subgroups are fairly well understood, see Mader's relatively recent book [77]. We will see that computable completely decomposable groups are neither among the "easily classifiable" objects (e.g., vector spaces) nor among the "unclassifiable" objects (e.g., linear orders). The results are intermediate in nature as well.

4.2.1. Subgroups of $\mathbb{Q}$. We first look at the degenerate case of only one elementary summand. Suppose $H \leqq \mathbb{Q}$ is a nonnull group, and let $p_0, p_1, \dots$ be the standard listing of primes. The *characteristic* of a nonzero element $h \in H$ is a sequence $(\alpha_0, \alpha_1, \dots)$, where $\alpha_i = \infty$ in case if $p_i^k | h$ for all k, and otherwise α_i is the largest k for which $p_i^k | h$ within H. We also say that α_i is

https://doi.org/10.1017/bsl.2014.32 Published online by Cambridge University Press

the p-height of h. Two characteristics $\chi = (\alpha_0, \alpha_1, \dots)$ and $\xi = (\beta_0, \beta_1, \dots)$ are *equivalent*, written $\chi \simeq \xi$, if

$$\sum_i |\alpha_i - \beta_i| < \infty.$$

The $\simeq$-equivalence class of characteristics corresponding to h is called the *type* of h, written $\mathbf{t}_H(h)$. It is clear that every two nonzero elements of H are of the same type. It thus makes sense to define the type of H, denoted by $\mathbf{t}(H)$, to be $\mathbf{t}_H(h)$ for some (equivalently, any) nonzero $h \in H$.

THEOREM 4.7 (Baer [6], after Levi [75]). *Suppose $A, B \leqq \mathbb{Q}$. Then $A \cong B$ if and only if $\boldsymbol{t}(A) = \boldsymbol{t}(B)$.*

To describe computable subgroups of $\mathbb{Q}$, we need to slightly adjust the standard invariants. Given a characteristic $\chi = (\alpha_0, \alpha_1, \dots)$, define

$$S_\chi = \{\langle i, k\rangle : \alpha_i \geq k \geq 0\}.$$

Clearly, $\chi \simeq \xi$ iff $S_\chi =^* S_\xi$, i.e., the sets agree up to a finite difference. We say that a type $\mathbf{t}$ is computably enumerable (c.e.) if for some (equivalently, for all) $\chi \in \mathbf{t}$ the set S_χ is c.e. We are ready to state:

THEOREM 4.8 (Mal'cev [80]). *Suppose $A \leqq \mathbb{Q}$ is of type $\mathbf{t}$. Then A has a computable presentation iff $\mathbf{t}$ is c.e.*

The proof of Theorem 4.8 is elementary and can be found in Melnikov [82] or Downey [26]. Given a rank 1 group that is not necessarily computable, what is the algorithmically simplest presentation of it? To answer this and similar questions, one typically uses degree spectra (Definition 2.3). Knight, Downey, Soskov, and Soskova independently observed:

FACT 4.9. *For any Turing degree $\mathbf{a}$ there exists a subgroup A of $\mathbb{Q}$ such that $DSp(A) = \{\boldsymbol{b} : \boldsymbol{a} \leq \boldsymbol{b}\}$.*

PROOF. For a set X of degree $\mathbf{a}$, encode $X \oplus \overline{X}$ (here $\oplus$ stands for the join of sets) into a type $\mathbf{t}$ by setting $\langle i, 1\rangle \in \mathbf{t}$ if $i \in X \oplus \overline{X}$, and keeping $\langle i, 0\rangle \in \mathbf{t}$ otherwise. Then use a relativized version of Theorem 4.8. ⊣

Not every rank 1 torsion-free group has a cone serving as its degree spectrum. Indeed, one can use the existence of nontotal enumeration degrees (see, e.g., Odifreddi [92]) to establish:

FACT 4.10. *There exists a subgroup A of $\mathbb{Q}$ such that $DSp(A)$ has no least element under Turing reducibility.*

Recall that $\mathbf{a}'$ stands for the *Turing jump* of a Turing degree $\mathbf{a}$ which is the degree of the halting problem for machines having access to some (any) set from $\mathbf{a}$. In contrast to Fact 4.10, Coles, Downey, and Slaman [19] discovered:

THEOREM 4.11. *For every subgroup A of $\mathbb{Q}$, the* jump degree spectrum *$DSp'(A) = \{\boldsymbol{a}' : \boldsymbol{a} \in DSp(A)\}$ of A has a least element.*

PROOF IDEA. The result follows at once from Theorem 4.8 and the computability-theoretic fact below:

PROPOSITION 4.12 (Coles et al. [19]). *For every set X, the collection $\{Y' : X \text{ is c.e. in } Y\}$ has a least element under $\leq_T$.*

https://doi.org/10.1017/bsl.2014.32 Published online by Cambridge University Press

Discussion. Soskov (see a discussion in [27]) observed that Proposition 4.12 can be derived from the theory of enumeration degrees, as follows. It is known that for every enumeration degree **a**, there exists a total enumeration degree **c** in the enumeration jump class of **a**, see Soskov [104]. One can show that $J(\mathbf{c})$ will be exactly the least jump enumeration of **a** in terms of Coles, Downey, and Slaman [19]. $\dashv$

4.2.2. A note on finite rank. It is clear that a completely decomposable group of finite rank is computably presentable if, and only if, all its elementary summands are computably presentable, and there is not much to say about these groups. Furthermore, all that is known about finite rank c.d. groups also holds for computable abelian groups of finite rank that are *not necessarily completely decomposable*. In this paragraph we briefly discuss the case of arbitrary torsion-free abelian groups of finite rank.

Mal'cev [78] gave a rather complex complete system of isomorphism invariants for torsion-free abelian groups of arbitrary finite rank in terms of sequences of matrices over p-adics. See Kurosh [73] for a statement and a proof. The invariants are so complex that algebraists are still looking for better invariants for groups of rank ≥ 2. The recent results of Thomas [106] suggest that a nice classification of such groups might not exist at all.

Dobrica [24] effectivized the above mentioned result of Mal'cev. As one would expect, computable groups correspond to effective sequences of matrices. Needless to say, the result of Dobrica is very difficult to apply. Nonetheless, some progress can be made when working with a group of finite rank directly. For instance, Theorem 4.11 can be easily generalized to torsion-free abelian groups of finite rank.

Fact 4.13 (Melnikov [82], Calvert, Harizanov and Shlapentokh [16]). *Every torsion-free abelian group of finite rank has a jump degree.*

Proof. Every torsion-free group of finite rank n be associated with a subset of a standard computable presentation of $\mathbb{Q}^n$, as follows. Pick the standard basis B_0 of $\mathbb{Q}^n$, and any basis B of the group. Given a presentation C of A, take the image B_1 of B in C and extend the embedding $B_1 \to B_0$ to an embedding ψ of C into $\mathbb{Q}^n$. Notice all isomorphic copies share the same image within $\mathbb{Q}^n$. The construction above and the usual padding imply that A has an X-computable copy if, and only if, its ψ-image is X-c.e. subgroup. The latter hold if and only if the image is an X-c.e. subset of $\mathbb{Q}^n$. It remains to apply Proposition 4.12. $\dashv$

A few further observations on finite rank torsion-free abelian groups can be found in Calvert's PhD thesis [13].

4.2.3. Homogeneous completely decomposable groups. We say that a completely decomposable group $\bigoplus_i H_i$ is *homogeneous* if all its elementary summands H_i are isomorphic. It follows at once from Theorem 4.8 that a homogeneous completely decomposable group is computably presentable if, and only if, its type is c.e.

The interesting algebraic and effective properties of such groups are related to their complete decompositions and Δ^0_n-categoricity. The nontrivial case

https://doi.org/10.1017/bsl.2014.32 Published online by Cambridge University Press

is when the rank of the group is infinite. It turns out that finding a full decomposition of a computable homogeneous c.d. group is closely related to Δ^0_α-categoricity (recall Definition 2.5).

THEOREM 4.14 (Downey and Melnikov [33]). *Every computable homogeneous completely decomposable group is Δ^0_3-categorical.*

Proof idea. The proof of Theorem 4.14 contained in [33] heavily relies on an effective analysis of full decompositions. It uses the following notion of independence:

DEFINITION 4.15. Let S be a nonempty set of primes. Then elements $g_0, \dots g_k$ of G are S-independent if for every $p \in S$ and each $m_i \in \mathbb{Z}$,

$$p \mid m_0 g_0 + \cdots + m_k g_k$$

implies $p|m_i$ for every $i \leq k$.

For $S = \emptyset$, we agree that S-independence is just linear independence. An S-basis is a maximal S-independent set. If S is a singleton $\{p\}$, we obtain the classical notions of p-independence and Kulikov basis (to be discussed in Subsection 5.1.6). An S-basis is *excellent* if it is also a maximal linearly independent set. There exist $\{p\}$-bases that are not excellent (see Downey and Melnikov [33]).

Using the notion of an excellent S-basis, we can show that every computable copy C of the given group H has a Δ^0_3-basis B such that $C = \bigoplus_{b \in B} Db$ for some fixed $D \leq \mathbb{Q}$. Clearly, having such a special Δ^0_3-basis is equivalent to saying that the group is Δ^0_3-categorical since the characteristic α of D has to be c.e. The special basis B can be chosen to be an excellent S-basis of $G_\alpha = \{g \in G : \chi(g) \geq \alpha\}$, where $S = \{p_i : \alpha_i \neq \infty\}$. This remark completes the proof idea. ⊣

One can show that Theorem 4.14 cannot be improved to Δ^0_2 in general. Furthermore, we can completely describe Δ^0_2-categoricity:

THEOREM 4.16 (Downey and Melnikov [33]). *A computable homogeneous completely decomposable group H is Δ^0_2-categorical if, and only if, the type* **t** *of H contains a characteristic χ which is an alternation of* 0 *and* ∞ *such that the positions of* 0 *in χ form a* semi-low *set.*

DISCUSSION. We briefly explain what *semi-lowness* is and how it can be used. Semi-lowness is a natural generalization of lowness. Recall that a set A is *low* if $A' = 0'$. It means that answers to Δ^A_2-questions are Δ^0_2 and thus can be effectively approximated using the Limit Lemma. A set S is *semi-low* if $\{e | W_e \cap S = \emptyset\}$ is Δ^0_2, and thus we have a similar effective approximation to Π^A_1-questions. Semi-low sets are used in the study of the automorphisms of the lattice of c.e. sets under finite difference, see Soare [102].

The proof of Theorem 4.16 is too technical to be discussed here. We mention that the proof splits into several substantially different cases depending on the isomorphism type of the group and its effective properties. ⊣

Since ω is semi-low, we have:

COROLLARY 4.17. *Every computable presentation of the free abelian group $\bigoplus_{i \in \omega} \mathbb{Z}$ admits a Δ^0_2-generating set.*

https://doi.org/10.1017/bsl.2014.32 Published online by Cambridge University Press

In Corollary 4.17, Δ_2^0 can be improved to Π_1^0 (see Downey and Melnikov [33]), but there clearly are computable copies of $\bigoplus_{i\in\omega}\mathbb{Z}$ having no computable generating base.

4.2.4. Completely decomposable groups that are not homogeneous. An explicit description of computably presentable completely decomposable groups by isomorphism invariants remains undiscovered. Nonetheless, following the usual approach of computable structure theory, we can derive a "weak" classification result in the sense described in Subsection 2.2.2. For the class of completely decomposable groups, the best that is known is:

THEOREM 4.18 (Downey and Melnikov [32]). *The index set and the isomorphism problem for completely decomposable groups are both* Σ_7^0.

The only known proof of the theorem above heavily relies on the fact that each computable completely decomposable group is Δ_5^0-categorical (same paper [32]). The latter can be derived using S-independence discussed in the previous paragraph, where S varies depending on the considered homogeneous subcomponent. The main difficulty is that these homogeneous subcomponents are not stable under automorphisms in general.

It is not yet known if Σ_7^0 is sharp. Nonetheless, using group presentations rather than constructivizations, one can show that the Δ_5^0-categoricity can not be improved to Δ_4^0-categoricity in general Downey and Melnikov [32]. Perhaps, similar techniques can be used to improve Σ_7^0 or show Σ_7^0 is sharp.

Not much is known towards the classification problem in the usual sense (i.e., finding an explicit description of the isomorphism types by algebraic and computability-theoretic invariants). In fact, the problem seems to be unexpectedly challenging even in the algebraically simplest cases. For instance, Khisamiev in the late 1990's asked for which sets S of primes the group

$$\bigoplus_{p\in S}[\mathbb{Z}]_p,$$

has a computable (decidable) presentation; here $[\mathbb{Z}]_p$ is the localization of the integers by p. Only recently Downey, Goncharov, Kach, Knight, Kudinov, Melnikov, and Turetsky [28] have discovered the exact answer:

THEOREM 4.19 (Downey et al. [28]). *The group* $\bigoplus_{p\in S}[\mathbb{Z}]_p$ *has a computable* (*decidable*) *presentation if, and only if,* S *is* Σ_3^0 (*respectively*, Σ_2^0).

The proof contained in [28] is one of the rare applications of infinite injury priority method in computable commutative algebra. The theorem above admits a generalization to a slightly broader class of completely decomposable groups (see the same paper [28]). Recently, Riggs (unpublished) has announced an extension of Theorem 4.19 to the case of finitely many symbols ∞ in each characteristic (the rest symbols are 0). Nonetheless, the general case is still an open problem.

4.2.5. Effective complete decompositions. Khismaiev [67, 70] suggested that it is natural to restrict ourselves to those computable presentations of completely decomposable groups that have an algorithm for a complete decomposition.

https://doi.org/10.1017/bsl.2014.32 Published online by Cambridge University Press

Definition 4.20. We say that a computable completely decomposable group A of infinite rank is *effectively completely decomposable* or *strongly decomposable* if it has a computable presentation of the form $\bigoplus_{b \in B} H_b b$, where B is a c.e. set.

Similarly to classifying computable completely decomposable groups, describing effectively completely decomposable groups seems to be a challenging task. The following result shows that such studies require new computability–theoretic notions.

Theorem 4.21 (Khisamiev [67]). *A group of the form $\bigoplus_{p \in S}[\mathbb{Z}]_p$ is effectively completely decomposable if, and only if, the set of primes S is Σ^0_2 and not quasi-hyperhyperimmune.*

As before, $[\mathbb{Z}]_p$ stands for the localization of the integers by a prime p. The technical notion of a quasi-h.h.-immune set can be found in [67]. Khismaiev showed that each h.h.-immune set is quasi-h.h.-immune, but that the converse fails.

A further discussion of effectively completely decomposable groups can be found in Downey and Melnikov [32]. Effectively completely decomposable groups have recently been used in the study of degree spectra of linear orders on computable abelian groups (to be discussed in Subsection 6.2.4).

4.3. Computable torsion-free abelian groups in general. As we mentioned at the beginning of the section, there is no satisfactory classification of countable torsion-free abelian groups. The recent results of Thomas [106] and Hjorth [58] suggest that countable torsion-free abelian groups of rank ≥ 2 might have no nice invariants at all. Computable torsion-free abelian groups can be very complicated as well. This subsection contains mostly "negative" results, i.e., results illustrating that computable torsion-free abelian groups are unclassifiable.

4.3.1. A method of constructing torsion-free abelian groups. Let $\Gamma = (V, E)$ be a countable graph. Suppose further that both edges and vertices of T are labeled by multisets of primes with at most countable multiplicity of elements. Write M_y for the multiset of primes that label $y \in V$, and write $M_{\{u,w\}}$ for the multisetlabeling $\{u, w\} \in E$. For a $p \in M_s$, where $s \in V \cup E$, write $r(M_s, p)$ for the multiplicity of p in the multiset M_s.

Definition 4.22. Let Γ be a countable graph labeled by multisets of primes. Define $A(\Gamma)$ to be the least subgroup of $\bigoplus_{v \in V} \mathbb{Q}v$ such that:

1. $p^{r(M_v,p)}|v$ for each $v \in V$;
2. $p^{r(M_{\{v,w\}},p)}|(u + w)$ for very $\{u, w\} \in E$.

The group $A(\Gamma)$ is clearly torsion-free. We may well have $\Gamma \not\cong \Xi$ but $A(\Gamma) \cong A(\Xi)$.

Remark 4.23. Note that the underlying tree Γ can be associated with a special basis of $A(\Gamma)$.

Hjorth [58] was the first to use this approach to "code" information into the isomorphism type of a countable group. The method was known to pure

https://doi.org/10.1017/bsl.2014.32 Published online by Cambridge University Press

algebraists long before Hjorth, but it had been used for different purposes, e.g., for constructing indecomposable groups, see Fuchs [45]. In the next paragraphs we will use this method to build complicated *computable* groups.

4.3.2. The isomorphism problem. Recall that in Subsection 2.2.2 we defined the isomorphism problem for a class K to be the set

$$E(K) = \{(i, j) : M_i, M_j \in K \text{ and } M_i \cong M_j\}.$$

Downey and Montalban [36] adjusted the construction of Hjorth [58] to obtain the following:

THEOREM 4.24 (Downey and Montalban [36]). *The isomorphism problem for computable torsion-free abelian groups is Σ^1_1-complete.*

Intuitively, Σ^1_1-completeness means that here is no simpler uniform way of checking if $A \cong B$ rather than just asking if there exists an isomorphism between A and B.

PROOF SKETCH. The key algebraic idea is rather straightforward. Given a rooted countable tree $T \subseteq \omega^{<\omega}$, label $\sigma \in T$ by $p_{2|\sigma|}$ with multiplicity ∞, and label an edge $\{\sigma, \sigma^+\}$ by $p_{2|\sigma|+1}$ with multiplicity ∞. We then use Definition 4.22 to obtain a torsion-free $A(T)$. Clearly, the process is effective in T. Let F be the tree which consists only of the infinite chain. Then T has a path iff $A(F) \leqq A(T)$. As a consequence, if T_0 is well-founded and T_1 is not, then $A(T_0) \ncong A(T_1)$. We would like to apply the well-known result of Harrison [53] about Σ^1_1-completeness of well-foundness for computable trees. To complete the proof we need an effective procedure that homogenizes trees that are *not* well-founded, so that we obtain $A(T_0) \cong A(T_1)$ for such trees. The latter can be done as in Goncharov and Knight [50]. ⊣

Theorem 4.24 implies that the isomorphism problem for torsion-free abelian groups is not hyperarithmetical. The latter had been earlier announced by Calvert [13], but his proof was incomplete.

4.3.3. An injectivity result and its application. The restricted functor of Definition 4.22 used in the proof of Theorem 4.24 is not injective (Melnikov [83]). Nonetheless, using a relatively technical combinatorial argument Fokina, Knight, Melnikov, Quinn, and Safranski [40] showed:

THEOREM 4.25 (Fokina et al. [40]). *The functor from the proof of Theorem* 4.24 *is injective when restricted to the class* rank homogeneous *trees.*

We do not give the formal definition of a rank homogeneous tree, we only note that such trees serve as technical tools for constructing structures of high Scott rank. See Calvert, Knight, and Millar [17] for more details. We also note that the proof of Theorem 4.25 that is contained in Fokina et al. [40] has a rather confusing but easily fixable flaw. A corrected proof can be found in the author's PhD thesis [84].

Theorem 4.25 stated above has recently been applied to obtain another strong evidence that computable torsion-free abelian groups are not classifiable in general. To state the result, we need a few more definitions.

https://doi.org/10.1017/bsl.2014.32 Published online by Cambridge University Press

Let E, R be equivalence relations on hyperarithmetical sets. Following Fokina et al. [39], we write $E \leq_{tc} R$ if $(x, y) \in E \Leftrightarrow (f(x), f(y)) \in R$ for a partial computable f defined on the domain of E. Using $\leq_{tc}$, we can define the notion of $\leq_{tc}$-completeness among members of a class as usual. Using the enumeration of all partial computable structures (Subsection 2.2.2), we measure the complexity of the isomorphism relation on a class using *tc* reducibility. For torsion-free abelian groups, we obtain:

THEOREM 4.26 (Fokina et al. [39]). *The isomorphism relation on computable torsion-free abelian groups is tc-complete among Σ^1_1 equivalence relations.*

PROOF SKETCH. It is clear that the isomorphism relation on computable torsion-free abelian groups is Σ^1_1. The isomorphism relation for rank-saturated trees is tc-complete among Σ^1_1 equivalence relations; see Fokina et al. [39] for a definition and a proof. What we need to know is that rank saturated trees are rank homogeneous. Since the coding from Theorem 4.24 is effective and 1-1 for rank homogeneous trees (Theorem 4.25), the theorem follows. ⊣

4.3.4. Degree spectra of torsion-free groups. It is not known if torsion-free abelian groups realize fewer degree spectra than arbitrary countable structures. We list below all that is currently known about degree spectra of torsion-free abelian groups.

Theorem 4.11 states that for every torsion-free abelian group A of rank 1, its *jump degree spectrum* $DSp'(A) = \{\mathbf{a}' : \mathbf{a} \in DSp(A)\}$ has the smallest element. We also discussed in Subsection 4.2.2 that the same result holds for groups of any finite rank. What about the case of infinite rank? As a corollary of the next result, the answer is negative even for completely decomposable groups.

THEOREM 4.27 (Melnikov [82]). *There exists a completely decomposable abelian group whose degree spectrum is $\{X : X' >_T 0'\}$ (i.e., the nonlow degrees).*

PROOF SKETCH. Given a finite set S, take the type $\mathbf{t}_S$ containing $(h_i)_{i\in\omega}$, where $h_i = 0$ if $D_i \subseteq S$ and $h_i = \infty$ otherwise. Let A_S be the subgroup of $\mathbb{Q}$ of type $\mathbf{t}_S$. Given a family of finite sets R, take

$$G_R = \bigoplus_{i\in\omega} \bigoplus_{S\in R} A_S.$$

We may assume that $\emptyset \in R$. Then we claim that G_R has an X-computable presentation if and only if R has a uniform Σ^X_2-enumeration (a little proof is required). To complete the theorem, relativize the result of Wehner [109] to obtain a family of finite sets that has no Σ^0_2-enumeration yet for every X with $X' >_T 0'$ has a Σ^X_2-enumeration. ⊣

We may apply the Turing jump operator to the degree spectrum of A, iterate this procedure over computable ordinals and seek for an α least so that $DSp^{(\alpha)}(A) = \{\mathbf{a}^{(\alpha)} : \mathbf{a} \in DSp(A)\}$ is just a cone above $\mathbf{b}$; if such an

α and a degree $\mathbf{b}$ can be found, then $\mathbf{b}$ called the (proper) *α'th jump degree* of the corresponding group. Andersen, Kach, Melnikov, and Solomon [1] proved:

THEOREM 4.28 (Andersen et al. [1]). *For every computable ordinal α there exists a torsion-free abelian group having a proper α'th jump degree.*

PROOF IDEA. As it follows from a computability-theoretic result of Ash, Jockusch, and Knight [3], it is sufficient to prove:

PROPOSITION 4.29. *For every Turing degree $\mathbf{a}$ and every infinite set S, a group $G(S)$ such that $G(S)$ has an $\mathbf{a}$-computable copy if, and only if, S is $\Sigma^{\mathbf{a}}_{\alpha}$.*

We discuss the proof of Proposition 4.29. The construction of $G(S)$ uses the functor from Definition 4.22. Using a transfinite inductive scheme, we define a computable sequence of infinite labeled trees naturally reflecting Σ^0_α- and Π^0_α-outcomes. We then uniformly produce groups G_i, each encoding a Σ^0_α- outcome if $i \in S$, and a Π^0_α-outcome if $i \notin S$. We then take a new fresh infinite computable set of primes $\{w_j\}$, and make G_i infinitely divisible by w_i; written $[G_i]_{w_i}$. We let $G(S) = \bigoplus_{i\in\mathbb{N}}[G_i]_{w_i}$. The main technical difficulty of the proof is reconstructing the isomorphism types of these specific labeled trees from an isomorphism type of the group. Not every choice of the coding components would do the job, as it is explained in Andersen et al. [1]. An effective reconstruction of S from the isomorphism type of $G(S)$ requires a development of a new machinery extending the earlier ideas contained in Hjorth [58], Downey and Montalban [36], and Fokina et al. [40]. The proof is too combinatorially involved to be further described here. $\dashv$

Despite of the mentioned above technical difficulties, we expect that the method used in the proof of Proposition 4.29 will find more applications in the future.

We also note that Proposition 4.29 is of some independent interest. It says that a computable torsion-free abelian group can have an arbitrarily complex hyperarithmetical set as its full isomorphism invariant. Thus, Proposition 4.29 itself is an anti-structure result in the sense of Goncharov and Knight [50].

4.3.5. Direct decompositions of torsion-free abelian groups. Recall that a group is *indecomposable* if it is not a direct sum of two nonnull groups. The standard construction of indecomposable groups of ranks $n \leq \omega$ uses the functor from Definition 4.22, and the classical examples of countable indecomposable groups are in fact computable. More can be said:

THEOREM 4.30 (Riggs [95]). *The index set of directly indecomposable abelian groups is Π^1_1-complete.*

Although the construction of Riggs [95] uses Definition 4.22 and is similar to those discussed in Subsection 4.3.4, its verification is less tricky since it uses decomposability analysis (e.g., Fuchs [45]) rather than definability analysis. The construction itself is more involved though. When restricted to finite rank groups, this index set becomes arithmetical (Riggs [95]).

§5. p-Groups. As it follows from Fact 3.5, the study of computable torsion abelian groups often (but not always) can be reduced to the study of p-groups. Countable abelian p-groups can be fully described up to isomorphism by their Ulm invariants and the dimension of their divisible component (to be discussed, see also Fuchs [44]). Nonetheless, from the algorithmic point of view these invariants are too complicated; the difficulty is rooted in the ability to compute ordinal p-heights of elements within a group. As a consequence, we should not hope for any reasonable classification of computable abelian p-groups up to isomorphism. However, we can fully describe, up to isomorphism, computable p-groups of finite Ulm type using algebraic and computability–theoretic invariants. Remarkably, already for the elementary case of sums of cyclic groups we need a new computability–theoretic notion.

In Subsection 5.1 we discuss the algorithmic content of direct sums of cyclic and quasi-cyclic p-groups. In Subsection 5.1.1 we look at computable sums of cyclic p-groups, and in Subsection 5.1.2 we study degree spectra of sums of cyclic p-groups. Quasi-cyclic summands first come into consideration in Subsection 5.1.3. In Subsection 5.1.4 we give an algebraic criterion for an arbitrary computable abelian p-group to be computably categorical; all such groups are direct sums of cyclic and quasi-cyclic summands. Categoricity relative to an oracle is discussed in Subsection 5.1.5, and Subsection 5.1.6 contains a result on Kulikov bases.

In Subsection 5.2 we introduce and apply a rather useful method of p-basic trees. Algebra necessary for the method is contained in Subsection 5.2.1, and the central computability–theoretic lemma about p-basic trees is stated and sketched in Subsection 5.2.2. In Subsection 5.2.3, we use the lemma to characterize computable reduced abelian p-groups of finite Ulm type. In the same paragraph we also discuss the case of Ulm type ω.

In Subsection 5.3 we consider computable abelian p-groups without any further restriction on their isomorphism type. In Subsection 5.3.1 we show that computable p-groups are "unclassifiable". In Subsection 5.3.2 we discuss Δ^0_α-categoricity for higher α.

5.1. Direct sums of cyclic and quasi-cyclic groups. We will need the following computability-theoretic notion.

Definition 5.1 (Khisamev [57], Ash et al. [5], Khoussainov et al. [71]). A total function $F : \omega \to \omega$ is *limitwise monotonic* if there is a computable function $f(x, y)$ of two arguments such that

$$F(x) = \sup_y f(x, y),$$

for every $x \in \omega$.

An infinite set is *limitwise monotonic* if it contains an infinite range of a limitwise monotonic function. Without loss of generality, we could also assume that the whole set is the range of an injective limitwise monotonic function, see e.g. Harris [52] and Kalimullin et al. [60].

5.1.1. Computable and c.e. presentations of sums of cyclic groups. Suppose a p-group A is isomorphic to $\bigoplus_i C_i$, where C_i are cyclic groups. Clearly, each C_i is isomorphic to Z_{p^k} for some k. Define the *characteristic* of A to be the set

$$S(A) = \left\{ \langle m, k \rangle : \bigoplus_{i \le m} Z_{p^k} \text{ is a direct summand of } A \right\},$$

and also consider its projection onto the second coordinate

$$\#A = \{k : \langle 1, k \rangle \in S(A)\}.$$

Limitwise monotonic functions describe computably presentable p-groups that are countable sums of cyclic groups.

Proposition 5.2 (Khisamiev [57]). *Suppose A is a direct sum of cyclic p-groups. Then A is computably presentable if, and only if, one of the following holds*:

1. *$\#A$ is finite, or*
2. *$\#A$ is infinite and limitwise monotonic, and $S(A)$ is Σ^0_2.*

Proof idea. The less elementary case is when $\#A$ is infinite. Suppose A is computable. We show that (2) holds. Recall the definition of p-height: $h_p(a) = \sup\{k : p^k | a\}$, allowing $h_p(a) = \infty$. We use $O(a)$ to denote the order of a. Notice that subsets $V_k = \{a \in A : O(a) = p \text{ and } h_p(a) \ge k\}$ are computably enumerable uniformly in k. Whence, $0'$ can uniformly enumerate bases of $\mathbb{Z}_p$-vector spaces V_k/V_{k+1}, showing that $S(A)$ is Σ^0_2. To see why $\#A$ is limitwise monotonic, note that our guess on the p-height of a chosen $a \in A$ of order p can only increase. At stage s, define $f(0, s)$ equal to our guess on $h_p(a) + 1$ of the first found nonzero a of order p. To define $f(1, s)$, pick b least in $V_{f(1,s)+1}$ of p-height $\ge f(1, s)$ and set $f(1, s) = h_p(b) + 1$, etc. The range of $\sup_s f(x, s)$ is an infinite set contained in $\#A$ witnessing that $\#A$ is limitwise monotonic.

Now suppose S is Σ^0_2 and contains the infinite range of $F(x) = \sup_y f(x, y)$, where f is computable. Then produce a copy of A as follows. Suppose at stage s we have enumerated only finitely many summands of the form Z_{p^z}, for various $z \le s$. If at stage $s + 1$ we discover that one of these summands has to be removed according to the new guess on $S(A)$, we take a large and fresh $c > z$ such that $u(c, s) = sup_{y \le s} f(c, y) > k$ and expand Z_{p^k} to $Z_{p^{u(c,s)}}$. We re-introduce a summand of size Z_{p^z} using new fresh generators. One can see that a summand isomorphic to Z_{p^z} will occur in the resulting group if and only if $z \in S$. Note that we also have to dynamically control the number of the cyclic summands. This remark completes the sketch. ⊣

A direct consequence of the proof of Proposition 5.2 is that every computable p-group that splits into a direct sum of cyclic groups has a presentation with a *computable* decomposition into cyclic summands.

We establish the following fact:

Proposition 5.3 (Khisamiev [69]). *Suppose $A \cong \bigoplus_{k \in S} Z_{p^k}$. Then A is c.e. presentable if, and only if, $S(A)$ is Σ^0_2.*

Proof. It is clear that any group of this form with $S(A) \in \Sigma^0_2$ has a c.e. presentation. Indeed, at stage s, keep exactly m copies of Z_{p^k} for the largest m such that $\langle n, k\rangle \in S(A)[s]$ for each $n \leq m$.

For the other direction, suppose $A = F_1/F_2$, where F_1 is a (computable) free abelian group, and F_2 is its c.e. subgroup. For every $k > 1$, define

$$U_k = \{x \in F_1 : px = 0 \bmod F_2 \;\&\; (\exists b \in F_1)\, p^k b = x \bmod F_2\}$$

and

$$V_k = \{x \in U_k : (\exists c \in F_1)\, p^{k+1} c = x \bmod F_2\}.$$

Both U_k and V_k are c.e. subgroups of F_1. We can pass to a c.e. presentation of U_k/V_k using padding. Since $F_2 \leqq V_k \leqq U_k$, by the Third Isomorphism Theorem

$$U_k/V_k \cong (U_k/F_2)/(V_k/F_2).$$

The abelian group $(U_k/F_2)/(V_k/F_2)$ is isomorphic to a vector space over $\mathbb{Z}_p$ whose dimension is equal to the number of $\mathbb{Z}_{p^k}$-components in A. This number is equal to $\sup\{m : \langle m, k\rangle \in S(A)\}$. Using $0'$ as an oracle, we can enumerate a $\mathbb{Z}_p$-basis of U_k/V_k, and thus approximate the $k'th$ column of $S(A)$. ⊣

Since there exist infinite Σ^0_2-sets that are not limitwise monotonic (Khismaiev [66], Khoussainov, Nies and Shore [71]), Propositions 5.2 and 5.3 imply:

Corollary 5.4 (Khisamiev). *There exists a c.e. presented p-group without elements of infinite height having no computable presentation.*

The corollary contrasts Theorem 4.6.

Corollary 5.5. *There exists a $0'$-computable p-group without elements of infinite height that has no c.e. presentation.*

Proof of Corollary. By the previous, it is sufficient to construct a Σ^0_3 set that is not Σ^0_2 but is the range of a function that possesses a $0'$-computable limitwise monotonic approximation. One can easily construct a Σ^0_2 (indeed, d-c.e.) set that is limitwise monotonic but is not Σ^0_1, and then relativize the construction to $0'$. ⊣

For a further analysis of Σ^0_n and Π^0_n-presentations of direct sums of cyclic groups, see Khisamiev [66].

5.1.2. Degree spectra of sums of cyclic groups. We open this paragraph with a general result that holds for arbitrary abelian p-groups. We say that a structure A satisfies the *effective extendability condition* (Richter [94]) if for every finite structure F isomorphic to a substructure of A, and every embedding $\phi : F \to A$, there is an algorithm that determines whether a given finite structure N extending F can be embedded into A by an embedding extending ϕ. Richter [94] showed that if a structure satisfies the effective extendability property, then its degree spectrum has no least element unless the structure is computable.

Proposition 5.6 (Khisamiev, Jr. [62]). *Countable abelian groups satisfy the effective extendability condition.*

https://doi.org/10.1017/bsl.2014.32 Published online by Cambridge University Press

Proof idea. It is sufficient to observe that, if a group A has arbitrarily large cyclic summands or its divisible subgroup has infinite rank, then extendability of an embedding

$$\phi : F \to A$$

of a finite F is completely regulated by the collection of p-heights of elements in $\phi(A)$ compared to their p-heights in $C \supset F$. If A does not have arbitrarily large cyclic summands and its divisible subgroup has finite rank, then we use the dimension of the divisible part, the highest order p^m of a cyclic summand, and also the dimensions of $\oplus_j \mathbb{Z}_p^k$ that detach as summands of A (here $k \leq m$) as nonuniform parameters. A formal proof can be reconstructed using only the first half of Kaplansky's little book [61]. $\dashv$

Corollary 5.7. *No countable, noncomputable abelian p-group has a degree.*

In particular, if a direct sum of cyclic p-groups has a degree, it must be $\mathbf{0}$ (compare with Fact 4.9). We would like to know which collections of degrees *can be realized* as degree spectra of direct sums of cyclic p-groups.

Proposition 5.2 reduces the study of degree spectra of groups that split into direct sums of cyclic groups to the study limitwise monotonicity relative to an oracle. Working directly with limitwise monotonic sets, Kalimullin, Khoussainov, and Melnikov [60] showed:

Theorem 5.8 (Kalimullin et al. [60]). *There exists an abelian p-group A such that A has an X-computable presentation relative to any noncomputable Δ^0_2-oracle, but does not possess a computable presentation. In fact, A splits into a direct sum of cyclic groups.*

The result resembles a similar fact for linear orders, see Russell Miller [88]. The proof of Theorem 5.8 combines the method of Δ^0_2-permitting with some specific properties of limitwise monotonicity relative to an oracle. In contrast, we have:

Theorem 5.9 (Kalimullin et al. [60]). *Let A be a direct sum of cyclic groups. Suppose A has a computable presentation relative to every degree except perhaps countably many degrees. Then A has a computable presentation.*

For instance, no group of this form may have the degree spectrum consisting exactly of noncomputable degrees. Countable structures with such spectra surprisingly exist as it was shown by Slaman [100] and Wehner [109]. The proof of Theorem 5.9 is a computability–theoretic forcing argument.

5.1.3. Computable sums of cyclic and quasi-cyclic summands. The quasi-cyclic p-group, or the Prüfer p-group, is the abelian group

$$\mathbb{Z}_{p^\infty} = \langle a_i, i \in \mathbb{N} | \, pa_0 = 0, pa_{i+1} = a_i : i > 0 \rangle.$$

These groups play a special role in abelian group theory since every divisible p-group splits into a direct sum of Prüfer p-groups. As the name suggests, algebraic properties of quasi-cyclic p-groups resemble properties of cyclic p-groups, see Fuchs [44]. In the following, the rank of a divisible p-group is the number of quasi-cyclic components in its full decomposition. The proposition below and its corollaries were known to Khisamiev.

PROPOSITION 5.10. *Suppose $A = R \oplus D$, where R is a direct sum of cyclic p-groups, and D is a direct sum of quasi-cyclic groups. Then*:

1. *If D has finite rank, then A has a computable presentation if and only if R has a computable presentation.*
2. *If D has infinite rank, then A has a computable presentation if and only if R has a c.e. presentation.*

PROOF SKETCH. For (1), note that every element of order p in A, except finitely many, has a nonzero projection onto R. Similarly to how it was done in the proof of Proposition 5.2, we can use these elements to show $S(R)$ is Σ_2^0 and obtain a limitwise monotonic function for $\#R$. Then the other direction of Proposition 5.2 implies (1) of the fact.

For (2), we can closely follow the proof of Proposition 5.3 and show $S(R)$ is Σ_2^0. By Proposition 5.3, R has a c.e. presentation. Now suppose R is c.e. presentable. A simple construction of a computable produces a copy of A using the Σ_2^0 set $S(R)$. ⊣

COROLLARY 5.11. *Suppose $A = R \oplus D$ is a c.e. presented group, where R is a direct sum of cyclic p-groups, and D is a direct sum of infinitely many quasi-cyclic groups. Then A has a computable presentation.*

The proof of the corollary above boils down to showing that $S(R)$ is Σ_2^0. Another corollary follows at once from Corollary 5.4 and Proposition 5.10(2)

COROLLARY 5.12. *There exists a computable abelian group whose reduced component has no computable presentation.*

Goncharov asked whether the corollary above holds for torsion-free abelian groups. Khisamiev and Khisamiev [69] claimed that the answer is negative. Nonetheless, the proof contained in Khisamiev and Khisamiev [69] seems incomplete[2]. As it stands, the question of Goncharov is still open.

5.1.4. Computable categoricity of p-groups. As a consequence of the following result, every computably categorical computable p-group is necessarily a direct sum of cyclic and quasi-cyclic p-groups. We choose to state it here.

THEOREM 5.13 (Smith [101], Goncharov [46]). *A computable abelian p-group A is computably categorical if and only if either $A \cong R \oplus D$ where D is divisible and F finite, or $G = D \oplus F \oplus \bigoplus_{i \in \omega} \mathbb{Z}_{p^m}$ where D is divisible of finite rank and F is finite.*

PROOF IDEA. Note that the result is stated incorrectly in Smith [101], while the proof of the most nontrivial case is missing in Goncharov [46]. However, an analysis of this case is contained in Smith [101]. This less elementary case is when A has arbitrarily large cyclic summands. We informally describe the main strategy in this case.

[2]More specifically, it is not clear how Lemma 4 of [69] helps in producing a desired single-valued computable enumeration, since the latter heavily relies on a specific presentation of the p-adics.

We construct a copy B of the group A and diagonalize against $\varphi_e : B \to B$ as follows. We pick an element a_0 in A_s, and make sure that $h_p^B(\varphi_e(a_0)) = h_p^A(a_0)+k$ for $k > 0$ using that A has arbitrarily large cyclic summands. The crucial subtlety is that the witness may well have infinite height, in which case the diagonalization is unsuccessful. If the height of the witness increases, i.e., $h_p^{A_{s+1}}(a_0) > h_p^{A_s}(a_0)$, then we pick a new witness a_1, but keep the previously defined witnesses a_0 as well. We pick s new a_2 only if the p-heights of both a_0 and a_1 have increased. Since we always pick a smallest possible witness, we eventually find a stable one of finite height. We can then split the main strategy into sub-strategies, each guessing the behavior of the $i'th$ witness' height, and then put them onto a (dynamic) tree of strategies. ⊣

Corollary 5.14 (Goncharov [46]). *Every computable abelian p-group has either one or infinitely many computable presentations up to computable isomorphism.*

Discussion. If a computable abelian p-group is not computably categorical, then the main strategy from the proof of Theorem can be used to produce infinitely many copies of the group. We leave details to the reader. ⊣

5.1.5. Categoricity relative to an oracle. Recall the notion of Δ_n^0-categoricity (Definition 2.5). It is not difficult to show that every p-group that is a direct sum of cyclic and quasi-cyclic groups is Δ_3^0-categorical. Calvert, Cenzer, Harizanov, and Morozov [15] raised a surpassingly challenging question of which direct sums of cyclic and quasi cyclic p-groups are Δ_2^0-categorical. Recently, Downey, Ng, and Melnikov [34, 35] announced several results that partially answer the question. For instance, the proposition below answers one of the two more specific questions left open in Calvert et al. [15]:

Proposition 5.15 (Downey, Melnikov, and Ng [35]). *Let A be a direct sum of cyclic p-groups and finitely many quasi-cyclic p-groups. Then A is Δ_2^0-categorical if and only if the orders of the cyclic summands are bounded.*

The proof of Proposition 5.15 relies on the technique of p-basic trees that will be explained in the next section. Modulo this technique, the proof is not difficult. In fact, using the technique of p-basic trees, we can lift the result to groups of arbitrary finite Ulm type.

In contrast, one needs a much more combinatorially involved analysis when dealing with the case of infinitely many quasi-cyclic summands. Computable Δ_2^0-categorical groups of this kind exist and seem difficult to explicitly describe. Although Downey, Melnikov, and Ng [34] contains a good deal of partial information about Δ_2^0-categorical groups of this kind, a complete description of Δ_2^0-categorical groups in this class remains undiscovered.

5.1.6. Kulikov basis. The notion of p-independence is the special case of S-independence that was defined in §4.2.3. We say that elements $a_0, \dots a_k$ are p-independent if $p^n | \sum_i m_i a_i$ implies $\bigwedge_i p^n | m_i$, for any choice of integers n and m_i. A maximal p-independent set of a group is called Kulikov basis of the group. The subgroup generated by a Kulikov basis is called a Kulikov subgroup of a p-basic subgroup of the group. A p-basic subgroup is always

https://doi.org/10.1017/bsl.2014.32 Published online by Cambridge University Press

a direct sum of cyclic groups. A detailed exposition of p-independence in the context of abelian p-groups can be found in Fuchs [44].

PROPOSITION 5.16 (Khisamiev [63]). *A computable presentation of an abelian p-group has a c.e. Kulikov basis if, and only if, the unary predicates* $p^k|$ *for* p^k*-divisibility are computable uniformly in k in this presentation.*

PROOF SKETCH. Suppose the unary predicates $p^k|$ are uniformly computable in a computable A. Then the basis is built by stages. At the first stage we pick the first found b such that $h_p = 0$, and $p^{k-1} = a$ for some a with the properties $pa = 0$ and $h_p(a) = k - 1$. Since $\langle b \rangle$ is pure in A, and pure cyclic subgroups detach, we have

$$A = \langle b \rangle \oplus C_1$$

for some C_1. The main difficulty in the construction is that C_1 is defined merely up to isomorphism, so at the next stage we will need to work modulo $\langle b \rangle$. We also have to calculate the p-height mod $\langle b \rangle$. We then iterate. Algebra needed for implementing this idea does not go beyond the first half of Kaplansky [61].

For the converse, suppose $(b_i)_{i \in I}$ is a c.e. Kulikov basis of A. Note that the factor-group $A/\langle b_i : i \in I \rangle$ is divisible (see Fuchs [44]). To verify whether $p^k|x$, find a presentation

$$x = \sum_i m_i b_i + p_k a,$$

where $a \in A$. From the definition of p-independence, we obtain that $p^k|x$ if, and only if, $p^k|m_i$ for each i. ⊣

Theorem 3.3 and Proposition 5.16 imply:

COROLLARY 5.17. *A computable abelian group A is decidable if, and only if, Th(A) is decidable and A has a computable Kulikov base.*

Theorem 3.3 and Proposition 5.16 are central in the proof of another interesting result that expresses decidability of a presentation in terms of *quasibasis*. Since decidable presentations are not in the scope of this survey, and since the statement is rather technical, we do not state the result. See Khisamiev [63, 66] for a formal statement and a proof.

5.2. Ulm invariants, and p-basic trees. In this subsection we discuss the technique of p-basic trees and give a proof sketch of one very important lemma that has various applications.

5.2.1. p-Basic trees. In mathematical practice, using tree-like diagrams representing abelian p-groups is rather common.

DEFINITION 5.18 (L. Rogers [97]). A p-basic tree is a set X together with a binary operation $p^n \cdot x$ of the sort $\{p^n : n \in \omega \setminus \{0\}\} \times X \to X$ such that:

(1) there is a unique element 0 in X for which $p \cdot 0 = 0$,
(2) $p^k \cdot (p^m \cdot g) = p^{k+m} \cdot g$, for every $g \in X$ and $k, m \in \omega$, and
(3) for each nonzero element x in X, there is a positive integer n such that $p^n \cdot x = 0$.

https://doi.org/10.1017/bsl.2014.32 Published online by Cambridge University Press

A p-basic tree can be visualized as a rooted tree, with 0 the root, as long as p is fixed. Given a p-basic tree X, we can pass to an abelian p-group $G(X)$ as follows. We use $X \setminus \{0\}$ as the set of generators, and put $px = y$ into the collection of relations if $p \cdot x = y$ in X. In fact, *every countable abelian p-group is generated by some p-basic tree*, see Rogers [97].

Remark 5.19. Every element of a p-group generated by a tree T can be uniquely expressed as $\sum_{v \in T} m_v v$, where $m_v \in \{0, 1, \dots, p-1\}$. Thus we are dealing with a special notion of independence. Also, the tree gives a way to control direct decompositions of the group.

Nonisomorphic rooted trees may give rise to isomorphic p-groups. The following operation gives a combinatorial description of trees corresponding to isomorphic groups. Suppose T is a rooted tree (a p-basic tree). We can "strip" T by detaching a simple chain of vertices from its original source node v, and then attaching the chain to the root of T. If the *tree rank*[3] of v, written $rk(v)$, does not change after this "stripping" then we get a tree corresponding to an isomorphic group. We can also strip the tree replacing infinitely many chains at once, as long as all ranks are preserved. Trees give rise to isomorphic groups if, and only if, they are equivalent up to stripping.

We will take for granted that any countable abelian p-group can be represented by some p-basic tree (Rogers [97]). In the next few lines we will define Ulm rank and Ulm type using p-basic trees. This is somewhat circular, since the proof of Rogers [97] requires a definition of Ulm rank without any reference to p-basic trees. See Kaplansky [61] and Fuchs [44] for a direct approach that does not refer to trees. This direct approach is equivalent to our approach.

Using any tree T representing an abelian p-group G, we define the Ulm type of G, and of T, as follows. Define $T' = \{v \in T : rk(v) \geq \omega \text{ or } rk(v) = \infty\}$, and define T^α by transfinite induction in the obvious way by taking an intersection at every limit stage. The least α such that $T^\alpha = T^{\alpha+1}$ is independent of the choice of T and is called the Ulm type of G. For example, every p-group that is a direct sum of cyclic and quasi-cyclic p-groups has Ulm type 1, and there are countable p-groups of arbitrarily large countable Ulm type. The Ulm factors $A_\alpha = A^{(\alpha)}/A^{(\alpha+1)}$ are isomorphism invariants of A that completely determine its isomorphism type. Note that $A_\alpha = A^{(\alpha)}/A^{(\alpha+1)}$ is a direct sum of cyclic groups.

5.2.2. Computable p-basic trees. So far, the lemma below has been the main technical tool in the study of computable p-groups. It is the crucial step in the proof of a generalization of Proposition 5.2 to any finite Ulm type (Theorem 5.21 below). Khisamiev [65] was the first to realize that there should be a way of passing from a Π^0_2-presented p-group H to a computable group G with $G' \cong H$ (according to [65], this approach was suggested to Khisamiev by Goncharov). Independently and slightly later, Ash, Knight and Oates [5] came up with a very clear procedure of passing from a Π^0_2

[3]The rank of 0 and of any node on an infinite path is ∞.

p-basic tree T to a "right" computable p-basic tree Γ (to be clarified in Lemma 5.20 below). Their paper [5] has never been published. All known proofs of the lemma below are too technical to be fully explained here. A sketch of Lemma 5.20 can be found in Ash and Knight [4], and see also Downey, Melnikov and Ng [35] for an extended sketch. We give only a proof idea.

Lemma 5.20 (Ash, Knight and Oates [5]). *Let T be a computable p-basic tree of Ulm type 1 in which 0 has tree-rank ω, and let C be any Π^0_2 subtree of $\omega^{<\omega}$ (C is viewed as a p-basic tree). There exists a computable p-basic tree U expanding C such that $U_0 \cong T$ and $U' = C$.*

Proof idea. It follows from Proposition 5.2 that there exists a computable limitwise monotonic function f such that

$$\mathit{range} \sup_y f(x, y)$$

is infinite and is contained in the set $\#T_0$ of finite lengths that occur in T. We also fix a computable predicate R such that $\sigma \in C$ if, and only if, $\exists^\infty y R(y, \sigma)$. Using f and R, we attach and "grow" more chains below a node x if we have more evidence $x \in C$. More specifically, if our current approximation to a Π^0_2-predicate $\exists^\infty y R(y, \sigma)$ "fires" on $\sigma \in \omega^{<\omega}$, as well as on all initial segments of σ, by providing new witnesses y for the corresponding strings, we start growing a few more longer simple chains below σ using f. The *main difficulty* is that some of the components we are constructing may become inactive forever, in this case we need to make sure these components do not produce chains of wrong sizes after stripping (recall we need $U_0 \cong T$). This is done again using f. If the predicate does not "fire" on σ, we may extend the chains below σ to slightly longer chains to make sure they not contribute any wrong length into U_0 after stripping (consider some elementary examples). If R fires again on σ, we attach a new fresh and very long chain to σ, and then we may have to extend the other (previously "slightly" extended) chains a bit more. In this case these other chains will not be ever extended again. Note that we have to care about repetitions of finite lengths; that is, in the notations of Proposition 5.2, we need to keep $S(T) = S(U_0)$. ⊣

5.2.3. Reduced groups of finite Ulm type. We say that a p-group A is *reduced* if it does not have a subgroup isomorphic to Z_{p^∞}. Using Lemma 5.20 and Proposition 5.2, we obtain:

Theorem 5.21 (Khisamiev [65], Ash, Knight and Oates [5]). *Let A be a reduced (abelian) p-group of Ulm type $n < \omega$. Then the following are equivalent:*

1. *A has a computable p-basic tree representing it;*
2. *A has a computable copy;*
3. (a) *for every $i < n$, the set*

$$S(A_i) = \{(m, k) : \text{ at least } k \text{ summands of } A_i \text{ are of order } p^m\}$$

is Σ^0_{2i+2}, and

https://doi.org/10.1017/bsl.2014.32 Published online by Cambridge University Press

(b) *for every $i < n$, the set*

$$\#A_i = \{m : Z_{p^m} \text{ is a summand of } A_i\}$$

is $\mathbf{0}^{(2i)}$-limitwise monotonic.

As far as we know, the following unexpected application of Theorem 5.21 and Lemma 5.20 is new:

PROPOSITION 5.22 (Melnikov). *Suppose G is a c.e. presented reduced abelian p-group having a nonzero element of infinite height. Then G has a computable presentation.*

The proposition may be compared to Corollary 5.4 and Theorem 4.6.

PROOF. Note that G' is generated by

$$\{h \in G : h \neq 0 \,\&\, (\forall k)(\exists x)\, p^k x = h\}.$$

Thus, $0''$ can list representatives of G'. Since the operation on G is computable and $=_G$ is $0'$-computable, we conclude that G' has a Δ^0_3-presentation. By Theorem 5.21 (relativized), G' can be represented by a Δ^0_3 p-basic tree T. Without loss of generality, we may assume that T is a Π^0_2-subtree of $\omega^{<\omega}$. (Hint: It is easy to show that every Δ^0_2-tree is isomorphic to a Π^0_1-subtree of $\omega^{<\omega}$.) We also nonuniformly pick a nonzero $g \in G$ such that (1) g has infinite height and (2) no element x such that $px = g$ has infinite height. We can effectively list all elements in $\{x : px = g\}$, possibly with repetitions. Let $x_0, x_1, x_2, \ldots$ be such a listing. It is crucial that none of the x_i can be equal to zero in G, since it would imply $g = 0$. Then the infinite set $\{h_p(x_i) + 1 : i \in \omega\}$ is the range of a limitwise monotonic function, since each of the $h_p(x_i)$ can be dynamically approximated from below. Indeed, $(\exists y)\,(p^k y - x_i = 0)$ is a c.e. relation on G, uniformly in k. It is not difficult to show that $\#G_0 \supseteq \{h_p(x_i) + 1 : i \in \omega\}$, and indeed $\#G_0 = \{h_p(x_i) + 1 : i \in \omega\}$. The proposition now follows from Lemma 5.20. ⊣

It has been an open problem for over 20 years whether Theorem 5.21 can be extended to reduced p-groups of Ulm type ω. The only known proof of Theorem 5.21 is not uniform. It is believed that the difficulty of extending the theorem to rank ω is rooted in this nonuniformity.

Note that in a computable reduced p-group, $\#G_i$ must be $\mathbf{0}^{(2i)}$-limitwise monotonic. A straightforward analysis shows that finding an index for a $\mathbf{0}^{(2i)}$-limitwise monotonic function ranging over $\#G_i$ takes at most three extra jumps on top of $\mathbf{0}^{(2i)}$. In fact, the property is $\Pi^0_3(\mathbf{0}^{(2i)})$ uniformly in i. This upper bound is sharp:

THEOREM 5.23 (Downey, Melnikov, and Ng [35]). *There exists a computable reduced abelian p-group G of Ulm type ω such that the (indices for) $\mathbf{0}^{(2i)}$-limiwise monotonic functions ranging over $\#G_i$ are not uniformly Σ^0_{2i+3}.*

Furthermore, the group G witnessing Theorem 5.23 has a computable p-basic tree. The proof of Theorem 5.23 relies on the technique of p-basic trees. Its proof can be viewed as an iterated $0'''$ argument. Although essentially $0'''$ in nature, it is sufficiently degenerate to make it work.

More specifically, since we control the group, we can significantly simplify many aspects of the construction. Theorem 5.23 gives evidence that characterizing computable p-groups of Ulm type $\geq \omega$ would have to use an iterated $0'''$-construction since limitwise monotonicity seems unavoidable in any such proof.

5.3. Computable p-groups in general. Countable p-groups can be fully classified, up to isomorphism, using Ulm invariants and the dimension of its maximal divisible subgroup. From the effective point of view, these invariants are very complicated. Even classically, having a sequence of dimensions indexed by transfinite numbers may be too difficult to handle. In the previous section, we have already seen that countable p-groups of Ulm type ω can be quite complicated algorithmically. This section contains several results that show how complex abelian p-groups can be in general.

5.3.1. The isomorphism problem for p-groups. The isomorphism problem for computable abelian p-groups is as hard as it could be:

THEOREM 5.24 (Folklore, see Goncharov and Knight [50]). *The isomorphism problem for computable abelian p-groups is Σ^1_1-complete.*

PROOF. It is well-known that there exists a computable sequence of computable trees $(T_i)_{i\in\omega}$ such that T_i is well-founded iff $i \in \mathcal{O}$, the Kleene's canonical Π^1_1-complete set. We may view the trees as p-basic trees, and obtain the corresponding groups G_i. Then we effectively pass to $A_i = \bigoplus_{k\in\omega} G_k$ and $B_i = A_i \oplus \bigoplus_{k\in\omega} \mathbb{Z}_{p^\infty}$. Then T_i has an infinite path iff $A_i \cong B_i$. ⊣

It is well-known that the Ulm type of a computable reduced abelian p-group has to be a computable ordinal, see e.g., Lin [11, 12] and Khisamiev [66]. Given a computable ordinal α, we could restrict the isomorphism problem to computable abelian p-groups of Ulm length $\leq \alpha$, and denote the resulting set of pairs by $E_{\alpha,p}$. The upper bound on the complexity of $E_{\alpha,p}$ can be calculated using $L^c_{\omega_1\omega}$ logic (see Calvert [14]). The precise upper bound depends on the form of α and is tedious. This obvious upper bound is sharp for every computable α (Calvert [14]).

5.3.2. Categoricity relative to an oracle. As we have seen before, every abelian p-group that is a direct sum of cyclic and quasi-cyclic summands is Δ^0_3-categorical. In contrast, abelian p-groups in general do not have any computable upper bound on categoricity (follows from Theorem 5.24). In [7], Barker produces examples of Δ^0_α-categorical but not Δ^0_β-categorical, for $\beta < \alpha$, computable p-groups. Barker's construction relies on the earlier work of Ash [2]. The machinery contained in Ash [2] is rather intricate, and in fact several (fixable) flaws have been found since the paper [2] was published. The author believes that the results of Barker [7] are correct and can be either proved directly or derived form any meta-theorem in the spirit of [2].

§6. Further topics. This section contains results that are not directly related to torsion and torsion-free abelian groups.

In Subsection 6.1 we discuss the little that is known about computable abelian groups that are not p-groups and not torsion-free. In Subsection 6.1.1 we look at torsion abelian groups that are not p-groups, and Subsection 6.1.1 contains an unexpected result on mixed abelian groups.

In Subsection 6.2 we survey results on computable ordered abelian groups. Standard definitions are contained in Subsection 6.2.1. Linear dependence is discussed in Subsection 6.2.2, and in Subsection 6.2.3 we give an algebraic criterion for computable categoricity and state results on computable dimension and Δ^0_α-categoricity. We discuss degrees of orders on computable abelian groups in Subsection 6.2.4, and Subsection 6.2.5 is devoted to degree spectra.

In Subsection 6.3, we briefly discuss some other subjects such as automatic and polynomial-time abelian groups and give references to the literature.

6.1. Other classes of abelian groups.

6.1.1. Torsion groups that are not p-groups. By Fact 3.5, a torsion group is computably presentable if and only if its p-components are uniformly computable.

FACT 6.1. *Let T be a computable torsion group. If T is computably categorical then T slits into a direct sum of cyclic and quasi-cyclic p-groups for various p.*

SKETCH. Recall every computably categorical p-group splits into a detract sum of cyclic and quasi-cyclic p-groups. The proof of Theorem 5.1.4 combined with Fact 3.5 implies that if the maximal p-subgroup of T is not computably categorical, then T is not computably categorical either. The fact now follows. ⊣

Computable categoricity in the class of torsion groups has not yet been characterized. The author expects that a notion similar to settling time (see Downey and Melnikov [33]) can be used to obtain a more elegant criterion.

Recall that there exist p-groups that have a $\mathbf{a}$-computable presentation if and only if $\mathbf{a} \in \Delta^0_2 \setminus \{0\}$ (Theorem 5.8). If we remove the restriction on the group to be a p-group, we can use a simpler argument to obtain a stronger result:

THEOREM 6.2 (Kalimullin, Khoussainov and Melnikov [60]). *There exists a torsion abelian group that admits a $\mathbf{a}$-computable presentation for every hyperimmune $\mathbf{a}$ but has no computable presentation.*

We also mention that, in contrast to torsion-free abelian groups, producing a torsion abelian group having a proper $\alpha' th$-jump degree is not too difficult, see Oates [91].

6.1.2. Mixed groups. Recall that an abelian group is *mixed* if it is neither torsion nor torsion-free. The algebraic class of countable mixed groups is much less understood that the classes of abelian torsion-free and p-groups. Even less is known about computable mixed groups. We discuss two results.

Recall that $T(A)$ stands for the maximal torsion subgroup of A. Using methods similar to the ones discussed in Subsection 4.1.1, Goncharov proved:

THEOREM 6.3 (Goncharov [46]). *Let A be a computable abelian group such that $A/T(A)$ is of infinite rank. Then A is not computably categorical. Furthermore, A has infinitely many computable presentations that are pairwise noncomputably isomorphic.*

In 1981, Goncharov [48] conjectured that every computable abelian group is either computably categorical or has infinitely many pairwise noncomputably isomorphic computable presentations. In 1983, Dobrica discovered the following surprising and counter-intuitive example:

PROPOSITION 6.4 (Dobrica [25]). *There exists a noncomputably categorical computable mixed group that splits into computably categorical torsion and torsion-free summands.*

PROOF SKETCH. The group can be chosen of the form $A \cong F \oplus \bigoplus_p C_p$, where $F \leqq (\mathbb{Q}, +)$ and for every prime p, $C_p \cong \mathbb{Z}_p$. We construct two copies of A, say B and C.

At stage 0 we put the multiplicative identity $\mathbf{1}$ of $\mathbb{Q}$ into F and make $\chi_F(\mathbf{1}) = (1, 1, 1, 1, \dots)$. We also initially make $A = B = C$, and we slightly obese our notations by not distinguishing between their elements. Note that at stage 0, for any generator $\mathbf{1}_p$ of $\mathbb{Z}_p$,

$$\left\{\frac{\mathbf{1}}{p} + n\mathbf{1}_p : n = 0, 1, \dots, p-1\right\}$$

is the automorphism orbit of $\frac{\mathbf{1}}{p}$ in A. We use this observation for diagonalization purposes.

To win against φ_e, we use the e'th prime. We suppress e. Wait for $\varphi : B \to C$ to converge on $\mathbf{1}/p \in B$. Then $\varphi(\mathbf{1}/p)$ must be of the form $\frac{\mathbf{1}}{p} + n\mathbf{1}_p$ in C, for some $n \in \{0, 1, \dots, p-1\}$, otherwise we do nothing. Make $\mathbf{1}/p$ divisible by p in B, while in C declare $p|(\frac{\mathbf{1}}{p} + m\mathbf{1}_p)$ for some $m \in \{0, 1, \dots, p-1\}$ such that $m \neq n$. (Notice: the latter does not imply $p^2|\mathbf{1}_p$ in C.) We leave the verification to the reader. ⊣

The author is not aware of any other published works related to categoricity of mixed groups. There is no proof of Goncharov's conjecture in the literature.

6.2. Ordered abelian groups.

6.2.1. Preliminaries. Standard references for algebra of ordered abelian groups are Fuchs [43] and Kokorin and Kopytov [72].

DEFINITION 6.5. An ordered abelian group is a triple $(G, +, \leq)$ such that $(G, +)$ is an abelian group and $\leq$ is a linear ordering such that $a \leq b$ implies $a + g \leq b + g$ for every $a, b, g \in G$.

When we say that an abelian group admits an order, we mean a linear order upon its domain that satisfies the definition above. A computable ordered abelian group is a computable abelian group with a computable order on it.

Recall that the absolute value of $a \in (G, +, \leq)$, written $|a|$, is equal to a if $a \geq 0$, and $|a| = -a$ otherwise. Two elements a, b of an ordered

https://doi.org/10.1017/bsl.2014.32 Published online by Cambridge University Press

abelian group are Archimedean equivalent if there exists $m \in \mathbb{Z}$ such that $|ma| \geq |b|$ and $|mb| \geq |a|$. Archimedean classes are equivalence classes mod Archimedean equivalence.

6.2.2. Linear independence v.s. order. It is well-known that an abelian group admits a linear ordering if and only if it is torsion-free, see e.g., Kokorin and Kopytov [72]. What is the effective content of this result? It is not difficult to show that there exists a computable presentation of the free abelian group of rank ω that is not computably orderable (Downey and Kurtz [31]). In contrast, Solomon observed that Proposition 4.2 implies that every computable torsion-free abelian group has a computable copy with a computable order on it. Indeed, using a computable basis we can, say, embed the group into an effectively ordered computable copy of $\bigoplus_{i\in\omega}\mathbb{Q}$; the latter of course admits various nonisomorphic computable orders.

Thus, given a computable basis we can produce a computable order. Conversely, suppose we are given a computable ordered abelian group. Can we produce a computable presentation of this ordered group that admits a computable base? Goncharov, Lempp, and Solomon gave a partial answer to this question:

THEOREM 6.6 (Goncharov, Lempp, and Solomon [51]). *If G is a computable ordered abelian group with finitely many Archimedean classes, then G has a computable presentation which admits a computable basis.*

In fact, Goncharov, Lempp, and Solomon showed that every such group has a base with a special nice property; we will not discuss this property here.

It is not known if Theorem 6.6 can be extended to the case of infinitely many Archimedean classes.

6.2.3. Computable categoricity, and beyond. The computable dimension of an algebraic structure is the number of its computable presentations up to computable isomorphism. Goncharov, Lempp, and Solomon applied their techniques of special bases to prove:

THEOREM 6.7 (Goncharov, Lempp, and Solomon [51]). *Every computable ordered abelian group has computable dimension* 1 *or* ω. *Furthermore, such a group is computably categorical if and only if it has finite rank.*

A detailed and well-presented proof of the theorem can be found in Goncharov, Lempp, and Solomon [51]. The paper [51] is highly recommended to everyone who is willing to learn the subject.

Recall the notion of Δ^0_α-categoricity.

THEOREM 6.8 (Melnikov [81]). *Suppose $\alpha = 2n + 1$ or $\alpha = \delta + 2n$ is a computable ordinal, where $\delta > 0$ a limit ordinal, and $n \in \omega$. Then there is a computable ordered abelian group which is Δ^0_α-categorical but not Δ^0_β-categorical for any $\beta < \alpha$.*

PROOF IDEA. For every Δ^0_2-linear order L having a left-most element we can uniformly produce a computable presentation of the free abelian group

https://doi.org/10.1017/bsl.2014.32 Published online by Cambridge University Press

F of rank ω so that the linear order on Archimedean classes of F is isomorphic to L. Using this functor, we can relativize and transfer the well-known results of Ash [2] to ordered abelian groups. $\dashv$

It follows from the main result of Section 6 in Solomon [103] that, in the proof idea above, the functor mapping an ordered group to a Δ^0_2-linear order on Archimedean classes cannot be replaced by a functor mapping a computable ordered group to a *computable* linear order. Indeed, there are computable abelian groups encoding *c.e.*-presented linear orders into Archimedean classes, and there exist c.e. presented linear orders having no computable copies (folklore).

6.2.4. Noncomputable orders on computable groups. Downey and Kurtz [31] demonstrated that the free abelian group of rank ω has a computable presentation that is not computably orderable. On the other hand, it is not difficult to show that the orders on a computable torsion-free abelian group can be represented by infinite paths through a computable subtree of 2^ω (i.e., they form a Π^0_1-class). The Low Basis Theorem implies that every computable presentation of a torsion-free abelian group admits a *low* linear ordering compatible with $+$. (Recall that a Turing degree $\mathbf{x}$ is low if $\mathbf{x}' = 0'$.) Can we say more?

It is well-known that there is an effective 1-1-correspondence between Π^0_1-classes and spaces of linear orders on computable orderable fields, see Metakides and Nerode [86]. In contrast, Solomon proved:

THEOREM 6.9 (Solomon [103]). *There exists a nonempty Π^0_1-class such that no computable torsion-free abelian group realizes this class as degrees of linear orders on the group.*

We also site Hatzikiriakou and Simpson [54] for the reverse mathematics of orderable abelian groups.

Recall that a completely decomposable group is effectively completely decomposable if it has a computable presentation with an algorithm for a complete decomposition (see Definition 4.20).

THEOREM 6.10 (Kach, Lange, Solomon [59]). *Every effectively completely decomposable group of infinite rank has a computable copy in which the set of degrees of orders is not closed upwards.*

The proof uses the method of c.e. permitting. See Kach, Lange, Solomon [59] for a detailed proof. We conjecture that the result can be extended to arbitrary computable torsion-free abelian groups of infinite rank.

6.2.5. A note on degree spectra. The functor from the proof of Theorem 6.8 can be used to transfer results on degree spectra of linear orders to ordered abelian groups.

FACT 6.11 (Melnikov [81]). *For every computable $\alpha \geq 3$ and every Turing degree $\mathbf{a} \geq 0^{(\alpha)}$ there exists an ordered abelian group having α'th proper jump degree.*

PROOF IDEA. Recall that the functor in the proof of Theorem 6.8 maps Δ^0_2-linear orders to computable ordered abelian groups, and the construction

https://doi.org/10.1017/bsl.2014.32 Published online by Cambridge University Press

defining the functor can be fully relativized to any Turing degree. The known results on degree spectra of linear orders that can be found in Frolov et al. [42] can be relativized to $0'$. We thus obtain the desired spectra of ordered abelian groups. $\dashv$

6.3. Further related subjects. It is natural to impose restrictions on computations in Definition 1.1. This way we obtain the notions of a polynomial-time presented abelian group (see Cenzer and Remmel [18]) and an automatic abelian group, see Tsankov [107] and Braun and Strüngmann [10]. Most of the computable groups that appeared in our survey have feasible presentations by default, but we have no formal explanation for this phenomenon.

An abelian group can be viewed as a generalization of a vector space, the algorithmic content of vector spaces has been intensively studied; see Metakides and Nerode [85] and Dekker [22, 23] and also more recent works of Shore [98] and Downey et al. [30]. On the other hand, abelian groups are the simplest nilpotent groups. Not much is known about computable nilpotent groups that are not abelian (e.g., Csima and Solomon [21] and Khismaiev [68]), and their systematic theory is still to be developed. We note that already a two-step computable nilpotent group may effectively encode in a definable way any other computable algebraic structure (Hirschfeldt et al. [56]), so one should not hope for a nicely structured theory of computable nilpotent groups.

Baumslag, Dyer, and Miller [8] discovered an interesting relation of computable abelian groups to homologies of finitely presented groups that we have already mentioned in Subsection 4.1.2.

They showed that the integral homology sequence of any finitely presented group is a sequence of c.e. presented abelian groups whose presentations are given uniformly. They also showed that any uniformly computable sequence of abelian groups with only first two terms necessarily finitely generated can be realized as a homology sequence of some finitely presented group. One of the results discussed in the survey, namely Theorem 4.6, solves a problem that was left open in Baumslag, Dyer, and Miller [8].

Finally, many proofs in computable abelian group theory can be transformed into proofs in reverse mathematics; see, e.g., Simpson [99]. Reverse mathematics examines and compares the proof-theoretic strengths of standard mathematical theorems. See Simpson [99] for a detailed exposition.

§7. Questions.

We pose several questions which we think are central to the theory at its present state. Some of these questions seem too general, and thus they should be viewed as research programs.

7.1. Torsion-free abelian groups. Goncharov posed the following question:

QUESTION 7.1 (Goncharov). *Is it true that for every $n \in \omega$ there exists a computable torsion-free abelian group that is Δ^0_{n+1}-categorical but not Δ^0_n-categorical?*

Recall that for $n \leq 4$ such examples have recently been found in the class of completely decomposable groups, see Downey and Melnikov [32]. It follows from Downey and Montalban [36] that no upper bound on categoricity can possibly be obtained in general. Furthermore, the main construction of [1] gives a uniform way of producing nonΔ^0_α-categorical examples for every computable α.

Goncharov asked whether the reduced part of a computable torsion-free abelian group has a computable presentation. As we already discussed in Subsection 5.1.3, this question of Goncharov is still open as well.

Question 7.2 (Downey and Melnikov). *Is the index set of computable completely decomposable groups Σ^0_7-complete*?

See Downey and Melnikov [32] for more questions on completely decomposable groups.

7.2. Computable p-groups. Recall the notion of a p-basic tree.

Question 7.3 (Ash, Knight, and Oates). *Is there a computable reduced abelian p-group that does not possess a computable p-basic tree representing it*?

If there are such computable reduced p-groups, they must be of Ulm type at least ω.

Question 7.4 (Khisamiev; Ash, Knight, and Oates). *Which reduced abelian p-groups of Ulm type ω admit computable presentations*?

See Ash and Knight [4] and Downey et al. [35] for a detailed discussion related to the two problems above.

7.3. Abelian groups in general. The question below was independently raised by Goncharov and Downey.

Question 7.5. *Is there an abelian group that has a* **b**-*computable presentation iff* $\mathbf{b} >_T 0$?

Such examples do not exist among direct sums of cyclic groups and finite rank torsion-free abelian groups. More generally, we would like to know if any degree spectrum of a countable structure can be realized as a degree spectrum of an abelian group.

The next question below goes back to Mal'cev.

Question 7.6. *Which computable abelian groups are computably categorical*?

The unknown cases are torsion groups in which p-components are computably categorical, and mixed groups of finite rank. Such a characterization, if it can be obtained, would help to prove Goncharov's conjecture (see Subsection 6.1.2).

7.4. Ordered abelian groups. A partial solution to the problem below was discussed in Subsection 6.2.2.

Question 7.7 (Goncharov, Lempp, and Solomon). *Does every computable ordered abelian group possess a computable presentation with a computable base*?

https://doi.org/10.1017/bsl.2014.32 Published online by Cambridge University Press

While the problem above seems approachable, the next problem looks more difficult.

QUESTION 7.8. *Which Π^0_1-classes can be realized as classes of linear orders on computable abelian groups*?

§8. **Acknowledgments.** I am thankful to my colleagues Rodney Downey, André Nies, Reed Solomon, and Matthew Harrison-Trainor for proofreading and many useful suggestions and corrections. The author was partially supported by Marsden Fund of New Zealand and the Packard Foundation.

REFERENCES

[1] B. ANDERSEN, A. KACH, A. MELNIKOV, and R. SOLOMON, *Jump degrees of torsion-free abelian groups*. ***Journal of Symbolic Logic***, vol. 77 (2012), no. 4, pp. 1067–1100.

[2] C. ASH, *Recursive labeling systems and stability of recursive structures in hyperarithmetical degrees*. ***Transactions of the American Mathematical Society***, vol. 298 (1986), pp. 497–514.

[3] C. ASH, C. JOCKUSCH, and J. KNIGHT, *Jumps of orderings*. ***Transactions of the American Mathematical Society***, vol. 319 (1990), no. 2, pp. 573–599.

[4] C. ASH and J. KNIGHT, ***Computable structures and the hyperarithmetical hierarchy***, Studies in Logic and the Foundations of Mathematics, vol. 144, North-Holland, Amsterdam, 2000.

[5] C. ASH, J. KNIGHT, and S. OATES, *Recursive abelian p-groups of small length*, Unpublished. An annotated manuscript: `https://dl.dropbox.com/u/4752353/Homepage/AKO.pdf`.

[6] R. BAER, *Abelian groups without elements of finite order*. ***Duke Mathematical Journal***, vol. 3 (1937), no. 1, pp. 68–122.

[7] E. BARKER, *Back and forth relations for reduced abelian p-groups*. ***Annals of Pure and Applied Logic***, vol. 75 (1995), no. 3, pp. 223–249.

[8] G. BAUMSLAG, E. DYER, and C. MILLER, III *On the integral homology of finitely presented groups*. ***Topology***, vol. 22 (1983), no. 1, pp. 27–46.

[9] W. BOONE, *The word problem*. ***Annals of Mathematics***, vol. 70 (1959), pp. 207–265.

[10] G. BRAUN and L. STRÜNGMANN, *Breaking up finite automata presentable torsion-free abelian groups*. ***International Journal of Algebra and Computation***, vol. 21x(2011), no. 8, pp. 1463–1472.

[11] C. LIN, *The effective content of Ulm's theorem*, ***Aspects of effective algebra (Clayton, 1979)***, Upside Down A Book, Yarra Glen, 1981, pp. 147–160.

[12] ———, *Recursively presented abelian groups: Effective p-group theory. I*. ***Journal of Symbolic Logic***, vol. 46 (1981), no. 3, pp. 617–624.

[13] W. CALVERT, ***Algebraic structure and computable structure***, ProQuest LLC, Ann Arbor, MI, 2005, Thesis (Ph.D.)–University of Notre Dame.

[14] ———, *The isomorphism problem for computable abelian p-groups of bounded length*. ***Journal of Symbolic Logic***, vol. 70 (2005), no. 1, pp. 331–345.

[15] W. CALVERT, D. CENZER, V. HARIZANOV, and A. MOROZOV, *Effective categoricity of abelian p-groups*. ***Annals of Pure and Applied Logic***, vol. 159 (2009), no. 1–2, pp. 187–197.

[16] W. CALVERT, V. HARIZANOV, and A. SHLAPENTOKH, *Turing degrees of isomorphism types of algebraic objects*. ***Journal of London Mathematical Society (2)***, vol. 75 (2007), no. 2, pp. 273–286.

[17] W. CALVERT, J. KNIGHT, and J. MILLAR, *Computable trees of Scott rank ω_1^{CK}, and computable approximation*. ***Journal of Symbolic Logic***, vol. 71 (2006), no. 1, pp. 283–298.

[18] D. CENZER and J. REMMEL, *Feasibly categorical abelian groups*, ***Feasible mathematics, II (Ithaca, NY, 1992)***, Progress in Computer Science and Applied Logic, vol. 13, Birkhäuser Boston, Boston, MA, 1995, pp. 91–153.

[19] R. Coles, R. Downey, and T. Slaman, *Every set has a least jump enumeration*. ***Journal of London Mathematical Society (2)***, vol. 62 (2000), no. 3, pp. 641–649.

[20] J. Crossley (editor) ***Aspects of Effective Algebra***, Upside Down A Book, Yarra Glen, 1981.

[21] B. Csima and R. Solomon, *The complexity of central series in nilpotent computable groups*. ***Annals of Pure and Applied Logic***, vol. 162 (2011), no. 8, pp. 667–678.

[22] J. Dekker, *Countable vector spaces with recursive operations. I*. ***Journal of Symbolic Logic***, vol. 34 (1969), pp. 363–387.

[23] ———, *Countable vector spaces with recursive operations. II*. ***Journal of Symbolic Logic***, vol. 36 (1971), pp. 477–493.

[24] V. Dobrica, *Constructivizable abelian groups*. ***Sibirskii Matematicheskii Zhurnal***, vol. 22 (1981), no. 3, pp. 208–213, 239.

[25] V. Dobritsa, *Some constructivizations of abelian groups*. ***Siberian Journal of Mathematics***, vol. 24 (1983), 167–173 (in Russian).

[26] R. Downey, *On presentations of algebraic structures*, ***Complexity, logic, and recursion theory***, Lecture Notes in Pure and Applied Mathematics, vol. 187, Dekker, New York, 1997, pp. 157–205.

[27] ———, *Computability, definability and algebraic structures*, ***Proceedings of the 7th and 8th Asian Logic Conferences***, Singapore University Press, Singapore, 2003, pp. 63–102.

[28] R. Downey, S. Goncharov, A. Kach, J. Knight, O. Kudinov, A. Melnikov, and D. Turetsky, *Decidability and computability of certain torsion-free abelian groups*. ***Notre Dame Journal of Formal Logic***, vol. 51 (2010), no. 1, pp. 85–96.

[29] R. Downey and D. Hirschfeldt, ***Algorithmic Randomness and Complexity***, Theory and Applications of Computability, Springer, New York, 2010.

[30] R. Downey, D. Hirschfeldt, A. Kach, S. Lempp, J. Mileti, and A. Montalbán *Subspaces of computable vector spaces*. ***Journal of Algebra***, vol. 314 (2007), no. 2, pp. 888–894.

[31] R. Downey and S. Kurtz, *Recursion theory and ordered groups*. ***Annals of Pure and Applied Logic***, vol. 32 (1986), no. 2, pp. 137–151.

[32] R. Downey and A. Melnikov, *Computable completely decomposable groups*. ***Transactions of the American Mathematical Society***, vol. 366 (2014), no. 8, pp. 4243–4266.

[33] ———, *Effectively categorical abelian groups*. ***Journal of Algebra***, vol. 373 (2013), pp. 223–248.

[34] R. Downey, A. Melnikov, and K. Ng, *Abelian p-groups and the halting problem*, submitted.

[35] ———, *Iterated effective embeddings of abelian p-groups*. ***International Journal of Algebra and Computation***, to appear.

[36] R. Downey and A. Montalbán, *The isomorphism problem for torsion-free abelian groups is analytic complete*. ***Journal of Algebra***, vol. 320 (2008), no. 6, pp. 2291–2300.

[37] Y. Ershov and S. Goncharov, ***Constructive models***, Siberian School of Algebra and Logic, Consultants Bureau, New York, 2000.

[38] Yu. Ershov, ***Problems of Solubility and Constructive Models [in Russian]***, Nauka, Moscow, 1980.

[39] E. Fokina, S. Friedman, V. Harizanov, J. Knight, C. McCoy, and A. Montalban, *Isomorphism and bi-embeddability relations on computable structures*. ***Journal of Symbolic Logic***, vol. 77 (2012), pp. 122–132.

[40] E. Fokina, J. Knight, A. Melnikov, S. Quinn, and C. Safranski, *Classes of Ulm type and coding rank-homogeneous trees in other structures*. ***Journal of Symbolic Logic***, vol. 76 (2011), no. 3, pp. 846–869.

[41] A. Fröhlich and J. Shepherdson, *Effective procedures in field theory*. ***Philosophical Transactions of the Royal Society of London, Series A***, vol. 248 (1956), pp. 407–432.

[42] A. Frolov, I. Kalimullin, V. Harizanov, O. Kudinov, and R. Miller, *Spectra of high$_n$ and non-low$_n$ degrees*. ***Journal of Logic and Computation***, vol. 22 (2012), no. 4, pp. 755–777.

[43] L. Fuchs, ***Partially Ordered Algebraic Systems***, Pergamon Press, Oxford, 1963.

[44] ———, ***Infinite Abelian Groups. Vol. I***, Pure and Applied Mathematics, Vol. 36, Academic Press, New York, 1970.

[45] ———, ***Infinite Abelian Groups. Vol. II***, Pure and Applied Mathematics, Vol. 36-II, Academic Press, New York, 1973.

[46] S. Goncharov, *Autostability of models and abelian groups*. ***Algebra i Logika***, vol. 19 (1980), no. 1, pp. 23–44, 132.

[47] ———, *The problem of the number of nonautoequivalent constructivizations*. ***Algebra i Logika***, vol. 19 (1980), no. 6, pp. 621–639, 745.

[48] ———, *Groups with a finite number of constructivizations*. ***Doklady Akademii Nauk SSSR***, vol. 256 (1981), no. 2, pp. 269–272.

[49] ———, ***Countable Boolean Algebras and Decidability***, Siberian School of Algebra and Logic, Consultants Bureau, New York,1997.

[50] S. Goncharov and J. Knight, *Computable structure and antistructure theorems*. ***Algebra Logika***, vol. 41 (2002), no. 6, pp. 639–681, 757.

[51] S. Goncharov, S. Lempp, and R. Solomon. *The computable dimension of ordered abelian groups*, ***Advances in Mathematics***, vol. 175 (2003), no. 1, pp. 102–143.

[52] K. Harris, *η-representation of sets and degrees*. ***Journal of Symbolic Logic***, vol. 73 (2008), no. 4, pp. 1097–1121.

[53] J. Harrison, *Recursive pseudo-well-orderings*. ***Transactions of the American Mathematical Society***, vol. 131 (1968), pp. 526–543.

[54] K. Hatzikiriakou and S. Simpson, WKL_0 *and orderings of countable abelian groups*, ***Logic and computation (Pittsburgh, PA, 1987)***, Contemporary Mathematics, vol. 106, American Mathematical Society, Providence, RI, 1990, pp. 177–180.

[55] G. Higman, *Subgroups of finitely presented groups*. ***Proceedings of the Royal Society, Series A***, vol. 262 (1961), pp. 455–475.

[56] D. Hirschfeldt, B. Khoussainov, R. Shore, and A. Slinko, *Degree spectra and computable dimensions in algebraic structures*. ***Annals of Pure and Applied Logic***, vol. 115 (2002), no. 1–3, pp. 71–113.

[57] N. Hisamiev, *Criterion for constructivizability of a direct sum of cyclic p-groups*. ***Izv. Akad. Nauk Kazakh. SSR Ser. Fiz.-Mat.***, (1981), no. 1, pp. 51–55, 86.

[58] G. Hjorth, *The isomorphism relation on countable torsion free abelian groups*. ***Fundamenta Mathematicae***, vol. 175 (2002), no. 3, pp. 241–257.

[59] A. Kach, K. Lange, and R. Solomon, *Degrees of orders on torsion-free Abelian groups*. ***Annals of Pure and Applied Logic***, vol. 164 (2013), no. 7–8, pp. 822–836.

[60] I. Kalimullin, B. Khoussainov, and A. Melnikov, *Limitwise monotonic sequences and degree spectra of structures*. ***Proceedings of the American Mathematical Society***, vol. 141 (2013), no. 9, pp. 3275–3289.

[61] I. Kaplansky, ***Infinite abelian groups***, Revised edition, The University of Michigan Press, Ann Arbor, MI, 1969.

[62] A. Khisamiev, *On the Ershov upper semilattice L_E*. ***Sibirskii Matematicheskii Zhurnal***, vol. 45 (2004), no. 1, pp. 211–228.

[63] N. Khisamiev, *Strongly constructive abelian p-groups*. ***Algebra i Logika***, vol. 22 (1983), no. 2, pp. 198–217.

[64] ———, *Hierarchies of torsion-free abelian groups*. ***Algebra i Logika***, vol. 25 (1986), no. 2, pp. 205–226, 244.

[65] ———, *Constructive abelian p-groups*. ***Siberian Advances in Mathematics***, vol. 2 (1992), no. 2, pp. 68–113.

[66] ———, ***Constructive Abelian Groups, Vol. 2***, Handbook of recursive mathematics, Studies in Logic and the Foundations of Mathematics, vol. 139, North-Holland, Amsterdam,1998, pp. 1177–1231.

[67] ———, *On a class of strongly decomposable abelian groups*. ***Algebra Logika***, vol. 41 (2002), no. 4, pp. 493–509, 511–512.

[68] ———, *On constructive nilpotent groups*. ***Sibirskii Matematicheskii Zhurnal***, vol. 48 (2007), no. 1, pp. 214–223.

[69] N. Khisamiev and Z. Khisamiev, *Nonconstructivizability of the reduced part of a strongly constructive torsion-free abelian group*. ***Algebra i Logika***, vol. 24 (1985), pp. 69–76.

[70] N. Khisamiev and A. Krykpaeva, *Effectively completely decomposable abelian groups*. ***Sibirskii Matematicheskii Zhurnal***, vol. 38 (1997), no. 6, pp. 1410–1412, iv.

[71] B. Khoussainov, A. Nies, and R. Shore, *Computable models of theories with few models*. ***Notre Dame Journal of Formal Logic***, vol. 38 (1997), no. 2, pp. 165–178.

[72] A. Kokorin and V. Kopytov, ***Fully ordered groups***, Halsted Press [John Wiley & Sons], New York-Toronto, 1974, Translated from the Russian by D. Louvish.

[73] A. Kurosh, ***The Theory of Groups***, Chelsea, New York, 1960, Translated from the Russian and edited by K. A. Hirsch. 2nd English ed. 2 volumes.

[74] S. Lempp, *The computational complexity of torsion-freeness of finitely presented groups*. ***Bulletin of the Australian Mathematical Society***, vol. 56 (1997), no. 2, pp. 273–277.

[75] F. Levi, ***Abelsche gruppen mit abzhlbaren elementen***, Habilitationsschrift, Leipzig, Teubner, 1917.

[76] Peter Loth, *Classifications of abelian groups and Pontrjagin duality*, ***Algebra, Logic and Applications***, vol. 10, Gordon and Breach Science Publishers, Amsterdam, 1998.

[77] A. Mader, *Almost completely decomposable groups*, ***Algebra, Logic and Applications***, vol. 13, Gordon and Breach Science Publishers, Amsterdam, 2000.

[78] A. Mal′cev, *Torsion-free abelian groups of finite rank*. ***Matematicheskii Sbornik***, vol. 4 (1938), no. 2, pp. 45–68.

[79] ———, *Constructive algebras. I*. ***Uspekhi Matematicheskikh Nauk***, vol. 16 (1961), no. 3 (99), pp. 3–60.

[80] ———, *On recursive Abelian groups*. ***Doklady Akademii Nauk SSSR***, vol. 146 (1962), pp. 1009–1012.

[81] A. Melnikov, *Effective properties of completely decomposable abelian groups*, ***CSc dissertation (2012)***.

[82] ———, *Enumerations and completely decomposable torsion-free abelian groups*. ***Theory of Computing Systems***, vol. 45 (2009), no. 4, pp. 897–916.

[83] ———, *Transforming trees into abelian groups*. ***New Zealand Journal of Mathematics***, vol. 41 (2011), pp. 75–81.

[84] ———, ***Computability and Structure***, The University of Auckland, 2012.

[85] G. Metakides and A. Nerode, *Recursively enumerable vector spaces*, ***Annals of Mathematical Logic***, vol. 11 (1977), no. 2, pp. 147–171.

[86] ———, *Effective content of field theory*. ***Annals of Mathematical Logic***, vol. 17 (1979), no. 3, pp. 289–320.

[87] C. Miller, III *Decision problems for groups—survey and reflections*, ***Algorithms and classification in combinatorial group theory (Berkeley, CA, 1989)***, Mathematical Sciences Research Institute Publications, vol. 23, Springer, New York,1992, pp. 1–59.

[88] R. Miller, *The Δ^0_2-spectrum of a linear order*. ***Journal of Symbolic Logic***, vol. 66 (2001), no. 2, pp. 470–486.

[89] P. Novikov, *On the algorithmic unsolvability of the word problem in group theory*. ***Trudy Matematicheskogo Instituta imeni VA Steklova***, vol. 44 (1955), pp. 1–143.

[90] A. Nurtazin, ***Computable classes and algebraic criteria of autostability***, Summary of Scientific Schools, Mathematics Institute, SB USSRAS, Novosibirsk,1974.

[91] S. Oates, ***Jump Degrees of Groups***, ProQuest LLC, Ann Arbor, MI, 1989, Thesis (Ph.D.)–University of Notre Dame.

[92] P. Odifreddi, ***Classical Recursion Theory. Vol. II***, Studies in Logic and the Foundations of Mathematics, vol. 143, North-Holland, Amsterdam, 1999.

[93] M. Rabin, *Computable algebra, general theory and theory of computable fields*. ***Transactions of the American Mathematical Society***, vol. 95 (1960), pp. 341–360.

[94] L. Richter, *Degrees of structures*. ***Journal of Symbolic Logic***, vol. 46 (1981), no. 4, pp. 723–731.

[95] K. Riggs, *The decomposability problem for torsion-free abelian groups is analytic complete*. ***Proceedings of the American Mathematical Society***, to appear.

[96] H. Rogers, ***Theory of Recursive Functions and Effective Computability***, second edition, MIT Press, Cambridge, MA, 1987.

[97] L. Rogers, *Ulm's theorem for partially ordered structures related to simply presented abelian p-groups*. ***Transactions of the American Mathematical Society***, vol. 227 (1977), pp. 333–343.

[98] R. Shore, *Controlling the dependence degree of a recursively enumerable vector space*. ***Journal of Symbolic Logic***, vol. 43 (1978), no. 1, pp. 13–22.

[99] S. Simpson, ***Subsystems of Second Order Arithmetic***, second edition, Perspectives in Logic, Cambridge University Press, Cambridge, 2009.

[100] T. Slaman, *Relative to any nonrecursive set*. ***Proceedings of the American Mathematical Society***, vol. 126 (1998), no. 7, pp. 2117–2122.

[101] R. Smith, *Two theorems on autostability in p-groups*, ***Logic Year 1979–80 (Proc. Seminars and Conf. Math. Logic, Univ. Connecticut, Storrs, Conn., 1979/80)***, Lecture Notes in Mathematics, vol. 859, Springer, Berlin,1981, pp. 302–311.

[102] R. Soare, ***Recursively Enumerable Sets and Degrees***, A study of computable functions and computably generated sets, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1987.

[103] R. Solomon, Π^0_1 *classes and orderable groups*. ***Annals of Pure and Applied Logic***, vol. 115 (2002), no. 1–3, pp. 279–302.

[104] I. Soskov, *A jump inversion theorem for the enumeration jump*. ***Archive for Mathematical Logic***, vol. 39 (2000), no. 6, pp. 417–437.

[105] W. Szmielew, *Elementary properties of Abelian groups*. ***Fundamenta Mathematicae***, vol. 41 (1955), pp. 203–271.

[106] S. Thomas, *The classification problem for torsion-free abelian groups of finite rank*. ***Journal of American Mathematical Society***, vol. 16 (2003), no. 1, pp. 233–258.

[107] T. Tsankov, *The additive group of the rationals does not have an automatic presentation*. ***Journal of Symbolic Logic***, vol. 76 (2011), no. 4, pp. 1341–1351.

[108] B. van der Waerden, *Eine Bemerkung über die Unzerlebarkeit von Polynomen*. ***Mathematische Annalen***, vol. 102 (1930), no. 1, pp. 738–739.

[109] S. Wehner, *Enumerations, countable structures and Turing degrees*. ***Proceedings of the American Mathematical Society***, vol. 126 (1998), no. 7, pp. 2131–2139.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
BERKELEY, CA, USA.