

ON m -COVERS AND m -SYSTEMS

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Abstract

Let $\mathcal{A} = \{a_s \pmod{n_s}\}_{s=0}^k$ be a system of residue classes. With the help of cyclotomic fields we obtain a theorem which unifies several previously known results related to the covering multiplicity of \mathcal{A} . In particular, we show that if every integer lies in more than $m_0 = \lfloor \sum_{s=1}^k 1/n_s \rfloor$ members of \mathcal{A} , then for any $a = 0, 1, 2, \dots$ there are at least $\binom{m_0}{\lfloor a/n_0 \rfloor}$ subsets I of $\{1, \dots, k\}$ with $\sum_{s \in I} 1/n_s = a/n_0$. We also characterize when any integer lies in at most m members of \mathcal{A} , where m is a fixed positive integer.

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1. Main results

For $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$, we simply denote the residue class

$$\{x \in \mathbb{Z} \mid x \equiv a \pmod{n}\}$$

by $a(n)$. For a finite system

$$A = \{a_s(n_s)\}_{s=1}^k \tag{1.1}$$

of residue classes, the function $w_A : \mathbb{Z} \rightarrow \mathbb{N} = \{0, 1, 2, \dots\}$ given by

$$w_A(x) = |\{1 \leq s \leq k \mid x \in a_s(n_s)\}| \tag{1.2}$$

is called the *covering function* of A . Obviously $w_A(x)$ is periodic modulo the least common multiple N of the moduli n_1, \dots, n_k , and it is easy to see that the average $\sum_{x=0}^{N-1} w_A(x)/N$ equals $\sum_{s=1}^k 1/n_s$. As in [13] we call $m(A) = \min_{x \in \mathbb{Z}} w_A(x)$ the *covering multiplicity* of system (1.1).

Let m be any positive integer. If $w_A(x) \geq m$ for all $x \in \mathbb{Z}$ (that is, $m(A) \geq m$), then (1.1) is said to be an m -cover of \mathbb{Z} as in [11, 12], and in this case $\sum_{s=1}^k 1/n_s \geq m$. Covers (that is, 1-covers) of \mathbb{Z} were first introduced by Erdős [2] and they are also called covering systems. If $w_A(x) = m$ for all $x \in \mathbb{Z}$, then we call (1.1) an *exact*

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m -cover of \mathbb{Z} as in [12, 13] (and in this case $\sum_{s=1}^k 1/n_s = m$). By [8, Theorem 1.3], when $m \geq 2$ there are exact m -covers of \mathbb{Z} that cannot split into two covers of \mathbb{Z} . If $w_A(x) \leq m$ for all $x \in \mathbb{Z}$, then we call (1.1) an m -system, and in this case $\sum_{s=1}^k 1/n_s \leq m$; any 1-system is said to be *disjoint*.

The reader may consult Guy [5, pp. 383–390] and Simpson [9] for some problems and results in covering theory. Covers of \mathbb{Z} have many surprising applications; see, for example [1], [5, Sections A19 and B21], [14, 20, 21]. Sun [19] showed that m -covers of \mathbb{Z} are related to zero-sum problems for abelian groups. Also, the topic of covering systems stimulated the birth of some new algebraic results (see [15, 17]).

Throughout this paper, for $a, b \in \mathbb{Z}$ we set $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$ and define $\lfloor a, b \rfloor$ and $\{a, b\}$ similarly. As usual, the integral part and the fractional part of a real number α are denoted by $\lfloor \alpha \rfloor$ and $\{\alpha\}$, respectively.

For system (1.1) we define its *dual system* A^* by

$$A^* = \{a_s + r(n_s) \mid 1 \leq r < n_s, 1 \leq s \leq k\}. \tag{1.3}$$

As $\{a_s + r(n_s)\}_{r=0}^{n_s-1}$ is a partition of \mathbb{Z} for any $s \in [1, k]$, we have $w_A(x) + w_{A^*}(x) = k$ for all $x \in \mathbb{Z}$. Thus $w_A(x) \leq m$ for all $x \in \mathbb{Z}$ if and only if $w_{A^*}(x) \geq k - m$ for all $x \in \mathbb{Z}$. This simple and new observation shows that we can study m -systems via covers of \mathbb{Z} , and construct covers of \mathbb{Z} via m -systems.

By a result in [12], if (1.1) is an m -cover of \mathbb{Z} then for any $m_1, \dots, m_k \in \mathbb{Z}^+$ there are at least m positive integers in the form $\sum_{s \in I} m_s/n_s$ with $I \subseteq [1, k]$. Applying this result to the dual A^* of an m -system (1.1), we obtain that there are more than $k - m$ integers in the form $\sum_{s=1}^k x_s/n_s$ with $x_s \in [0, n_s]$; equivalently, at most $m - 1$ of the numbers in $[1, k]$ cannot be written in the form $\sum_{s=1}^k m_s/n_s = k - \sum_{s=1}^k (n_s - m_s)/n_s$ with $m_s \in [1, n_s]$. This implies the following result stated in [16, Remark 1.3]: if (1.1) is an m -system, then there are $m_1, \dots, m_k \in \mathbb{Z}^+$ such that $\sum_{s=1}^k m_s/n_s = m$.

The following theorem unifies and generalizes several known results.

THEOREM 1.1. *Let $\mathcal{A} = \{a_s(n_s)\}_{s=0}^k$ be a finite system of residue classes with $m(\mathcal{A}) > m = \lfloor \sum_{s=1}^k m_s/n_s \rfloor$, where $m_1, \dots, m_k \in \mathbb{Z}^+$. Then, for any $0 \leq \alpha < 1$, either*

$$\sum_{\substack{I \subseteq [1, k] \\ \sum_{s \in I} m_s/n_s = (\alpha + a)/n_0}} (-1)^{|I|} \exp\left(2\pi i \sum_{s \in I} \frac{a_s m_s}{n_s}\right) = 0 \tag{1.4}$$

for any $a \in \mathbb{N}$, or

$$\left| \left\{ I \subseteq [1, k] : \sum_{s \in I} \frac{m_s}{n_s} = \frac{\alpha + a}{n_0} \right\} \right| \geq \binom{m}{\lfloor a/n_0 \rfloor} \tag{1.5}$$

for all $a = 0, 1, 2, \dots$

EXAMPLE 1.2. Erdős observed that $\{0(2), 0(3), 1(4), 5(6), 7(12)\}$ is a cover of \mathbb{Z} with the moduli

$$n_0 = 2, \quad n_1 = 3, \quad n_2 = 4, \quad n_3 = 6, \quad n_4 = 12$$

distinct. As $\lfloor \sum_{s=1}^4 1/n_s \rfloor = 0$, by Theorem 1.1 in the case $\alpha = 0$ we have $\sum_{s \in I} 1/n_s = 1/n_0 = 1/2$ for some $I \subseteq [1, 4]$; we can actually take $I = \{1, 3\}$. Since $\sum_{s=1}^4 1/n_s < (5/6 + 1)/n_0 = 11/12$, by Theorem 1.1 in the case $\alpha = 5/6$ the set $\mathcal{I} = \{I \subseteq [1, 4] : \sum_{s \in I} 1/n_s = 5/12\}$ cannot have a single element; in fact, $\mathcal{I} = \{\{1, 4\}, \{2, 3\}\}$ and

$$\begin{aligned} & (-1)^{|(1,4)|} \exp(2\pi i(0/n_1 + 7/n_4)) + (-1)^{|(2,3)|} \exp(2\pi i(1/n_2 + 5/n_3)) \\ &= -\exp(\pi i/6) + \exp(\pi i/6) = 0. \end{aligned}$$

COROLLARY 1.3. *If $\mathcal{A} = \{a_s(n_s)\}_{s=0}^k$ is a finite system of residue classes with $w_{\mathcal{A}}(x) > m = \lfloor \sum_{s=1}^k 1/n_s \rfloor$ for all $x \in \mathbb{Z}$, then*

$$\left| \left\{ I \subseteq [1, k] : \sum_{s \in I} \frac{1}{n_s} = \frac{a}{n_0} \right\} \right| \geq \binom{m}{\lfloor a/n_0 \rfloor} \text{ for all } a \in \mathbb{N}. \tag{1.6}$$

In particular, if (1.1) has covering multiplicity $m(A) = \lfloor \sum_{s=1}^k 1/n_s \rfloor$, then

$$\left| \left\{ I \subseteq [1, k] : \sum_{s \in I} \frac{1}{n_s} = n \right\} \right| \geq \binom{m(A)}{n} \text{ for each } n \in \mathbb{N}. \tag{1.7}$$

PROOF. Observe that the left-hand side of (1.4) is nonzero in the case $\alpha = a = 0$. So (1.6) follows from Theorem 1.1 immediately. In the case $n_0 = 1$ this yields the latter result in Corollary 1.3. \square

REMARK 1.4. Let (1.1) be an exact m -cover of \mathbb{Z} . Then $\sum_{s=1}^k 1/n_s = m$ and $\lfloor \sum_{s \in [1, k] \setminus \{t\}} 1/n_s \rfloor = m - 1$ for any $t = 1, \dots, k$. So Corollary 1.3 implies the following result in [13]: for any $t \in [1, k]$ and $a \in \mathbb{N}$,

$$\left| \left\{ I \subseteq [1, k] \setminus \{t\} : \sum_{s \in I} \frac{1}{n_s} = \frac{a}{n_t} \right\} \right| \geq \binom{m-1}{\lfloor a/n_t \rfloor}.$$

As $m(A) = \sum_{s=1}^k 1/n_s$, the inequality $|\{I \subseteq [1, k] : \sum_{s \in I} 1/n_s = n\}| \geq \binom{m}{n}$ also holds for all $n = 0, 1, \dots, m$, which was first established in [10] by means of the Riemann zeta function.

COROLLARY 1.5. *Let (1.1) be an m -system with $m = \lceil \sum_{s=1}^k 1/n_s \rceil$, where $\lceil \alpha \rceil$ denotes the least integer not smaller than a real number α . Then*

$$\left| \left\{ \langle m_1, \dots, m_k \rangle \in \mathbb{Z}^k : m_s \in [1, n_s], \sum_{s=1}^k \frac{m_s}{n_s} = n \right\} \right| \geq \binom{k-m}{n-m} \tag{1.8}$$

for every $n = m, \dots, k$.

PROOF. Let $n \in [m, k]$. Clearly the left-hand side of (1.8) coincides with

$$L := \left| \left\{ \langle x_1, \dots, x_k \rangle : x_s \in [0, n_s - 1], \sum_{s=1}^k \frac{x_s}{n_s} = \sum_{s=1}^k \frac{n_s}{n_s} - n = k - n \right\} \right|.$$

Since $\sum_{s=1}^k 1/n_s > m - 1$, $w_A(x) = m$ for some $x \in \mathbb{Z}$. As the dual A^* of (1.1) has covering multiplicity $m(A^*) = k - m$, applying Corollary 1.3 to A^* leads to $L \geq \binom{k-m}{k-n} = \binom{k-m}{n-m}$. This concludes the proof. \square

REMARK 1.6. When (1.1) is an exact m -cover of \mathbb{Z} , it was proved in [13] (by a different approach) that for each $n \in \mathbb{N}$ the equation $\sum_{s=1}^k x_s/n_s = n$ with $x_s \in [0, n_s)$ has at least $\binom{k-m}{n}$ solutions.

COROLLARY 1.7. Let $\mathcal{A} = \{a_s(n_s)\}_{s=0}^k$ be a finite system of residue classes with $m(\mathcal{A}) > m = \lfloor \sum_{s=1}^k m_s/n_s \rfloor$, where $m_1, \dots, m_k \in \mathbb{Z}^+$. Suppose that $J \subseteq [1, k]$ and $\sum_{s \in I} m_s/n_s = \sum_{s \in J} m_s/n_s$ for no $I \subseteq [1, k]$ with $I \neq J$. Then

$$\left\{ n_0 \sum_{s \in J} \frac{m_s}{n_s} \right\} + \left\{ n_0 \sum_{s \in \bar{J}} \frac{m_s}{n_s} \right\} < 1, \tag{1.9}$$

where $\bar{J} = [1, k] \setminus J$. Also

$$\sum_{s \in J} \frac{m_s}{n_s} \geq m \quad \text{or} \quad \sum_{s \in \bar{J}} \frac{m_s}{n_s} \geq m. \tag{1.10}$$

PROOF. Let $v = \sum_{s \in J} m_s/n_s$, $\alpha = \{n_0 v\}$ and $b = \lfloor n_0 v \rfloor$. Then $(\alpha + b)/n_0 = v$ and

$$\begin{aligned} & \sum_{\substack{I \subseteq [1, k] \\ \sum_{s \in I} m_s/n_s = v}} (-1)^{|I|} \exp\left(2\pi i \sum_{s \in I} \frac{a_s m_s}{n_s}\right) \\ &= (-1)^{|J|} \exp\left(2\pi i \sum_{s \in J} \frac{a_s m_s}{n_s}\right) \neq 0. \end{aligned}$$

By Theorem 1.1, inequality (1.5) holds for any $a \in \mathbb{N}$. Applying (1.5) with $a = mn_0 + n_0 - 1$ we find that $\sum_{s \in I} m_s/n_s = (\alpha + mn_0 + n_0 - 1)/n_0$ for some $I \subseteq [1, k]$, therefore $\sum_{s=1}^k m_s/n_s \geq m + (\alpha + n_0 - 1)/n_0$. As $\lfloor \sum_{s=1}^k m_s/n_s \rfloor = m$, we must have

$$\left\{ \sum_{s=1}^k \frac{m_s}{n_s} \right\} \geq \frac{\alpha + n_0 - 1}{n_0}, \text{ that is, } n_0 - 1 + \alpha \leq n_0 \left\{ \sum_{s=1}^k \frac{m_s}{n_s} \right\} < n_0.$$

Therefore $\alpha \leq \{n_0 \{\sum_{s=1}^k m_s/n_s\}\} = \{n_0 \sum_{s=1}^k m_s/n_s\}$, which is equivalent to (1.9).

Inequality (1.5) in the case $a = b$ gives that $\binom{m}{\lfloor b/n_0 \rfloor} \leq 1$, thus $\lfloor v \rfloor \in \{0, m\}$. As $n_0 \{v\} - \alpha = \lfloor n_0 \{v\} \rfloor \leq n_0 - 1$, $\{v\} \leq (\alpha + n_0 - 1)/n_0 \leq \{\sum_{s=1}^k m_s/n_s\}$. If $\lfloor v \rfloor = 0$, then $m + v \leq m + \{\sum_{s=1}^k m_s/n_s\} = \sum_{s=1}^k m_s/n_s$ and hence $\sum_{s \in \bar{J}} m_s/n_s \geq m$. Therefore (1.10) is valid. We are done. \square

REMARK 1.8. Let (1.1) be an exact m -cover of \mathbb{Z} . [11, Theorem 4(ii)] asserts that if $\emptyset \neq J \subset [1, k]$ then $\sum_{s \in I} 1/n_s = \sum_{s \in J} 1/n_s$ for some $I \subseteq [1, k]$ with $I \neq J$. This follows from Corollary 1.7, for, $\mathcal{A} = \{a_s(n_s)\}_{s=0}^k$ (where $a_0 = 0$ and $n_0 = 1$) is an $(m + 1)$ -cover of \mathbb{Z} with $\sum_{s \in J \cup \bar{J}} 1/n_s = \sum_{s=1}^k 1/n_s = m$.

In the 1960s Erdős made the following conjecture: for any system (1.1) with $1 < n_1 < \dots < n_k$, if it is a cover of \mathbb{Z} then $\sum_{s=1}^k 1/n_s > 1$, in other words it cannot be a disjoint cover of \mathbb{Z} . This was later confirmed by H. Davenport, L. Mirsky, D. Newman and R. Radó who proved that if (1.1) is a disjoint cover of \mathbb{Z} with $1 < n_1 \leq \dots \leq n_{k-1} \leq n_k$ then $n_{k-1} = n_k$.

COROLLARY 1.9. Let (1.1) be an m -cover of \mathbb{Z} with

$$n_1 \leq \dots \leq n_{k-l} < n_{k-l+1} = \dots = n_k \quad (0 < l < k). \tag{1.11}$$

Then, for any $r \in [0, l]$ with $r < n_k/n_{k-l}$, either $\sum_{s=1}^{k-r} 1/n_s \geq m$ or

$$\binom{l}{r} \in D(n_k) = \left\{ \sum_{p|n_k} p x_p \mid x_p \in \mathbb{N} \text{ for any prime divisor } p \text{ of } n_k \right\}.$$

PROOF. Set $\mathcal{A} = \{a_s(n_s)\}_{s=0}^k$ where $a_0 = 0$ and $n_0 = 1$. Suppose that $\sum_{s=1}^{k-r} 1/n_s < m$. Then $\sum_{s=1}^k 1/n_s < m + r/n_k < m + 1 \leq m(\mathcal{A})$. Since

$$\left| \left\{ I \subseteq [1, k] : \sum_{s \in I} \frac{1}{n_s} = m + \frac{r}{n_k} \right\} \right| = 0 < \binom{m}{m},$$

by Theorem 1.1 we must have

$$\sum_{\substack{I \subseteq [1, k] \\ \sum_{s \in I} 1/n_s = r/n_k}} (-1)^{|I|} \exp\left(2\pi i \sum_{s \in I} \frac{a_s}{n_s}\right) = 0.$$

Observe that $r/n_k < 1/n_{k-l} = \min\{1/n_s \mid 1 \leq s \leq k-l\}$. Therefore,

$$0 = \sum_{\substack{I \subseteq (k-l, k] \\ \sum_{s \in I} 1/n_s = r/n_k}} (-1)^{|I|} \exp\left(2\pi i \sum_{s \in I} \frac{a_s}{n_s}\right) = (-1)^r \Sigma_r,$$

where

$$\Sigma_r = \sum_{\substack{I \subseteq (k-l, k] \\ |I|=r}} \exp\left(2\pi i \sum_{s \in I} \frac{a_s}{n_s}\right).$$

By [16, Lemma 3.1], $\Sigma_r = 0$ implies that

$$\binom{l}{r} = |\{I \subseteq (k-l, k] : |I| = r\}| \in D(n_k).$$

This concludes the proof. □

REMARK 1.10. Let (1.1) be an m -cover of \mathbb{Z} with (1.11). By Corollary 1.9 in the case $r = l$, either $l \geq n_k/n_{k-l} > 1$ or $\sum_{s=1}^{k-l} 1/n_s \geq m$; this is one of the main results in [12]. Corollary 1.9 in the case $r = 1$ yields that either $\sum_{s=1}^{k-1} 1/n_s \geq m$ or $l \in D(n_k)$; this implies the extended Newman–Znám result (see [7]) which asserts that if (1.1) is an exact m -cover of \mathbb{Z} (and hence $\sum_{s=1}^{k-1} 1/n_s < \sum_{s=1}^k 1/n_s = m$) then l is not smaller than the least prime divisor of n_k .

Let (1.1) be an m -system with (1.11), and let $r \in \mathbb{N}$ and $r < n_k/n_{k-l}$. With the help of the dual system of (1.1), we can also show that either $\sum_{s=1}^k 1/n_s \leq m - r/n_k$ or

$$\binom{l+r-1}{r} = |\{(x_{k-l+1}, \dots, x_k) \in \mathbb{N}^l \mid x_{k-l+1} + \dots + x_k = r\}| \in D(n_k).$$

If (1.1) is disjoint with $1 < n_1 < \dots < n_k$, then $\sum_{s=1}^k 1/n_s < 1$ since (1.1) is not a disjoint cover of \mathbb{Z} ; Erdős [3] showed further that $\sum_{s=1}^k 1/n_s \leq 1 - 1/2^k$. Now we give a generalization of this result.

THEOREM 1.11. Let (1.1) be an m -system with $k \geq m$, $\sum_{s=1}^k 1/n_s \neq m$ and $n_1 \leq \dots \leq n_k$. Then

$$\sum_{s=1}^k \frac{1}{n_s} \leq m - \frac{1}{2^{k-m+1}}, \tag{1.12}$$

and equality holds if and only if $n_s = 2^{\max\{s-m+1, 0\}}$ for all $s = 1, \dots, k$.

REMARK 1.12. Let $k \geq m \geq 1$ be integers. Then $m - 1$ copies of $0(1)$, together with the $k - m + 1$ residue classes

$$1(2), 2(2^2), \dots, 2^{k-m}(2^{k-m+1}),$$

form an m -system with the moduli $2^{\max\{s-m+1, 0\}}$ ($s = 1, \dots, k$).

We will prove Theorems 1.1 and 1.11 in the next section. Section 3 deals with two characterizations of m -systems one of which is as follows.

THEOREM 1.13. System (1.1) is an m -system if and only if, for any $n \in [m, k)$,

$$S(n, \alpha) = \begin{cases} (-1)^k & \text{if } \alpha = 0, \\ 0 & \text{if } 0 < \alpha < 1, \end{cases} \tag{1.13}$$

where $S(n, \alpha)$ represents the sum

$$\sum_{\substack{m_1, \dots, m_k \in \mathbb{Z}^+ \\ \{\sum_{s=1}^k m_s/n_s\} = \alpha}} (-1)^{\lfloor \sum_{s=1}^k m_s/n_s \rfloor} \binom{n}{\lfloor \sum_{s=1}^k m_s/n_s \rfloor} \exp\left(2\pi i \sum_{s=1}^k \frac{a_s m_s}{n_s}\right).$$

Theorem 1.13 in the case $m = 1$ yields the following result.

COROLLARY 1.14. *If (1.1) is disjoint, then*

$$\sum_{\substack{m_1, \dots, m_k \in \mathbb{Z}^+ \\ \sum_{s=1}^k m_s/n_s=1}} \exp\left(2\pi i \sum_{s=1}^k \frac{a_s m_s}{n_s}\right) = (-1)^{k-1}. \tag{1.14}$$

A residue class $a(n) = a + n\mathbb{Z}$ is a coset of $n\mathbb{Z}$ in the additive group \mathbb{Z} with $[\mathbb{Z} : n\mathbb{Z}] = n$. In [18] the author conjectured that if $\{a_s G_s\}_{s=1}^k$ ($1 < k < \infty$) is a disjoint system of left cosets in a group G with all the indices $n_s = [G : G_s]$ finite, then $\gcd(n_s, n_t) \geq k$ for some $1 \leq s < t \leq k$.

2. Proofs of Theorems 1.1 and 1.11

LEMMA 2.1. *Let $N \in \mathbb{Z}^+$ be a common multiple of the moduli n_1, \dots, n_k in (1.1). And let $m, m_1, \dots, m_k \in \mathbb{Z}^+$. If (1.1) is an m -cover of \mathbb{Z} , then $(1 - z^N)^m$ divides the polynomial $\prod_{s=1}^k (1 - z^{Nm_s/n_s} \exp(2\pi i a_s m_s/n_s))$. When m_1, \dots, m_k are relatively prime to n_1, \dots, n_k respectively, the converse also holds.*

PROOF. For any $r = 0, 1, \dots, N - 1$, clearly $\exp(2\pi i r/N)$ is a zero of the polynomial $\prod_{s=1}^k (1 - z^{Nm_s/n_s} \exp(2\pi i a_s m_s/n_s))$ with multiplicity $M_r = |\{s \in [1, k] : n_s \mid m_s(r + a_s)\}|$. Observe that $M_r \geq w_A(-r)$. If m_s is relatively prime to n_s for each $s \in [1, k]$, then $M_r = w_A(-r)$. As $(1 - z^N)^m = \prod_{r=0}^{N-1} (1 - z \exp(-2\pi i r/N))^m$, the desired result follows from the above. \square

PROOF OF THEOREM 1.1. Set $m_0 = 1$, and let N_0 be the least common multiple of n_0, n_1, \dots, n_k . In light of Lemma 2.1, we can write

$$P(z) = \prod_{s=0}^k \left(1 - z^{N_0 m_s/n_s} \exp\left(2\pi i \frac{a_s m_s}{n_s}\right)\right)$$

in the form $(1 - z^{N_0})^{m+1} Q(z)$ where $Q(z) \in \mathbb{C}[z]$. Clearly

$$\deg Q = \deg P - (m + 1)N_0 = N_0 \left(\sum_{s=0}^k \frac{m_s}{n_s} - m - 1\right) < \frac{N_0}{n_0}.$$

Also

$$\begin{aligned} & \prod_{s=1}^k \left(1 - z^{N_0 m_s/n_s} \exp\left(2\pi i \frac{a_s m_s}{n_s}\right)\right) \\ &= \sum_{n=0}^m (-1)^n \binom{m}{n} z^{nN_0} \sum_{r=0}^{n_0-1} z^{rN_0/n_0} \exp\left(2\pi i r \frac{a_0}{n_0}\right) Q(z) \end{aligned} \tag{2.1}$$

since

$$\frac{1 - z^{N_0}}{1 - z^{N_0/n_0} \exp(2\pi i a_0/n_0)} = \sum_{r=0}^{n_0-1} z^{rN_0/n_0} \exp\left(2\pi i r \frac{a_0}{n_0}\right).$$

Let $a \in \mathbb{N}$ and

$$C_a = (-1)^{\lfloor a/n_0 \rfloor} \sum_{\substack{I \subseteq [1, k] \\ \sum_{s \in I} m_s/n_s = (\alpha+a)/n_0}} (-1)^{|I|} \exp\left(2\pi i \sum_{s \in I} (a_s - a_0) \frac{m_s}{n_s}\right).$$

By comparing the coefficients of $z^{N_0(\alpha+a)/n_0}$ on both sides of (2.1) we obtain that

$$\begin{aligned} & \sum_{\substack{I \subseteq [1, k] \\ \sum_{s \in I} m_s/n_s = (\alpha+a)/n_0}} (-1)^{|I|} \exp\left(2\pi i \sum_{s \in I} \frac{a_s m_s}{n_s}\right) \\ &= (-1)^{\lfloor a/n_0 \rfloor} \binom{m}{\lfloor a/n_0 \rfloor} \exp(2\pi i a_0 \{a/n_0\}) [z^{\alpha N_0/n_0}] Q(z), \end{aligned}$$

where $[z^{\alpha N_0/n_0}] Q(z)$ denotes the coefficient of $z^{\alpha N_0/n_0}$ in $Q(z)$. Therefore

$$C_a = \exp\left(-2\pi i \alpha \frac{a_0}{n_0}\right) \binom{m}{\lfloor a/n_0 \rfloor} [z^{\alpha N_0/n_0}] Q(z) = \binom{m}{\lfloor a/n_0 \rfloor} C_0. \tag{2.2}$$

For an algebraic integer ω in the field $K = \mathbb{Q}(\exp(2\pi i/N_0))$, the norm

$$N(\omega) = \prod_{1 \leq r \leq N_0, \gcd(r, N_0)=1} \sigma_r(\omega)$$

(with respect to the field extension K/\mathbb{Q}) is a rational integer, where σ_r is the automorphism of K (in the Galois group $\text{Gal}(K/\mathbb{Q})$) induced by $\sigma_r(\exp(2\pi i/N_0)) = \exp(2\pi i r/N_0)$. (See, for example, [6, Ch. 1].) As $N((-1)^{\lfloor a/n_0 \rfloor} C_a)$ equals

$$\prod_{\substack{1 \leq r \leq N_0 \\ \gcd(r, N_0)=1}} \sum_{\substack{I \subseteq [1, k] \\ \sum_{s \in I} m_s/n_s = (\alpha+a)/n_0}} (-1)^{|I|} \exp\left(2\pi i r \sum_{s \in I} (a_s - a_0) \frac{m_s}{n_s}\right),$$

we have

$$\begin{aligned} |N(C_a)| &= \prod_{\substack{1 \leq r \leq N_0 \\ \gcd(r, N_0)=1}} \left| \sum_{\substack{I \subseteq [1, k] \\ \sum_{s \in I} m_s/n_s = (\alpha+a)/n_0}} (-1)^{|I|} \exp\left(2\pi i r \sum_{s \in I} (a_s - a_0) \frac{m_s}{n_s}\right) \right| \\ &\leq \left| \left\{ I \subseteq [1, k] : \sum_{s \in I} \frac{m_s}{n_s} = \frac{\alpha+a}{n_0} \right\} \right|^{\varphi(N_0)}, \end{aligned}$$

where φ is Euler's totient function. Also

$$|N(C_a)| = \left| N\left(\binom{m}{\lfloor a/n_0 \rfloor}\right) \right| \times |N(C_0)| = \binom{m}{\lfloor a/n_0 \rfloor}^{\varphi(N_0)} |N(C_0)|.$$

Suppose that $C_b \neq 0$ for some $b \in \mathbb{N}$. Then $N(C_b) \neq 0$, and hence $N(C_0) \in \mathbb{Z}$ is nonzero. For any $a \in \mathbb{N}$,

$$\left| \left\{ I \subseteq [1, k] : \sum_{s \in I} \frac{m_s}{n_s} = \frac{\alpha + a}{n_0} \right\} \right|^{\varphi(N_0)} \geq |N(C_a)| \geq \binom{m}{\lfloor a/n_0 \rfloor}^{\varphi(N_0)},$$

and hence (1.5) holds. This concludes the proof. □

PROOF OF THEOREM 1.11. We use induction on k .

In the case $k = m$, we have $n_k > 1$ and hence

$$\sum_{s=1}^k \frac{1}{n_s} \leq k - 1 + \frac{1}{n_k} \leq m - \frac{1}{2} = m - \frac{1}{2^{k-m+1}};$$

also $\sum_{s=1}^k 1/n_s = m - 1/2$ if and only if $n_1 = \dots = n_{k-1} = 1$ and $n_k = 2$.

Now let $k > m$. Clearly $\sum_{s=1}^{k-1} 1/n_s < \sum_{s=1}^k 1/n_s < m$. Assume that

$$\sum_{s=1}^{k-1} \frac{1}{n_s} \leq m - \frac{1}{2^{(k-1)-m+1}} = m - \frac{1}{2^{k-m}}$$

and that equality holds if and only if $n_s = 2^{\max\{s-m+1, 0\}}$ for all $s \in [1, k - 1]$. When $n_k > 2^{k-m+1}$,

$$\sum_{s=1}^k \frac{1}{n_s} = \sum_{s=1}^{k-1} \frac{1}{n_s} + \frac{1}{n_k} < \left(m - \frac{1}{2^{k-m}} \right) + \frac{1}{2^{k-m+1}} = m - \frac{1}{2^{k-m+1}}.$$

If $\sum_{s=1}^k 1/n_s > m - 1/n_k$, then $\lceil \sum_{s=1}^k 1/n_s \rceil = m$, thus $\sum_{s=1}^k m_s/n_s = m$ for some $m_1, \dots, m_k \in \mathbb{Z}^+$ (by Corollary 1.5), and hence

$$m - \sum_{s=1}^k \frac{1}{n_s} \geq \min \left\{ \frac{1}{n_s} \mid 1 \leq s \leq k \right\} = \frac{1}{n_k}.$$

This shows that indeed $\sum_{s=1}^k 1/n_s \leq m - 1/n_k$. Provided that $n_k \leq 2^{k-m+1}$, inequality (1.12) holds, and also

$$\begin{aligned} \sum_{s=1}^k \frac{1}{n_s} = m - \frac{1}{2^{k-m+1}} &\iff n_k = 2^{k-m+1} \quad \text{and} \quad \sum_{s=1}^{k-1} \frac{1}{n_s} = m - \frac{1}{2^{k-m}} \\ &\iff n_s = 2^{\max\{s-m+1, 0\}} \quad \text{for } s = 1, \dots, k-1, k. \end{aligned}$$

This concludes the induction step and we are done. □

3. Characterizations of m -systems

PROOF OF THEOREM 1.13. Like Lemma 2.1, system (1.1) is an m -system if and only if $f(z) = (1 - z^N)^m / \prod_{s=1}^k (1 - z^{N/n_s} \exp(2\pi i a_s/n_s))$ is a polynomial, where N is the least common multiple of n_1, \dots, n_k .

Set $c = m - \sum_{s=1}^k 1/n_s$. If $f(z)$ is a polynomial, then $\deg f = cN$ and $[z^{cN}]f(z) = (-1)^{k-m} \exp(-2\pi i \sum_{s=1}^k a_s/n_s)$.

For $|z| < 1$,

$$f(z) = \sum_{n=0}^m \binom{m}{n} (-1)^n z^{nN} \prod_{s=1}^k \sum_{x_s=0}^{\infty} \exp\left(2\pi i \frac{a_s x_s}{n_s}\right) z^{N x_s/n_s}.$$

Let $\alpha \geq 0$. Then

$$\begin{aligned} [z^{(c+\alpha)N}]f(z) &= \sum_{n=0}^m (-1)^n \binom{m}{n} \sum_{\substack{x_1, \dots, x_k \in \mathbb{N} \\ \sum_{s=1}^k x_s/n_s = c+\alpha-n}} \exp\left(2\pi i \sum_{s=1}^k \frac{a_s x_s}{n_s}\right) \\ &= \sum_{n=0}^m (-1)^n \binom{m}{n} \sum_{\substack{m_1, \dots, m_k \in \mathbb{Z}^+ \\ \sum_{s=1}^k m_s/n_s = \alpha+m-n}} \exp\left(2\pi i \sum_{s=1}^k \frac{a_s (m_s - 1)}{n_s}\right) \\ &= (-1)^m \exp\left(-2\pi i \sum_{s=1}^k \frac{a_s}{n_s}\right) S(m, \alpha), \end{aligned}$$

where $S(n, \alpha)$ ($n \in \mathbb{N}$) represents the sum

$$\sum_{\substack{m_1, \dots, m_k \in \mathbb{Z}^+ \\ \sum_{s=1}^k m_s/n_s - \alpha \in \mathbb{N}}} (-1)^{\sum_{s=1}^k m_s/n_s - \alpha} \binom{n}{\sum_{s=1}^k m_s/n_s - \alpha} \exp\left(2\pi i \sum_{s=1}^k \frac{a_s m_s}{n_s}\right)$$

which agrees with its definition in the case $0 \leq \alpha < 1$ given in Theorem 1.13.

(i) Suppose that (1.1) is an m -system. Then $f(z)$ is a polynomial of degree cN and hence

$$S(m, \alpha) = (-1)^m \exp\left(2\pi i \sum_{s=1}^k \frac{a_s}{n_s}\right) [z^{(c+\alpha)N}]f(z) = \begin{cases} (-1)^k & \text{if } \alpha = 0, \\ 0 & \text{if } \alpha > 0. \end{cases}$$

For any integer $n \geq m$, (1.1) is also an n -system and so (1.13) holds.

(ii) Now assume that (1.13) holds for all $n \in [m, k]$. For any $n \geq k$, we also have (1.13) by (i) because (1.1) is a k -system. If $0 < \alpha < 1$ then $S(n, \alpha) = 0$ for any integer $n \geq m$. Fix $\alpha > 0$. If $S(n, \alpha) = 0$ for all integers $n \geq m$, then for any integer $n \geq m$,

$$S(n, \alpha + 1) = S(n, \alpha) - S(n + 1, \alpha) = 0$$

because $\binom{n}{j-1} = \binom{n+1}{j} - \binom{n}{j}$ for $j = 1, 2, \dots$. Thus, by induction, $S(n, \alpha) = 0$ for

all $\alpha > 0$ and $n = m, m + 1, \dots$. It follows that $[z^{(c+\alpha)N}]f(z) = 0$ for any $\alpha > 0$. So $f(z)$ is a polynomial and (1.1) is an m -system.

The proof of Theorem 1.13 is now complete. □

The following characterization of m -covers plays important roles in [11, 12].

LEMMA 3.1 (Sun [11]). *Let $m, m_1, \dots, m_k \in \mathbb{Z}^+$. If (1.1) forms an m -cover of \mathbb{Z} , then*

$$\sum_{\substack{I \subseteq [1, k] \\ \{\sum_{s \in I} m_s/n_s\} = \theta}} (-1)^{|I|} \binom{\lfloor \sum_{s \in I} m_s/n_s \rfloor}{n} \exp\left(2\pi i \sum_{s \in I} \frac{a_s m_s}{n_s}\right) = 0 \tag{3.1}$$

for all $0 \leq \theta < 1$ and $n = 0, 1, \dots, m - 1$. The converse also holds if m_1, \dots, m_k are relatively prime to n_1, \dots, n_k , respectively.

We can provide a new proof of Lemma 3.1 in a way similar to the proof of Theorem 1.13.

LEMMA 3.2. *Let $n \in \mathbb{Z}^+$ and $l \in [0, n - 1]$. Then*

$$\sum_{\substack{J \subseteq [1, n] \\ |J| = l}} \exp\left(2\pi i \sum_{j \in J} \frac{j}{n}\right) = (-1)^l. \tag{3.2}$$

PROOF. Clearly we have the identity

$$\prod_{0 < j < n} (1 - ze^{2\pi i j/n}) = \frac{1 - z^n}{1 - z} = 1 + z + \dots + z^{n-1}.$$

Comparing the coefficients of z^l we then obtain (3.2). □

Using Lemmas 3.1 and 3.2 we can deduce another characterization of m -systems.

THEOREM 3.3. *System (1.1) is an m -system if and only if*

$$\sum_{\substack{x_s \in [0, n_s) \text{ for } s \in [1, k] \\ \{\sum_{s=1}^k x_s/n_s\} = \theta}} \binom{\lfloor \sum_{s=1}^k x_s/n_s \rfloor}{n} \exp\left(2\pi i \sum_{s=1}^k \frac{a_s x_s}{n_s}\right) = 0 \tag{3.3}$$

for all $0 \leq \theta < 1$ and $n \in [0, k - m)$.

PROOF. The case $k \leq m$ is trivial, so we just let $k > m$. Recall that (1.1) is an m -system if and only if its dual A^* is a $(k - m)$ -cover of \mathbb{Z} .

By Lemma 3.1 in the case $m_1 = \dots = m_k = 1$, A^* forms a $(k - m)$ -cover of \mathbb{Z} if and only if for any $0 \leq \theta < 1$ and $n \in [0, k - m)$ the sum

$$\sum_{\substack{x_s \in [0, n_s) \text{ for } s \in [1, k] \\ \{\sum_{s=1}^k x_s/n_s\} = \theta}} (-1)^{\sum_{s=1}^k x_s} \binom{\lfloor \sum_{s=1}^k x_s/n_s \rfloor}{n} \exp\left(2\pi i \sum_{s=1}^k \frac{a_s x_s}{n_s}\right) \prod_{s=1}^k f_s(x_s)$$

vanishes, where

$$f_s(x_s) = \sum_{\substack{J \subseteq [1, n_s] \\ |J|=x_s}} \exp\left(2\pi i \sum_{j \in J} \frac{j}{n_s}\right) = (-1)^{x_s}$$

by Lemma 3.2. This concludes the proof. □

The following consequence extends Corollary 1.14.

COROLLARY 3.4. *Let (1.1) be an m -system. Then we have*

$$\sum_{\substack{m_s \in [1, n_s] \text{ for } s \in [1, k] \\ m - \sum_{s=1}^k m_s/n_s \in \mathbb{N}}} \binom{k-1 - \sum_{s=1}^k m_s/n_s}{m - \sum_{s=1}^k m_s/n_s} \exp\left(2\pi i \sum_{s=1}^k \frac{a_s m_s}{n_s}\right) = (-1)^{k-m}.$$

PROOF. If $k \leq m$, then the left-hand side of the last equality coincides with

$$\binom{k-1 - \sum_{s=1}^k n_s/n_s}{m - \sum_{s=1}^k n_s/n_s} \exp\left(2\pi i \sum_{s=1}^k \frac{a_s n_s}{n_s}\right) = \binom{-1}{m-k} = (-1)^{m-k}.$$

Now let $k > m$. As $\{-a_s(n_s)\}_{s=1}^k$ is an m -system, by Theorem 3.3 and the identity

$$(-1)^{k-m-1} \binom{x-1}{k-m-1} = \sum_{n=0}^{k-m-1} (-1)^n \binom{x}{n}$$

(see [4, (5.16)]) we have

$$\begin{aligned} 0 &= \sum_{\substack{x_s \in [0, n_s] \text{ for } s \in [1, k] \\ \{\sum_{s=1}^k x_s/n_s\} = 0}} \binom{\lfloor \sum_{s=1}^k x_s/n_s \rfloor - 1}{k-m-1} \exp\left(2\pi i \sum_{s=1}^k \frac{-a_s x_s}{n_s}\right) \\ &= \sum_{\substack{m_s \in [1, n_s] \text{ for } s \in [1, k] \\ \sum_{s=1}^k (n_s - m_s)/n_s \in \mathbb{N}}} \binom{\sum_{s=1}^k (n_s - m_s)/n_s - 1}{k-m-1} \exp\left(-2\pi i \sum_{s=1}^k \frac{a_s (n_s - m_s)}{n_s}\right) \\ &= \sum_{\substack{m_s \in [1, n_s] \text{ for } s \in [1, k] \\ \sum_{s=1}^k m_s/n_s \in [0, k-1]}} \binom{k-1 - \sum_{s=1}^k m_s/n_s}{k-1-m} \exp\left(2\pi i \sum_{s=1}^k \frac{a_s m_s}{n_s}\right) \\ &\quad + \binom{k-1 - \sum_{s=1}^k n_s/n_s}{k-1-m} \exp\left(2\pi i \sum_{s=1}^k \frac{a_s n_s}{n_s}\right). \end{aligned}$$

So the desired equality follows. □

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