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A NOTE ON A RESIDUAL SUBSET OF LIPSCHITZ FUNCTIONS ON METRIC SPACES

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Abstract Let (X, d) be a quasi-convex, complete and separable metric space with reference probability measure m. We prove that the set of real-valued Lipschitz functions with non-zero pointwise Lipschitz constant m-almost everywhere is residual, and hence dense, in the Banach space of Lipschitz and bounded functions. The result is the metric analogous to a result proved for real-valued Lipschitz maps defined on $\mathbb{R}^2$ by Alberti *et al*.

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1. Introduction

In the context of metric spaces, say (X, d), it is possible to look at the pointwise variation of a real-valued map by considering

$$\operatorname{Lip} f(x) := \limsup_{y\to x, y\neq x} \frac{|f(x) - f(y)|}{d(x, y)}, \tag{1.1}$$

which is called the *pointwise Lipschitz constant*. In the smooth framework, $\operatorname{Lip} f$ corresponds to the modulus of ∇f: if (X, d) is an open subset of $\mathbb{R}^d$ endowed with the Euclidean norm and f is locally Lipschitz, then $\operatorname{Lip} f = |\nabla f|$ almost everywhere with respect to the Lebesgue measure. Or, more generally, if (X, d, m) is a metric measure space admitting a differentiable structure in the sense of Cheeger (see [**4**, **6**] for the definitions) and f is Lipschitz, then $\operatorname{Lip} f = |\mathrm{d}f|$ m-almost everywhere, where $\mathrm{d}f$ is the Cheeger differential of f.

Once the pointwise information is given, we are interested in looking at those points where the 'differential' vanishes. Define the singular set of f as

$$S(f) := \{x \in X : \operatorname{Lip} f(x) = 0\}.$$

The classical Sard theorem states that if $f : \mathbb{R}^n \to \mathbb{R}$ is sufficiently smooth, then the Lebesgue measure of $f(S(f))$ is 0. As soon as the regularity assumption on f is dropped, the conclusion of Sard's theorem does not hold anymore and one may look for weaker properties to hold.

The question, inspired by a similar problem in [**3**, § 6], is whether it is possible to approximate any Lipschitz function with functions having negligible $S(f)$ with respect to a given reference measure.

For real-valued Lipschitz functions defined on $\mathbb{R}^2$ with the Lebesgue measure playing the role of the reference measure, a positive answer is contained in [**1**, Proposition 4.10]. We prove the following theorem.

Theorem 1.1. *Assume that (X, d) is a quasi-convex, complete and separable metric space and let m be a Borel probability measure over it. The set of those $f \in D^\infty(X)$ such that $m(S(f)) = 0$ is residual, and therefore dense, in $D^\infty(X)$.*

The Banach space $D^\infty(X)$ will be the space of bounded functions, with bounded pointwise Lipschitz constant, endowed with the uniform norm. (See below for a precise definition.) Recall that a set in a topological space is residual if it contains a countable intersection of open dense sets. By Baire's category theorem, a residual set in a complete metric space is dense.

2. Setting

Let (X, d) be a metric space and let m be a Borel probability measure over X so that X coincides with its support. For $f\colon X \to \mathbb{R}$, the *Lipschitz constant* of f is defined as usual by

$$\mathrm{LIP}(f) := \sup_{x,y\in X, x\neq y} \frac{|f(x) - f(y)|}{d(x, y)}$$

and we say that f is Lipschitz if $\mathrm{LIP}(f)$ is a finite number. Accordingly, denote by $\mathrm{LIP}^\infty(X)$ the space of bounded Lipschitz functions. The natural norm on $\mathrm{LIP}^\infty(X)$ is given by

$$\|f\|_{\mathrm{LIP}^\infty(X)} = \|f\|_\infty + \mathrm{LIP}(f),$$

where $\|\cdot\|_\infty$ is the uniform norm. The space of bounded Lipschitz functions endowed with $\|f\|_{\mathrm{LIP}^\infty(X)}$ turns out to be a Banach space. The pointwise version of $\mathrm{LIP}(f)$ is given by the pointwise Lipschitz constant, as defined in (1.1). The corresponding space of bounded functions with bounded pointwise Lipschitz constant can be considered:

$$D^\infty(X) := \{f\colon X \to \mathbb{R}\colon \|f\|_\infty + \|\mathrm{Lip}\, f\|_\infty < \infty\}.$$

A study of $D^\infty(X)$ and $\mathrm{LIP}^\infty(X)$ can be found in [**5**]. The following results are taken from [**5**].

It is straightforward to note that $\mathrm{LIP}^\infty(X) \subset D^\infty(X)$ and for a general metric space this is the only valid inclusion. Examples of metric spaces and functions in $D^\infty(X)$ not satisfying a global Lipschitz bound can be constructed (see [**5**]). If (X, d) is quasi-convex, the other inclusion also holds and $\mathrm{LIP}^\infty(X) = D^\infty(X)$ and the two semi-norms are comparable, i.e. there exists $C \geqslant 1$ such that

$$\|\mathrm{Lip}\, f\|_\infty \leqslant \mathrm{LIP}(f) \leqslant C\|\mathrm{Lip}\, f\|_\infty.$$

Hence, $D^\infty(X)$, or equivalently $\mathrm{LIP}^\infty(X)$, endowed with the norm $\|\cdot\|_\infty + \|\mathrm{Lip}(\cdot)\|_\infty$ is a Banach space. We will denote this norm by $\|\cdot\|_{D^\infty}(X)$.

Recall that a metric space (X,d) is quasi-convex if there exists a constant $C \geqslant 1$ such that for each pair of points $x, y \in X$ there exists a curve γ connecting the two points such that $l(\gamma) \leqslant Cd(x,y)$, where $l(\gamma)$ denotes the length of γ defined with the usual 'affine' approximation: for $\gamma\colon [a,b] \to X$ its length $l(\gamma)$ is defined as

$$l(\gamma) := \sup\left\{\sum_{i=1}^{n} d(x_i, x_{i+1})\colon a = x_1 < x_2 < \cdots < x_{n+1} = b,\ n \in \mathbb{N}\right\}.$$

Associated with the length $l(\gamma)$ there is the distance obtained from minimizing it:

$$d_L(x,y) = \inf\{l(\gamma)\colon \gamma_0 = x,\ \gamma_1 = y\}.$$

Indeed, the function d_L is a distance on each component of accessibility by rectifiable paths, i.e. those paths having finite l. By quasi-convexity it follows that

$$d(x,y) \leqslant d_L(x,y) \leqslant Cd(x,y)$$

with $C > 1$. Hence, (X, d_L) is a complete and separable metric space that is also a length space. Clearly, (X, d_L) has the same open sets as (X,d). For a more detailed discussion on length spaces see [**2**].

We will use the following notation. For $r > 0$ and $z \in X$, we will denote by $B_r(z)$ the ball of radius r centred on z. The complement in X of a set A will be denoted by A^c and ∂A denotes the topological boundary of A. The closure of A is $\mathrm{cl}(A)$ and the interior part $\mathrm{int}(A)$. Given a set we can consider the distance from it: for $x \in X$ and $A \subset X$

$$d(x,A) := \inf_{w \in A} d(x,w).$$

3. The result

Lemma 3.1. *For any Borel function $f\colon X \to \mathbb{R}$, the function $\mathrm{Lip}\, f\colon X \to \bar{\mathbb{R}}$ is universally measurable.*

Proof. In order to prove the claim we just have to show that the set $\{x \in X\colon \mathrm{Lip}\, f(x) \geqslant a\}$ is Souslin for any $a \in \mathbb{R}$. Since f is a Borel map, it follows that

$$\bigcap_{n\in\mathbb{N}} \left\{(x,y) \in X \times X\colon 0 < d(x,y) \leqslant \frac{1}{n},\ \frac{|f(x) - f(y)|}{d(x,y)} \geqslant a\right\}$$

is a Borel set. Note that

$$\{x \in X\colon \mathrm{Lip}\, f(x) \geqslant a\} = P_1\left(\bigcap_{n\in\mathbb{N}} \left\{(x,y) \in X \times X\colon 0 < d(x,y) \leqslant \frac{1}{n},\ \frac{|f(x) - f(y)|}{d(x,y)} \geqslant a\right\}\right),$$

where $P_1\colon X \times X \to X$ denotes the projection on the first element. It follows from the definition of Souslin sets that $\{x \in X\colon \mathrm{Lip}\, f(x) \geqslant a\}$ is Souslin and the claim follows. □

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After Lemma 3.1 it then makes sense to look at those functions f such that $m(S(f))=0$. We will need the following lemma.

Lemma 3.2. *Let $K\subset X$ be a closed set and consider the length distance function from K, that is $g(x):=d_L(x,K)$. Then*

$$1\leqslant \operatorname{Lip} g(x)\leqslant C\quad \text{for } x\in K^c.$$

Proof.

Step 1. Assume that $d=d_L$ so that (X,d) is a length space and $g=d(x,K)$. Then fix $x\in K^c$: for any $z\in K$ and $y\in K^c$ it holds that

$$d(x,z)-d(y,z)\leqslant d(x,y),$$

and hence trivially $\operatorname{Lip} g(x)\leqslant 1$.

Consider now a minimizing sequence $z_n\in K$ for x, that is $g(x)\geqslant d(x,z_n)-1/n$. From the length structure it follows that for any n there exists $\gamma^n\colon[0,1]\to X$, a rectifiable curve starting in x and arriving in z_n, such that $d(x,z_n)\geqslant l(\gamma^n)-1/n$. So for any y_n in the image of γ^n,

$$\frac{g(x)-g(y_n)}{d(x,y_n)}\geqslant\frac{l(\gamma_n)-d(y_n,z_n)-2/n}{d(x,y_n)}.$$

Since $l(\gamma^n)\geqslant d(x,y_n)+d(y_n,z_n)$ it follows that

$$\frac{g(x)-g(y_n)}{d(x,y_n)}\geqslant\frac{d(x,y_n)-2/n}{d(x,y_n)}.$$

Since the only constraint made on y_n was to belong to the image of γ^n, we can choose y_n such that the previous ratio converges to 1. Hence $\operatorname{Lip} g(x)=1$.

Step 2. We now drop the assumption on the length structure of the space. Let (X,d) be quasi-convex and $g(x)=d_L(x,K)$. Since (X,d_L) is a length space for any $x\in K^c$,

$$\limsup_{y\to x,y\neq x}\frac{|g(x)-g(y)|}{d_L(x,y)}=1.$$

Since (X,d_L) and (X,d) have the same open sets, K^c does not depend on the metric. Since $d\leqslant d_L\leqslant Cd$, the claim follows. □

We can now prove Theorem 1.1. The proof uses the ideas contained in [**1**, Proposition 4.10].

Theorem 3.3. *Assume that (X,d) is a quasi-convex, complete and separable space and let m be a Borel probability measure over it. The set of those $f\in D^\infty(X)$ such that $m(S(f))=0$ is residual in $D^\infty(X)$ and is therefore dense.*

Proof. Consider the sets

$$G:=\{f\in D^\infty(X)\colon m(S(f))=0\}\quad\text{and}\quad G_r:=\{f\in D^\infty(X)\colon m(S(f))<r\}.$$

The claim is then to prove that G is a residual set. Since $G=\bigcap G_r$, where the intersection runs over a sequence of r converging to 0, the claim is proved once it is proved that each G_r is open and dense in $D^\infty(X)$.

Step 1. The set G_r is open in $D^\infty(X)$. Fix $f \in G_r$. Then there exists $\delta > 0$ such that

$$m(\{x \in X \colon \operatorname{Lip} f(x) \leqslant \delta\}) < r.$$

Since for any $g \in D^\infty(X)$ it holds that

$$\operatorname{Lip} f(x) \leqslant \operatorname{Lip} g(x) + \operatorname{Lip}(f-g)(x),$$

for any $g \in D^\infty(X)$ such that $\|g - f\|_{D^\infty}(X) \leqslant \delta$ it holds that

$$S(g) \subset \{x \in X \colon \operatorname{Lip} f(x) \leqslant \delta\}.$$

Therefore, $m(S(g)) < r$ and consequently $g \in G_r$.

Step 2. The set G_r is dense in $D^\infty(X)$. Given $f \in D^\infty(X)$ and $\delta > 0$, we have to find $g \in G_r$ such that $\|f - g\|_{D^\infty(X)} \leqslant \delta$. Without loss of generality, we can assume that $m(S(f)) \geqslant r$.

For every $\varepsilon > 0$ denote by $S(f)^\varepsilon$ the ε-neighbourhood of the set of singular points of f, i.e.

$$S(f)^\varepsilon = \{z \in X \colon d(z, S(f)) < \varepsilon\}.$$

The set $S(f)^\varepsilon$ is open and denote by K its complementary in X. Associated with K we consider the distance function $\hat{g}$ as defined in Lemma 3.2, that is $\hat{g}(x) := d_L(x, K)$. A rough bound on $\hat{g}(x)$ can be given in terms of the 'diameter' of $S(f)$:

$$\hat{g}(x) \leqslant C \sup\{d(x,z) \colon \operatorname{cl}(S(f)^\varepsilon)\},$$

where $\operatorname{cl}(S(f)^\varepsilon)$ stands for the closure of $S(f)^\varepsilon$. Since, in approximating with functions in G_r, we can make an error in measure strictly less than r and since m is a probability measure, we can assume $S(f)$ to have finite diameter and by inner regularity we can even assume it to be closed. Therefore,

$$\|\hat{g}\|_\infty \leqslant M, \quad M > 0.$$

From Lemma 3.2 we have $\operatorname{Lip} \hat{g}(x) > 0$ for $x \in S(f)^\varepsilon$ and clearly $\operatorname{Lip} \hat{g}(x) = 0$ for $x \in \operatorname{int}(K)$, where $\operatorname{int}(K)$ stands for the interior part of K.

Note that the boundary of $S(f)^\varepsilon$ is contained in the set $\{z \colon d(z, S(f)) = \varepsilon\}$. Indeed, $z \in \partial S(f)^\varepsilon$ if and only if $d(z, S(f)) \geqslant \varepsilon$ and for every $\eta > 0$ there exists a point $w \in X$ such that

$$d(z,w) \leqslant \eta \quad \text{and} \quad d(w, S(f)) < \varepsilon.$$

Let η_n be a sequence converging to 0 and let w_n be the corresponding sequence converging to z. With each w_n associate $x_n \in S(f)$ such that $d(w_n, x_n) < \varepsilon$. Then

$$d(z, x_n) \leqslant d(z, w_n) + d(w_n, x_n) < \eta_n + \varepsilon.$$

Passing to the limit, $d(z, S(f)) \leqslant \varepsilon$ and therefore necessarily $d(z, S(f)) = \varepsilon$.

Moreover, for $\varepsilon \neq \varepsilon'$

$$\{z\colon d(z, S(f)) = \varepsilon\} \cap \{z\colon d(z, S(f)) = \varepsilon'\} = \emptyset,$$

and hence there exists at most countably many ε so that $m(\{z\colon d(z, S(f)) = \varepsilon\}) > 0$. Hence, for any $r > 0$ there exists $\varepsilon > 0$ such that

$$m(\{z\colon d(z, S(f)) = \varepsilon\}) = 0 \quad \text{and} \quad m(S(f)^{\varepsilon} \setminus S(f)) < r,$$

where the second expression holds because $S(f)$ is closed. From what has been said so far, we define $g := f + (\delta/2M)\hat{g}$ such that

$$\|f - g\|_{D^{\infty}(X)} \leqslant \delta.$$

To conclude the proof, observe that $S(g) \subset S(f)^{\varepsilon} \setminus S(f)$, and hence by construction $g \in G_r$. □

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