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Joint Mean Oscillation and Local Ideals in the Toeplitz Algebra II: Local Commutivity and Essential Commutant

Jingbo Xia

Abstract. A well-known theorem of Sarason [11] asserts that if $[T_f, T_h]$ is compact for every $h \in H^\infty$, then $f \in H^\infty + C(T)$. Using local analysis in the full Toeplitz algebra $\mathcal{T} = \mathcal{T}(L^\infty)$, we show that the membership $f \in H^\infty + C(T)$ can be inferred from the compactness of a much smaller collection of commutators $[T_f, T_h]$. Using this strengthened result and a theorem of Davidson [2], we construct a proper C^*-subalgebra $\mathcal{T}(\mathcal{L})$ of $\mathcal{T}$ which has the same essential commutant as that of $\mathcal{T}$. Thus the image of $\mathcal{T}(\mathcal{L})$ in the Calkin algebra does not satisfy the double commutant relation [12], [1]. We will also show that no *separable* subalgebra $\mathcal{S}$ of $\mathcal{T}$ is capable of conferring the membership $f \in H^\infty + C(T)$ through the compactness of the commutators $\{[T_f, S] : S \in \mathcal{S}\}$.

1 Introduction

In this sequel to our earlier work [13], we continue to explore the C^*-algebraic implications of various local oscillatory behaviors of functions. As it is a sequel, we will follow the notation of [13]. Thus T denotes the unit circle and dm the Lebesgue measure on T normalized so that $m(T) = 1$. We write L^p for $L^p(T, dm)$ and H^p for the Hardy subspace of L^p, $1 \le p \le \infty$. Let $P\colon L^2 \to H^2$ denote the orthogonal projection. Given $f \in L^\infty$, the Toeplitz operator T_f and the Hankel operator H_f are defined by the formulas $T_f\varphi = Pf\varphi$ and $H_f\varphi = (1-P)f\varphi$ respectively, $\varphi \in H^2$. We have $T_{\bar{g}f} - T_{\bar{g}}T_f = H_g^* H_f$. Let $\mathcal{T}$ denote the *full* Toeplitz algebra. That is, $\mathcal{T}$ is the C^*-algebra generated by $\{T_f : f \in L^\infty\}$. Let $\mathcal{K}$ be the collection of compact operators on H^2. It is well known that $\mathcal{K} \subset \mathcal{T}$.

For each $\tau \in T$, let $\mathcal{K}_\tau$ denote the ideal in $\mathcal{T}$ generated by $\mathcal{K}$ and $\{T_\eta : \eta \in C(T), \eta(\tau) = 0\}$. Recall that the usual *localization* in $\mathcal{T}$ is simply the fact that $\bigcap_{\tau \in T} \mathcal{K}_\tau = \mathcal{K}$ [3, p. 198].

Recall from [9] that, for $f \in \mathrm{BMO}$ and $\tau \in T$, the *local mean oscillation* of f at τ is

$$\mathrm{LMO}(f)(\tau) = \lim_{\delta \downarrow 0} \sup \left\{ \frac{1}{|I|} \int_I |f - f_I|\, dm : |\lambda - \tau| \le \delta \text{ for all } \lambda \in I \right\}.$$

Here and in what follows, I always denotes an arc in T with $|I| = m(I) > 0$, and $f_I = \int_I f\, dm/|I|$. Recall from [13] that, given $f, g \in \mathrm{BMO}$ and $\tau \in T$, the *joint local*

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mean oscillation of f and g at τ is defined to be

$$\mathrm{LMO}(f,g)(\tau) = \lim_{\delta\downarrow 0} \sup\left\{ \frac{1}{|I|}\int_I |f-f_I|\,dm \frac{1}{|I|}\int_I |g-g_I|\,dm : |\lambda-\tau| \le \delta \text{ for all } \lambda \in I \right\}.$$

Both $\mathrm{LMO}(f)$ and $\mathrm{LMO}(f,g)$ are useful invariants in the study of $\mathcal{T}$ [9], [13].

Given any $\tau \in T$, we let $\mathcal{L}(\tau)$ denote the collection of *bounded* functions ξ on T which are continuous on $T\setminus\{\tau\}$. For any such τ, we also define $\mathcal{H}(\tau) = H^\infty \cap \mathcal{L}(\tau)$. If $\mathcal{G}$ is a subset of L^∞, $\mathcal{T}(\mathcal{G})$ denotes the norm-closed operator algebra generated by $\{T_g : g \in \mathcal{G}\}$. In the case $\mathcal{G}$ is L^∞ itself, we will simply write $\mathcal{T}$ instead of $\mathcal{T}(L^\infty)$.

The results contained in this paper are motivated by, and can be viewed as a natural extension of, a number of previous investigations [2], [7], [9], [11], [13]. Recall that a well-known theorem of Sarason [11] asserts that, if $f \in L^\infty$ and if $[T_f, T_h]$ is compact for every $h \in H^\infty$, then $f \in H^\infty + C(T)$. Throughout the paper, we will write $Q = 1 - P$. It is well known that $Q\eta \in \mathrm{VMO}$ if $\eta \in C(T)$. Also, because T is compact, for any $f \in \mathrm{BMO}$, we have $f \in \mathrm{VMO}$ if and only if $\mathrm{LMO}(f)(\tau) = 0$ for every $\tau \in T$. Thus our first result is a local version of Sarason's theorem:

Theorem 1 *Let $f \in L^\infty$ and let $\tau \in T$.*

(a) *If $[T_f, T_h] \in \mathcal{K}_\tau$ for every $h \in \mathcal{H}(\tau)$, then* $\mathrm{LMO}(Qf)(\tau) = 0$.
(b) *If* $\mathrm{LMO}(Qf)(\tau) = 0$, *then $[T_f, T_g] \in \mathcal{K}_\tau$ for every $g \in H^\infty$.*

An immediate consequence of this is a stronger version of Sarason's theorem: The membership $f \in H^\infty + C(T)$ can be inferred from the compactness of a much smaller collection of commutators $[T_f, T_h]$.

Corollary 2 *Let $\mathcal{H}$ denote the subalgbra of H^∞ generated by $\bigcup_{\tau\in T} \mathcal{H}(\tau)$. If $f \in L^\infty$ is such that $[T_f, T_h]$ is compact for every $h \in \mathcal{H}$, then $f \in H^\infty + C(T)$.*

A key motivating factor for our consideration of the subalgebras $\mathcal{H}(\tau)$ of H^∞ is the following remarkable result of Davidson [2].

Theorem 3 **[2]** *If S is a bounded operator on H^2 which is not the sum of a bounded Toeplitz operator and a compact operator, then there is an $h \in H^\infty$ such that $[S, T_h]$ is not compact. Furthermore, h may be required to have at most one discontinuity.*

In other words, one may require the h in Theorem 3 to belong to some $\mathcal{H}(\tau)$ in the notation of the present paper.

Let H be a Hilbert space and let $\mathcal{S}$ be a subset of $\mathcal{B}(H)$. Recall that the *essential commutant* of $\mathcal{S}$ is the subalgebra $\{T \in \mathcal{B}(H) : [T, S] \text{ is compact for every } S \in \mathcal{S}\}$ of $\mathcal{B}(H)$. Using Theorem 3 and Sarason's theorem mentioned earlier, Davidson proved in [2] that the essential commutant of $\mathcal{T}$ is $\mathcal{T}(\mathrm{QC})$, where $\mathrm{QC} = \big(H^\infty + C(T)\big) \cap \overline{\big(H^\infty + C(T)\big)} = \mathrm{VMO} \cap L^\infty$. Using Theorem 3 and Corollary 2 in place of Sarason's theorem, we can produce an algebra smaller than $\mathcal{T}$ whose essential commutant also equals $\mathcal{T}(\mathrm{QC})$.

Corollary 4 *Let $\mathcal{L}$ be the norm-closed subalgebra of L^∞ generated by $\bigcup_{\tau\in T} \mathcal{L}(\tau)$. Then the essential commutant of $\mathcal{T}(\mathcal{L})$ equals $\mathcal{T}(\mathrm{QC})$.*

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As we will show in Section 3, $\mathcal{T}(\mathcal{L})$ is strictly contained in $\mathcal{T}$. It is well known that $\mathcal{T}$ is contained in the essential commutant of $\mathcal{T}(\mathrm{QC})$. Thus it follows from Corollary 4 that the second essential commutant of $\mathcal{T}(\mathcal{L})$ differs from $\mathcal{T}(\mathcal{L})$. This brings Voiculescu's double commutant relation [12] into the picture.

Given a separable Hilbert space H, let $\mathcal{Q}$ denote the Calkin algebra $\mathcal{B}(H)/\mathcal{K}(H)$ and let $\pi\colon \mathcal{B}(H) \to \mathcal{Q}$ denote the quotient map. Voiculescu proved in [12] that if $\mathcal{A}$ is a separable unital C^*-subalgebra of $\mathcal{Q}$, then $\mathcal{A}$ coincides with its double commutant in $\mathcal{Q}$, *i.e.*, $\mathcal{A} = \mathcal{A}''$. The same is also true if $\mathcal{A} = \pi(\mathcal{N})$, where $\mathcal{N}$ is any von Neumann algebra [8], [10]. In [1], Berger and Coburn constructed a simple, non-separable, unital C^*-subalgebra $\mathcal{A}$ of $\mathcal{Q}$ for which the double commutant relation fails, *i.e.*, $\mathcal{A} \neq \mathcal{A}''$. Their construction used Toeplitz operators on the Segal-Bargmann space. Corollary 4 leads to another example of a C^*-subalgebra $\mathcal{A}$ of $\mathcal{Q}$ with the property $\mathcal{A} \neq \mathcal{A}''$. Whereas the $\mathcal{A}$ in the Berger-Coburn example is a simple C^*-algebra, the $\mathcal{A}$ in our example below obviously has a non-trivial ideal.[1]

Theorem 5 *Let $\pi\colon \mathcal{B}(H^2) \to \mathcal{Q} = \mathcal{B}(H^2)/\mathcal{K}$ denote the quotient homomorphism and let $\mathcal{L}$ be the same as in Corollary 4. Then $\mathcal{A} = \pi\big(\mathcal{T}(\mathcal{L})\big)$ is a unital C^*-subalgebra of $\mathcal{Q}$ for which the double commutant relation fails,* i.e.*, $\mathcal{A} \neq \mathcal{A}''$.*

Let us now consider a *separable* unital C^*-subalgebra $\mathcal{S}$ of $\mathcal{T}$. Since, by Voiculescu's theorem, the double essential commutant of $\mathcal{S}$ must coincide with $\mathcal{S} + \mathcal{K}$ and since $\mathcal{T}$ is contained in the essential commutant of $\mathcal{T}(\mathrm{QC})$, the essential commutant of $\mathcal{S}$ must properly contain $\mathcal{T}(\mathrm{QC})$. That is, there is a bounded operator A on H^2 such that $A \notin \mathcal{T}(\mathrm{QC})$ and such that $[A, S]$ is compact for every $S \in \mathcal{S}$. This naturally invites the question, can such an A be found within the Toeplitz algebra $\mathcal{T}$? Better yet, is there such an A in the form of a Toeplitz operator T_f with some $f \notin H^\infty + C(T)$?

Another look at Sarason's original theorem and its improved version, Corollary 2, also leads to the same questions. That is, now that we know there is a closed proper subalgebra $\mathcal{H}$ of H^∞ such that the compactness of the commutators $\{[T_f, T_h] : h \in \mathcal{H}\}$ implies $f \in H^\infty + C(T)$, is there a *separable* subalgebra of H^∞ which has the same property? More generally, does there exist a *separable* subalgebra $\mathcal{S}$ of $\mathcal{T}$ which has the property that the compactness of the commutators $\{[T_f, S] : S \in \mathcal{S}\}$ necessitates the membership $f \in H^\infty + C(T)$? Our last theorem answers these very natural questions.

Theorem 6 *Suppose that $\mathcal{S}$ is a subset of $\mathcal{T}$ and suppose that $\mathcal{S}$ is separable in the operator-norm topology. Then there is a real-valued $f \in L^\infty$ such that $f \notin H^\infty + C(T)$ and such that $[T_f, S]$ is compact for every $S \in \mathcal{S}$. Moreover, given such an $\mathcal{S}$, there is a $\tau = \tau(\mathcal{S}) \in T$ such that there is an $f \in \mathcal{L}(\tau)$ which satisfies the above requirements.*

The rest of the paper consists of the proofs of these results. More specifically, the proofs of Theorems 1 and 6 and Corollaries 2 and 4 will be given in Section 2. Section 3 contains the proof of Theorem 5 along with some remarks.

[1] Since the initial submission of this paper, the author has learned a great deal more about the relation $\mathcal{A} \neq \mathcal{A}''$ for C^*-subalgebras $\mathcal{A}$ of $\mathcal{Q}$. First of all, in the literature there is an example of a C^*-subalgebra $\mathcal{A}$ of $\mathcal{Q}$ with $\mathcal{A} \neq \mathcal{A}''$ dating back to 1972, namely Example 2.4 in [8]. Furthermore, the relation $\mathcal{A} \neq \mathcal{A}''$ appears to be ubiquitous among non-separable C^*-subalgebras of $\mathcal{Q}$ in at least the following sense: The author has managed to show that if B is any von Neumann algebra whose dimension as a linear space is infinite, then B contains a C^*-subalgebra A such that $\pi(A) \neq \pi(B)$ and $\{\pi(A)\}'' = \pi(B)$ [14].

2 Local Commutivity

To prove Theorem 1, we need to recall a result from our earlier work [13].

Theorem 7 **[13, Theorem 2]** *Let* $f, g \in \mathrm{BMO}$ *and* $\tau \in T$. *Then* $H_g^* H_f \in \mathcal{K}_\tau$ *if and only if* $\mathrm{LMO}(Qf, Qg)(\tau) = 0$. *If, in addition,* f *and* g *are real-valued, then* $H_g^* H_f \in \mathcal{K}_\tau$ *if and only if* $\mathrm{LMO}(f, g)(\tau) = 0$.

Theorem 7 takes care of the operator-theoretical portion of the proof of Theorem 1; what remains is a function-theoretical construction.

Proposition 8 *Suppose that* $f \in \mathrm{BMO}$ *and that* τ *is a point in* T *such that* $\mathrm{LMO}(f)(\tau) > 0$. *Then there exists an* $h \in \mathcal{H}(\tau)$ *such that* $\mathrm{LMO}(f, h)(\tau) > 0$.

Proof By the obvious circular symmetry, it suffices to consider the case where $\tau = 1$. That is, assuming $\mathrm{LMO}(f)(1) > 0$, we need to find an $h \in \mathcal{H}(1)$ such that $\mathrm{LMO}(f, h)(1) > 0$.

We start by picking a C^∞-function ζ on $\mathbf{R}$ with the properties that $0 \le \zeta \le 1$ on $\mathbf{R}$, that $\zeta = 1$ on $[1/3, 2/3]$, and that $\zeta = 0$ on $\mathbf{R} \setminus (1/6, 5/6)$. Since $\mathrm{LMO}(f)(1) > 0$, there is a sequence $\{I_n\}$ of open arcs in T and a $\delta > 0$ such that $\lim_{n\to\infty} \sup\{|1-\lambda| : \lambda \in I_n\} = 0$ and such that

$$\frac{1}{|I_n|} \int_{I_n} |f - f_{I_n}|\, dm \ge \delta \quad \text{for every } n \ge 1. \tag{2.1}$$

Because $|I_n| \to 0$, passing to a subsequence if necessary, we may further assume

(i) $I_n = \{e^{it} : \alpha_n < t < \beta_n\}$, where $-\pi/2 < \alpha_n < \beta_n < \pi/2$;
(ii) $|I_{n+1}| \le 2^{-n} \cdot 10^{-1} \cdot \|\zeta'\|_\infty^{-1} \cdot |I_n|$ for every $n \ge 1$.

(By the definition of ζ, it is obvious that $\|\zeta'\|_\infty \ge 1$.) Now, for each $n \ge 1$, define the function ξ_n on T by the formula

$$\xi_n(e^{it}) = \zeta\left(\frac{t - \alpha_n}{\beta_n - \alpha_n}\right), \quad |t| \le \pi.$$

Thus each ξ_n is a C^∞-function on T and vanishes outside I_n.

Next we use induction to produce a sequence $\{s_n\}$, where each s_n is either 1 or -1, such that

$$-2 \le \sum_{j=1}^{n} s_j \xi_j(\lambda) \le 2 \quad \text{for all } \lambda \in T \text{ and } n \ge 1. \tag{2.2}$$

We start by picking $s_1 = 1$. Suppose that $n \ge 1$ and that $s_1, \dots, s_n \in \{1, -1\}$ have been chosen such that

$$-2 \le \sum_{j=1}^{m} s_j \xi_j(\lambda) \le 2 \quad \text{for all } \lambda \in T \text{ and } 1 \le m \le n.$$

Then define s_{n+1} as follows. If $\sum_{j=1}^{n} s_j\xi_j(\lambda) \geq 0$ for *every* $\lambda \in I_{n+1}$, then we set $s_{n+1} = -1$. Otherwise, *i.e.*, if $\sum_{j=1}^{n} s_j\xi_j(\lambda) < 0$ for *at least one* $\lambda \in I_{n+1}$, we set $s_{n+1} = 1$. Since $\xi_{n+1} = 0$ on $T \setminus I_{n+1}$, it is clear that we still have $-2 \leq \sum_{j=1}^{n+1} s_j\xi_j(\lambda) \leq 2$ for all $\lambda \in T$ in the case that s_{n+1} is chosen to be -1. On the other hand, we claim that

$$\sum_{j=1}^{n} s_j\xi_j(\lambda) < \frac{1}{10} \quad \text{for every } \lambda \in I_{n+1} \text{ if } \sum_{j=1}^{n} s_j\xi_j(\lambda^*) < 0 \text{ for some } \lambda^* \in I_{n+1}.$$

Indeed from the definition of ξ_j it is easy to see that $|\xi_j(\lambda)-\xi_j(\lambda^*)| \leq (\|\zeta'\|_\infty/|I_j|)\cdot|I_{n+1}|$ for all $\lambda \in I_{n+1}$. By condition (ii), $(\|\zeta'\|_\infty/|I_j|)\cdot|I_{n+1}| \leq 2^{-n}\cdot 10^{-1}$ for every $j \leq n$. Since $\sum_{j=1}^{n} s_j\xi_j(\lambda) \leq \sum_{j=1}^{n} s_j\xi_j(\lambda^*) + \sum_{j=1}^{n} |\xi_j(\lambda) - \xi_j(\lambda^*)|$, our claim is verified. Thus, in the case s_{n+1} is chosen to be 1, we also have $-2 \leq \sum_{j=1}^{n+1} s_j\xi_j(\lambda) \leq 2$ for all $\lambda \in T$. By induction, we have the desired sequence $\{s_n\}$.

Define $\xi = 3 + \sum_{j=1}^{\infty} s_j\xi_j$. It is obvious that, if U is an open arc containing 1, then all but a finite number of terms in $\sum_{j=1}^{\infty} s_j\xi_j$ vanish on $T \setminus U$. Hence ξ is a C^∞-function on $T \setminus \{1\}$. Furthermore, it follows from (2.2) that

$$1 \leq \xi(\lambda) \leq 5 \quad \text{for every } \lambda \in T \setminus \{1\}. \tag{2.3}$$

Next we show that

$$\liminf_{n\to\infty} \frac{1}{|I_n|}\int_{I_n} |\xi - \xi_{I_n}|\, dm \geq \frac{1}{3}. \tag{2.4}$$

Indeed, because $|s_j| = 1$, for any $n \geq 2$, we have

$$|\xi - \xi_{I_n}| \geq |\xi_n - (\xi_n)_{I_n}| - \sum_{k>n}(|\xi_k| + |\xi_k|_{I_n}) - \sum_{1\leq j<n} |\xi_j - (\xi_j)_{I_n}|. \tag{2.5}$$

When $k > n$, $\int_{I_n} |\xi_k|\, dm/|I_n| \leq |I_k|/|I_n| \leq 2^{-k}$ by condition (ii). Thus

$$\frac{1}{|I_n|}\int_{I_n} \sum_{k>n}(|\xi_k| + |\xi_k|_{I_n})\, dm \leq \sum_{k>n} 2^{-k}\cdot 2 = 2^{-n+1}. \tag{2.6}$$

Now, if $j < n$ and $\lambda \in I_n$, then $|\xi_j(\lambda) - (\xi_j)_{I_n}| \leq \int_{I_n} |\xi_j(\lambda) - \xi_j(w)|\, dm(w)/|I_n| \leq \sup_{w\in I_n} |\xi_j(\lambda) - \xi_j(w)| \leq (\|\zeta'\|_\infty/|I_j|)\cdot|I_n| \leq (\|\zeta'\|_\infty/|I_{n-1}|)\cdot|I_n| \leq 2^{-n}$ by the definition of ξ_j and (ii). Therefore

$$\frac{1}{|I_n|}\int_{I_n} \sum_{1\leq j<n} |\xi_j - (\xi_j)_{I_n}|\, dm \leq (n-1)2^{-n}. \tag{2.7}$$

Finally, by the definition of ζ and ξ_n, we can write $I_n = E \cup F \cup G$ such that $|E| = |F| = |G| = |I_n|/3$ and such that $\xi_n = 0$ on E and $\xi_n = 1$ on F. Hence

$$\frac{1}{|I_n|}\int_{I_n} |\xi_n - \xi_{I_n}|\, dm \geq \frac{|E|}{|I_n|}|0 - \xi_{I_n}| + \frac{|F|}{|I_n|}|1 - \xi_{I_n}| = \frac{1}{3}\{|\xi_{I_n}| + |1 - \xi_{I_n}|\} \geq \frac{1}{3}. \tag{2.8}$$

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Obviously, (2.4) follows from (2.5)–(2.8).

Let u be the harmonic extension of $\log \xi$ to the unit disc D by the Poisson formula and let v be the harmonic conjugate of u define by the conjugate formula. That is,

$$u(z) + iv(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log \xi(e^{it})\, dt, \quad z \in D.$$

(2.3) ensures that $\log \xi$ is also C^∞ on $T \setminus \{1\}$. Thus, by a well-known theorem about conjugate functions (see, *e.g.*, [5, p. 106]), the boundary value of v is continuous on $T \setminus \{1\}$. Therefore, if we set

$$h = \exp(u + iv),$$

then the outer function h is bounded and continuous on $T \setminus \{1\}$. In other words, $h \in \mathcal{H}(1)$. Since $|h| = \xi$ on T, we have $|h - h_{I_n}| \geq \big|\xi - |h_{I_n}|\big|$. Note that $\int_{I_n} |\xi - \xi_{I_n}|\, dm \leq 2 \int_{I_n} |\xi - r|\, dm$ for any $r \in \mathbf{R}$. Thus it follows from (2.4) that

$$\text{(2.9)} \qquad \liminf_{n\to\infty} \frac{1}{|I_n|} \int_{I_n} |h - h_{I_n}|\, dm \geq \liminf_{n\to\infty} \frac{1}{|I_n|} \int_{I_n} \big|\xi - |h_{I_n}|\big|\, dm \geq \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}.$$

Since the sequence $\{I_n\}$ of arcs converges to the point 1, combining (2.1) and (2.9) and recalling the definition of LMO, we now have

$$\text{LMO}(f, h)(1) \geq \liminf_{n\to\infty} \frac{1}{|I_n|} \int_{I_n} |f - f_{I_n}|\, dm \frac{1}{|I_n|} \int_{I_n} |h - h_{I_n}|\, dm \geq \frac{\delta}{6} > 0$$

as desired. This completes the proof. ■

Proof of Theorem 1 (a) Let $f \in L^\infty$ and $\tau \in T$ be such that $[T_f, T_h] \in \mathcal{K}_\tau$ for every $h \in \mathcal{H}(\tau)$. By the analyticity of h, we have $[T_f, T_h] = T_{fh} - T_h T_f = H^*_{\bar{h}} H_f$. Thus Theorem 7 tells us that $\text{LMO}(Qf, Q\bar{h})(\tau) = 0$ for every $h \in \mathcal{H}(\tau)$. The analyticity of h also means that $Q\bar{h} = \bar{h} - \bar{h}(0)$. By the definition of LMO, it is clear that $\text{LMO}(Qf, h)(\tau) = \text{LMO}\big(Qf, \bar{h} - \bar{h}(0)\big)(\tau)$. That is, the condition $[T_f, T_h] \in \mathcal{K}_\tau$ implies $\text{LMO}(Qf, h)(\tau) = 0$ for every $h \in \mathcal{H}(\tau)$. Proposition 8 now tells us that $\text{LMO}(Qf)(\tau) = 0$.

(b) Suppose that $\text{LMO}(Qf)(\tau) = 0$. For any $g \in H^\infty$, it follows from the definition of LMO and the boundedness of g that $\text{LMO}(Qf, Q\bar{g})(\tau) = 0$. Thus, by Theorem 7, $[T_f, T_g] = H^*_{\bar{g}} H_f \in \mathcal{K}_\tau$. This completes the proof. ■

Proof of Corollary 2 If $f \in L^\infty$ is such that $[T_f, T_h] \in \mathcal{K}$ for every $h \in \mathcal{H}$, then it follows from Theorem 1 that $\text{LMO}(Qf)(\tau) = 0$ for every $\tau \in T$. By the compactness of T, this means that $Qf \in \text{VMO}$. Thus the Hankel operator $H_f = H_{Qf}$ is compact, which implies that $f \in H^\infty + C(T)$ (see [4] or [15, p. 198]). ■

Proof of Corollary 4 Let S be an operator in the essential commutant of $\mathcal{T}(\mathcal{L})$. Since $\mathcal{L}$ contains every $\mathcal{H}(\tau)$, Theorem 3 tells us that $S = T_f + K$ with $f \in L^\infty$ and $K \in \mathcal{K}$. Since $\mathcal{T}(\text{QC}) \supset \mathcal{K}$, it suffices to show that $f \in \text{QC}$. Because $\mathcal{L} \supset \mathcal{H}$, it follows from Corollary 2 that $f \in H^\infty + C(T)$. Since $\mathfrak{T}(\mathcal{L})$ is $*$-symmetric, $T_{\bar f} = T_f^*$ also commutes with $\mathfrak{T}(\mathcal{L})$ modulo compact operators. Thus $\bar f$ also belongs to $H^\infty + C(T)$. Hence $f \in \mathrm{QC}$. ■

Lemma 9 *For any given $\tau \in T$, there is a real-valued $f \in \mathcal{L}(\tau)$ which satisfies the following two conditions:*

(i) $\mathrm{LMO}(f)(\tau) > 0$.
(ii) *If $g \in L^\infty$ and if τ is a Lebesgue point for g, then* $\mathrm{LMO}(f, g)(\tau) = 0$.

Proof As was the case for the proof of Proposition 8, it suffices to consider the case that $\tau = 1$. Define the function f on T by the formula

$$f(e^{it}) = \begin{cases} 1 & \text{if } 0 \le t \le 1/2 \\ 2(1-t) & \text{if } 1/2 < t \le 1 \\ 0 & \text{if } t \in [-\pi, \pi] \setminus [0, 1]. \end{cases}$$

It is obvious that f is continuous on $T \setminus \{1\}$, *i.e.*, $f \in \mathcal{L}(1)$. Now if we set $I_n = \{e^{it} : -2^{-n} \le t \le 2^{-n}\}$ for $n \ge 1$, then it is also obvious that $\int_{I_n} |f - f_{I_n}|\, dm/|I_n| = 1/2$. Since the sequence $\{I_n\}$ of arcs converges to 1, it follows that $\mathrm{LMO}(f)(1) \ge 1/2$, which verifies property (i).

Next we show that f also satisfies condition (ii). Let $g \in L^\infty$ be such that 1 is a Lebesgue point for this function. Let $\epsilon > 0$ be given. Then there is a $0 < \delta < 1/2$ such that

$$\frac{1}{2r} \int_{-r}^{r} |g(e^{it}) - g(1)|\, dt \le \frac{\epsilon}{4} \quad \text{whenever } 0 < r \le \delta. \tag{2.10}$$

Now consider any arc $I = \{e^{it} : a \le t \le b\}$ such that $-\delta \le a < b \le \delta$. Write

$$L(I) = \frac{1}{|I|} \int_I |f - f_I|\, dm \frac{1}{|I|} \int_I |g - g(1)|\, dm.$$

Since $f(e^{it}) = 1$ when $0 \le t \le 1/2$ and $f(e^{it}) = 0$ when $-1/2 \le t < 0$, it is clear that $L(I) = 0$ if either $0 \le a$ or $b \le 0$. Thus it suffices to consider the case where $a < 0 < b$. But, when $a < 0 < b$, it is clear from (2.10) that $\int_I |g - g(1)|\, dm/|I| \le 2 \cdot (\epsilon/4) = \epsilon/2$. Obviously, $|f - f_I| \le 1$. Therefore

$$L(I) \le \epsilon/2 \quad \text{whenever } -\delta \le a < b \le \delta. \tag{2.11}$$

Note that $\int_I |g - g_I|\, dm/|I| \le 2 \int_I |g - c|\, dm/|I|$ for any $c \in \mathbf{C}$. Hence it follows from (2.11) that, if $I = \{e^{it} : a \le t \le b\}$ and $-\delta \le a < b \le \delta$, then

$$\frac{1}{|I|} \int_I |f - f_I|\, dm \frac{1}{|I|} \int_I |g - g_I|\, dm \le 2L(I) \le \epsilon.$$

This proves that $\mathrm{LMO}(f, g)(1) = 0$ if 1 is a Lebesgue point for g. ■

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Proof of Theorem 6 The separability of $\mathcal{S}$ means that $\mathcal{S}$ is contained in the operator-norm closure of a countable subset $\{A_1, \ldots, A_j, \ldots\}$ of $\mathcal{T}$. Now each A_j is the limit in operator norm of a sequence of operators of the form $\sum_{k=1}^{K} T_{g_{k1}} \cdots T_{g_{kM}}$, where $g_{km} \in L^\infty$. Hence there is a countable set $G = \{g_1, \ldots, g_n, \ldots\}$ of real-valued functions in L^∞ such that $\mathcal{S} \subset \mathcal{T}(G)$.

For each g_n, almost every point in T is a Lebesgue point. Therefore there is a $\tau \in T$ which is a Lebesgue point for *every* g_n, $n = 1, 2, \ldots$. For this τ, let $f \in \mathcal{L}(\tau)$ be the real-valued function provided by Lemma 9. The membership in $\mathcal{L}(\tau)$ means that $\mathrm{LMO}(f)(u) = 0$ when $u \in T \setminus \{\tau\}$. Thus $\mathrm{LMO}(f, g_n)(u) = 0$ for all n and $u \in T \setminus \{\tau\}$. Since τ is a Lebesgue point for every g_n, Lemma 9 yields that $\mathrm{LMO}(f, g_n)(\tau) = 0$, $n = 1, 2, \ldots$. Therefore $\mathrm{LMO}(f, g_n)(u) = 0$ for all n and $u \in T$. Thus Theorem 7 tells us that $H^*_{g_n} H_f \in \bigcap_{u \in T} \mathcal{K}_u = \mathcal{K}$. That is, for every n, $H^*_{g_n} H_f$ is compact, which clearly implies the compactness of $[T_f, T_{g_n}]$. Hence $[T_f, S]$ is compact for every $S \in \mathcal{T}(G)$. Now Lemma 9 also yields that $\mathrm{LMO}(f)(\tau) > 0$, which obviously implies $f \notin \mathrm{VMO}$. Since f is real-valued, we have $f \notin H^\infty + C(T)$ as promised. ∎

3 The Double Commutant Relation

Recall that $\mathcal{L}$ is the norm-closed subalgebra of L^∞ generated by $\bigcup_{\tau \in T} \mathcal{L}(\tau)$, where $\mathcal{L}(\tau)$ is the collection of functions on T which are bounded and continuous on $T \setminus \{\tau\}$. The proof of Theorem 5 starts in the obvious way.

Lemma 10 *$\mathcal{L} \neq L^\infty$. More specifically, if E is a measurable, nowhere dense set in T such that $|E| > 0$, then $\chi_E \notin \mathcal{L}$.*

Proof Let $\mathcal{L}_0$ be the collection of functions of the form $\sum_{j=1}^{n} f_{1j} \cdots f_{mj}$, where $f_{ij} \in \mathcal{L}(\tau_{ij})$. Then $\mathcal{L}$ is the closure of $\mathcal{L}_0$ with respect to the essential-supremum norm $\|.\|_\infty$. To show that $\chi_E \notin \mathcal{L}$, it suffices to show that $\|\chi_E - f\|_\infty \geq 1/3$ for any $f \in \mathcal{L}_0$. That is, it suffices to show that

$$\sup_{\tau \in T \setminus N} |\chi_E(\tau) - f(\tau)| \geq 1/3 \quad \text{whenever } |N| = 0 \text{ and } f \in \mathcal{L}_0. \tag{3.1}$$

Observe that each $f \in \mathcal{L}_0$ has at most a finite number of discontinuities. Thus for each $f \in \mathcal{L}_0$ there is a finite set F such that $T \setminus F = \bigcup_{j \in J} I_j$, where J is countable and where each I_j is an open arc in T such that $\sup_{\tau, \tau' \in I_j} |f(\tau) - f(\tau')| \leq 1/3$. Since $|F| = 0$ and $|E| > 0$, there is a $j_0 \in J$ such that $|E \cap I_{j_0}| > 0$. Because E is nowhere dense, we have $|(T \setminus E) \cap I_{j_0}| > 0$. Thus for any set N with $|N| = 0$ we also have $|E \cap (I_{j_0} \setminus N)| > 0$ and $|(T \setminus E) \cap (I_{j_0} \setminus N)| > 0$. Now if we let $\tau \in E \cap (I_{j_0} \setminus N)$ and $\tau' \in (T \setminus E) \cap (I_{j_0} \setminus N)$, since $|f(\tau) - f(\tau')| \leq 1/3$, the inequalities $|1 - f(\tau)| < 1/3$ and $|0 - f(\tau')| < 1/3$ cannot hold simultaneously. This proves (3.1). ∎

Let $\mathcal{C}_{1/2}$ denote the ideal in the C^*-algebra $\mathcal{T}$ generated by the semi-commutators $\{T_{fg} - T_f T_g : f, g \in L^\infty\}$ of Toeplitz operators. Now, because the linear span of $\{\varphi\psi : \varphi, \psi \in H^\infty\}$ is dense in L^∞ (see [3, p. 163]), $\mathcal{C}_{1/2}$ coincides with the ideal $\mathcal{C}$ in

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$\mathcal{T}$ generated by the commutators $\{[A,B] : A, B \in \mathcal{T}\}$. Hence we have the short exact sequence

$$\{0\} \to \mathcal{C}_{1/2} \to \mathcal{T} \to L^\infty \to \{0\}. \tag{3.2}$$

See [3, p. 179].

Proof of Theorem 5 Corollary 4 states that $\pi\big(\mathcal{T}(\mathrm{QC})\big)$ is the commutant of $\pi\big(\mathcal{T}(\mathcal{L})\big)$ in the Calkin algebra $\mathcal{Q} = \mathcal{B}(H^2)/\mathcal{K}$. It is well known that $\mathcal{T}$ is contained in the essential commutant of $\mathcal{T}(\mathrm{QC})$, *i.e.*, $\pi(\mathcal{T}) \subset \big\{\pi\big(\mathcal{T}(\mathrm{QC})\big)\big\}' = \big\{\pi\big(\mathcal{T}(\mathcal{L})\big)\big\}''$. Thus it suffices to show that $\pi(\mathcal{T}) \neq \pi\big(\mathcal{T}(\mathcal{L})\big)$. Let $s\colon \mathcal{T} \to L^\infty$ be the symbol map in (3.2), *i.e.*, $s(T_\varphi) = \varphi$. Since $s(\mathcal{T}) = L^\infty$ and $s\big(\mathcal{T}(\mathcal{L})\big) = \mathcal{L}$, and since $\mathcal{L} \neq L^\infty$ by Lemma 10, we must have $\mathcal{T}(\mathcal{L}) \neq \mathcal{T}$. Since $\ker \pi = \mathcal{K}$, this and the relation $\mathcal{K} \subset \mathcal{T}(\mathcal{L}) \subset \mathcal{T}$ together imply $\pi(\mathcal{T}) \neq \pi\big(\mathcal{T}(\mathcal{L})\big)$. ■

Remark 11 Let $\mathcal{C}_{1/2}(\mathcal{L})$ be the ideal in $\mathcal{T}(\mathcal{L})$ generated by $\{T_{fg} - T_f T_g : f, g \in \mathcal{L}\}$. It is well known that, for any arc I in T with $0 < |I| < |T|$, $T_{\chi_I} - T_{\chi_I}^2$ is not compact [6]. Obviously, $\chi_I \in \mathcal{L}$. Therefore $\pi\big(\mathcal{C}_{1/2}(\mathcal{L})\big) \neq \{0\}$. On the other hand, (3.2) tells us that $\pi\big(\mathcal{C}_{1/2}(\mathcal{L})\big) \neq \pi\big(\mathcal{T}(\mathcal{L})\big)$. Hence $\pi\big(\mathcal{C}_{1/2}(\mathcal{L})\big)$ is a proper ideal in $\pi\big(\mathcal{T}(\mathcal{L})\big)$.

Remark 12 If S is an operator that essentially commutes with the essential commutant of $\mathcal{T}(\mathrm{QC})$, then S essentially commutes with $\mathcal{T}$. By Davidson's theorem, such an S belongs to $\mathcal{T}(\mathrm{QC})$. Therefore $\pi\big(\mathcal{T}(\mathrm{QC})\big)$ satisfies the double commutant relation $\mathcal{A} = \mathcal{A}''$ in $\mathcal{Q}$. On the other hand, it is well known that $\pi\big(\mathcal{T}(\mathrm{QC})\big) \cong \mathrm{QC}$ is not separable. Therefore the fact that $\pi\big(\mathcal{T}(\mathrm{QC})\big)$ satisfies the double commutant relation does not follow from Voiculescu's theorem. Also, it is an elementary exercise in measure theory to show that QC contains no projections other than 0 and 1. In particular, $\pi\big(\mathcal{T}(\mathrm{QC})\big)$ is not the image of any von Neumann algebra under π. Therefore the results of [8], [10] cannot be applied to $\pi\big(\mathcal{T}(\mathrm{QC})\big)$ either. Nevertheless, Davidson's theorem tells us that the double commutant relation $\mathcal{A} = \mathcal{A}''$ can also be satisfied by a subalgebra of $\mathcal{Q}$ which is neither separable nor close to being the image of any von Neumann algebra.

Finally, the results of [2] and the above discussion lead to the obvious:

Problem 13 What is the essential commutant of $\mathcal{T}(\mathrm{QC})$? In particular, does the essential commutant of $\mathcal{T}(\mathrm{QC})$ coincide with $\mathcal{T}$?

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Department of Mathematics
State University of New York at Buffalo
Buffalo, NY 14260
USA
email: jxia@acsu.buffalo.edu