APPROXIMATING THE DENSITY OF THE TIME TO RUIN VIA FOURIER-COSINE SERIES EXPANSION

BY

ZHIMIN ZHANG

ABSTRACT

In this paper, the density of the time to ruin is studied in the context of the classical compound Poisson risk model. Both one-dimensional and two-dimensional Fourier-cosine series expansions are used to approximate the density of the time to ruin, and the approximation errors are also obtained. Some numerical examples are also presented to show that the proposed method is very efficient.

KEYWORDS

Density of the time to ruin, COS, Fourier transform, Ruin.

1. INTRODUCTION

In this paper, we suppose that the surplus flow of an insurance company evolves as the classical risk model

$$U_t = u + ct - \sum_{j=1}^{N_t} X_j, \quad t \geq 0,$$

where $u \geq 0$ is the initial surplus level, and $c > 0$ is the constant premium rate. The claim number process $\{N_t\}_{t\geq 0}$ is a homogeneous Poisson process with intensity $\lambda > 0$. The random variables $X_1, X_2, \ldots$, representing the individual claim sizes, are independent and identically distributed like a generic variable X with density function f_X and mean μ_X. Throughout this paper, we assume that the net profit condition $c > \lambda\mu_X$ holds true, so that ruin is not a certain event.

For the surplus process U_t, we define the time of ruin by

$$\tau = \inf\{t \geq 0 : U_t < 0\},$$

Astin Bulletin 47(1), 169-198. doi: 10.1017/asb.2016.27

and let $\tau = \infty$ if $U_t \geq 0$ for all $t \geq 0$. Given the initial surplus u, the Laplace transform of the ruin time is defined by

$$\phi_\delta(u) = E[e^{-\delta\tau} I(\tau < \infty) | U_0 = u],$$

where $\delta \geq 0$ is the interest force, and $I(A)$ is the indicator function of an event A. When $\delta = 0$, $\phi_\delta(u)$ reduces to the ultimate ruin probability

$$\psi(u) = P(\tau < \infty | U_0 = u), \quad u \geq 0.$$

Let $f_\tau(u, t)$ denote the (defective) density of the time to ruin so that

$$\phi_\delta(u) = \int_0^\infty e^{-\delta t} f_\tau(u, t) dt.$$

Define the Laplace transform of $\phi_\delta(u)$ by $\widehat{\phi}_\delta(s) = \int_0^\infty e^{-su}\phi_\delta(u)du$. It is well known that (see, e.g., Gerber and Shiu, 1998)

$$\widehat{\phi}_\delta(s) = \frac{\lambda\left(\frac{1-\widehat{f}_X(\rho)}{\rho} - \frac{1-\widehat{f}_X(s)}{s}\right)}{cs - (\lambda + \delta) + \lambda \widehat{f}_X(s)}, \tag{1.1}$$

where $\widehat{f}_X(s) = \int_0^\infty e^{-sx} f_X(x)dx$ is the Laplace transform of f_X, and ρ is the unique positive root of the following generalized Lundberg equation

$$cs - (\lambda + \delta) + \lambda \widehat{f}_X(s) = 0. \tag{1.2}$$

Furthermore, the initial value is given by the simple formula

$$\widehat{\phi}_\delta(0) = \frac{\lambda}{c} \frac{1 - \widehat{f}_X(\rho)}{\rho}. \tag{1.3}$$

Formula (1.1) is useful for deriving the explicit expression for $\phi_\delta(u)$. However, in order to find some closed-form expressions for $f_\tau(u, t)$, we need to invert the Laplace transform of the time of ruin $\phi_\delta(u)$ w.r.t. to the argument δ. Usually, this is very hard to achieve for most classes of claim size densities.

The finite time ruin problem has been a challenging research topic in ruin theory for a long time. Recently, more and more attention has been paid to the identification of a closed-form expression for the density of the time of ruin. Existing results in the literature are mainly based on the inversion of the Laplace transform of the time of ruin. In the classical compound Poisson risk model, Drekic and Willmot (2003) studied the density of the ruin time for exponential claim size density; Dickson and Willmot (2005) followed the same approach to derive an expression for the density of the ruin time, and applied the result to the case when the claim size distribution is a mixture of Erlang distributions with the same scale parameter. It follows from Willmot and Woo (2007) that the class of Erlang mixtures is very large and any density on the positive half line can be approximated by this class of density functions. Hence, the results in

Dickson and Willmot (2005) are very useful for computing the density of ruin time. Garcia (2005) studied the survival probability through analytic inversion of Laplace transform when the claim sizes followed exponential, a mixture of two exponentials and Erlang (2) distributions. Dickson (2007) used probabilistic arguments to derive the joint densities of the time to ruin, the surplus before ruin and the deficit at ruin. Dickson (2008) applied Laplace inversion method to find some explicit solutions for the joint density of the time to ruin and the deficit at ruin when the individual claim sizes were distributed as Erlang (2) and as a mixture of two exponential distributions.

Recently, some finite time ruin problems have also been solved in some Sparre Andersen risk models in the past few years. Dickson et al. (2005) derived the density of the time to ruin for a Sparre Andersen model with Erlang inter-claim times and exponential claim sizes. This result was extended by Borovkov and Dickson (2008) to a general Sparre Andersen risk model with exponential claim amounts. In the Erlang (2) risk model, Dickson and Li (2010, 2012) derived some explicit formulae for the density of the time to ruin and the joint density of the time to ruin and the deficit at ruin. Landriault et al. (2011) studied some joint densities involving the time to ruin through the use of Lagrange's expansion theorem. Shi and Landriault (2013) made use of a multivariate version of Lagrange's expansion theorem to obtain a series expansion for the density of the time of ruin when the claim size density is a combination of exponentials.

Compared with the infinite time ruin problems, the finite time ruin problems are very hard to tackle since very few explicit results can be found. Although some formulae for the density of the rime of ruin have been developed by some researchers, they are based on some specific assumptions of the claim size density. In particular, exponential, Erlang and combination of exponential densities are widely used. The major difficulty comes from the explicit Laplace inversion of (1.1). To relax the restriction of claim size distributions, we develop an efficient method for computing the density of the time of ruin. Our method is on the ground of Fourier transform of the density of the time of ruin, which is easily applied as long as the Fourier transform of the claim size density is available. Furthermore, in comparison with Laplace transform, Fourier transform has the advantage of allowing for the use of some fast computation algorithms.

Fang and Oosterlee (2008) proposed a Fourier-cosine series expansion method (also known as COS method) for pricing European options. Since then, the COS method has been widely used in option pricing theory (see, e.g., Fang and Oosterlee, 2009, Zhang and Oosterlee, 2013, 2014). Ruijter and Oosterlee (2012) and Meng and Ding (2013) extended the COS method to higher dimensions with a multi-dimensional asset price process. The COS method can be easily used as long as the corresponding Fourier transform (or characteristic function) is available. Because the Fourier transforms of the ruin probability and Gerber–Shiu function are easily obtained, Chau et al. (2015a, 2015b) capitalized on the COS method to compute the ultimate ruin probability and Gerber–Shiu function in a class of Lévy risk models. The Fourier transform method has also been used by Zhang and Yang (2013, 2014) to estimate the ruin probability in the

Lévy risk model, where the FFT algorithm is used for computation. Different from Zhang and Yang (2013, 2014), we use the COS method to approximate the density of the time of ruin. Note that the COS method has $O(n)$ computational complexity in comparison with that of $O(n\log n)$ complexity via the FFT algorithm.

The remainder of this paper is organized as follows. In Section 2, we present some preliminaries on the Fourier transform of the time to ruin. In Section 3, supposing that the initial surplus is fixed, we use one-dimensional Fourier-cosine series expansion (1-COS) method to approximate $f_\tau(u,t)$. The two-dimensional Fourier-cosine series expansion (2-COS) method is applied for approximation of $f_\tau(u,t)$ in Section 4. Several numerical examples are given in Section 5 to show the efficiency of the proposed method, and some conclusions are given in Section 6.

2. Fourier transform of the ruin time

For $\omega \in \mathbb{R}$, we define the following one-dimensional Fourier transforms

$$\mathcal{F}_1 f_\tau(u,\omega) = \int_0^\infty e^{i\omega t} f_\tau(u,t)dt, \quad \mathcal{F}_1 f_X(\omega) = \int_0^\infty e^{i\omega x} f_X(x)dx.$$

Furthermore, for $\omega_1, \omega_2 \in \mathbb{R}$, we define the two-dimensional Fourier transform of $f_\tau(u,t)$ by

$$\mathcal{F}_2 f_\tau(\omega_1,\omega_2) = \int_0^\infty \int_0^\infty e^{i\omega_1 u + i\omega_2 t} f_\tau(u,t)dtdu = \int_0^\infty e^{i\omega_1 u}\mathcal{F}_1 f_\tau(u,\omega_2)du.$$

By analytic continuation, we have

$$\mathcal{F}_1 f_\tau(u,\omega) = E[e^{i\omega\tau} I(\tau < \infty)|U_0 = u] = \phi_{-i\omega}(u)$$

and

$$\mathcal{F}_2 f_\tau(\omega_1,\omega_2) = \int_0^\infty e^{i\omega_1 u}\phi_{-i\omega_2}(u)du = \widehat{\phi}_{-i\omega_2}(-i\omega_1).$$

It follows from (1.1) that $\mathcal{F}_1 f_\tau(u,\omega)$ has Laplace transform (w.r.t. u)

$$\int_0^\infty e^{-su}\mathcal{F}_1 f_\tau(u,\omega)du = \frac{\lambda\left(\frac{1-\widehat{f}_X(\rho(\omega))}{\rho(\omega)} - \frac{1-\widehat{f}_X(s)}{s}\right)}{cs - (\lambda - i\omega) + \lambda\widehat{f}_X(s)}, \tag{2.1}$$

where $\rho(\omega)$ is the unique root with positive real part of equation

$$cs - (\lambda - i\omega) + \lambda\widehat{f}_X(s) = 0. \tag{2.2}$$

This can be proved by Rouche's theorem. When $\omega = 0$, we have $\rho(0) = 0$. Applying analytic continuation to (2.1), we get

$$\mathcal{F}_2 f_\tau(\omega_1, \omega_2) = \frac{\lambda \left(\frac{1-\widehat{f}_X(\rho(\omega_2))}{\rho(\omega_2)} - \frac{1-\mathcal{F}_1 f_X(\omega_1)}{-i\omega_1} \right)}{-ic\omega_1 - (\lambda - i\omega_2) + \lambda \mathcal{F}_1 f_X(\omega_1)}. \quad (2.3)$$

When $u = 0$, formula (1.3) gives

$$\mathcal{F}_1 f_\tau(0, \omega) = \frac{\lambda}{c} \frac{1 - \widehat{f}_X(\rho(\omega))}{\rho(\omega)}. \quad (2.4)$$

In the following two examples, we present some more explicit formulae for the Fourier transforms of $f_\tau(u, t)$.

Example 1. *Assume that X follows a combination-of-exponentials distribution with density function*

$$f_X(x) = \sum_{j=1}^{m} A_j \alpha_j e^{-\alpha_j x}, \quad x > 0, \quad (2.5)$$

where m is a positive integer, $\sum_{j=1}^m A_j = 1$, $0 < \alpha_1 < \cdots < \alpha_m < \infty$. *It is easily seen that*

$$\widehat{f}_X(s) = \sum_{j=1}^{m} A_j \frac{\alpha_j}{s + \alpha_j}, \quad \mathcal{F}_1 f_X(\omega) = \sum_{j=1}^{m} A_j \frac{\alpha_j}{\alpha_j - i\omega}.$$

Then, (2.3) *becomes*

$$\mathcal{F}_2 f_\tau(\omega_1, \omega_2) = \frac{\lambda \sum_{j=1}^m \frac{A_j(\rho(\omega_2)+i\omega_1)}{(\alpha_j+\rho(\omega_2))(\alpha_j - i\omega_1)}}{ic\omega_1 + \lambda - i\omega_2 - \lambda \sum_{j=1}^m A_j \frac{\alpha_j}{\alpha_j - i\omega_1}}. \quad (2.6)$$

We find that equation (2.2) *reduces to*

$$cs + i\omega - \lambda \sum_{j=1}^{m} A_j \frac{s}{s + \alpha_j} = 0,$$

https://doi.org/10.1017/asb.2016.27 Published online by Cambridge University Press

which has $m+1$ roots. One of these roots is $\rho(\omega)$. We denote the other m roots by $-R_1(\omega), \dots, -R_m(\omega)$ which have negative real parts. By (2.1)*, we have*

$$\begin{aligned}
\int_0^\infty e^{-su}\mathcal{F}_1 f_\tau(u,\omega)du &= \frac{\frac{\lambda}{c}\sum_{j=1}^m A_j \frac{s-\rho(\omega)}{(\rho(\omega)+\alpha_j)(s+\alpha_j)}}{\frac{1}{\prod_{k=1}^m(s+\alpha_k)}(s-\rho(\omega))\prod_{k=1}^m(s+R_k(\omega))} \\
&= \frac{\lambda}{c}\sum_{j=1}^m \frac{A_j}{\rho(\omega)+\alpha_j}\frac{\prod_{k=1,k\neq j}^m(s+\alpha_k)}{\prod_{k=1}^m(s+R_k(\omega))} \\
&= \frac{H_c(s,\omega)}{\prod_{j=1}^m(s+R_j(\omega))},
\end{aligned} \tag{2.7}$$

where

$$H_c(s,\omega) = \frac{\lambda}{c}\sum_{j=1}^m \frac{A_j}{\rho(\omega)+\alpha_j}\prod_{k=1,k\neq j}^m(s+\alpha_k)$$

is a polynomial function of s with degree $m-1$. For convenience, suppose that $R_1(\omega), \dots, R_m(\omega)$ are distinct for each ω. Then, applying partial fractions to (2.7) *gives*

$$\int_0^\infty e^{-su}\mathcal{F}_1 f_\tau(u,\omega)du = \sum_{j=1}^m \frac{H_{c,j}(\omega)}{s+R_j(\omega)},$$

where

$$H_{c,j}(\omega) = \frac{H_c(-R_j(\omega),\omega)}{\prod_{k=1,k\neq j}^m(R_k(\omega)-R_j(\omega))}, \quad j=1,\dots,m.$$

Upon Laplace inversion we find that the one-dimensional Fourier transform of the time to ruin takes the following form:

$$\mathcal{F}_1 f_\tau(u,\omega) = \sum_{j=1}^m H_{c,j}(\omega)e^{-R_j(\omega)u}, \quad u \geq 0. \tag{2.8}$$

Example 2. *Assume that X follows Erlang distribution with density function*

$$f_X(x) = \frac{\alpha^m x^{m-1}}{(m-1)!}e^{-\alpha x}, \quad x > 0, \tag{2.9}$$

where m is the shape parameter which is a positive integer and $\alpha > 0$ is the scale parameter. The Laplace transform and Fourier transform of f_X are given by

$$\widehat{f}_X(s) = \left(\frac{\alpha}{\alpha+s}\right)^m, \quad \mathcal{F}_1 f_X(\omega) = \left(\frac{\alpha}{\alpha-i\omega}\right)^m,$$

https://doi.org/10.1017/asb.2016.27 Published online by Cambridge University Press

respectively. In this case, (2.3) *becomes*

$$\mathcal{F}_2 f_\tau(\omega_1, \omega_2) = \frac{\lambda \left(\frac{1-\left(\frac{\alpha}{\alpha+\rho(\omega_2)}\right)^m}{\rho(\omega_2)} + \frac{1-\left(\frac{\alpha}{\alpha-i\omega_1}\right)^m}{i\omega_1} \right)}{-ic\omega_1 - \lambda + i\omega_2 + \lambda \left(\frac{\alpha}{\alpha-i\omega_1}\right)^m}. \tag{2.10}$$

The generalized Lundberg equation (2.2) *becomes*

$$cs - (\lambda - i\omega) + \lambda \left(\frac{\alpha}{s+\alpha} \right)^m = 0.$$

Again, we denote the other m roots (with negative real parts) by $-R_1(\omega), \ldots, -R_m(\omega)$*, which are assumed to be distinct. It follows from* (2.1) *that*

$$\int_0^\infty e^{-su} \mathcal{F}_1 f_\tau(u, \omega) du = \frac{H_g(s, \omega)}{\prod_{k=1}^m (s + R_k(\omega))}, \tag{2.11}$$

where

$$H_g(s, \omega) = \frac{\lambda}{c} \frac{(s+\alpha)^m}{s - \rho(\omega)} \left(\frac{1 - \left(\frac{\alpha}{\rho(\omega)+\alpha}\right)^m}{\rho(\omega)} - \frac{1 - \left(\frac{\alpha}{s+\alpha}\right)^m}{s} \right)$$

is a polynomial function of s with degree $m - 1$*. By partial fractions, we find that* (2.11) *becomes*

$$\int_0^\infty e^{-su} \mathcal{F}_1 f_\tau(u, \omega) du = \sum_{j=1}^m \frac{H_{g,j}(\omega)}{s + R_j(\omega)},$$

where

$$H_{g,j}(\omega) = \frac{H_g(-R_j(\omega), \omega)}{\prod_{k=1, k\neq j}^m (R_k(\omega) - R_j(\omega))}, \quad j = 1, \ldots, m.$$

Finally, by Laplace inversion, we get

$$\mathcal{F}_1 f_\tau(u, \omega) = \sum_{j=1}^m H_{g,j}(\omega) e^{-R_j(\omega)u}, \quad u \geq 0. \tag{2.12}$$

Remark 1. In the above two examples, we show that explicit expressions for the one-dimensional Fourier transform of the density of the time to ruin can be obtained for combination-of-exponentials and Erlang claim size densities. We assert that the corresponding result is also readily obtained when the claim size density belongs to the rational family. The detailed solution procedure is the same, and hence we omit it here.

3. One-dimensional Fourier-cosine Approximation

In this section, we apply the 1-COS method to approximate the density of the time to ruin. It is well known that for a function with a finite support $[a_1, a_2]$, it has cosine series expansion

$$f(x) = {\sum_{k=0}^{\infty}}' B_k \cos\left(k\pi \frac{x-a_1}{a_2-a_1}\right), \tag{3.1}$$

with

$$B_k = \frac{2}{a_2-a_1}\int_{a_1}^{a_2} f(x)\cos\left(k\pi \frac{x-a_1}{a_2-a_1}\right)dx, \quad k = 0, 1, 2, \ldots \tag{3.2}$$

where $\sum'$ indicates that the first term in the summation is weighted by one-half (see, e.g., Fang and Oosterlee, 2008).

Let us consider the density of the ruin time $f_\tau(u, t)$ with fixed initial surplus u. For $a > 0$, define the following auxiliary function:

$$f_{\tau,a}(u, t) = f_\tau(u, t) I(t \le a).$$

As a function of t, $f_{\tau,a}(u, t)$ has a finite domain $[0, a]$, it follows from formula (3.1) that

$$f_{\tau,a}(u, t) = {\sum_{k=0}^{\infty}}' \widehat{B}_{a,k}(u)\cos\left(k\pi\frac{t}{a}\right), \quad t \in [0, a], \tag{3.3}$$

where

$$\widehat{B}_{a,k}(u) = \frac{2}{a}\int_0^a f_{\tau,a}(u, t)\cos\left(k\pi\frac{t}{a}\right)dt, \quad k = 0, 1, 2, \ldots \tag{3.4}$$

Furthermore, truncating the series summation in (3.3) yields

$$f_{\tau,a}(u, t) \approx f_{\tau,a,1}(u, t) := {\sum_{k=0}^{K-1}}' \widehat{B}_{a,k}(u)\cos\left(k\pi\frac{t}{a}\right), \tag{3.5}$$

where K is a sufficiently large integer.

It is not convenient to compute the cosine coefficients $\widehat{B}_{a,k}(u)$ by formula (3.4). Note that $f_{\tau,a}(u, t) = f_\tau(u, t)$ for $t \in [0, a]$. When a is large enough, we have

$$\widehat{B}_{a,k}(u) \approx B_{a,k}(u) := \frac{2}{a}\int_0^\infty f_\tau(u, t)\cos\left(k\pi\frac{t}{a}\right)dt = \frac{2}{a}\mathrm{Re}\left(\mathcal{F}_1 f_\tau(u, k\pi/a)\right),$$

where $\mathrm{Re}(\cdot)$ denotes taking the real part of the argument. The coefficients $B_{a,k}(u)$ are easy to obtain when f_X is either a combination of exponentials or Erlang, since the Fourier transform $\mathcal{F}_1 f_\tau(u, \omega)$ is available. See formulae (2.8) and (2.12). Replacing $\widehat{B}_{a,k}(u)$ by $B_{a,k}(u)$ in (3.5), we obtain the following approximant:

$$f_\tau(u,t) = f_{\tau,a}(u,t) \approx f_{\tau,a,2}(u,t) := \sum_{k=0}^{K-1}{}' B_{a,k}(u)\cos\left(k\pi\frac{t}{a}\right), \quad t\in[0,\ a]. \tag{3.6}$$

Remark 2. Define the finite time ruin probability by

$$\psi(u,t) = P(\tau < t|U_0 = u), \quad t>0, \quad u\geq 0.$$

Integrating (3.6) and interchanging the order of integration and summation, we obtain for $0\leq t\leq a$

$$\begin{aligned}\psi(u,t) &= \int_0^t f_\tau(u,s)ds \approx \int_0^t \sum_{k=0}^{K-1}{}' B_{a,k}(u)\cos\left(k\pi\frac{s}{a}\right)ds\\ &= \sum_{k=0}^{K-1}{}' B_{a,k}(u)t\cdot\text{sinc}\left(k\pi\frac{t}{a}\right),\end{aligned} \tag{3.7}$$

where

$$\text{sinc}(x) = \begin{cases}\frac{\sin(x)}{x}, & x\neq 0,\\ 1, & x=0.\end{cases}$$

Now, we study the approximation error of (3.6). Fix A such that $0 < A\leq a$, we measure the error by

$$Er[f_{\tau,a,2}](u) := \left(\int_0^A (f_\tau(u,t) - f_{\tau,a,2}(u,t))^2 dt\right)^{\frac{1}{2}}.$$

Since $f_\tau(u,t) = f_{\tau,a}(u,t)$ for $0\leq t\leq a$, by triangle inequality, we have

$$\begin{aligned}Er[f_{\tau,a,2}](u) &\leq \left(\int_0^A (f_{\tau,a}(u,t) - f_{\tau,a,1}(u,t))^2 dt\right)^{\frac{1}{2}}\\ &\quad + \left(\int_0^A (f_{\tau,a,1}(u,t) - f_{\tau,a,2}(u,t))^2 dt\right)^{\frac{1}{2}}\\ &\leq \left(\int_0^a (f_{\tau,a}(u,t) - f_{\tau,a,1}(u,t))^2 dt\right)^{\frac{1}{2}}\\ &\quad + \left(\int_0^a (f_{\tau,a,1}(u,t) - f_{\tau,a,2}(u,t))^2 dt\right)^{\frac{1}{2}}\\ &:= \epsilon_{a,K,1}(u) + \epsilon_{a,K,2}(u),\end{aligned}$$

where $\epsilon_{a,K,1}(u)$ is the series truncation error for including only the first K terms, and $\epsilon_{a,K,2}(u)$ is the error related to approximating $\widehat{B}_{a,k}(u)$ by $B_{a,k}(u)$.

First, we consider the error $\epsilon_{a,K,1}(u)$. Using the well-known result

$$\int_0^a \cos\left(k_1\pi\frac{x}{a}\right)\cos\left(k_2\pi\frac{x}{a}\right)dx = \begin{cases} a, & k_1 = k_2 = 0, \\ \frac{a}{2}, & k_1 = k_2 \neq 0, \\ 0, & k_1 \neq k_2, \end{cases} \tag{3.8}$$

we have

$$[\epsilon_{a,K,1}(u)]^2 = \sum_{k=K}^{\infty} |\widehat{B}_{a,k}(u)|^2 \int_0^a \left(\cos\left(k\pi\frac{t}{a}\right)\right)^2 dt = \frac{a}{2}\sum_{k=K}^{\infty} |\widehat{B}_{a,k}(u)|^2. \tag{3.9}$$

Using integration by parts, we have

$$\begin{aligned}\widehat{B}_{a,k}(u) &= \frac{2}{a}\int_0^a f_{\tau,a}(u,t)\cos\left(k\pi\frac{t}{a}\right)dt = \frac{2}{a}\int_0^a f_\tau(u,t)\cos\left(k\pi\frac{t}{a}\right)dt \\ &= -\frac{2}{k\pi}\int_0^a \sin\left(k\pi\frac{t}{a}\right)\frac{\partial}{\partial t}f_\tau(u,t)dt,\end{aligned}$$

which yields

$$|\widehat{B}_{a,k}(u)| \leq \frac{2}{k\pi}\int_0^a \left|\frac{\partial}{\partial t}f_\tau(u,t)\right| dt = \frac{C(u)}{k}, \tag{3.10}$$

where

$$C(u) = \frac{2}{\pi}\int_0^\infty \left|\frac{\partial}{\partial t}f_\tau(u,t)\right| dt.$$

Hence, using the upper bound (3.10), we find that (3.9) gives

$$[\epsilon_{a,K,1}(u)]^2 \leq \frac{a}{2}\sum_{k=K}^{\infty}\frac{(C(u))^2}{k^2} \leq \frac{(C(u))^2}{2}\frac{a}{K-1}. \tag{3.11}$$

Next, we study $\epsilon_{a,K,2}(u)$. For $n = 1, 2, \ldots$, let

$$m_n(u) = E[\tau^n I(\tau < \infty)|U_0 = u]$$

be the nth moment of the ruin time. By Markov inequality, we have

$$\begin{aligned}|\widehat{B}_{a,k}(u) - B_{a,k}(u)| &= \left|-\frac{2}{a}\int_a^\infty f_\tau(u,t)\cos\left(k\pi\frac{t}{a}\right)dt\right| \\ &\leq \frac{2}{a}\int_a^\infty f_\tau(u,t)dt \leq \frac{2m_1(u)}{a^2}.\end{aligned}$$

https://doi.org/10.1017/asb.2016.27 Published online by Cambridge University Press

Hence, using formula (3.8), we have

$$
\begin{aligned}
[\epsilon_{a,K,2}(u)]^2 &= \sum_{k=0}^{K-1}{}'[\widehat{B}_{a,k}(u) - B_{a,k}(u)]^2 \int_0^a \left(\cos\left(k\pi\frac{t}{a}\right)\right)^2 dt \\
&\leq \frac{a}{2}\sum_{k=0}^{K-1}[\widehat{B}_{a,k}(u) - B_{a,k}(u)]^2 \\
&\leq 2(m_1(u))^2\frac{K}{a^3}.
\end{aligned} \tag{3.12}
$$

Finally, combining (3.11) and (3.12), we obtain the following result.

Proposition 1. *The approximation error* $Er[f_{\tau,a,2}](u)$ *has the following upper bound,*

$$
Er[f_{\tau,a,2}](u) \leq \frac{C(u)}{\sqrt{2}}\sqrt{\frac{a}{K-1}} + \sqrt{2}m_1(u)\sqrt{\frac{K}{a^3}}. \tag{3.13}
$$

Remark 3. The cut-off parameter a plays an important role in calculating $f_{\tau,a,2}$. To find the optimal cut-off parameter a^*, we minimize the right-hand side of the above equation w.r.t. a by setting

$$
\frac{d}{da}\left(\frac{C(u)}{\sqrt{2}}\sqrt{\frac{a}{K-1}} + \sqrt{2}m_1(u)\sqrt{\frac{K}{a^3}}\right) = 0
$$

to obtain

$$
a^* = \left(\frac{6m_1(u)}{C(u)}\sqrt{K(K-1)}\right)^{\frac{1}{2}} = O(K^{\frac{1}{2}})
$$

for a fixed initial surplus level u. Replacing a by a^* in (3.13), we obtain the following convergence rate

$$
Er[f_{\tau,a,2}](u) = O(K^{-\frac{1}{4}})
$$

for a fixed initial surplus level u.

4. Two-dimensional Fourier-cosine Approximation

In this section, we apply the 2-COS method to approximate the density of the time to ruin. If $f(x_1, x_2)$ has a finite domain $[a_1, b_1] \times [a_2, b_2]$, we can express f in terms of two-dimensional Fourier series (see, e.g., Pivato, 2010 and Meng and Ding, 2013). In particular, f has the following cosine series expansion,

$$
f(x_1, x_2) = \sum_{k_1=0}^{\infty}{}'\sum_{k_2=0}^{\infty}{}'\widehat{C}_{k_1,k_2}\cos\left(k_1\pi\frac{x_1-a_1}{b_1-a_1}\right)\cos\left(k_2\pi\frac{x_2-a_2}{b_2-a_2}\right), \tag{4.1}
$$

https://doi.org/10.1017/asb.2016.27 Published online by Cambridge University Press

where

$$\widehat{C}_{k_1,k_2} = \frac{2}{b_1 - a_1} \frac{2}{b_2 - a_2} \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x_1, x_2) \cos\left(k_1 \pi \frac{x_1 - a_1}{b_1 - a_1}\right) \times \cos\left(k_2 \pi \frac{x_2 - a_2}{b_2 - a_2}\right) dx_1 dx_2.$$

For $a_1, a_2 > 0$, we define the auxiliary function

$$f_{\tau,a_1,a_2}(u,t) = f_\tau(u,t) I(u \le a_1, t \le a_2).$$

Then, $f_{\tau,a_1,a_2}(u,t)$ has a finite domain $[0, a_1] \times [0, a_2]$. By formula (4.1), we have for $(u,t) \in [0, a_1] \times [0, a_2]$,

$$f_\tau(u,t) = f_{\tau,a_1,a_2}(u,t) = \sum_{k_1=0}^{\infty}{}' \sum_{k_2=0}^{\infty}{}' \widehat{C}_{a_1,a_2,k_1,k_2} \cos\left(k_1 \pi \frac{u}{a_1}\right) \cos\left(k_2 \pi \frac{t}{a_2}\right), \quad (4.2)$$

where

$$\begin{aligned} \widehat{C}_{a_1,a_2,k_1,k_2} &= \frac{4}{a_1 a_2} \int_0^{a_1} \int_0^{a_2} f_{\tau,a_1,a_2}(u,t) \cos\left(k_1 \pi \frac{u}{a_1}\right) \cos\left(k_2 \pi \frac{t}{a_2}\right) dt du \\ &= \frac{4}{a_1 a_2} \int_0^{a_1} \int_0^{a_2} f_\tau(t,u) \cos\left(k_1 \pi \frac{u}{a_1}\right) \cos\left(k_2 \pi \frac{t}{a_2}\right) dt du. \end{aligned} \quad (4.3)$$

Truncating the series summation in (4.2) yields

$$f_\tau(u,t) \approx f_{\tau,a_1,a_2,3}(u,t) := \sum_{k_1=0}^{K_1-1}{}' \sum_{k_2=0}^{K_2-1}{}' \widehat{C}_{a_1,a_2,k_1,k_2} \cos\left(k_1 \pi \frac{u}{a_1}\right) \cos\left(k_2 \pi \frac{t}{a_2}\right), \quad (4.4)$$

where K_1 and K_2 are sufficiently large integers.

The coefficients $\widehat{C}_{a_1,a_2,k_1,k_2}$ can be approximated via characteristic function $\mathcal{F}_2 f_\tau(\omega_1, \omega_2)$ as follows. Using the following trigonometric relation

$$2\cos(\alpha)\cos(\beta) = \cos(\alpha + \beta) + \cos(\alpha - \beta),$$

https://doi.org/10.1017/asb.2016.27 Published online by Cambridge University Press

we have

$$\begin{aligned}\widehat{C}_{a_1,a_2,k_1,k_2} &= \frac{2}{a_1a_2}\int_0^{a_1}\int_0^{a_2} f_\tau(u,t)\cos\left(k_1\pi\frac{u}{a_1}+k_2\pi\frac{t}{a_2}\right)dtdu \\ &\quad+\frac{2}{a_1a_2}\int_0^{a_1}\int_0^{a_2} f_\tau(u,t)\cos\left(k_1\pi\frac{u}{a_1}-k_2\pi\frac{t}{a_2}\right)dtdu \\ &= \frac{2}{a_1a_2}\mathrm{Re}\left(\int_0^{a_1}\int_0^{a_2} f_\tau(u,t)\exp\left(ik_1\pi\frac{u}{a_1}+ik_2\pi\frac{t}{a_2}\right)dtdu\right) \\ &\quad+\frac{2}{a_1a_2}\mathrm{Re}\left(\int_0^{a_1}\int_0^{a_2} f_\tau(u,t)\exp\left(ik_1\pi\frac{u}{a_1}-ik_2\pi\frac{t}{a_2}\right)dtdu\right). \end{aligned} \tag{4.5}$$

When both a_1 and a_2 are large enough, we have

$$\begin{aligned}\widehat{C}_{a_1,a_2,k_1,k_2} &\approx C_{a_1,a_2,k_1,k_2} \\ &:= \frac{2}{a_1a_2}\mathrm{Re}\left(\int_0^{\infty}\int_0^{\infty} f_\tau(u,t)\exp\left(ik_1\pi\frac{u}{a_1}+ik_2\pi\frac{t}{a_2}\right)dtdu\right) \\ &\quad+\frac{2}{a_1a_2}\mathrm{Re}\left(\int_0^{\infty}\int_0^{\infty} f_\tau(u,t)\exp\left(ik_1\pi\frac{u}{a_1}-ik_2\pi\frac{t}{a_2}\right)dtdu\right) \\ &= \frac{2}{a_1a_2}\mathrm{Re}\left(\mathcal{F}_2 f_\tau\left(\frac{k_1\pi}{a_1},\frac{k_2\pi}{a_2}\right)\right)+\frac{2}{a_1a_2}\mathrm{Re}\left(\mathcal{F}_2 f_\tau\left(\frac{k_1\pi}{a_1},-\frac{k_2\pi}{a_2}\right)\right). \end{aligned} \tag{4.6}$$

Finally, replacing $\widehat{C}_{a_1,a_2,k_1,k_2}$ by C_{a_1,a_2,k_1,k_2} in (4.4), we obtain for $(u,t) \in [0,a_1] \times [0,a_2]$

$$f_\tau(u,t) \approx f_{\tau,a_1,a_2,4}(u,t) := \sum_{k_1=0}^{K_1-1}{}' \sum_{k_2=0}^{K_2-1}{}' C_{a_1,a_2,k_1,k_2}\cos\left(k_1\pi\frac{u}{a_1}\right)\cos\left(k_2\pi\frac{t}{a_2}\right). \tag{4.7}$$

Remark 4. *Except for the 2-COS formula* (4.7)*, there are also some other possibly feasible trigonometric series expansions. The following are some examples.*

- *Sine–Sine expansion:*

$$f_\tau(u,t) \approx \sum_{k_1=1}^{K_1-1}\sum_{k_2=1}^{K_2-1} C^{ss}_{a_1,a_2,k_1,k_2}\sin\left(k_1\pi\frac{u}{a_1}\right)\sin\left(k_2\pi\frac{t}{a_2}\right), \tag{4.8}$$

where

$$C^{ss}_{a_1,a_2,k_1,k_2} = \frac{2}{a_1a_2}\mathrm{Re}\left(\mathcal{F}_2 f_\tau\left(\frac{k_1\pi}{a_1},-\frac{k_2\pi}{a_2}\right)\right)-\frac{2}{a_1a_2}\mathrm{Re}\left(\mathcal{F}_2 f_\tau\left(\frac{k_1\pi}{a_1},\frac{k_2\pi}{a_2}\right)\right).$$

https://doi.org/10.1017/asb.2016.27 Published online by Cambridge University Press

- *Sine–Cosine expansion:*

$$f_\tau(u,t) \approx \sum_{k_1=1}^{K_1-1} \sum_{k_2=0}^{K_2-1}{}' C^{sc}_{a_1,a_2,k_1,k_2} \sin\left(k_1\pi \frac{u}{a_1}\right) \cos\left(k_2\pi \frac{t}{a_2}\right), \tag{4.9}$$

where

$$C^{sc}_{a_1,a_2,k_1,k_2} = \frac{2}{a_1a_2} \mathrm{Im}\left(\mathcal{F}_2 f_\tau\left(\frac{k_1\pi}{a_1}, \frac{k_2\pi}{a_2}\right)\right) + \frac{2}{a_1a_2} \mathrm{Im}\left(\mathcal{F}_2 f_\tau\left(\frac{k_1\pi}{a_1}, -\frac{k_2\pi}{a_2}\right)\right).$$

- *Cosine–Sine expansion:*

$$f_\tau(u,t) \approx \sum_{k_1=0}^{K_1-1}{}' \sum_{k_2=1}^{K_2-1} C^{cs}_{a_1,a_2,k_1,k_2} \cos\left(k_1\pi \frac{u}{a_1}\right) \sin\left(k_2\pi \frac{t}{a_2}\right), \tag{4.10}$$

where

$$C^{cs}_{a_1,a_2,k_1,k_2} = \frac{2}{a_1a_2} \mathrm{Im}\left(\mathcal{F}_2 f_\tau\left(\frac{k_1\pi}{a_1}, \frac{k_2\pi}{a_2}\right)\right) - \frac{2}{a_1a_2} \mathrm{Im}\left(\mathcal{F}_2 f_\tau\left(\frac{k_1\pi}{a_1}, -\frac{k_2\pi}{a_2}\right)\right).$$

Here, $\mathrm{Im}(\cdot)$ *denotes taking the imaginary part of the argument. The above series expansions can be obtained by some basic calculations as in the derivation of* (4.7)*. We omit the detailed procedure. The interested readers are referred to Section 2 in Meng and Ding (2013).*

Remark 5. *Usually, when we expand a function in terms of cosine series, we should first extend it to an even function; while for sine series expansion, we should first make an odd extension. Because density function* $f_\tau(u,0) \neq 0$ *and* $f_\tau(0,t) \neq 0$*, the odd extension of* f_τ *is not continuous at the zero point. If we use sine series expansion, the approximation will lead to a large bias in the neighborhood of* $(0,0)$*. Hence, the cosine series expansion is more preferable to sine series expansion. This is the reason why we use the COS method to approximate the density of the time to ruin.*

Remark 6. *For the finite time ruin probability, integrating* (4.7) *and interchanging the order of integration and summation, we obtain for* $0 \le t \le a_2$

$$\begin{aligned}\psi(u,t) &\approx \int_0^t \sum_{k_1=0}^{K_1-1}{}' \sum_{k_2=0}^{K_2-1}{}' C_{a_1,a_2,k_1,k_2} \cos\left(k_1\pi \frac{u}{a_1}\right) \cos\left(k_2\pi \frac{s}{a_2}\right) ds \\ &:= \sum_{k_1=0}^{K_1-1}{}' \sum_{k_2=0}^{K_2-1}{}' C_{a_1,a_2,k_1,k_2} t \cdot \mathrm{sinc}\left(k_2\pi \frac{t}{a_2}\right) \cos\left(k_1\pi \frac{u}{a_1}\right). \end{aligned} \tag{4.11}$$

Now, we study the approximation error of (4.7). For convenience, we consider the special case $a_1 = a_2 = a$, $K_1 = K_2 = K$. We measure the approximate

https://doi.org/10.1017/asb.2016.27 Published online by Cambridge University Press

error over a finite domain $[0, A] \times [0, A]$ by the L^2 distance, where $0 < A \le a$. Define

$$Er[f_{\tau,a,a,4}] = \left(\int_0^A \int_0^A (f_\tau(u,t) - f_{\tau,a,a,4}(u,t)^2 dtdu \right)^{\frac{1}{2}}.$$

We shall show that the above error converges to zero as $a, K \to \infty$. To this end, we need the following lemmas.

Lemma 1. *Suppose that*

$$\max_{u \ge 0} \int_0^\infty \left| \frac{\partial}{\partial t} f_\tau(u,t) \right| dt < \infty, \quad \max_{t \ge 0} \int_0^\infty \left| \frac{\partial}{\partial u} f_\tau(u,t) \right| du < \infty,$$
$$\int_0^\infty \int_0^\infty \left| \frac{\partial^2}{\partial u \partial t} f_\tau(u,t) \right| dtdu < \infty, \tag{4.12}$$

then

$$|\widehat{C}_{a,a,k_1,k_2}| \le \begin{cases} \frac{D_1}{k_1}, & k_1 \ne 0, \ k_2 = 0, \\ \frac{D_2}{k_2}, & k_1 = 0, \ k_2 \ne 0, \\ \frac{D_3}{k_1 k_2}, & k_1 \ne 0, \ k_2 \ne 0, \end{cases} \tag{4.13}$$

where

$$D_1 = \frac{4}{\pi} \max_{t \ge 0} \int_0^\infty \left| \frac{\partial}{\partial u} f_\tau(u,t) \right| du, \quad D_2 = 2C,$$
$$D_3 = \frac{4}{\pi^2} \int_0^\infty \int_0^\infty \left| \frac{\partial^2}{\partial u \partial t} f_\tau(u,t) \right| dtdu.$$

Proof. By (4.3), we have

$$\widehat{C}_{a,a,k_1,k_2} = \frac{4}{a^2} \int_0^a \int_0^a f_\tau(u,t) \cos\left(k_1 \pi \frac{u}{a}\right) \cos\left(k_2 \pi \frac{t}{a}\right) dtdu.$$

Using integration by parts, we obtain

$$\int_0^a f_\tau(u,t) \cos\left(k_1 \pi \frac{u}{a}\right) du = -\frac{a}{k_1 \pi} \int_0^a \sin\left(k_1 \pi \frac{u}{a}\right) \frac{\partial}{\partial u} f_\tau(u,t) du, \quad k_1 \ne 0,$$
$$\int_0^a f_\tau(u,t) \cos\left(k_2 \pi \frac{t}{a}\right) dt = -\frac{a}{k_2 \pi} \int_0^a \sin\left(k_2 \pi \frac{t}{a}\right) \frac{\partial}{\partial t} f_\tau(u,t) dt, \quad k_2 \ne 0,$$

and when $k_1 \neq 0,\ k_2 \neq 0$,

$$\int_0^a \int_0^a f_\tau(u,t) \cos\left(k_1\pi \frac{u}{a}\right) \cos\left(k_2\pi \frac{t}{a}\right) dt du$$
$$= -\frac{a}{k_2\pi} \int_0^a \int_0^a \sin\left(k_2\pi \frac{t}{a}\right) \frac{\partial}{\partial t} f_\tau(u,t) dt \cos\left(k_1\pi \frac{u}{a}\right) du$$
$$= \frac{a^2}{k_1 k_2 \pi^2} \int_0^a \int_0^a \sin\left(k_1\pi \frac{u}{a}\right) \sin\left(k_2\pi \frac{t}{a}\right) \frac{\partial^2}{\partial u \partial t} f_\tau(u,t) du dt.$$

Hence, when $k_1 \neq 0, k_2 = 0$,

$$|\widehat{C}_{a,a,k_1,0}| = \frac{4}{k_1\pi a} \left| \int_0^a \int_0^a \cos\left(k_2\pi \frac{t}{a}\right) \sin\left(k_1\pi \frac{u}{a}\right) \frac{\partial}{\partial u} f_\tau(u,t) du dt \right|$$
$$\leq \frac{4}{k_1\pi a} \int_0^a \int_0^a \left| \frac{\partial}{\partial u} f_\tau(u,t) \right| du dt \leq \frac{D_1}{k_1};$$

when $k_1 = 0, k_2 \neq 0$,

$$|\widehat{C}_{a,a,0,k_2}| = \frac{4}{k_2\pi a} \left| \int_0^a \int_0^a \cos\left(k_1\pi \frac{u}{a}\right) \sin\left(k_2\pi \frac{t}{a}\right) \frac{\partial}{\partial t} f_\tau(u,t) dt du \right|$$
$$\leq \frac{4}{k_2\pi a} \int_0^a \int_0^a \left| \frac{\partial}{\partial t} f_\tau(u,t) \right| dt du \leq \frac{D_2}{k_2};$$

when $k_1 \neq 0, k_2 \neq 0$,

$$|\widehat{C}_{a,a,k_1,k_2}| = \frac{4}{k_1 k_2 \pi^2} \left| \int_0^a \int_0^a \sin\left(k_1\pi \frac{u}{a}\right) \sin\left(k_2\pi \frac{t}{a}\right) \frac{\partial^2}{\partial u \partial t} f_\tau(u,t) dt du \right|$$
$$\leq \frac{4}{k_1 k_2 \pi^2} \int_0^a \int_0^a \left| \frac{\partial^2}{\partial u \partial t} f_\tau(u,t) \right| dt du \leq \frac{D_3}{k_1 k_2}.$$

This completes the proof. ■

Lemma 2. *Suppose that*

$$\int_0^\infty m_n(u) du < \infty, \quad \psi(u) \leq D_4 u^{-(n+1)}, \tag{4.14}$$

where D_4 is a positive constant. Then,

$$|C_{a,a,k_1,k_2} - \widehat{C}_{a,a,k_1,k_2}| \leq \frac{D_5}{a^{n+2}}, \tag{4.15}$$

where $D_5 = 4\int_0^\infty m_n(u)du + \frac{4}{n} D_4$.

https://doi.org/10.1017/asb.2016.27 Published online by Cambridge University Press

Proof. Because

$$
\begin{aligned}
&C_{a,a,k_1,k_2} - \widehat{C}_{a,a,k_1,k_2} \\
&= \frac{4}{a^2}\int_0^\infty \int_0^\infty f_\tau(u,t)\cos\left(k_1\pi\frac{u}{a}\right)\cos\left(k_2\pi\frac{t}{a}\right)dtdu \\
&\quad -\frac{4}{a^2}\int_0^a \int_0^a f_\tau(u,t)\cos\left(k_1\pi\frac{u}{a}\right)\cos\left(k_2\pi\frac{t}{a}\right)dtdu \\
&= \frac{4}{a^2}\int_0^\infty \int_a^\infty f_\tau(u,t)\cos\left(k_1\pi\frac{u}{a}\right)\cos\left(k_2\pi\frac{t}{a}\right)dtdu \\
&\quad +\frac{4}{a^2}\int_a^\infty \int_0^a f_\tau(u,t)\cos\left(k_1\pi\frac{u}{a}\right)\cos\left(k_2\pi\frac{t}{a}\right)dtdu,
\end{aligned}
$$

we have

$$
|C_{a,a,k_1,k_2} - \widehat{C}_{a,a,k_1,k_2}| \le \frac{4}{a^2}\int_0^\infty \int_a^\infty f_\tau(u,t)dtdu + \frac{4}{a^2}\int_a^\infty \int_0^a f_\tau(u,t)dtdu.
$$

By Markov inequality, we have

$$
\begin{aligned}
\int_0^\infty \int_a^\infty f_\tau(u,t)dtdu &= \int_0^\infty P(a < \tau < \infty | U_0 = u)du \\
&\le \int_0^\infty \frac{1}{a^n} E[\tau^n I(\tau < \infty)|U_0 = u]du \\
&= \frac{1}{a^n}\int_0^\infty m_n(u)du.
\end{aligned}
$$

By the second inequality in (4.14), we have

$$
\begin{aligned}
&\int_a^\infty \int_0^a f_\tau(u,t)dtdu \le \int_a^\infty \int_0^\infty f_\tau(u,t)dtdu \\
&= \int_a^\infty \psi(u)du \le \int_a^\infty D_4 u^{-(n+1)}du = \frac{D_4}{na^n}.
\end{aligned}
$$

Hence,

$$
|C_{a,a,k_1,k_2} - \widehat{C}_{a,a,k_1,k_2}| \le \frac{4}{a^2}\frac{1}{a^n}\int_0^\infty m_n(u)du + \frac{4}{a^2}\frac{D_4}{na^n} \le \frac{D_5}{a^{n+2}}.
$$

This completes the proof. ■

Remark 7. *The conditions in* (4.14) *are not very restrictive. If the claim size density* f_X *is light-tailed, both the* n*th moment of the time of ruin and the ultimate ruin probability have exponential upper bounds, which implies that conditions* (4.14) *hold true. Furthermore, it follows from Lemma 1 of Pitts and Politis (2008) that*

https://doi.org/10.1017/asb.2016.27 Published online by Cambridge University Press

the condition $\int_0^\infty m_n(u)du < \infty$ *in* (4.14) *is satisfied whenever* $EX^{n+2} < \infty$. *We note that the error bounds in the 1-COS and 2-COS methods are both dependent on the moments of the time to ruin. It follows from Lee and Willmot (2014, 2016) that the moments of the time to ruin can be obtained for Coxian claim sizes. Some other useful references on the moments of the time to ruin are Delbaen (1990) and Lin and Willmot (2000).*

Proposition 2. *Suppose that the conditions in* (4.12) *and* (4.14) *hold true, then*

$$Er[f_{\tau,a,a,4}] \le \left(\sqrt{D_1^2+2D_3^2}+\sqrt{D_2^2+2D_3^2}\right)\frac{a}{\sqrt{K-1}}+D_5\frac{K}{a^{n+1}}. \tag{4.16}$$

Proof. Since $f_\tau(u,t) = f_{\tau,a,a}(u,t)$ for $(u,t) \in [0,a]\times[0,a]$, by triangle inequality, we have

$$\begin{aligned}
&Er[f_{\tau,a,a,4}]\\
&\le \left(\int_0^A\int_0^A (f_{\tau,a,a}(u,t)-f_{\tau,a,a,3}(u,t))^2dtdu\right)^{\frac{1}{2}}\\
&\quad+\left(\int_0^A\int_0^A (f_{\tau,a,a,3}(u,t)-f_{\tau,a,a,4}(u,t))^2dtdu\right)^{\frac{1}{2}}\\
&\le \left(\int_0^a\int_0^a (f_{\tau,a,a}(u,t)-f_{\tau,a,a,3}(u,t))^2dtdu\right)^{\frac{1}{2}}\\
&\quad+\left(\int_0^a\int_0^a (f_{\tau,a,a,3}(u,t)-f_{\tau,a,a,4}(u,t))^2dtdu\right)^{\frac{1}{2}}\\
&:= \epsilon_{a,K,3}+\epsilon_{a,K,4}.
\end{aligned}$$

First, we consider the error $\epsilon_{a,K,3}$. Since

$$\begin{aligned}
&f_{\tau,a,a}(u,t)-f_{\tau,a,a,3}(u,t)\\
&=\sum_{k_1=0}^{\infty}{}'\sum_{k_2=K}^{\infty}\widehat{C}_{a,a,k_1,k_2}\cos\left(k_1\pi\frac{u}{a}\right)\cos\left(k_2\pi\frac{t}{a}\right)\\
&\quad+\sum_{k_1=K}^{\infty}\sum_{k_2=0}^{K-1}{}'\widehat{C}_{a,a,k_1,k_2}\cos\left(k_1\pi\frac{u}{a}\right)\cos\left(k_2\pi\frac{t}{a}\right),
\end{aligned}$$

https://doi.org/10.1017/asb.2016.27 Published online by Cambridge University Press

then by triangle inequality, we have

$$\epsilon_{a,K,3} \le \left(\int_0^a\int_0^a\left(\sum_{k_1=0}^{\infty}{}' \sum_{k_2=K}^{\infty}\widehat{C}_{a,a,k_1,k_2}\cos\left(k_1\pi\frac{u}{a}\right)\cos\left(k_2\pi\frac{t}{a}\right)\right)^2 dtdu\right)^{\frac{1}{2}}$$
$$+\left(\int_0^a\int_0^a\left(\sum_{k_1=K}^{\infty}\sum_{k_2=0}^{K-1}{}'\widehat{C}_{a,a,k_1,k_2}\cos\left(k_1\pi\frac{u}{a}\right)\cos\left(k_2\pi\frac{t}{a}\right)\right)^2 dtdu\right)^{\frac{1}{2}}$$
$$:= \epsilon_{a,K,3,1} + \epsilon_{a,K,3,2}. \tag{4.17}$$

It follows from formula (3.8) that

$$[\epsilon_{a,K,3,1}]^2 = \int_0^a\int_0^a\left(\sum_{k_1=0}^{\infty}{}' \sum_{k_2=K}^{\infty}\widehat{C}_{a,a,k_1,k_2}\cos\left(k_1\pi\frac{u}{a}\right)\cos\left(k_2\pi\frac{t}{a}\right)\right)^2 dtdu$$
$$= \sum_{k_1=0}^{\infty}{}' \sum_{k_2=K}^{\infty}|\widehat{C}_{a,a,k_1,k_2}|^2\int_0^a\left(\cos\left(k_1\pi\frac{u}{a}\right)\right)^2 du\cdot\int_0^a\left(\cos\left(k_2\pi\frac{t}{a}\right)\right)^2 dt$$
$$\le a^2\sum_{k_1=0}^{\infty}\sum_{k_2=K}^{\infty}|\widehat{C}_{a,a,k_1,k_2}|^2. \tag{4.18}$$

Furthermore, by Lemma 2, we have

$$[\epsilon_{a,K,3,1}]^2 \le a^2\sum_{k_2=K}^{\infty}|\widehat{C}_{a,a,0,k_2}|^2 + a^2\sum_{k_1=1}^{\infty}\sum_{k_2=K}^{\infty}|\widehat{C}_{a,a,k_1,k_2}|^2$$
$$\le a^2\sum_{k_2=K}^{\infty}\left(\frac{D_2}{k_2}\right)^2 + a^2\sum_{k_1=1}^{\infty}\sum_{k_2=K}^{\infty}\left(\frac{D_3}{k_1k_2}\right)^2$$
$$\le a^2\left(D_2^2+2D_3^2\right)\sum_{k_2=K}^{\infty}\frac{1}{k_2^2}$$
$$\le (D_2^2+2D_3^2)\frac{a^2}{K-1}. \tag{4.19}$$

Similarly,

$$[\epsilon_{a,K,3,2}]^2 \le a^2\sum_{k_1=K}^{\infty}\sum_{k_2=0}^{\infty}|\widehat{C}_{a,a,k_1,k_2}|^2 \le (D_1^2+2D_3^2)\frac{a^2}{K-1}. \tag{4.20}$$

https://doi.org/10.1017/asb.2016.27 Published online by Cambridge University Press

Hence,

$$\epsilon_{a,K,3} \le \left(\sqrt{D_1^2 + 2D_3^2} + \sqrt{D_2^2 + 2D_3^2}\right) \frac{a}{\sqrt{K-1}}. \qquad (4.21)$$

Next, for $\epsilon_{a,K,4}$, we have

$$\begin{aligned}
&[\epsilon_{a,K,4}]^2 \\
&= \int_0^a \int_0^a \left({\sum_{k_1=0}^{K-1}}' {\sum_{k_2=0}^{K-1}}' [\widehat{C}_{a,a,k_1,k_2} - C_{a,a,k_1,k_2}] \cos\left(k_1\pi \frac{u}{a}\right) \cos\left(k_2\pi \frac{t}{a}\right)\right)^2 dtdu \\
&\le a^2 \sum_{k_1=0}^{K-1} \sum_{k_2=0}^{K-1} |\widehat{C}_{a,a,k_1,k_2} - C_{a,a,k_1,k_2}|^2 \\
&\le D_5^2 \frac{K^2}{a^{2n+2}}. \qquad (4.22)
\end{aligned}$$

Finally, combining (4.17), (4.21) and (4.22) we complete the proof. ■

Remark 8. *Suppose that the conditions in* (4.14) *hold true for* $n \ge 1$. *We minimize the right-hand side of the inequality* (4.16) *w.r.t.* a *by setting*

$$\frac{d}{da}\left(\left(\sqrt{D_1^2 + 2D_3^2} + \sqrt{D_2^2 + 2D_3^2}\right) \frac{a}{\sqrt{K-1}} + D_5 \frac{K}{a^{n+1}}\right) = 0$$

to obtain optimal cut-off parameter

$$a^\bullet = \left(\frac{D_5(n+1)K\sqrt{K-1}}{\sqrt{D_1^2 + 2D_3^2} + \sqrt{D_2^2 + 2D_3^2}} \right)^{\frac{1}{n+2}} = O\left(K^{\frac{3}{2(n+2)}}\right).$$

Replacing a *by* $a^\bullet$ *in* (4.16)*, we obtain*

$$Er[f_{\tau,a,a,4}] = O(K^{-\frac{n-1}{2(n+2)}}).$$

5. Numerical examples

In this section, we present some numerical examples to check the approximation performance of the COS method. The computer used for all experiments has an Intel Core(TM) i5-4690 CPU, 3.50GHz with cache size 8.00 GB; the code is written in MATLAB 2013b. In the sequel, we set $c = 1.1$ and $\lambda = 1$, and we consider the following claim size densities:

1. Exponential: $f_X(x) = e^{-x}$, $x > 0$;
2. Erlang(2,2): $f_X(x) = 4xe^{-2x}$, $x > 0$;

https://doi.org/10.1017/asb.2016.27 Published online by Cambridge University Press

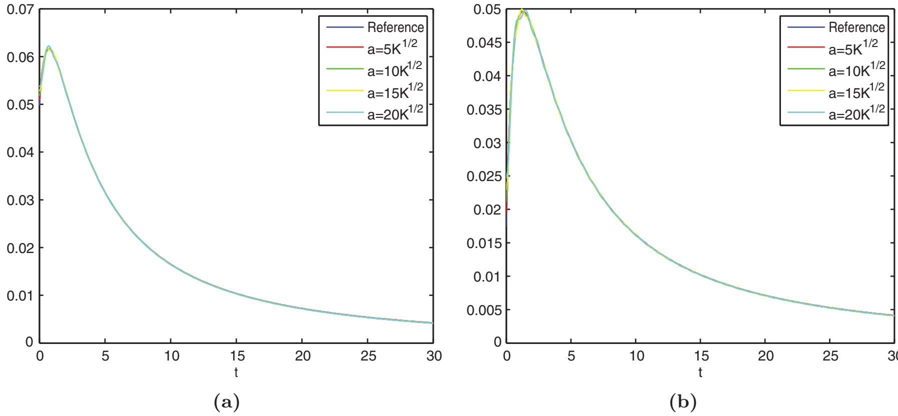


FIGURE 1: The 1-COS approximation of $f_\tau(u, t)$ for $u = 3$, $K = 2048$ and $a = 5K^{\frac{1}{2}}, 10K^{\frac{1}{2}}, 15K^{\frac{1}{2}}, 20K^{\frac{1}{2}}$. (a) Exponential claim size density; (b) Erlang (2,2) claim size density. (Color online)

3. Combination of two exponentials: $f_X(x) = 3e^{-1.5x} - 3e^{-3x},\ x > 0$;
4. Mixture of two exponentials: $f_X(x) = \frac{1}{6}e^{-\frac{1}{2}x} + \frac{4}{3}e^{-2x},\ x > 0$;
5. Erlang (4,4): $f_X(x) = \frac{4^4x^3e^{-4x}}{3!},\ x > 0$;
6. Mixture of two Erlangs: $f_X(x) = 0.4 \cdot 4xe^{-2x} + 0.6 \cdot \frac{3^3x^2}{2!}e^{-3x},\ x > 0$;
7. Mixture of four exponentials: $f_X(x) = 0.2 \cdot \frac{1}{2}e^{-\frac{x}{2}} + 0.3 \cdot 2e^{-2x} + 0.1 \cdot \frac{1}{3}e^{-\frac{x}{3}} + 0.4 \cdot 3e^{-3x},\ x > 0$;
8. Phase-type: $f_X(x) = \boldsymbol{\alpha} \exp(\boldsymbol{T}x)\boldsymbol{t},\ x > 0$, with $\boldsymbol{\alpha} = (\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$ and

$$\boldsymbol{T} = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{1}{3} \\ 1 & \frac{3}{2} & -3 \end{pmatrix}, \quad \boldsymbol{t} = \begin{pmatrix} 1 \\ \frac{7}{6} \\ \frac{1}{2} \end{pmatrix};$$

9. Inverse-Gaussian: $f_X(x) = \sqrt{\frac{1}{2\pi x^3}} \exp\left(-\frac{(x-1)^2}{2x}\right),\ x > 0$.

It can be checked that the means associated with the above density functions are all smaller than 1.1, so that the net profit condition $c > \lambda\mu_X$ holds true.

First, we conduct some numerical studies by using the 1-COS method to approximate the density of the time to ruin. We consider exponential and Erlang (2,2) claim size densities. In these two cases, explicit formulae for $f_\tau(u, t)$ can be expressed via the modified Bessel function and the generalized hypergeometric function. We refer the readers to Dickson (2007) and Dickson (2008). It follows from Remark 3 that the optimal cut-off parameter $a^* = O(K^{1/2})$. In Figure 1, we illustrate the effect of the parameter a by setting $K = 2048$ and $a = 5K^{\frac{1}{2}}, 10K^{\frac{1}{2}}, 15K^{\frac{1}{2}}, 20K^{\frac{1}{2}}$. For both exponential and Erlang (2,2) claim size densities, we find that the approximation is not very sensitive to the coefficient before $K^{\frac{1}{2}}$. Next, we study the effect of the parameter K. In Figure 2, we plot the approximated density curves by setting $a = 10K^{\frac{1}{2}}$ and

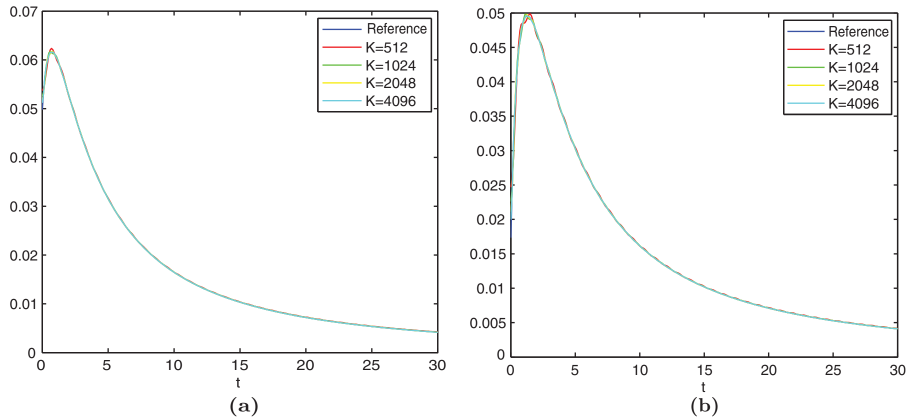


FIGURE 2: The 1-COS approximation of $f_\tau(u, t)$ for $u = 3$, $a = 10K^{\frac{1}{2}}$ and $K = 512, 1024, 2048, 4096$. (a) Exponential claim size density; (b) Erlang (2,2) claim size density. (Color online)

$K = 512, 1024, 2048, 4096$. We observe that the larger the value of K, the better the approximation. It follows from Figures 1 and 2 that the density of the time to ruin is harder to approximate when t is in the neighborhood of 2. This is possibly due to that the density curve is relatively complicated when $t = 2$ and it is relatively smooth when t is large. Furthermore, we present some approximation errors in Tables 1 and 2, where we set $K = 4096$ and $a = 10K^{\frac{1}{2}}$. Again, we find that the errors are very small.

Next, we compare the 1-COS method with the existing methods in the literature. Dickson and Willmot (2005) derive an infinite series expression for the density of the time to ruin when the claim size distribution is a mixture of Erlang distributions with the same scale parameter. We call their method D&W's method. As is shown in Willmot and Woo (2007), any countable mixture of Erlang distributions can be written in the form of a countable mixture of Erlang distributions with the same scale parameter, as long as the supremum of the set of scale parameters is finite. Hence, D&W's method can be used to compute the density of the time to ruin for a wide variety of claim size distributions. Now, we compare the 1-COS method with D&W's method. We shall consider the claim size densities (2)–(8). Set $u = 3$, $K = 4096$ and $a = 10K^{\frac{1}{2}}$. In Figure 3, we plot the densities of the ruin time for different claim size densities, where we use blue color to indicate D&W's method and use red color to indicate the 1-COS method. We can observe that all the blue curves are well covered by the red curves, which implies that the 1-COS method performs very well from the point view of accuracy. In Table 3, we also report some values of absolute error, where the absolute error is defined as the absolute value of the difference between the 1-COS solution and the reference value that is computed by D&W's method. Here, C(M)-2(4)-Exps means combination (mixture) of 2(4) exponentials; M-2-Erlangs means mixture of two Erlangs. We observe that the absolute errors are very small. In Table 4, we report the computation times when using the 1-COS

TABLE 1

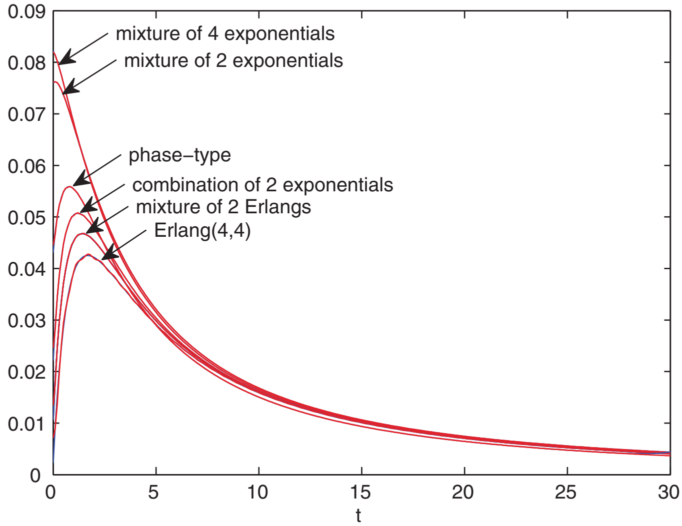
THE 1-COS APPROXIMATION ERRORS FOR $f_\tau(u, t)$ WITH EXPONENTIAL CLAIM SIZE DENSITY.

	$t = 0$	$t = 5$	$t = 10$	$t = 15$	$t = 20$	$t = 25$	$t = 30$
$u = 1$	1.28×10^{-2}	3.01×10^{-6}	1.14×10^{-6}	7.97×10^{-7}	6.75×10^{-7}	6.17×10^{-7}	5.85×10^{-7}
$u = 2$	4.39×10^{-4}	1.31×10^{-6}	1.38×10^{-6}	1.40×10^{-6}	1.40×10^{-6}	1.41×10^{-6}	1.41×10^{-6}
$u = 3$	1.41×10^{-3}	2.29×10^{-6}	2.09×10^{-6}	2.05×10^{-6}	2.04×10^{-6}	2.04×10^{-6}	2.04×10^{-6}
$u = 4$	1.10×10^{-3}	2.54×10^{-6}	2.38×10^{-6}	2.36×10^{-6}	2.35×10^{-6}	2.35×10^{-6}	2.35×10^{-6}
$u = 5$	6.21×10^{-4}	2.70×10^{-6}	2.61×10^{-6}	2.60×10^{-6}	2.59×10^{-6}	2.60×10^{-6}	2.60×10^{-6}

TABLE 2

THE 1-COS APPROXIMATION ERRORS FOR $f_\tau(u, t)$ WITH ERLANG(2,2) CLAIM SIZE DENSITY.

	$t = 0$	$t = 5$	$t = 10$	$t = 15$	$t = 20$	$t = 25$	$t = 30$
$u = 1$	1.73×10^{-2}	5.07×10^{-6}	2.58×10^{-6}	2.12×10^{-6}	1.95×10^{-6}	1.88×10^{-6}	1.84×10^{-6}
$u = 2$	2.80×10^{-3}	1.38×10^{-6}	9.54×10^{-7}	8.77×10^{-7}	8.57×10^{-7}	8.52×10^{-7}	8.52×10^{-7}
$u = 3$	2.64×10^{-3}	1.20×10^{-6}	3.81×10^{-7}	5.44×10^{-7}	2.58×10^{-7}	4.36×10^{-7}	2.81×10^{-7}
$u = 4$	9.65×10^{-4}	3.19×10^{-6}	1.76×10^{-5}	1.78×10^{-5}	1.38×10^{-5}	1.01×10^{-5}	7.23×10^{-5}
$u = 5$	2.66×10^{-4}	2.59×10^{-5}	1.02×10^{-5}	9.98×10^{-6}	7.74×10^{-6}	5.71×10^{-6}	4.19×10^{-6}



(a)

FIGURE 3: Comparison of the 1-COS method and D&W's method for $u = 3$, $K = 4096$ and $a = 10K^{\frac{1}{2}}$. Red curve: 1-COS method; blue curve: D&W's method. (Color online)

method and D&W's method, where the same parameter settings are used as above. The computation time is counted based on computing the density curves illustrated in Figure 3. We find that the computation time of the 1-COS method is less than one second, and overall, the 1-COS method is more than 100 times faster than D&W's method.

Finally, we use the 2-COS method to approximate the density of the time to ruin. We set $u = 3$ and consider exponential and phase-type claim size densities. It follows from Remark 8 that the optimal cut-off parameter $a^{\bullet} = O(K^{\frac{3}{2(n+2)}})$. In Figure 4, we study the impact of the cut-off parameter a by setting $a = 10K^{\frac{3}{10}}, 20K^{\frac{3}{10}}, 30K^{\frac{3}{10}}, 40K^{\frac{3}{10}}$. When K is large enough, we find that the approximation is not very sensitive to the coefficient before $K^{\frac{3}{10}}$. In Figure 5, we show the approximation results for varying values of K with $a = 30K^{\frac{3}{10}}$, we find that the approximated curves perform well as K increases. When the true density curve has a large curvature, the approximation is not good enough. This fact has also been observed in the application of the 1-COS method. For the approximated curve in Figure 5 with $K = 4096$, the computation time is about 90 seconds. In a comparison with the 1-COS method, the computation of the 2-COS method is not only slower, but also lead to a lower accuracy. However, the 2-COS method can be used to approximate $f_{\tau}(u, t)$ for a larger class of claim size density functions. Let us consider the Inverse-Gaussian claim size density as illustrated at the beginning of this section. In this case, both the 1-COS method and D&W's method cannot be used, but we can use the 2-COS method because f_X has an explicit Fourier transform

TABLE 3

ABSOLUTE ERRORS BETWEEN APPROXIMATES BY THE 1-COS METHOD AND D&W METHOD.

	$t = 0$	$t = 5$	$t = 10$	$t = 15$	$t = 20$	$t = 25$	$t = 30$
C-2-Exps	2.42×10^{-3}	1.22×10^{-6}	2.38×10^{-7}	3.51×10^{-7}	7.31×10^{-8}	2.09×10^{-7}	4.29×10^{-7}
M-2-Exps	1.58×10^{-4}	5.45×10^{-6}	5.33×10^{-6}	5.37×10^{-6}	5.42×10^{-6}	5.46×10^{-6}	5.52×10^{-6}
Erlang(4,4)	4.83×10^{-3}	1.55×10^{-5}	1.03×10^{-5}	9.39×10^{-6}	9.81×10^{-6}	1.04×10^{-5}	1.09×10^{-5}
M-2-Erlangs	2.72×10^{-3}	1.05×10^{-6}	9.64×10^{-7}	1.21×10^{-6}	8.34×10^{-7}	4.31×10^{-7}	1.15×10^{-7}
M-4-Exps	2.15×10^{-4}	4.22×10^{-6}	4.22×10^{-6}	4.26×10^{-6}	4.29×10^{-6}	4.31×10^{-6}	4.33×10^{-6}
phase-type	1.43×10^{-3}	8.91×10^{-7}	3.14×10^{-7}	1.38×10^{-7}	1.36×10^{-7}	2.56×10^{-7}	7.77×10^{-4}

TABLE 4

COMPUTATION TIME (SECONDS) COMPARISON BETWEEN D&W'S METHOD AND THE 1-COS METHOD.

	D&W's method	1-COS
C-2-Exps	105.0872	0.1247
M-2-Exps	52.1508	0.1136
Erlang(4,4)	20.4540	0.1614
M-2-erlangs	105.1909	0.1957
M-4-Exps	110.1276	0.1876
phase-type	52.2192	0.1566

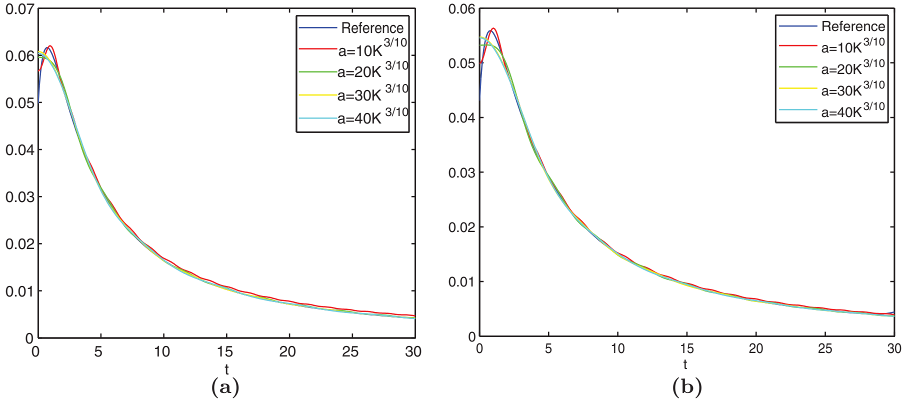


FIGURE 4: The 2-COS approximation of $f_\tau(u, t)$ for $u = 3$, $K = 2048$ and $a = 10K^{\frac{3}{10}}, 20K^{\frac{3}{10}}, 30K^{\frac{3}{10}}, 40K^{\frac{3}{10}}$. (a) Exponential claim size density; (b) phase-type claim size density. (Color online)

$\mathcal{F}_1 f_X(\omega) = \exp\left(1 - \sqrt{1 - 2i\omega}\right)$. We compare the 2-COS method with the path Monte Carlo method. Again, we set $u = 3$. We generate 10^7 sample paths of the surplus process to estimate the density function of the time to ruin and plot the estimated result in Figure 6 with light blue color. When we use the 2-COS method, we set $K = 4096$, $a = 30K^{\frac{3}{10}}$, and plot the approximated curve in Figure 6 with red color. It can be observed that the 2-COS method also performs well in this case. When we use path Monte Carlo method, it takes more than 20 hours; when we use the 2-COS method, it takes about 90 seconds. Hence, the 2-COS method is more efficient than the Monte Carlo method from the point view of computation time.

6. CONCLUDING REMARKS

In this article, the density of the ruin time is studied in the classical compound Poisson risk model. We use the COS method to approximate the density function of the time to ruin. Both error analysis and numerical experiments are made

https://doi.org/10.1017/asb.2016.27 Published online by Cambridge University Press

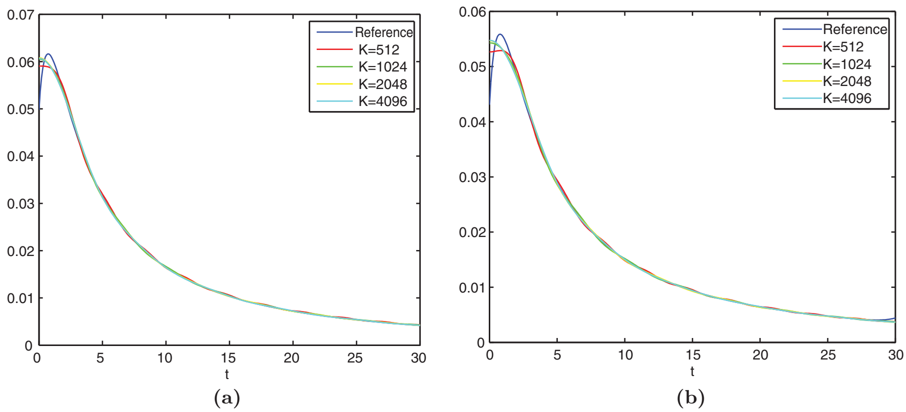


FIGURE 5: The 2-COS approximation of $f_\tau(u, t)$ for $u = 3$, $a = 30K^{\frac{3}{10}}$ and $K = 512, 1024, 2048, 4096$. (a) Exponential claim size density; (b) phase-type claim size density. (Color online)

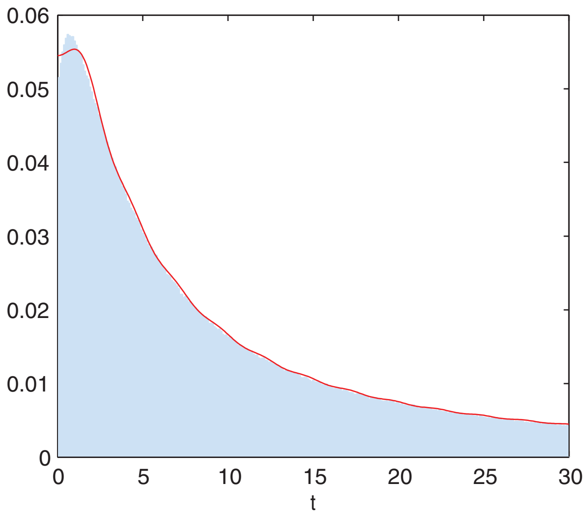


FIGURE 6: Comparison of the path Monte Carlo method and the 2-COS method. Light blue color: path Monte Carlo; red color: 2-COS. (Color online)

to check the efficiency of this method. It is shown that the COS method is easily applied as long as the corresponding Fourier transform is available. In particular, the 1-COS method is applicable as long as the claim size density belongs to the rational family, and the 2-COS method is applicable when the expression of the Fourier transform of the claim size density exists. The 2-COS method can be used for approximation for a larger class of density functions, however, the 1-COS method can lead to a faster computation speed and a higher accuracy.

There are some open problems that can be done in the future. First, we can study the statistical estimation of the density of the ruin time. We may suppose that both the Poisson intensity λ and the claim size density are unknown, but samples on the claim numbers and claim sizes are available. Because in this case we cannot obtain the one-dimensional Fourier transform $\mathcal{F}_1 f_\tau(u, \omega)$, we have to use the 2-COS method. The difficulty comes from the analysis of the consistency of the estimator. Next, we can use the COS method to study the expected discounted dividends before ruin in a risk model with constant dividend barrier. It is known that the expected discounted dividends before ruin can be expressed in terms of the q-scale function of the surplus process in the Lévy risk model. Although the q-scale function is not integrable, we can multiply it by an exponential decay factor so that its Fourier transform exists. Hence, we can recover the q-scale function by the COS method. Finally, the COS method can also be applied to compute the density of ruin time in some more general risk models, such as the Lévy risk model and the Sparre Andersen risk model. Furthermore, some more general density functions involving the number of claims and the deficit at ruin can also be computed by this method.

Acknowledgements

The author would like to thank the Editor and two referees for their valuable suggestions which significantly improved the paper. This research was supported by the National Natural Science Foundation of China [11471058, 11101451], the Natural Science Foundation Project of CQ CSTC of China [cstc2014jcyjA00007] and MOE (Ministry of Education in China) Project of Humanities and Social Sciences (Project No.16YJC910005).

References

Borovkov, K.A. and Dickson, D.C.M. (2008) On the ruin time distribution for a Sparre Andersen process with exponential claim sizes. *Insurance: Mathematics and Economics*, **42**(3), 1104–1108.

Chau, K.W., Yam S.C.P. and Yang, H. (2015a) Modern Fourier-cosine method for Gerber-Shiu function. *Insurance: Mathematics and Economics*, **61**, 170–180.

Chau, K.W., Yam S.C.P. and Yang, H. (2015b) Modern Fourier-cosine method for ruin probabilities. *Journal of Computational and Applied Mathematics*, **281**, 94–106.

Delbaen, F. (1990) A remark on the moments of ruin time in classical risk theory. *Insurance: Mathematics and Economics*, **9**, 121–126.

Dickson, D.C.M. (2007) Some finite time ruin problems. *Annals of Actuarial Science*, **2**, 217–232.

Dickson, D.C.M. (2008) Some explicit solutions for the joint density of the time of ruin and the deficit at ruin. *ASTIN Bulletin*, **38**(1), 259–276.

Dickson, D.C.M., Hughes, B.D. and Zhang, L. (2005) The density of the time to ruin for a Sparre Andersen process with Erlang arrivals and exponential claims. *Scandinavian Actuarial Journal*, **2005**(5), 358–376.

Dickson, D.C.M. and Li, S. (2010) Finite time ruin problems for the Erlang(2) risk model. *Insurance: Mathematics and Economics*, **46**(1), 12–18.

https://doi.org/10.1017/asb.2016.27 Published online by Cambridge University Press

DICKSON, D.C.M. and LI, S. (2012) Erlang risk models and finite time ruin problems. *Scandinavian Actuarial Journal*, **3**, 183–202.
DICKSON, D.C.M. and WILLMOT, G.E. (2005) The density of the time to ruin in the classical Poisson risk model. *ASTIN Bulletin*, **35**(1), 45–60.
DREKIC, S. and WILLMOT, G.E. (2003) On the density and moments of the time of ruin with exponential claims. *ASTIN Bulletin*, **33**, 11–21.
FANG, F. and OOSTERLEE, C.W. (2008) A novel option pricing method based on Fourier cosine series expansions. *SIAM Journal on Scientific Computing*, **31**(2), 826–848.
FANG, F. and OOSTERLEE, C.W. (2009) Pricing early-exercise and discrete barrier options by Fourier cosine series expansions. *Numerische Mathematik*, **114**, 27–62.
GARCIA, J.M.A. (2005) Explicit solutions for survival probabilities in the classical risk model. *ASTIN Bulletin*, **35**(1), 113–130.
GERBER, H.U. and SHIU, E.S.W. (1998) On the time value of ruin; with discussion and a reply by the authors. *North American Actuarial Journal*, **2**(1), 48–78.
LANDRIAULT, D., SHI, T. and WILLMOT, G.E. (2011) Joint densities involving the time to ruin in the Sparre-Andersen risk model with exponential claim sizes. *Insurance: Mathematics and Economics*, **49**, 371–379.
LEE, W.Y. and WILLMOT, G.E. (2014) On the moments of the time to ruin in dependent Sparre Andersen models with emphasis on Coxian interclaim times. *Insurance: Mathematics and Economics*, **59**, 1–10.
LEE, W.Y. and WILLMOT, G.E. (2016) The moments of the time to ruin in dependent Sparre Andersen models with Coxian claim sizes. *Scandinavian Actuarial Journal*, **2016**(6), 550–564.
LIN, X.S. and WILLMOT, G.E. (2000) The moments of the time of ruin, the surplus before ruin, and the deficit at ruin. *Insurance: Mathematics and Economics*, **27**(1), 19–44.
MENG, Q.-J. and DING D. (2013) An efficient pricing method for rainbow options based on two-dimensional modified sine-sine series expansions. *International Journal of Computer Mathematics*, **90**(5), 1096–1113.
PITTS, S.M. and POLITIS, K. (2008) Approximations for the moments of ruin time in the compound Poisson model. *Insurance: Mathematics and Economics*, **42**, 668–679.
PIVATO, M. (2010) *Linear Partial Differential Equations and Fourier Theory*. New York: Cambridge University Press.
RUIJTER, M.J. and OOSTERLEE, C.W. (2012) Two-dimensional Fourier cosine series expansion method for pricing financial options. *SIAM Journal on Scientific Computing*, **34**, 642–671.
SHI, T. and LANDRIAULT, D. (2013) Distribution of the time to ruin is some Sparre Andersen risk models. *ASTIN Bulletin*, **43**(1), 39–59.
WILLMOT, G.E. and WOO, J.-K. (2007) On the class of Erlang mixtures with risk theoretic applications. *North American Actuarial Journal*, **11**(2), 2007.
ZHANG, B. and OOSTERLEE, C.W. (2013) Efficient pricing of European-style Asian options under exponential Lévy processes based on Fourier cosine expansions. *SIAM Journal on Financial Mathematics*, **4**(1), 399–426.
ZHANG, B. and OOSTERLEE, C.W. (2014) Pricing of early-exercise Asian options under Lévy processes based on Fourier cosine expansions. *Applied Numerical Mathematics*, **78**, 14–30.
ZHANG, Z. and YANG, H. (2013) Nonparametric estimate of the ruin probability in a pure-jump Lévy risk model. *Insurance: Mathematics and Economics*, **53**, 24–35.
ZHANG, Z. and YANG, H. (2014) Nonparametric estimation for the ruin probability in a Lévy risk model under low-frequency observation. *Insurance: Mathematics and Economics*, **59**, 168–177.

ZHIMIN ZHANG (Corresponding author)
College of Mathematics and Statistics
Chongqing University, Chongqing
401331, P.R. China
E-Mail: zmzhang@cqu.edu.cn

https://doi.org/10.1017/asb.2016.27 Published online by Cambridge University Press