The Treatment of Arithmetic Progressions by Archimedes.

By Professor GIBSON.

The following paper was written last summer, and was submitted to Dr Mackay with a view to eliciting his opinion particularly on the curious passage referred to in § 3, and on the remarks contained in § 8. I was not aware of the intention of Mr T. L. Heath to follow up his excellent edition of Apollonius by an edition of Archimedes on similar lines, and when I saw the announcement of his Archimedes in the month of October, I at once concluded that the notes I had made would have been anticipated by him. Since reading his masterly work, however, I am disposed to think there is still sufficient interest in the notes I have written to justify me in laying them before the Society; I therefore submit them in their original form, although I should have omitted certain details had I been acquainted with Mr Heath's work before writing the paper.

1. In his books *On Helices* and *On Conoids and Spheroids*, Archimedes has effected the evaluation of areas and of volumes by methods which are very closely analogous to the modern algebraical methods, depending as they do largely on the sum of arithmetical progressions. The main difference is to be found in the almost exclusive use by Archimedes of inequality theorems which are required for the application of the method of exhaustion, while the modern treatment replaces this by a more or less rigorous use of convergent series. This peculiarity has to a certain extent obscured the really complete command he had of such progressions, but a careful study of his works is sufficient to show that, where he did not enunciate his theorems in forms giving the sum of the series in closed terms, he took this course, not from inability to give the closed form, but chiefly because the inequality theorems were those of which almost exclusively he was in need, and also because their statement was much more concise. The undoubted prolixity of enunciation and demonstration is due much more to the want of an algebraic symbolism than to anything in the method of proof; and

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one can hardly be surprised that he should have chosen the simpler form for his theorems, when nothing was to be gained but rather something would have been lost by enunciating them as equalities.

2. In presenting the methods of Archimedes it is, I think, a mistake to adopt the modern method of using symbols merely for the first term, the common difference, and the number of terms, as the real simplicity of the proofs is thereby obscured. The chief defect of his notation—and it is one that causes great prolixity both in statement and in demonstration—is the absence of a symbol for the number of terms. In this paper, therefore, I use distinct letters to represent the terms of the series; in Archimedean language the terms are straight lines, and these are specified sometimes by one letter only (as in this paper) and sometimes by two; I employ, however, a symbol for the number of terms as well as the modern algebraic or geometric symbolism. Throughout the paper the first term of the arithmetic progression will be denoted by a, the second by b, etc., the last by l, the second last by k, etc., while, unless it be otherwise stated, the number of terms will be n; it will also be supposed, unless otherwise specified, that a is the greatest and l the least term.

3. Though Archimedes enunciates (*Opera* I., p. 290*) and repeatedly uses the theorem that when l is the common difference twice the sum of the n lines $a, b, c, \ldots l$ is greater than na and twice the sum of the $n-1$ lines $b, c, \ldots l$ is less than na, he gives no distinct proposition in proof of these inequalities. But in the 11th Proposition of the Book *On Helices* he indicates how they may be proved, and in the course of the 10th Proposition of the same book he states and proves the exact theorem, namely—

$$2(a+b+\ldots+l)=n(a+l);$$

further, in this same proposition he uses this value for the sum repeatedly, so that he was evidently quite familiar with it.

On the other hand, it is rather curious that in the only proposition where he requires to use the exact value, he has, if we accept the text of Heiberg, fallen into error. In *Conoids and Spheroids*, Prop. 21 (*Opera*, I., p. 392) he has to compare the sum of the

* The references are to Heiberg's edition.

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$n-1$ lines $b, c, \ldots l$ with $(n-1)a$, and he says that $(n-1)a$ is greater than twice the sum of $b, c, \ldots l$, though his own diagram shows clearly the equality of the two expressions. The two sentences in lines 14–18 are quite conclusive as to the mistake, and the editor, by referring to page 290, where the inequalities are stated, leads one to infer that he, too, has failed to notice the slip. Archimedes would seem not to have observed that the number of lines $b, c, \ldots l$, is the same as the number of lines a with which he was comparing them. The slip is no doubt a trivial one, and does not affect the final conclusion, but it appears to indicate the subordinate position which the exact theorem occupied in his collection of results. *

It will be noticed that the least term is assumed to be equal to the common difference; it will be seen later how Archimedes gets over that restriction when a series occurs not satisfying that condition.

4. The most important theorems are those dealing with the sums of squares, and it was possibly the summation of the series that constituted a portion of the difficulties referred to in the letters to Dositheus, prefixed to the books on *Helices* and *Conoids and Spheroids.*

The use to which these series are put may help to explain their origin. In finding the volume of a segment of a conoid or spheroid, Archimedes employed three sets of cylinders.

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* The text of Heiberg in this passage differs considerably from that of Torelli, but it is hardly possible that the latter can be correct. The sentence (*Torelli*, p. 287, at foot) "Ἆρα καὶ ὁ ὅλος κύλινδρος κ.τ.λ" is a mere repetition of that preceding it, while the position of ἄρα at the beginning of the sentence is at variance with Greek usage. The deletion of πολλῷ before ἄρα is stated in Heiberg's note to be due to Commandine, and it is easy to understand the deletion, for in Torelli's text the inscribed figure is only compared with the *whole* of the circumscribed cylinder and not also with a part of it as in Heiberg's text. In all the texts there is a certain ambiguity as to the precise meaning of the phrases "all the lines" and "all the lines cut off between AB, BΔ, but lines 14–16 in Heiberg make the meaning quite clear, for there it is explicitly stated that the circumscribed cylinder diminished by one of its elementary cylinders is more than double of the inscribed figure while it is obviously exactly double. Heath's rendering of the proposition is, of course, quite accurate in its mathematics, but in the condensation of the original text the erroneous statement has apparently been overlooked. So far as I am aware, the slip has not been previously pointed out.

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The first set consisted of a single cylinder, K, whose axis was that of the segment, whose lower base was the base of the segment and whose upper base was in the tangent plane parallel to the base of the segment.

The second set, C, formed a figure circumscribing the segment; it was built up of cylinders of equal altitudes, with generators parallel to the axis of the segment, whose lower bases were the base of the segment and the sections in which the segment was cut by planes drawn parallel to its base through points dividing its axis into any number, say n, of equal parts.

The third set, I, formed a figure inscribed in the segment in the same way as the circumscribed figure C.

C, containing n cylinders, is greater than the segment, and I, containing $n-1$, is less. Archimedes shows that C – I can be made less than any given solid, and he finds limits for the ratios of C to K and of I to K. In finding these limits he has to sum the series
$$a^2+b^2+\quad . \quad . \quad +l^2.$$

It is perhaps worth remarking that if the method just described be applied to the known theorem (*Euclid* XII., 7) that a pyramid on a triangular base is a third of the prism on the same base and of the same altitude, the inequality theorems may be at once deduced. The pyramid and prism being supposed to have a common edge divided into n equal parts, and planes being drawn through the points of section parallel to the base, the sets C, I may be taken as prisms with edges parallel to the common edge, while the whole prism will represent K. If a be one side of the triangular base, the sides of the triangular sections parallel to a will with a form an A, P, $a, b, \ldots l$, and we shall have

$$\frac{\mathrm{C}}{\mathrm{K}}=\frac{a^2+b^2+\ldots+l^2}{na^2}, \qquad \frac{\mathrm{I}}{\mathrm{K}}=\frac{b^2+\ldots+l^2}{na^2}.$$

But $$\mathrm{C/K}>\tfrac{1}{3} \quad \text{and} \quad \mathrm{I/K}<\tfrac{1}{3};$$

hence $$3(a^2+b^2+\ldots+l^2).>na^2>3(b^2+\ldots+l^2).$$

Whether this theorem on the relation between the pyramid and prism which Archimedes himself cites (*Opera* I., p. 4) as one established by Eudoxus in a manner generally accepted as sound, may have led him to these inequalities, can only be matter of conjecture. In any case, his proof is quite different, as will now be shown.

5. The 10th Proposition of the Book *On Helices* is as follows, l being the common difference :—

$$3(a^2+b^2+\ldots+l^2)=(n+1)a^2+l(a+b+\ldots+l)$$

and the corollary is $3(a^2+b^2+\ldots+l^2)>na^2>3(b^2+\ldots+l^2)$

The proof is very peculiar. It is obvious that

$$a=b+l=c+k=d+j=\text{ etc. }=l+b$$

Hence squaring the $n-1$ values of a, adding results and increasing each side of the equation by $2a^2$ he gets

$$(n+1)a^2=2(a^2+b^2+\ldots+l^2)+2(bl+ck+\ldots+kc+lb).$$

He next shows that

$$a^2+b^2+\ldots+l^2=l(a+b+\ldots+l)+2(bl+ck+\ldots+kc+lb)$$

by proving that each side of the equation is equal to

$$l(a+3b+5c+\ldots+(2n-1)l).$$

Now this transformation is certainly very artificial, but it seems to me not impossible that this last step was really the first in order of discovery.

It may be assumed (Cantor, *Gesch der Math.*, I., p. 153) that Archimedes was familiar with the process of building up a square of side a by starting with a square of side l and adding successively the gnomons $(k^2-l^2),\ (j^2-k^2)\ldots(a^2-b^2),$

and hence that

$$\begin{aligned} a^2 &= (a^2-b^2)+(b^2-c^2)+\ldots+(k^2-l^2)+l^2 \\ &= l[(a+b)+(b+c)+\quad+(k+l)+l\ ] \\ &= l[\ a+2(b+c+\quad\ldots\quad+k+l)] \end{aligned}$$

since $l=a-b=b-c=\ldots=k-l.$

Indeed this is the form in which he expresses the value of a^2 in the transformation, though his proof of this value is no doubt quite different. If the supposition be made that he started from this value of a^2, and the corresponding values for the other squares, he would get for the sum of the n squares

$$\begin{aligned} & l[a+2(b+c+d+\ldots+k+l)] \\ &+l[b+2(\quad c+d+\quad+k+l)] \\ &+l[c+2(\quad\quad d+\quad+k+l)] \\ &+\quad\ldots\quad\ldots\quad\ldots \\ &+l[k+2(\quad\quad\quad\quad l)] \\ &+l[l\quad\quad\quad\quad\quad\ ] \end{aligned}$$

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that is,

$$l(a+b+\ldots+l)+2l(\ b+2\ c+3\ d+\ldots+(n-2)\ k+(n-1)l\ )$$

or $\quad l(a+b+\quad +l)+2\ (lb+2lc+3ld+\ldots+(n-2)lk+(n-1)ll)$

or since $\quad 2l=k,\quad 3l=j\ldots(n-2)l=c,\quad (n-1)l=b$

$$l(a+b+\quad +l)+2(lb+kc+jd+\ldots+ck+bl)$$

If this were the form first found for the sum of the squares, the property that each term in the product consisted of two factors whose sum was a would lead to the squaring of the $n-1$ values $b+l$, $c+k$, etc.

The actual demonstration given by Archimedes is no doubt quite different, but the artificiality of the transformation referred to above leads to the suspicion that the traditional method of representing a square as a sum of gnomons may have played a more important part than the completed proof suggests.

The fact that in the enunciation the series

$$a+b+\ldots+l$$

is not summed can not be due to ignorance of that sum, seeing that in the course of the demonstration the summation is repeatedly effected; the reason for the form given seems to be simply that he had no need for the exact sum of the squares in any part of his work, as the inequalities of the corollary contained all he required. Besides, the series are only auxiliary to the determination of areas and volumes; it should not therefore surprise us that he does not put the expression for the sum into a form which his whole discussion shows he might have done had he been treating the series for their own sake.

At the same time, it is to be observed that his substitution of the inequality theorems of the corollary for the exact theorem of the proposition obliges him to treat the ellipsoid in a different way from the hyperboloid, as will be seen in § 8.

6. The theorem of the preceding paragraph assumes the common difference to be equal to the least term, but obviously cases arise where that condition is not satisfied, and Archimedes provides for such cases in the 11th proposition of the same book (*Opera* II., 42–50). The diagram to that proposition makes the common difference equal to the least term, but the enunciation omits the characteristic phrase expressive of this condition and the

demonstration is also independent of it, while the repeated applications of the theorem in the Book *On Helices* show that he understood it in its most general form.

It will be convenient to take the number of terms as $n+1$, and the proposition may then be put in the form

$$\frac{na^2}{a^2+b^2+\ldots+k^2} < \frac{a^2}{al+\frac{1}{3}(a-l)^2} < \frac{na^2}{b^2+\ldots+k^2+l^2}.$$

The theorem will be proved, he says, if it be proved that

$$a^2+b^2+\ldots+k^2 > nal+\tfrac{1}{3}n(a-l)^2 > b^2+\ldots+k^2+l^2.$$

To effect the proof, he subtracts l from each term and thus reduces

the progression $\quad a, b, \ldots k, l \quad [n+1 \text{ terms}]$

to the progression $\quad a-l, b-l \ldots k-l \quad [n \text{ terms}]$

in which the least term $k-l$ is equal to the common difference, and to which therefore the results of Prop. 10 are applicable.

Thus $$a^2+b^2+\ldots+k^2 = (a-l)^2+\ldots+(k-l)^2+nl^2 + 2l[(a-l)+\ldots+(k-l)]$$

and $$nal+\tfrac{1}{3}n(a-l)^2 = nl(a-l)+nl^2+\tfrac{1}{3}n(a-l)^2$$

But $$(a-l)^2+\ldots+(k-l)^2 > \tfrac{1}{3}n(a-l)^2$$

and $$(a-l)+\ldots+(k-l) > \tfrac{1}{2}n(a-l)$$

Hence $$a^2+b^2+\ldots+k^2 > nal+\tfrac{1}{3}n(a-l)^2$$

and in the same way the other inequality is established.

The transformation here adopted brings the general A.P. within the range of his methods, and would have enabled him to sum the squares of any number of terms even when the least term is not equal to the common difference. It would, however, have been rather troublesome to work out the details and express the sum in a purely geometrical form, though Archimedes certainly shows remarkable skill in dealing with complicated cases like this.

7. Another extension of the theorem of § 5 is needed for his cubatures in *Conoids and Spheroids*, and it is found in the 2nd proposition of that book.

Let $a, b, \ldots l$ be n lines in A.P. of which the common difference is l, and p any other line and let S denote the sum

$$(pa+a^2)+(pb+b^2+\ \ldots\ +(pl+l^2);$$

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then the following inequalities hold, namely,

$$\frac{S}{n(pa+a^2)} > \frac{\frac{1}{2}p+\frac{1}{3}a}{p+a} > \frac{S-(pa+a^2)}{n(pa+a^2)} \quad \text{- - (A)}$$

The proof is effected by considering separately the sums

$$p(a+b+\ldots+l) \quad \text{and} \quad a^2+b^2+\ldots+l^2$$

and applying to these the proper inequality theorems.

In order to make the obervations in the next section more easily understood, I will indicate the bearing of this theorem on the cubature of the hyperboloid of revolution. Suppose a segment cut off by a plane at right angles to the axis at distance a from the vertex; let the distance a be divided into n equal parts, the distances from the vertex of the points of section forming the A.P. $a, b, \ldots l$, and let the figures described in §4 be constructed. Then if p be the transverse axis of the hyperboloid, the bases of the cylinders forming the set C are proportional to

$$(pa+a^2), \quad (pb+b^2), \ldots (pl+l^2)$$

and of those forming the set I, to

$$(pb+b^2), \ldots (pl+l^2).$$

It is easy then to see that

$$\frac{C}{K} = \frac{S}{n(pa+a^2)} \quad \text{and} \quad \frac{I}{K} = \frac{S-(pa+a^2)}{n(pa+a^2)}$$

and the theorem of this article proves that

$$\frac{C}{K} > \frac{\frac{1}{2}p+\frac{1}{3}a}{p+a} > \frac{I}{K}$$

and the application of the method of exhaustion then shows that

$$\frac{\text{segment}}{K} = \frac{\frac{1}{2}p+\frac{1}{3}a}{p+a}$$

8. In the case of the ellipsoid of revolution the corresponding bases are proportional to

$$(pa-a^2), \quad (pb-b^2), \ldots (pl-l^2)$$

and Zeuthen [*Kegelschnitte im Altertum*, p. 450] expresses surprise that Archimedes did not proceed in this case on the same lines as the above treatment of the hyperboloid. But it is not, I think, hard to understand the difference of treatment; it is simply impossible by means of the inequalities alone to establish the proper relations for the ellipsoid. If σ represent the sum

$$(pa-a^2)+(pb-b^2)+\ .\ .\ .\ +(pl-l^2)\ *$$

the relations required are

$$\frac{\sigma}{n(pa-a^2)}>\frac{\frac{1}{2}p-\frac{1}{3}a}{p-a}>\frac{\sigma-(pa-a^2)}{n(pa-a^2)}; \qquad (B)$$

but from the inequalities

$$p(a+b+\ldots+l)>\tfrac{1}{2}npa>p(b+\ldots+l)$$
$$a^2+b^2+\ldots+l^2)>\tfrac{1}{3}na^2>\ b^2+\ldots+l^2$$

it is only possible to conclude

$$pa+(pb-b^2)+\ldots+(pl-l^2)>na(\tfrac{1}{2}p-\tfrac{1}{3}a)$$

and $$-a^2+(pb-b^2)+\ldots+(pl-l^2)<na(\tfrac{1}{2}p-\tfrac{1}{3}a)$$

or $$\frac{\sigma+a^2}{n(pa-a^2)}>\frac{\frac{1}{2}p-\frac{1}{3}a}{p-a}>\frac{\sigma-ap}{n(pa-a^2)}$$

and this form is absolutely unsuitable. To get the proper form by this method, we have to take the exact value of σ, namely,

$$\sigma=na(\tfrac{1}{2}p-\tfrac{1}{3}a)+\tfrac{1}{2}a(p-a-\tfrac{1}{3}l)$$

and therefore

$$\sigma-(pa-a^2)=na(\tfrac{1}{2}p-\tfrac{1}{3}a)-\tfrac{1}{2}a(p-a+\tfrac{1}{3}l)$$

In order that (B) may be true, therefore, it is necessary to have $p>a+\frac{1}{3}l$, and though this condition is satisfied, it could not be established by means of the inequalities alone.

From formula (A) of § 7 we get

$$lS>nla(\tfrac{1}{2}p+\tfrac{1}{3}a)>lS-l(pa+a^2)$$

Hence the limit of lS for $n=\infty$ (or $l=0$, since $nl=a$) is $\frac{1}{2}a^2p+\frac{1}{3}a$, so that the result is equivalent to the integration

$$\int_0^a(px+x^2)dx=\tfrac{1}{2}a^2p+\tfrac{1}{3}a^3$$

and in the same way the formula (B) is equivalent to

$$\int_0^a(px-x^2)dx=\tfrac{1}{2}a^2p-\tfrac{1}{3}a^3$$

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* In the diagram of Archimedes (*Opera* I., p. 462),

$p=\mathrm{BZ},\quad a=\mathrm{B\Delta},\quad b=\mathrm{BE}$ etc.

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The fact, however, that Archimedes did not establish the theorem (B), which would have taken the place of the integral last written, seems to be due to his preference for inequalities, which in its turn was probably a consequence of his geometrical methods with their prolix enunciations rather than, as Zeuthen seems to think (p. 452), to the absence of a theorem corresponding to

$$\int[\phi(x)+\psi(x)]dx=\int\phi(x)dx+\int\psi(x)dx$$

for negative as well as positive values of $\psi(x)$. There is no doubt a great amount of truth in the general remarks of Zeuthen in the passage referred to, but the difficulty of establishing the inequality theorem (A) by a process equally applicable to theorem (B) or in general of establishing theorems that shall be equally applicable to positive and negative quantities is more than a difficulty of language. There is unquestionably a difficulty of language, but there is also a special difficulty arising from the use of inequalities, as in the case of theorem (B). The difference between ancient and modern methods introduced by the employment of negative quantities or negative operators seems to me to go deeper than is sometimes realised.

9. Had Archimedes first investigated the inequalities (B) he might have treated the cubature of the ellipsoid much more concisely. It may be noticed, however, that the transformation required in the case of a segment of the ellipsoid is at bottom identical with that of § 6. In dealing with the ellipsoid he requires to sum the series

$$u=(a^2-b^2)+(a^2-c^2)+\quad .\ .\ .\quad+(a^2-l^2)$$

where a, b, c,...l are $n+1$ lines in A.P. of which the common difference is not equal to the least line l. To effect the summation he puts

$$a^2-l^2=(a^2-x^2)+2l(x-l)+(x-l)^2$$

where x is any of the lines a, b, c, etc. Applying the theorem of § 7 to the series of n terms with

$$2l(x-l)+(x-l)^2$$

as general term, he gets

$$\frac{u}{n(a^2-l^2)}>\frac{2\ a+l}{3(a+l)}>\frac{u-(a^2-l^2)}{n(a^2-l^2)}$$

But since $$u = na^2 - (b^2 + \ . \ . \ . \ + l^2)$$

this pair of inequalities is equivalent to

$$a^2 + b^2 + \ldots + k^2 > \tfrac{1}{3}n(a^2 + al + l^2) > b^2 + \ldots + k^2 + l^2$$

and these are the inequalities established in § 6. *

10. Nearly all the theorems referred to in this paper seem to be due to Archimedes himself, and the whole treatment shows an originality of conception and execution that is somewhat difficult for us to recognise. The so-called geometrical algebra of the Greeks, valuable and important as it is for many purposes, is but a clumsy instrument compared with modern algebra in dealing with the summations discussed above, and in reading Archimedes one cannot fail to be struck with the prolixity of the enunciations and the length of the demonstrations caused in part by the absence of mere technical terms, but chiefly by the purely geometrical form in which his work is cast. It is, however, only an additional testimony to his genius that he triumphed over such difficulties and was able to carry the mensuration of the more common surfaces and solids to a stage which is even now the limit of instruction that does not involve the Integral Calculus.

* As another illustration of the application of the inequality theorems, I had worked out the value of the area of a segment of a parabola from the figure used in the mechanical quadrature, but as the method I followed is identical with that given by Heath (p. cliv.), I omit my investigation.