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Constructive Packings by Linear Hypergraphs

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For k-graphs F_0 and H, an F_0-packing of H is a family $\mathscr{F}$ of pairwise edge-disjoint copies of F_0 in H. Let $\nu_{F_0}(H)$ denote the maximum size $|\mathscr{F}|$ of an F_0-packing of H. Already in the case of graphs, computing $\nu_{F_0}(H)$ is NP-hard for most fixed F_0 (Dor and Tarsi [6]).

In this paper, we consider the case when F_0 is a fixed linear k-graph. We establish an algorithm which, for $\zeta > 0$ and a given k-graph H, constructs in time polynomial in $|V(H)|$ an F_0-packing of H of size at least $\nu_{F_0}(H) - \zeta|V(H)|^k$. Our result extends one of Haxell and Rödl, who established the analogous algorithm for graphs.

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1. Introduction

For k-uniform hypergraphs (k-graphs, for short) F_0 and H, an F_0-*packing* of H is a family $\mathscr{F}$ of pairwise edge-disjoint copies of F_0 in H. Let $\nu_{F_0}(H)$ denote the maximum size $|\mathscr{F}|$ of an F_0-packing in H. Already in the case of graphs, computing $\nu_{F_0}(H)$ is NP-hard for any fixed graph F_0 having a component with three or more edges (Dor and Tarsi [6]). Haxell and Rödl proved, however, that nearly optimal F_0-packings can be polynomially constructed for graphs H satisfying $\nu_{F_0}(H) = \Omega(n^2)$.

Theorem 1.1 (Haxell and Rödl [12]). *For every graph F_0 and for all $\zeta > 0$, there exists $n_0 = n_0(F_0, \zeta)$ and an algorithm which, for a given graph H on $n > n_0$ vertices, constructs in time polynomial in n an F_0-packing of H of size at least $\nu_{F_0}(H) - \zeta n^2$.*

Note that Theorem 1.1 remains true when $n \leqslant n_0$, but it is not interesting. In this case, one exhaustively searches for the optimal F_0-packing of H in time $O(1)$.

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The aim of this paper is to provide an extension of Theorem 1.1 to the case of linear hypergraphs F_0. A k-graph F_0 is said to be *linear* if every pair of its edges meet in at most one vertex – which is true of all (simple) graphs F_0.

Theorem 1.2. *For every linear k-graph F_0 and for all $\zeta > 0$, there exists an integer $n_0 = n_0(F_0, \zeta)$ and an algorithm which, for a given k-graph H on $n > n_0$ vertices, constructs in time polynomial in n an F_0-packing of H of size at least $\nu_{F_0}(H) - \zeta n^k$.*

The proofs of Theorems 1.1 and 1.2 both rely on the following well-known relaxation of an F_0-packing. A function $\psi : \binom{H}{F_0} \to [0,1]$ is a *fractional F_0-packing* of H if, for each edge $e \in H$,

$$\sum\left\{\psi(F) : F \in \binom{H}{F_0} \text{ satisfies } e \in F\right\} = \sum\left\{\psi(F) : F \in \binom{H}{F_0}_e\right\} \leqslant 1, \tag{1.1}$$

where $\binom{H}{F_0}$ denotes the family of all copies of F_0 in H and $\binom{H}{F_0}_e$ denotes the family of all such copies containing the edge e. The *size* $|\psi|$ of a fractional F_0-packing ψ is given by

$$|\psi| = \sum\left\{\psi(F) : F \in \binom{H}{F_0}\right\}$$

and $\nu^*_{F_0}(H)$ denotes the maximum size $|\psi|$ of a fractional F_0-packing ψ of H. Note that the characteristic function of an F_0-packing is a fractional F_0-packing, and hence $\nu_{F_0}(H) \leqslant \nu^*_{F_0}(H)$. It is known that building a fractional F_0-packing ψ of maximum size $\nu^*_{F_0}(H)$ is a linear programming problem, and hence constructable in time polynomial in $|V(H)|$.

Theorem 1.2 is not the first partial hypergraph extension of Theorem 1.1 (*cf.* Remark 1.4).

Theorem 1.3 ([12, 13, 21, 26]). *For every k-graph F_0 and for all $\zeta > 0$, there exists $n_0 = n_0(F_0, \zeta)$ so that for every k-graph H on $n > n_0$ vertices,*

$$\nu^*_{F_0}(H) - \nu_{F_0}(H) \leqslant \zeta n^k.$$

Theorem 1.3 implies that the parameter $\nu_{F_0}(H)$, when large enough, can be approximated in polynomial time by the parameter $\nu^*_{F_0}(H)$. When $k = 2$, Theorem 1.3 was a corollary of Theorem 1.1 since Haxell and Rödl, in fact, built F_0-packings of H of size $\nu^*_{F_0}(H) - \zeta n^2$. An alternative proof of Theorem 1.3 when $k = 2$ was later given by Yuster [26], which allowed F_0 to be replaced with a family of graphs. Theorem 1.3 when $k = 3$ was proved by Haxell, Rödl and the second author [13]. Finally, for $k \geqslant 2$, Theorem 1.3 was established by Rödl, Schacht, Siggers and Tokushige [21]. For future reference, we make the following remark, indicating the main difference between Theorems 1.2 and 1.3.

Remark 1.4. Theorem 1.3 is not restricted to the case that F_0 is linear, but claims no algorithm for building a nearly optimal F_0-packing of H. Theorem 1.2 provides such an algorithm, but only in the case when F_0 is linear. We explain the reason for this difference in upcoming Remarks 2.8 and 2.9.

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The proofs of Theorems 1.1–1.3 all depend heavily on graph and hypergraph versions of the *regularity method*, which relates to the celebrated Szemerédi Regularity Lemma. We shall next present the regularity tools we need for this paper. More generally, we proceed along the following itinerary.

Itinerary of paper. In Section 2, we present five algorithmic tools we need, each of which has a graph analogue in Haxell and Rödl [12]. In particular, we present three *regularity tools*: a *Regularity Lemma* (Theorem 2.1, due to Czygrinow and Rödl [5]), a *Slicing Lemma* (Lemma 2.3), and a *Packing Lemma* (Lemma 2.6). We also present two *supplemental* (non-regularity) *tools*: a *Crossing Lemma* (Lemma 2.10, due to Haxell and Rödl [12]) and a *Bounding Lemma* (Lemma 2.12). In Section 3, we use these tools to prove Theorem 1.2. In Section 4, we prove the Packing Lemma. In Section 5, we prove the Slicing Lemma. In Section 6, we prove the Bounding Lemma.

2. Algorithmic tools: regular and supplemental

In this section, we present the regularity and supplemental tools advertised above.

2.1. Regularity, Slicing and Packing Lemmas

We require the following concepts. For a k-graph H, let non-empty pairwise disjoint subsets $U_1, \dots, U_k \subset V(H)$ be given. Write $H[U_1, \dots, U_k]$ for the edges of H which intersect each $U_i, 1 \leqslant i \leqslant k$. The *density* of $(U_1, \dots, U_k)$ is defined as

$$d(U_1, \dots, U_k) = \frac{|H[U_1, \dots, U_k]|}{|U_1| \cdots |U_k|}.$$

For $d \in [0,1]$ and $\varepsilon > 0$, we say that $(U_1, \dots, U_k)$ is (d, ε)*-regular* if, for all $U_i' \subseteq U_i, 1 \leqslant i \leqslant k$, where $|U_i'| > \varepsilon |U_i|$, we have

$$|d(U_1', \dots, U_k') - d| < \varepsilon.$$

We say that $(U_1, \dots, U_k)$ is ε*-regular* if it is (d, ε)-regular for some $d \in [0,1]$, and ε*-irregular* otherwise.

When $k = 2$, the celebrated Szemerédi Regularity Lemma [23, 24] guarantees that, for all $\varepsilon > 0$, there exist integers $T_0 = T_0(\varepsilon)$ and $N_0 = N_0(\varepsilon)$ so that every graph H on $n \geqslant N_0$ vertices admits a vertex partition $V(H) = V_1 \cup \cdots \cup V_t$ into $t \leqslant T_0$ parts where all but $\varepsilon\binom{t}{2}$ pairs (V_i, V_j), $1 \leqslant i < j \leqslant t$, are ε-regular. (Moreover, these parts can be arranged to have nearly the same size $|V_1| \leqslant \cdots \leqslant |V_t| \leqslant |V_1| + 1$.) Alon, Duke, Lefmann, Rödl and Yuster [2] showed that the partition $V(H) = V_1 \cup \cdots \cup V_t$ in Szemerédi's Regularity Lemma can be constructed in time $O(M(n)) = O(n^{2.3727})$, where $M(n)$ is the time needed to multiply two $n \times n$ matrices with 0,1-entries over the integers (see [25]). Kohayakawa, Rödl and Thoma [18] improved this running time to $O(n^2)$.

For $k \geqslant 2$, the following hypergraph version of Szemerédi's Regularity Lemma was established by Frankl and Rödl [7], where the algorithmic assertion was established by Czygrinow and Rödl [5]. (In the following statement, the input k-graph H is equipped with a vertex partition $V(H) = V_1 \cup \cdots \cup V_\ell$, which is refined into a regular partition – a common ability of any regularity lemma.)

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Theorem 2.1 (Regularity Lemma [5, 7]). *For all $\varepsilon > 0$ and all positive integers k and ℓ, there exist integers $T_0 = T_0(\varepsilon, k, \ell)$ and $N_0 = N_0(\varepsilon, k, \ell)$ so that the following holds.*

Let a k-graph H on $n \geqslant N_0$ vertices be given with a vertex partition $V(H) = V_1 \cup \cdots \cup V_\ell$ satisfying $|V_1| \leqslant \cdots \leqslant |V_\ell| \leqslant |V_1| + 1$. Then, one may construct, in time $O(n^{2k-1} \log^2 n)$, a refined partition

$$V_i = V_{i0} \cup V_{i1} \cup \cdots \cup V_{it}, \quad \text{with } m \stackrel{\text{def}}{=} |V_{i1}| = \cdots = |V_{it}|,$$

$1 \leqslant i \leqslant \ell$, where $t \leqslant T_0$, where $V_0 = V_{10} \cup \cdots \cup V_{\ell 0}$ has size $|V_0| < \varepsilon n$, and where all but $\varepsilon\binom{\ell}{k} t^k$ many k-tuples $(V_{i_1 j_1}, \ldots, V_{i_k j_k})$, $1 \leqslant i_1 < \cdots < i_k \leqslant \ell$, $1 \leqslant j_1, \ldots, j_k \leqslant t$, are ε-regular and labelled as such.

Remark 2.2. The 'labelling' assertion of Theorem 2.1 is not explicitly stated in [5], but is implicit in their proof [4]. For completeness, we mention a recent result of Conlon, Hàn, Person and Schacht [3] which would make the labelling easy to see (but at the cost of producing a larger polynomial running time). The authors in [3] established a k-graph M_k with 2^k edges and $k2^{k-1}$ vertices for which the following equivalence holds with $d = d_H(V_{i_1 j_1}, \ldots, V_{i_k j_k})$.

(a) If $\delta > 0$ is sufficiently smaller than $\varepsilon > 0$, and if $H[V_{i_1 j_1}, \ldots, V_{i_k j_k}]$ has within $d^{2^k} m^{k2^{k-1}} (1 \pm \delta)$ copies of M_k, then $(V_{i_1 j_1}, \ldots, V_{i_k j_k})$ is (d, ε)-regular.

(b) If $\varepsilon > 0$ is sufficiently smaller than $\delta > 0$, and if $(V_{i_1 j_1}, \ldots, V_{i_k j_k})$ is (d, ε)-regular, then $H[V_{i_1 j_1}, \ldots, V_{i_k j_k}]$ has within $d^{2^k} m^{k2^{k-1}} (1 \pm \delta)$ copies of M_k.

(In fact, when $k = 2$, M_2 turns out to be C_4 (the 4-cycle), and the equivalence above is precisely the one devised and used by Alon, Duke, Lefmann, Rödl and Yuster [2] for their algorithmic version of Szemerédi's Regularity Lemma.) Now, employing the above result in the proof of Theorem 2.1 would render the promised labelling. The running time would increase to $O(k2^{k-1})$, but for the purpose of proving Theorem 1.2 it would not matter.

We shall now present the Slicing Lemma.

Lemma 2.3 (Slicing Lemma). *For every integer $k \geqslant 2$ and for all $d_0, \varepsilon' > 0$, there exists $\varepsilon = \varepsilon_{\text{Lem.2.3}}(k, d_0, \varepsilon') > 0$ so that the following holds.*

Let G be an ε-regular k-partite k-graph with vertex partition $V(G) = V_1 \cup \cdots \cup V_k$, where $|V_1| = \cdots = |V_k| = m$ is sufficiently large. Suppose that $p_1, \ldots, p_s \geqslant d_0$ are given with

$$\sum_{i=1}^{s} p_i \leqslant d_G(V_1, \ldots, V_k).$$

Then, there exists an algorithm which, in time $O(m^k)$, constructs an edge-partition $G = G_0 \cup G_1 \cup \cdots \cup G_s$, where each G_i, $1 \leqslant i \leqslant s$, is (p_i, ε')-regular.

Remark 2.4. In the context of the Slicing Lemma, it is an easy consequence that the class G_0 is $(p_0, s\varepsilon')$-regular, where $p_0 = D - \sum_{i=1}^{s} p_i$. (In this paper, however, we do not use this feature.)

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Our final regularity tool is the Packing Lemma, which considers the following setup.

Setup 2.5 (Packing Setup). *Let F_0 be a linear k-graph with vertex set*

$$V(F_0) = [f] = \{1, \dots, f\},$$

and let G be an f-partite k-graph with vertex partition $V(G) = V_1 \cup \cdots \cup V_f$ satisfying $|V_1| = \cdots = |V_f| = m$. Suppose, moreover, that for some $d, \varepsilon > 0$, G has the following property. For each $\{i_1, \dots, i_k\} \in \binom{[f]}{k}$,

(a) *if $\{i_1, \dots, i_k\} \in F_0$, then $(V_{i_1}, \dots, V_{i_k})$ is (d, ε)-regular,*
(b) *if $\{i_1, \dots, i_k\} \notin F_0$, then $G[V_{i_1}, \dots, V_{i_k}] = \emptyset$.*

In the context of Setup 2.5, a subhypergraph F' of G on vertices $v_1, \dots, v_f$ is a *partite-isomorphic copy* of F_0 if $v_i \in V_i$ for all $1 \leqslant i \leqslant f$, and if $v_i \to i$ defines an isomorphism from F' to F_0.

Lemma 2.6 (Packing Lemma). *Let F_0 be a fixed linear k-graph with $V(F_0) = [f]$. For all $d_0, \mu > 0$, there exists $\varepsilon = \varepsilon_{\text{Lem.2.6}}(d_0, \mu) > 0$ so that the following holds.*

Let G be a k-graph satisfying the hypothesis of Setup 2.5 with F_0 above, with some $d > d_0$, with $\varepsilon = \varepsilon_{\text{Lem.2.6}}$ above, and with m sufficiently large. Then, there exists an algorithm which, in time polynomial in m, constructs an F_0-packing $\mathscr{F}_G$ of G covering all but $\mu|G|$ edges of G, and which consists entirely of partite-isomorphic copies of F_0 in G. In particular,

$$|\mathscr{F}_G| \geqslant (1 - \mu)(d - \varepsilon)m^k.$$

Remark 2.7. The last assertion of the Packing Lemma is an easy consequence of its predecessor. Indeed, in the context above, let $G' \subseteq G$ denote the set of edges covered by $\mathscr{F}_G$. Every element $F \in \mathscr{F}_G$ covers precisely $|F_0|$ edges of G', and every edge of G' is covered by precisely one element $F \in \mathscr{F}_G$. Thus,

$$\begin{aligned} |\mathscr{F}_G| \times |F_0| = |G'| &\geqslant (1 - \mu)|G| \\ &= (1 - \mu) \sum \{|G[V_{i_1}, \dots, V_{i_k}]| : \{i_1, \dots, i_k\} \in F_0\} \\ &\geqslant (1 - \mu)|F_0|(d - \varepsilon)m^k, \end{aligned}$$

where the last inequality follows from the definition of (d, ε)-regularity. The result now follows.

Remark 2.8. For $k \geqslant 3$, the conclusion of Lemma 2.6 is false when F_0 is not linear. Indeed, for example, consider when $k = 3$, $f = 4$, F_0 consists of the triples $\{1, 2, 3\}$ and $\{2, 3, 4\}$, and G is defined as follows. Take the random bipartite graph $\mathbb{G}(V_2, V_3, 1/2)$. For each $v_2 \in V_2$ and $v_3 \in V_3$, if $\{v_2, v_3\} \in \mathbb{G}(V_2, V_3, 1/2)$, put $\{v_1, v_2, v_3\} \in G$ for every $v_1 \in V_1$. Otherwise, put $\{v_2, v_3, v_4\} \in G$ for every $v_4 \in V_4$. Clearly, G contains no copies of F_0. However, by the Chernoff inequality, with high probability, both of (V_1, V_2, V_3) and (V_2, V_3, V_4) are $(1/2, o(1))$-regular.

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Remark 2.9. The papers [13, 21] proving Theorem 1.3 use hypergraph regularity lemmas from [8, 20] (see also [9, 10]) which allow an analogue of the Packing Lemma when F_0 is not necessarily linear. Unfortunately, algorithmic versions of these regularity lemmas are not known for $k \geqslant 4$, although, for $k = 3$, such an algorithm has been given [15] (see also [14, 19]).

2.2. Crossing and Bounding Lemmas

In what follows, let H and F_0 be k-graphs, and suppose H has vertex partition Π: $V(H) = V_1 \cup \cdots \cup V_\ell$. We say a copy $F \in \binom{H}{F_0}$ *crosses* Π if $|V(F) \cap V_i| \leqslant 1$ for every $1 \leqslant i \leqslant \ell$. Let $\binom{H}{F_0}_\Pi$ denote the subcollection of copies $F \in \binom{H}{F_0}$ which cross Π. The Crossing Lemma, due to Haxell and Rödl [12] (see Remark 2.11), then states that if H has a fractional F_0-packing ψ, then one may construct a relatively small partition Π whose crossing copies of F_0 comprise most of the value of ψ.

Lemma 2.10 (Crossing Lemma [12]). *For every k-graph F_0 on f vertices, and for all $\mu > 0$, there exists $L_0 = L_0(\mu, F_0)$ so that the following holds.*

Let H be a k-graph on n vertices, and let ψ be a fractional F_0-packing of H. There exists an algorithm which constructs, in time $O(n^f)$, a vertex partition $\Pi : V(H) = V_1 \cup \cdots \cup V_\ell$, $\ell \leqslant L_0$, satisfying that $\lfloor n/\ell \rfloor \leqslant |V_i| \leqslant \lceil n/\ell \rceil$ for all $1 \leqslant i \leqslant \ell$, and satisfying that

$$|\psi_\Pi| \stackrel{\text{def}}{=} \sum \left\{ \psi(F) : F \in \binom{H}{F_0}_\Pi \right\} \geqslant (1 - \mu)|\psi|.$$

Remark 2.11. Haxell and Rödl proved Lemma 2.10 in the following more general setting (see Lemma 11 in [12]): with $V = V(H)$, H is replaced by $\binom{V}{f}$, where $f = |V(F_0)|$, and ψ is replaced by an arbitrary function $g : \binom{V}{f} \to [0, \infty)$. Their lemma then constructs a partition Π so that $|g_\Pi| \geqslant (1 - \mu)|g|$, where

$$|g| = \sum \left\{ g(S) : S \in \binom{V}{f} \right\} \quad \text{and} \quad |g_\Pi| = \sum \left\{ g(S) : S \in \binom{V}{f}_\Pi \right\},$$

where $\binom{V}{f}_\Pi$ is the set of f-tuples S which cross the partition Π. We could not find an explicit mention of the time complexity of Lemma 11 in [12], although $O(n^f)$ is clear from the proof. Indeed, in time $O(n^f)$, they define a weight function w on $\binom{V}{2}$ by

$$w(\{x, y\}) = \sum \left\{ g(S) : x, y \in S \in \binom{V}{f} \right\}.$$

Then, they apply Lemma 10 in [12] to V and w to construct in time $O(n^2)$ (with running time $O(n^2)$ explicitly stated in Lemma 10) an equitable bipartition $V = V_1 \cup V_2$ so that

$$\sum \{ w(\{x, y\}) : x \in V_1, \ y \in V_2 \} \geqslant (1/2) \sum \left\{ w(\{x, y\}) : \{x, y\} \in \binom{V}{2} \right\}.$$

They then apply Lemma 10 to V_1 and V_2, and so on, so that after at most $\log_2(f^2/\mu) = O(1)$ iterations, they reach the promised partition.

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We now present the Bounding Lemma, which considers weighted hypergraphs H_0 and the following concepts. Let F_0 be a k-graph, and let H_0 be an edge-weighted k-graph with weight function $\omega : H_0 \to [0,1]$. A *fractional* (ω, F_0)*-packing* of H_0 is a function $\hat{\psi} : \binom{H_0}{F_0} \to [0,1]$ satisfying that, for each $e \in H_0$,

$$\sum\left\{\hat{\psi}(F) : F \in \binom{H_0}{F_0}_e\right\} \leqslant \omega(e)$$

(recall the notation in (1.1)). (If $\omega \equiv 1$ is the constant function on H_0, then $\hat{\psi}$ is a fractional F_0-packing of H_0.) As before, set

$$|\hat{\psi}| = \sum\left\{\hat{\psi}(F) : F \in \binom{H_0}{F_0}\right\} \quad \text{and}$$

$$\nu^*_{F_0}(H_0) = \max\{|\hat{\psi}| : \hat{\psi} \text{ is a fractional } (\omega, F_0)\text{-packing of } H_0\}.$$

Finally, we say that a fractional (ω, F_0)-packing $\hat{\psi}$ is δ*-bounded* if, for each $F \in \binom{H_0}{F_0}$, $\hat{\psi}(F) \in \{0\} \cup [\delta, 1]$. The Bounding Lemma then states that the parameter $\nu^*_{F_0}(H_0)$ can be approximated by a δ-bounded fractional (ω, F_0)-packing $\hat{\psi}$.

Lemma 2.12 (Bounding Lemma). *For every k-graph F_0 and for all $\xi > 0$, there exists a positive constant $\delta = \delta_{\text{Lem.2.12}}(F_0, \xi)$ so that the following holds.*

Let H_0 be a weighted k-graph on r vertices with weight function $\omega : H_0 \to [0,1]$. Then, there exists a δ-bounded fractional (ω, F_0)-packing $\hat{\psi}$ of H_0 such that $|\hat{\psi}| \geqslant \nu^*_{F_0}(H_0) - \xi r^k$. Moreover, the function $\hat{\psi}$ can be found, in time depending on r, by an exhaustive search.

We conclude this section by stating specific versions of some familiar tools.

2.3. Some familiar tools

Fact 2.13 (Cauchy–Schwarz inequality: see, *e.g.*, [22]). *For $a_1, \dots, a_t \geqslant 0$ and $\tau \geqslant 0$, suppose $\sum_{i=1}^t a_i \geqslant (1-\tau)at$ and $\sum_{i=1}^t a_i^2 \leqslant (1+\tau)a^2 t$. Then, for all but $2\tau^{1/3}t$ terms $1 \leqslant i \leqslant t$, we have $a_i = a(1 \pm 2\tau^{1/3})$.*

Fact 2.14 (Chernoff inequality: see, *e.g.*, [1, 16]). *Let X have binomial distribution. Then, for any $0 < \delta < 3/2$, $\mathbb{P}[X \neq (1 \pm \delta)\mathbb{E}[X]] \leqslant 2\exp\{-\delta^2 \mathbb{E}[X]/3\}$.*

3. Proof of Theorem 1.2

Let F_0 be a given linear k-graph on f vertices and let $\zeta > 0$ be given. Our first step is to define some auxiliary constants with respect to which the size of the input hypergraph H needs to be large.

Step 0: auxiliary constants and input *H*. Set

$$\mu = \xi = \frac{\zeta}{6}. \tag{3.1}$$

With ξ given above, let

$$\delta = \delta_{\text{Lem.2.12}}(F_0, \xi) > 0 \tag{3.2}$$

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be the constant guaranteed by the Bounding Lemma (Lemma 2.12). Set

$$d_0 = \delta. \tag{3.3}$$

With μ in (3.1) and d_0 in (3.3), let $\varepsilon_{\text{Lem.2.6}} = \varepsilon_{\text{Lem.2.6}}(F_0, d_0, \mu) > 0$ be the constant guaranteed by the Packing Lemma (Lemma 2.6). Set

$$\varepsilon' = (d_0\mu)\varepsilon_{\text{Lem.2.6}}, \tag{3.4}$$

and let $\varepsilon_{\text{Lem.2.3}} = \varepsilon_{\text{Lem.2.3}}(k, d_0, \varepsilon') > 0$ be the constant guaranteed by the Slicing Lemma (Lemma 2.3). Define

$$\varepsilon = \min\{\varepsilon_{\text{Lem.2.3}}, \varepsilon_{\text{Lem.2.6}}\} \tag{3.5}$$

(which is achieved by $\varepsilon_{\text{Lem.2.3}}$).

In all that follows, the integer n_0 is assumed to be sufficiently large with respect to all constants discussed above. In particular, n_0 is large with respect to the following additional constants. With $\mu > 0$ given in (3.1), let $L_0 = L_0(\mu)$ be the constant guaranteed by the Crossing Lemma (Lemma 2.10). With $\varepsilon > 0$ given in (3.5) and L_0 given above, let $T_0 = T_0(\varepsilon, k, L_0)$ and $N_0 = N_0(\varepsilon, k, L_0)$ be the constants given by the Regularity Lemma (Theorem 2.1). The integer n_0 is larger than N_0 and T_0.

Now, let H be a given k-graph on $n > n_0$ vertices. We construct, in time polynomial in n, an F_0-packing $\mathscr{F}_H$ of H of size

$$|\mathscr{F}_H| \geqslant \nu^*_{F_0}(H) - \zeta n^k. \tag{3.6}$$

Since $\nu^*_{F_0}(H) \geqslant \nu_{F_0}(H)$, this will prove Theorem 1.2. We proceed to the first step of our algorithm.

Step 1: preprocessing *H*. First, equip H with a maximum fractional F_0-packing ψ^*, *i.e.*, one for which $|\psi^*| = \nu^*_{F_0}(H)$. Constructing ψ^* is a linear programming problem with running time polynomial in n.

We now apply the Crossing Lemma (Lemma 2.10) to H and ψ^*. With $\mu > 0$ given in (3.1), Lemma 2.10 guarantees the constant $L_0 = L_0(\mu)$ (discussed in Step 0) and constructs, in time $O(n^2)$, a vertex partition $\Pi : V(H) = V_1 \cup \cdots \cup V_\ell$ where $\ell \leqslant L_0$, $\lfloor n/\ell \rfloor \leqslant |V_i| \leqslant \lceil n/\ell \rceil$, and where

$$|\psi^*_\Pi| \stackrel{\text{def}}{=} \sum \left\{ \psi^*(F) : F \in \binom{H}{F_0}_\Pi \right\} \geqslant (1-\mu)|\psi^*|. \tag{3.7}$$

We mention that we build ψ^* so that we may apply the Crossing Lemma, and we need the Crossing Lemma in order to prove Proposition 3.1 below.

Step 2: regularizing *H* and building *H*$_0$. Our next step is to apply the Regularity Lemma (Theorem 2.1) to H (and Π) and to construct, as usual, the resulting 'cluster' hypergraph H_0. To that end, with $\varepsilon > 0$ given in (3.5), ℓ obtained in Step 1 (with $\ell \leqslant L_0$), Theorem 2.1 guarantees the constant $T_0 = T_0(\varepsilon, k, \ell)$ (discussed in Step 0) and constructs, in time $O(n^{2k-1}\log^2 n)$, a refined vertex partition

$$\hat{\Pi} : V(H) = V_0 \cup \bigcup \{V_{ij} : 1 \leqslant i \leqslant \ell,\ 1 \leqslant j \leqslant t\},$$

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where

(i) $t \leqslant T_0$ and $m \stackrel{\text{def}}{=} |V_{11}| = \cdots = |V_{\ell t}|$ and $|V_0| < \varepsilon n$,

(ii) all but $\varepsilon\binom{\ell}{k}t^k$ many k-tuples $(V_{i_1 j_1}, \ldots, V_{i_k j_k})$, $1 \leqslant i_1 < \cdots < i_k \leqslant \ell$, $1 \leqslant j_1, \ldots, j_k \leqslant t$, are ε-regular and labelled as such.

We now construct the cluster hypergraph H_0 which will, in fact, be a weighted hypergraph. To begin, H_0 will have vertex set $V(H_0) = \{u_{ij} : 1 \leqslant i \leqslant \ell, 1 \leqslant j \leqslant t\}$. Consider the set of all $\binom{\ell}{k}t^k$ many k-tuples of the form $\{u_{i_1 j_1}, \ldots, u_{i_k j_k}\}$, where $1 \leqslant i_1 < \cdots < i_k \leqslant \ell$ and $1 \leqslant j_1, \ldots, j_k \leqslant t$. For each such k-tuple $\{u_{i_1 j_1}, \ldots, u_{i_k j_k}\}$, define

$$\omega(\{u_{i_1 j_1}, \ldots, u_{i_k j_k}\}) = \begin{cases} d_H(V_{i_1 j_1}, \ldots, V_{i_k j_k}) & (V_{i_1 j_1}, \ldots, V_{i_k j_k}) \text{ is (labelled to be) } \varepsilon\text{-regular}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.8}$$

Then H_0 will consist of all k-tuples above whose weight is non-zero. (Note that H_0 consists only of k-tuples $\{u_{i_1 j_1}, \ldots, u_{i_k j_k}\}$ where $(V_{i_1 j_1}, \ldots, V_{i_k j_k})$ 'crosses' the partition $V_1 \cup \cdots \cup V_\ell$.) Together with the function ω, H_0 is a weighted k-graph on ℓt vertices, and since $\ell \leqslant L_0$ and $t \leqslant T_0$, the construction of H_0 is complete in time $O(1)$.

While we do not use it yet, we note that $\nu^*_{F_0}(H_0)$ is essentially a $1/m^k$ portion of $|\psi^*| = \nu^*_{F_0}(H)$.

Proposition 3.1.

$$m^k \nu^*_{F_0}(H_0) \geqslant |\psi^*_\Pi| - 2\varepsilon n^k \stackrel{(3.7)}{\geqslant} (1-\mu)|\psi^*| - 2\varepsilon n^k = (1-\mu)\nu^*_{F_0}(H) - 2\varepsilon n^k.$$

We will prove Proposition 3.1 at the end of this section.

Step 3: bounding H_0. We now apply the Bounding Lemma (Lemma 2.12) to the weighted hypergraph H_0. To that end, with $\xi > 0$ given in (3.1) and δ given in (3.2), we apply Lemma 2.12 to H_0 to guarantee a δ-bounded fractional (ω, F_0)-packing $\hat{\psi}$ of H_0 satisfying

$$|\hat{\psi}| \geqslant \nu^*_{F_0}(H_0) - \xi(\ell t)^k. \tag{3.9}$$

The Bounding Lemma also ensures that $\hat{\psi}$ can be constructed by an exhaustive search in time $O(1)$ (since H_0 has $\ell t \leqslant L_0 T_0 = O(1)$ many vertices).

We establish some notation related to the fractional (ω, F_0)-packing $\hat{\psi}$ of H_0. Set (*cf.* (3.3))

$$\binom{H_0}{F_0}^+ = \left\{F \in \binom{H_0}{F_0} : \hat{\psi}(F) \neq 0\right\} = \left\{F \in \binom{H_0}{F_0} : \hat{\psi}(F) \geqslant \delta \stackrel{(3.3)}{=} d_0\right\},$$

where the last equality follows from the fact that $\hat{\psi}$ is δ-bounded. For a fixed $e \in H_0$, we write

$$\binom{H_0}{F_0}^+_e = \binom{H_0}{F_0}_e \cap \binom{H_0}{F_0}^+.$$

Step 4: slicing H. We now run the Slicing Lemma (Lemma 2.3), repeatedly, over the hypergraph H. To that end, fix $e = \{u_{i_1 j_1}, \ldots, u_{i_k j_k}\} \in H_0$, which fixes the corresponding

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hypergraph $H[V_{i_1j_1},\dots,V_{i_kj_k}]$. For each $F \in \binom{H_0}{F_0}_e^+$, we wish to cut (using Lemma 2.3) a 'regular' slice from $H[V_{i_1j_1},\dots,V_{i_kj_k}]$ of density $p_F = \hat{\psi}(F)$. Let us first check that it is appropriate to do so. First, every $p_F = \hat{\psi}(F) \geqslant d_0$ on account of $F \in \binom{H_0}{F_0}_e^+$, as is required by the Slicing Lemma. Second, since $\hat{\psi}$ is an (ω, F_0)-packing of H_0, we have

$$\sum\left\{p_F : F \in \binom{H_0}{F_0}_e^+\right\} = \sum\left\{\hat{\psi}(F) : F \in \binom{H_0}{F_0}_e^+\right\} \leqslant \omega(e) \overset{(3.8)}{=} d_H(V_{i_1j_1},\dots,V_{i_kj_k}),$$

as is also required by the Slicing Lemma. Finally, by (3.5), $\varepsilon \leqslant \varepsilon_{\text{Lem.2.3}}(d_0, \varepsilon')$ was chosen to be sufficient for an application of the Slicing Lemma (Lemma 2.3). Consequently, Lemma 2.3 constructs, in time $O(m^k)$, a partition

$$H[V_{i_1j_1},\dots,V_{i_kj_k}] = H_*[V_{i_1j_1},\dots,V_{i_kj_k}] \cup \bigcup\left\{H_F[V_{i_1j_1},\dots,V_{i_kj_k}] : F \in \binom{H_0}{F_0}_e^+\right\}, \tag{3.10}$$

where each slice $H_F[V_{i_1j_1},\dots,V_{i_kj_k}]$, $F \in \binom{H_0}{F_0}_e^+$, is $(\hat{\psi}(F), \varepsilon')$-regular. (We use H_* notation to denote the remainder, which we henceforth ignore.)

Step 5: packing H (locally). We now run the Packing Lemma (Lemma 2.6), repeatedly, over the hypergraph H. To that end, fix $F \in \binom{H_0}{F_0}^+$, and construct the following f-partite subhypergraph $G_F \subseteq H$ (recall $f = |V(F_0)|$):

$$\begin{aligned} V(G_F) &= \bigcup\{V_{ij} : u_{ij} \in V(F)\} \quad \text{and} \\ G_F = E(G_F) &= \bigcup\{H_F[V_{i_1j_1},\dots,V_{i_kj_k}] : \{u_{i_1j_1},\dots,u_{i_kj_k}\} \in F\}, \end{aligned} \tag{3.11}$$

where for each edge $e = \{u_{i_1j_1},\dots,u_{i_kj_k}\} \in F$, $H_F[V_{i_1j_1},\dots,V_{i_kj_k}]$ is the slice (from Step 4) from $H[V_{i_1j_1},\dots,V_{i_kj_k}]$ corresponding to F. Note that the hypergraph G_F is constructed in time $O(m^k)$.

We now apply the Packing Lemma (Lemma 2.6) to the hypergraph G_F, but first check that it is appropriate to do so. Observe that G_F and F satisfy the hypothesis of Setup 2.5. Indeed, for each edge $e = \{u_{i_1j_1},\dots,u_{i_kj_k}\} \in F$, the corresponding hypergraph $G_F[V_{i_1j_1},\dots,V_{i_kj_k}]$ is $(\hat{\psi}(F), \varepsilon')$-regular, where $\hat{\psi}(F) \geqslant d_0 = \delta$ on account that $F \in \binom{H_0}{F_0}^+$. Otherwise, for each $\{u_{i_1j_1},\dots,u_{i_kj_k}\} \in \binom{V(F)}{k} \setminus F$, the corresponding hypergraph $G_F[V_{i_1j_1},\dots,V_{i_kj_k}] = \emptyset$. Finally, recall from (3.4) that $\varepsilon' \leqslant \varepsilon_{\text{Lem.2.6}}(d_0, \mu)$ was chosen in accordance with the Packing Lemma (Lemma 2.6). Lemma 2.6 therefore constructs, in time polynomial in m, an F_0-packing $\mathscr{F}_{G_F}$ of G_F satisfying

$$|\mathscr{F}_{G_F}| \geqslant (1-\mu)\big(\hat{\psi}(F) - \varepsilon'\big)m^k \geqslant (1-\mu)\left(1 - \frac{\varepsilon'}{d_0}\right)\hat{\psi}(F)m^k. \tag{3.12}$$

Step 6: constructing the promised $\mathscr{F}_H$. We define

$$\mathscr{F}_H = \left\{\mathscr{F}_{G_F} : F \in \binom{H_0}{F_0}^+\right\}, \tag{3.13}$$

which amounts to collecting the 'local' packings $\mathscr{F}_{G_F}$ over all $F \in \binom{H_0}{F_0}^+$. The remainder of this section checks that $\mathscr{F}_H$ is an F_0-packing of H, that $\mathscr{F}_H$ was constructed in time polynomial in n, and that $\mathscr{F}_H$ has the size promised in (3.6).

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$\mathscr{F}_H$ is an F_0-packing of H. Indeed, let $F \neq F' \in \mathscr{F}_H$ be fixed. Note that, by construction of $\mathscr{F}_H$ (*cf.* (3.13)), there exist $\hat{F}, \hat{F}' \in \binom{H_0}{F_0}^+$ so that $F \in \mathscr{F}_{G_{\hat{F}}}$ and $F' \in \mathscr{F}_{G_{\hat{F}'}}$. Now, let us assume, for contradiction, that $F \cap F' \neq \emptyset$.

If $\hat{F} = \hat{F}'$, then $F \cap F' \neq \emptyset$ contradicts the Packing Lemma (Lemma 2.6) since $\mathscr{F}_{G_{\hat{F}}} = \mathscr{F}_{G_{\hat{F}'}}$ was an F_0-packing of $G_{\hat{F}} = G_{\hat{F}'}$. Henceforth, we assume $\hat{F} \neq \hat{F}'$.

Let $e \in F \cap F'$, and write $e \in H[V_{i_1 j_1}, \dots, V_{i_k j_k}]$ for some $1 \leqslant i_1 < \cdots < i_k \leqslant \ell$ and $1 \leqslant j_1, \dots, j_k \leqslant t$. It follows from $e \in F \in \mathscr{F}_{G_{\hat{F}}}$ and similarly $e \in F' \in \mathscr{F}_{G_{\hat{F}'}}$ that

$$e \in G_{\hat{F}}[V_{i_1 j_1}, \dots, V_{i_k j_k}] \cap G_{\hat{F}'}[V_{i_1 j_1}, \dots, V_{i_k j_k}],$$

or equivalently (*cf.* (3.11)),

$$e \in H_{\hat{F}}[V_{i_1 j_1}, \dots, V_{i_k j_k}] \cap H_{\hat{F}'}[V_{i_1 j_1}, \dots, V_{i_k j_k}]. \tag{3.14}$$

But (3.14) contradicts the Slicing Lemma, since $H_{\hat{F}}[V_{i_1 j_1}, \dots, V_{i_k j_k}]$ and $H_{\hat{F}'}[V_{i_1 j_1}, \dots, V_{i_k j_k}]$ are distinct classes of a partition (distinct because $\hat{F} \neq \hat{F}'$).

$\mathscr{F}_H$ was constructed in time polynomial in n. Indeed, in Step 1, we constructed a maximum fractional F_0-packing ψ^* of H, which as a linear programming problem is done in time polynomial in n. We then applied the Crossing Lemma (Lemma 2.10) to H and ψ^*, which was done in time $O(n^f)$. In Step 2, we applied the Regularity Lemma (Theorem 2.1) to H and Π, which was done in time $O(n^{2k-1} \log^2 n)$, and we constructed the weighted cluster H_0 in time $O(1)$. In Step 3, we applied the Bounding Lemma (Lemma 2.12) to H_0, which constructed $\hat{\psi}$ in time $O(1)$. In Step 4, we applied the Slicing Lemma (Lemma 2.3) to H at most $\binom{\ell t}{k} \leqslant (L_0 T_0)^k = O(1)$ times, where each such application took time $O(m^k) = O(n^k)$. In Step 5, we applied the Packing Lemma at most $(\ell t)^f \leqslant (L_0 T_0)^f = O(1)$ times, where each such application took time polynomial in m (and so polynomial in n).

$\mathscr{F}_H$ has size promised in (3.6). From (3.13), we have

$$\begin{aligned}
|\mathscr{F}_H| &= \sum \left\{ |\mathscr{F}_{G_F}| : F \in \binom{H_0}{F_0}^+ \right\} \\
&\overset{(3.12)}{\geqslant} (1-\mu)\left(1 - \frac{\varepsilon'}{d_0}\right) m^k \sum \left\{ \hat{\psi}(F) : F \in \binom{H_0}{F_0}^+ \right\} \\
&= (1-\mu)\left(1 - \frac{\varepsilon'}{d_0}\right) m^k |\hat{\psi}| \\
&\overset{(3.4)}{\geqslant} (1-\mu)^2 m^k |\hat{\psi}| \overset{(3.9)}{\geqslant} (1-\mu)^2 m^k \left(\nu^*_{F_0}(H_0) - \xi (\ell t)^k \right) \\
&\overset{\text{Prop.3.1}}{\geqslant} (1-\mu)^2 \left((1-\mu) \nu^*_{F_0}(H) - 2\varepsilon n^k - \xi (m \ell t)^k \right) \\
&\overset{(3.1)}{\geqslant} (1-2\mu)\left(\nu^*_{F_0}(H) - 4\mu n^k \right) \overset{(3.1)}{\geqslant} \nu^*_{F_0}(H) - 6\mu n^k \overset{(3.1)}{=} \nu^*_{F_0}(H) - \zeta n^k,
\end{aligned}$$

where the second equality holds since $\hat{\psi}$ vanishes outside $\binom{H_0}{F_0}^+$ (and where we used $m\ell t \leqslant n$ and $\nu^*_{F_0}(H) \leqslant n^k$). All that remains is to prove Proposition 3.1.

Proof of Proposition 3.1. It suffices to produce a fractional packing $\psi_0 : \binom{H_0}{F_0} \to [0,1]$ for which $m^k |\psi_0|$ has the lower bound of Proposition 3.1. To produce ψ_0, we use the

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following notation. Define

$$H_{\hat{\Pi}} = \bigcup \{H[V_{i_1 j_1}, \ldots, V_{i_k j_k}] : \{u_{i_1 j_1}, \ldots, u_{i_k j_k}\} \in H_0\}.$$

Thus, $H_{\hat{\Pi}}$ consist of all edges $\{v_{i_1 j_1}, \ldots, v_{i_k j_k}\} \in H$ for which $v_{i_1 j_1} \in V_{i_1 j_1}, \ldots, v_{i_k j_k} \in V_{i_k j_k}$, for some $1 \leqslant i_1 < \cdots < i_k \leqslant \ell$, $1 \leqslant j_1, \ldots, j_k \leqslant t$, where $(V_{i_1 j_1}, \ldots, V_{i_k j_k})$ is (labelled to be) ε-regular. Since each edge of $H_{\hat{\Pi}}$ crosses the partition $\Pi : V(H) = V_1 \cup \cdots \cup V_\ell$ (*cf.* the Crossing Lemma (Lemma 2.10)), every element $F \in \binom{H_{\hat{\Pi}}}{F_0}$ also crosses Π, and so

$$\binom{H_{\hat{\Pi}}}{F_0} \subseteq \binom{H}{F_0}_{\Pi}. \tag{3.15}$$

Note that the mapping

$$\pi : V(H_{\hat{\Pi}}) \to V(H_0) \quad \text{given by} \quad v \mapsto u_{ij} \iff v \in V_{ij}$$

defines a homomorphism from $H_{\hat{\Pi}}$ to H_0. As such, since each $F' \in \binom{H_{\hat{\Pi}}}{F_0}$ crosses the partition Π, we have that $F = \pi(F')$ defines a copy of F_0 in H_0, *i.e.*, $F = \pi(F') \in \binom{H_0}{F_0}$. We shall call $F = \pi(F')$ the *projection* of F' in H_0 and say that $F' \in \binom{H_{\hat{\Pi}}}{F_0}$ *projects* to $F = \pi(F') \in \binom{H_0}{F_0}$.

Now, define the function $\psi_0 : \binom{H_0}{F_0} \to [0,1]$ by setting, for $F \in \binom{H_0}{F_0}$,

$$\psi_0(F) = \frac{1}{m^k} \sum \left\{ \psi^*(F') : F' \in \binom{H_{\hat{\Pi}}}{F_0} \text{ projects to } F \right\}. \tag{3.16}$$

To show that ψ_0 is a fractional (ω, F_0)-packing of H_0, fix

$$e = \{u_{i_1 j_1}, \ldots, u_{i_k j_k}\} \in H_0.$$

From (3.16),

$$\sum \left\{ \psi_0(F) : F \in \binom{H_0}{F_0}_e \right\} = \frac{1}{m^k} \sum \left\{ \sum \left\{ \psi^*(F') : F' \in \binom{H_{\hat{\Pi}}}{F_0} \text{ projects to } F \right\} : F \in \binom{H_0}{F_0}_e \right\}.$$

Every $F' \in \binom{H_{\hat{\Pi}}}{F_0}$ projects to some $F \in \binom{H_0}{F_0}_e$ if and only if $F' \cap H[V_{i_1 j_1}, \ldots, V_{i_k j_k}] \neq \emptyset$ (recall $e = \{u_{i_1 j_1}, \ldots, u_{i_k j_k}\}$). Therefore,

$$\begin{aligned}
\sum \left\{ \psi_0(F) : F \in \binom{H_0}{F_0}_e \right\}
&= \frac{1}{m^k} \sum \left\{ \psi^*(F') : F' \in \binom{H_{\hat{\Pi}}}{F_0} \text{ satisfies } F' \cap H[V_{i_1 j_1}, \ldots, V_{i_k j_k}] \right\} \\
&= \frac{1}{m^k} \sum \left\{ \sum \left\{ \psi^*(F') : F' \in \binom{H_{\hat{\Pi}}}{F_0}_{e'} \right\} : e' \in H[V_{i_1 j_1}, \ldots, V_{i_k j_k}] \right\} \\
&\leqslant \frac{1}{m^k} \sum \left\{ \sum \left\{ \psi^*(F') : F' \in \binom{H}{F_0}_{e'} \right\} : e' \in H[V_{i_1 j_1}, \ldots, V_{i_k j_k}] \right\} \\
&\leqslant \frac{1}{m^k} |H[V_{i_1 j_1}, \ldots, V_{i_k j_k}]| = d_H(V_{i_1 j_1}, \ldots, V_{i_k j_k}) \overset{(3.8)}{=} \omega(e),
\end{aligned}$$

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where in the last inequality we used that ψ^* is a fractional F_0-packing of H, *i.e.*, the final inner sum is at most 1.

To finish the proof of Proposition 3.1, consider the quantity $|\psi^*_\Pi| - m^k|\psi_0|$. From (3.16), we have that

$$\begin{aligned} m^k|\psi_0| &= \sum\left\{\sum\left\{\psi^*(F') : F' \in \binom{H_{\hat\Pi}}{F_0} \text{ projects to } F\right\} : F \in \binom{H_0}{F_0}\right\} \\ &= \sum\left\{\psi^*(F') : F' \in \binom{H_{\hat\Pi}}{F_0}\right\}, \end{aligned}$$

where the last equality holds from the fact that every $F' \in \binom{H_{\hat\Pi}}{F_0}$ projects to some $F \in \binom{H_0}{F_0}$. Therefore, we have (*cf.* (3.7) and (3.15))

$$\begin{aligned} |\psi^*_\Pi| - m^k|\psi_0| &= \sum\left\{\psi^*(F) : F \in \binom{H}{F_0}_\Pi\right\} - \sum\left\{\psi^*(F') : F' \in \binom{H_{\hat\Pi}}{F_0}\right\} \\ &= \sum\left\{\psi^*(F) : F \in \binom{H}{F_0}_\Pi \setminus \binom{H_{\hat\Pi}}{F_0}\right\} \\ &= \sum\left\{\psi^*(F) : F \in \binom{H}{F_0}_\Pi \text{ satisfies } F \cap \left(H \setminus H_{\hat\Pi}\right) \neq \emptyset\right\} \\ &\leqslant \sum\left\{\sum\left\{\psi^*(F) : e \in F \in \binom{H}{F_0}_\Pi\right\} : e \in H \setminus H_{\hat\Pi}\right\} \\ &\leqslant \sum\left\{\sum\left\{\psi^*(F) : F \in \binom{H}{F_0}_e\right\} : e \in H \setminus H_{\hat\Pi}\right\} \leqslant |H \setminus H_{\hat\Pi}|, \end{aligned}$$

where in the last inequality we used that ψ^* is a fractional F_0-packing of H. Note that $H \setminus H_{\hat\Pi}$ consists of edges e for which $e \cap V_0 \neq \emptyset$, or else, $e \in H[V_{i_1 j_1}, \ldots, V_{i_k j_k}]$ for some $1 \leqslant i_1 < \cdots < i_k \leqslant \ell$ and $1 \leqslant j_1, \ldots, j_k \leqslant t$ where $(V_{i_1 j_1}, \ldots, V_{i_k j_k})$ is not (labelled to be) ε-regular. However, at most $\varepsilon n \cdot n^{k-1} + \varepsilon\binom{\ell}{k} t^k m^k \leqslant 2\varepsilon n^k$ edges $e \in H$ can have these properties, which completes the proof. □

4. Proof of the Packing Lemma

Our proof of the Packing Lemma (Lemma 2.6) is a hypergraph analogue of the proof of Lemma 5 in Haxell and Rödl [12]. The Packing Lemma will follow nearly immediately from Theorem 4.1 and Lemma 4.2 below.

The following statement is a well-known result of Grable [11] which concerns hypergraph packings. A *packing* $\mathscr{P}$ in a hypergraph P is a family of pairwise disjoint edges. In a hypergraph P and $x \in V(P)$, let $N_P(x) = \{Q : Q \cup x \in P\}$ denote the neighbourhood of x in P, and for $x, x' \in V(P)$, write $N_P(x, x') = N_P(x) \cap N_P(x')$. Further, write $\deg_P(x) = |N_P(x)|$ and $\deg_P(x, x') = |N_P(x, x')|$.

Theorem 4.1 (Grable [11]). *For every integer $p \geqslant 2$ and for all $\lambda > 0$, there exists*

$$\beta = \beta_{\text{Thm.4.1}}(p, \lambda) > 0$$

so that the following holds. Let P be a p-graph with sufficiently large vertex set $X = V(P)$ satisfying that, for some $\Delta > 0$,

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(a) *for all* $x \in X$, $\deg_P(x) = (1 \pm \beta)\Delta$,
(b) *for all distinct* $x, x' \in X$, $\deg_P(x, x') < \frac{\Delta}{(\log |X|)^4}$.

Then, there exists a packing $\mathscr{P}$ *of* P *covering all but* $\lambda|X|$ *vertices of* X. *Moreover,* $\mathscr{P}$ *can be constructed in time polynomial in* $|X|$.

We call the following result the *Extension Lemma*, which we prove later in this section.

Lemma 4.2 (Extension Lemma). *For all integers* $f \geqslant k \geqslant 2$ *and all* $d_0, \gamma > 0$, *there exists* $\delta = \delta_{\text{Lem.4.2}}(f, k, d_0, \gamma) > 0$ *so that the following holds.*

Let a linear k*-graph* F_0 *with vertex set* $[f]$ *be given, and let* G *be given as in Setup 2.5 with some* $d \geqslant d_0$, *with* $\varepsilon = \delta$ *above, and with a sufficiently large integer* m. *Then, there exists* $G' \subseteq G$, *where* $|G'| > (1-\gamma)|G|$, *so that for each* $\{i_1, \ldots, i_k\} \in F_0$, *every* $\{v_{i_1}, \ldots, v_{i_k}\} \in G'[V_{i_1}, \ldots, V_{i_k}]$ *belongs to within* $(1 \pm \gamma)d^{|F_0|-1}m^{f-k}$ *many partite-isomorphic copies of* F_0 *in* G'. *Moreover, the subhypergraph* G' *can be found in time* $O(m^f)$.

4.1. Proof of the Packing Lemma

Let F_0 (on f vertices), d_0, and $\mu > 0$ be given as in Lemma 2.6. To define the promised constant $\varepsilon = \varepsilon_{\text{Lem.2.6}}(d_0, \mu) > 0$, we first consider some auxiliary constants. Let $\beta = \beta_{\text{Thm.4.1}}(p = f, \lambda = \mu/2) > 0$ be the constant guaranteed by Theorem 4.1. Let $\delta = \delta_{\text{Lem.4.2}}(f, k, d_0, \gamma = \beta) > 0$ by the constant guaranteed by Lemma 4.2. We set $\varepsilon = \delta$, and take m to be sufficiently large whenever needed.

Now, let G be given as in the hypothesis of the Packing Lemma (Lemma 2.6). We apply the Extension Lemma (Lemma 4.2) to G to construct, in time $O(m^f)$, the subhypergraph $G' \subseteq G$ guaranteed there. As in Theorem 4.1, set $X = G'$ and define P to be the family of all partite-isomorphic copies of F_0 in G'. Note that a packing $\mathscr{P}$ of P corresponds to an F_0-packing of G'.

We now apply Theorem 4.1 to P, but first check that it is appropriate to do so. From the application of the Extension Lemma, every vertex $x \in X = V(P) = G'$ satisfies $\deg_P(x) = (1 \pm \gamma)d^{|F_0|-1}m^{f-k}$. Setting $\Delta = d^{|F_0|-1}m^{f-k}$ and recalling that $\gamma = \beta$ was chosen to be sufficient for an application of Theorem 4.1, we see $\deg_P(x) = (1 \pm \beta)\Delta$. Note that, easily, for each $x \neq x' \in X$, $\deg_P(x, x') \leqslant m^{f-(k+1)} = O(\frac{1}{m}\Delta)$. Moreover, $|X| = \Theta(m^k)$, so $\deg_P(x, x') < \Delta/\log^4|X|$. Thus, Theorem 4.1 constructs, in time polynomial in $|X| = \Theta(m^k)$, a packing $\mathscr{P}$ covering all but $\lambda|X|$ vertices $x \in X$. This corresponds to an F_0-packing $\mathscr{F}$ covering all but $\lambda|G'|$ edges in G'. Together with the edges $G \backslash G'$, the F_0-packing $\mathscr{F}$ covers all but $2\lambda|G| = \mu|G|$ edges of G, which completes the proof.

4.2. Proof of Lemma 4.2

To prove Lemma 4.2, we will use its following seemingly 'weaker' version.

Lemma 4.3 ('Weak' Extension Lemma). *For all integers* $f \geqslant k \geqslant 2$ *and all* $d_0, \zeta > 0$, *there exists* $\varepsilon = \varepsilon_{\text{Lem.4.3}}(f, k, d_0, \zeta) > 0$ *so that the following holds.*

Let a linear k*-graph* F_0 *with vertex set* $[f]$ *be given, and let* G *be given as in Setup 2.5 with some* $d \geqslant d_0$, *with* ε *above, and with a sufficiently large integer* m. *Then, for each* $\{i_1, \ldots, i_k\} \in$

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F_0, all but ζm^k elements $\{v_{i_1},\dots,v_{i_k}\} \in G[V_{i_1},\dots,V_{i_k}]$ belong to within $(1 \pm \zeta)d^{|F_0|-1}m^{f-k}$ many partite-isomorphic copies of F_0 in G.

We prove Lemma 4.3 at the end of the section.

It is clear that Lemma 4.2 implies Lemma 4.3, but we need the converse to hold. The equivalence between Lemmas 4.2 and 4.3 is not clear, as we now indicate.

Remark 4.4. To form G', it would be natural to delete from G all $|F_0|\zeta m^k$ edges which are 'bad' in the sense of Lemma 4.3. In this case, all remaining edges in G' clearly extend to at most $(1+\zeta)d^{|F_0|-1}m^{f-k}$ many copies of F_0 in G'. The concern is that each such edge may not extend to at least $(1-\zeta)d^{|F_0|-1}m^{f-k}$ many copies of F_0 in G' (on account of deletion).

We now prove that Lemma 4.3 implies Lemma 4.2.

Proof of Lemma 4.2. Let integers $f \geqslant k \geqslant 2$ and $d_0, \gamma > 0$ be given. To define the promised constant $\delta = \delta_{\text{Lem 4.2}}(f,k,d_0,\gamma) > 0$, we first define an auxiliary constant $\zeta > 0$ to satisfy

$$4f^{3k}\frac{\sqrt{\zeta}}{d_0^{f^k}} < \gamma. \tag{4.1}$$

Now, let $\varepsilon = \varepsilon_{\text{Lem 4.3}}(f,k,d_0,\zeta) > 0$ be the constant guaranteed by Lemma 4.3, and set $\delta = \varepsilon$. Let a linear k-graph F_0 and G be given as in Setup 2.5 with some constant $d \geqslant d_0$, with $\delta = \varepsilon$ above, and with a sufficiently large integer m. To define the promised hypergraph $G' \subseteq G$, we make two considerations (that of a 'good edge' and that of a 'good vertex').

First, for a fixed $\{i_1,\dots,i_k\} \in F_0$, we shall call an edge $\{v_{i_1},\dots,v_{i_k}\} \in G[V_{i_1},\dots,V_{i_k}]$ a *good edge* if it belongs to within $(1 \pm \zeta)d^{|F_0|-1}m^{f-k}$ many partite-isomorphic copies of F_0 in G. Otherwise, we call $\{v_{i_1},\dots,v_{i_k}\}$ a *bad edge*. The first step in defining G' is to delete all bad edges from G, across all $\{i_1,\dots,i_k\} \in F_0$. Upon doing so, we shall call the resulting (intermediate) hypergraph $G_1 \subseteq G$, where Lemma 4.3 implies $|G_1| \geqslant |G| - |F_0|\zeta m^k \geqslant |G| - f^k\zeta m^k$. Note that G_1 is identified in time $O(m^f)$.

Second, fix $1 \leqslant i \leqslant f$ and fix $\{i_1,\dots,i_k\} = K \in F_0$ for which $i \in K$. We shall call a vertex $v_i \in V_i$ a *K-bad vertex* if v_i belongs to at least $\sqrt{\zeta}m^{k-1}$ bad edges

$$\{v_{i_1},\dots,v_{i_k}\} \in G[V_{i_1},\dots,V_{i_k}].$$

Note that, for K fixed above, at most $\sqrt{\zeta}m$ vertices $v_i \in V_i$ can be K-bad, since otherwise, we would have ζm^k bad edges within $G[V_{i_1},\dots,V_{i_k}]$, contradicting Lemma 4.3. Now, call a vertex $v_i \in V_i$ a *bad vertex* if there exists any $K \in F_0$ for which v_i is a K-bad vertex, and call v_i a *good vertex* otherwise. Then there are at most $\sqrt{\zeta}f^{k-1}m$ bad vertices $v_i \in V_i$ and at most $\sqrt{\zeta}f^k m$ bad vertices in all of G. Note, moreover, that bad vertices in G are clearly identified in time $O(m^k)$.

Now, to define G', we simply induce the hypergraph G_1, defined above, on the good vertices of G (which takes time $O(m^k)$). Since each bad vertex of G can belong to at most

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$f^{k-1}m^{k-1}$ edges of G_1, we have that

$$|G'| > |G_1| - \sqrt{\zeta} f^{2k-1} m^k > |G| - \zeta f^k m^k - \sqrt{\zeta} f^{2k-1} m^k > |G| - 2\sqrt{\zeta} f^{2k} m^k. \tag{4.2}$$

Since $|G| \geqslant |F_0|(d-\varepsilon)m^k > (d_0/2)m^k$, we thus have

$$|G'| > \left(1 - 4f^{2k}\frac{\sqrt{\zeta}}{d_0}\right)|G| \overset{(4.1)}{>} (1-\gamma)|G|.$$

Thus, G' is as large as promised by Lemma 4.2, and was constructed in time $O(m^f)$. It remains to verify that each of its elements extends to within the promised number of copies of F_0 in G'.

To that end, we establish some notation needed for the remainder of the section. Suppose hypergraphs A_0 and B are defined in the context of Setup 2.5. For an edge $b \in B$, define

$$\mathrm{ext}_{A_0,B}(b) = \left| \left\{ A \in \binom{B}{A_0}_b \ : \ A \text{ is a partite-isomorphic copy of } A_0 \right\} \right| \tag{4.3}$$

for the number of *extensions* of the edge b to partite-isomorphic copies of A_0 in B.

Now, fix $\{i_1,\dots,i_k\} = K \in F_0$, and without loss of generality, assume that $\{i_1,\dots,i_k\} = \{1,\dots,k\}$. Fix an edge $\{v_1,\dots,v_k\} \in G'[V_1,\dots,V_k]$. Since $\{v_1,\dots,v_k\}$ is a good edge in G,

$$\mathrm{ext}_{F_0,G}(\{v_1,\dots,v_k\}) = (1 \pm \zeta) d^{|F_0|-1} m^{f-k}, \tag{4.4}$$

and clearly,

$$\mathrm{ext}_{F_0,G'}(\{v_1,\dots,v_k\}) \leqslant \mathrm{ext}_{F_0,G}(\{v_1,\dots,v_k\}) \leqslant (1+\zeta) d^{|F_0|-1} m^{f-k}. \tag{4.5}$$

It remains to verify that $\mathrm{ext}_{F_0,G'}(\{v_1,\dots,v_k\})$ is not much smaller than $\mathrm{ext}_{F_0,G}(\{v_1,\dots,v_k\})$. To that end, fix $\{j_1,\dots,j_k\} = K_1 \in F_0$ where $K_1 \neq K$. We consider two cases.

Case 1: $K \cap K_1 = \emptyset$. It follows from (4.2) that

$$|(G \setminus G')[V_{j_1},\dots,V_{j_k}]| \leqslant 2\sqrt{\zeta} f^{2k} m^k. \tag{4.6}$$

Fix $\{v_{j_1},\dots,v_{j_k}\} \in (G \setminus G')[V_{j_1},\dots,V_{j_k}]$. Clearly, at most m^{f-2k} copies of F_0 in G can contain both $\{v_1,\dots,v_k\}$ and $\{v_{j_1},\dots,v_{j_k}\}$, and all of these copies are lost in G'. Thus, (4.6) implies that, summing over all $\{v_{j_1},\dots,v_{j_k}\} \in (G \setminus G')[V_{j_1},\dots,V_{j_k}]$, the edge $\{v_1,\dots,v_k\}$ lost at most

$$2\sqrt{\zeta} f^{2k} m^k \times m^{f-2k} = 2\sqrt{\zeta} f^{2k} m^{f-k}$$

many copies of F_0 from G.

Case 2: $K \cap K_1 \neq \emptyset$. Since F_0 is a linear hypergraph, it must be the case that $|K \cap K_1| = 1$. Set $\{i\} = K \cap K_1$, and without loss of generality, assume $i = 1$. Fix $\{v_{j_1},\dots,v_{j_k}\} \in (G \setminus G')[V_{j_1},\dots,V_{j_k}]$, where for sake of argument, we assume $v_1 \in \{v_{j_1},\dots,v_{j_k}\}$. Since v_1 is a K_1-good vertex, $\{v_{j_1},\dots,v_{j_k}\}$ can be one of only at most $\sqrt{\zeta} m^{k-1}$ edges deleted from G which contain v_1. Since $\{v_1,\dots,v_k\}$ and $\{v_{j_1},\dots,v_{j_k}\}$ constitute $2k-1$ distinct vertices, there can be at most m^{f-2k+1} many copies of F_0 in G containing both these edges, and all of these copies are lost in G'. Thus, summing over all $\{v_{j_1},\dots,v_{j_k}\} \in (G \setminus G')[V_{j_1},\dots,V_{j_k}]$ containing v_1, the edge $\{v_1,\dots,v_k\}$ lost at most

$$\sqrt{\zeta} m^{k-1} \times m^{f-2k+1} = \sqrt{\zeta} m^{f-k}$$

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many copies of F_0 from G.

Over all $\{j_1,\dots,j_k\} = K_1 \in F_0$ distinct from $\{1,\dots,k\} = K \in F_0$, Cases 1 and 2 imply that

$$\begin{aligned}\mathrm{ext}_{F_0,G'}(\{v_1,\dots,v_k\}) &\geqslant \mathrm{ext}_{F_0,G}(\{v_1,\dots,v_k\}) - \big((|F_0|-1)\big(2\sqrt{\zeta}f^{2k}m^{f-k} + \sqrt{\zeta}m^{f-k}\big)\big)\\ &\overset{(4.4)}{\geqslant} (1-\zeta)d^{|F_0|-1}m^{f-k} - 3\sqrt{\zeta}f^{3k}m^{f-k}\\ &\geqslant \left(1-\zeta-3f^{3k}\frac{\sqrt{\zeta}}{d_0^{f^k}}\right)d^{|F_0|-1}m^{f-k}\\ &\overset{(4.1)}{>} (1-\gamma)d^{|F_0|-1}m^{f-k}.\end{aligned}$$

The above inequality and (4.5) imply $\mathrm{ext}_{F_0,G'}(\{v_1,\dots,v_k\}) = (1\pm\gamma)d^{|F_0|-1}m^{f-k}$, which concludes the proof of Lemma 4.2. □

4.3. Proof of Lemma 4.3

To prove Lemma 4.3, we shall use the following result from [17].

Theorem 4.5 (Counting Lemma for Linear Hypergraphs). *For all integers $f_1 \geqslant k \geqslant 2$ and all $d_0,\tau > 0$, there exists $\delta = \delta_{\mathrm{Thm.4.5}}(f_1,k,d_0,\tau) > 0$ so that the following holds.*

Let a linear k-graph F_1 with vertex set $[f_1]$ be given, and let G be given as in Setup 2.5 with some $d \geqslant d_0$, with $\varepsilon = \delta$, and with a sufficiently large integer m. Then, the number of partite-isomorphic copies of F_1 in G, which we write as $\#\{F_1 \subset_{\mathrm{p.i.}} G\}$, satisfies

$$\#\{F_1 \subset_{\mathrm{p.i.}} G\} = (1\pm\tau)d^{|F_1|}m^{f_1}.$$

Let integers $f \geqslant k \geqslant 2$ be given and let $d_0,\zeta > 0$ be given. Define auxiliary constant $\tau = \zeta^3/6$. Let $\delta_1 = \delta_{\mathrm{Thm.4.5}}(f_1 = f,k,d_0,\tau) > 0$ be the constant guaranteed by Theorem 4.5. Let $\delta_2 = \delta_{\mathrm{Thm.4.5}}(f_1 = 2f-k,k,d_0,\tau) > 0$ be the constant guaranteed by Theorem 4.5. Let $\varepsilon_0 > 0$ be sufficiently small that each of the following inequalities holds:

$$(1+\tau)\big(1-\varepsilon_0 d_0^{-1}\big)^{-1} \leqslant 1+2\tau \quad \text{and} \quad (1-\tau)\big(1+\varepsilon_0 d_0^{-1}\big)^{-1} \geqslant 1-2\tau. \tag{4.7}$$

Define $\varepsilon = \min\{\varepsilon_0,\delta_1,\delta_2\}$. Let F_0 and G be given as in Setup 2.5 with some $d \geqslant d_0$, with ε given above, and with a sufficiently large integer m.

Fix $\{i_1,\dots,i_k\} \in F_0$, and assume without loss of generality that $\{i_1,\dots,i_k\} = \{1,\dots,k\} = [k]$. Our proof will make a joint appeal to the Counting Lemma (Theorem 4.5) and the Cauchy–Schwarz inequality (Fact 2.13). For that purpose, we make the following considerations.

Define hypergraph $F_0^2 \supseteq F_0$ as follows. Let

$$V(F_0^2) = \{1,\dots,k,k+1,\dots,f\} \cup \{(k+1)',\dots,f'\}$$

so that F_0^2 has $2f-k$ vertices. Include every edge of F_0 in F_0^2. More generally, suppose $[k] \neq K = \{i_1,\dots,i_k\} \in F_0$. Since F_0 is linear, $|K \cap [k]| \in \{0,1\}$, and without loss of generality, assume $K \cap [k] \subseteq \{i_1\}$. Write, for some $\ell \in \{0,1\}$,

$$K \setminus [k] = \{i_{\ell+1},\dots,i_k\} \quad \text{and define} \quad K' = \{i_1,\dots,i_\ell,i'_{\ell+1},\dots,i'_k\}.$$

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Now, put $K' \in F_0^2$. We repeat this procedure over all $[k] \neq K \in F_0$, which completes the definition of F_0^2. Then, F_0^2 is a linear k-graph on $2f - k$ vertices and $2|F_0| - 1$ edges.

Define hypergraph $G^2 \supseteq G$ similarly. For $k+1 \leqslant t \leqslant f$, let V_t' be a copy of the class V_t. Let

$$V(G^2) = V_1 \cup \cdots \cup V_k \cup V_{k+1} \cup \cdots \cup V_f \cup V_{k+1}', \dots, V_f'$$

be a $(2f-k)$-partition. Include every edge of G in G^2. More generally, suppose $K \in F_0^2$ has the form (for some $j \geqslant 0$) $K = \{i_1, \dots, i_j, i_{j+1}', \dots, i_k'\}$ where $K \cap [f] = \{i_1, \dots, i_j\}$. Let

$$G_K^2 = G^2[V_{i_1}, \dots, V_{i_j}, V_{i_{j+1}}', \dots, V_f'] \quad \text{be a copy of} \quad G[V_{i_1}, \dots, V_{i_j}, V_{i_{j+1}}, \dots, V_f].$$

Define

$$G^2 = \bigcup \left\{ G_K^2 : K \in \binom{V(F_0^2)}{k} \right\}.$$

We now make the following observations (see (4.8) and (4.10)). To begin (recall that we assume $\{1, \dots, k\} \in F_0$),

$$\#\{F_0 \subset_{\text{p.i.}} G\} = \sum_{\{v_1,\dots,v_k\} \in G[V_1,\dots,V_k]} \text{ext}_{F_0,G}(\{v_1, \dots, v_k\}).$$

Then, Theorem 4.5 (with $F_1 = F_0$) implies that

$$\sum_{\{v_1,\dots,v_k\} \in G[V_1,\dots,V_k]} \text{ext}_{F_0,G}(\{v_1, \dots, v_k\}) \geqslant d^{|F_0|} m^f (1 - \tau).$$

Since, by the hypothesis of Setup 2.5, we have $|G[V_1, \dots, V_k]| = (d \pm \varepsilon) m^k$, where $d \geqslant d_0$, the inequality above implies

$$\begin{aligned} \sum_{\{v_1,\dots,v_k\} \in G[V_1,\dots,V_k]} & \text{ext}_{F_0,G}(\{v_1, \dots, v_k\}) \\ & \geqslant d^{|F_0|-1} m^{f-k} |G[V_1, \dots, V_k]| (1 - \tau) \big(1 + \varepsilon d_0^{-1}\big)^{-1} \\ & \overset{(4.7)}{\geqslant} d^{|F_0|-1} m^{f-k} |G[V_1, \dots, V_k]| (1 - 2\tau). \end{aligned} \tag{4.8}$$

Similarly,

$$\#\{F_0^2 \subset_{\text{p.i.}} G^2\} = \sum_{\{v_1,\dots,v_k\} \in G[V_1,\dots,V_k]} \text{ext}_{F_0^2,G^2}(\{v_1, \dots, v_k\}),$$

and Theorem 4.5 (applied with $F_1 = F_0^2$) implies that

$$\sum_{\{v_1,\dots,v_k\} \in G[V_1,\dots,V_k]} \text{ext}_{F_0^2,G^2}(\{v_1, \dots, v_k\}) \leqslant d^{|F_0^2|} m^{|V(F_0^2)|} (1 + \tau). \tag{4.9}$$

However, $|F_0^2| = 2|F_0| - 1$, $|V(F_0^2)| = 2f - k$, and for each fixed $\{v_1, \dots, v_k\} \in G[V_1, \dots, V_k]$ we have

$$\text{ext}_{F_0^2,G^2}(\{v_1, \dots, v_k\}) = \text{ext}_{F_0,G}^2(\{v_1, \dots, v_k\}).$$

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Since $|G[V_1,\ldots,V_k]| = (d \pm \varepsilon)m^k$, inequality (4.9) implies

$$\sum_{\{v_1,\ldots,v_k\}\in G[V_1,\ldots,V_k]} \mathrm{ext}^2_{F_0,G}(\{v_1,\ldots,v_k\})$$
$$\leqslant d^{2|F_0|-2}m^{2f-2k}|G[V_1,\ldots,V_k]|(1+\tau)\big(1-\varepsilon d_0^{-1}\big)^{-1}$$
$$\overset{(4.7)}{\leqslant}\big(d^{|F_0|-1}m^{f-k}\big)^2|G[V_1,\ldots,V_k]|(1+2\tau). \qquad (4.10)$$

Comparing (4.8) and (4.10) and using the Cauchy–Schwarz inequality (Fact 2.13), we see that all but $6\tau^{1/3}|G[V_1,\ldots,V_k]| \leqslant \zeta m^k$ elements $\{v_1,\ldots,v_k\} \in G[V_1,\ldots,V_k]$ satisfy the conclusion of Lemma 4.3, as promised.

5. Proof of the Slicing Lemma

Our proof of the Slicing Lemma (Lemma 2.3) is a hypergraph analogue of the proof of Lemma 6 in Haxell and Rödl [12]. In what follows, we shall use the following variation of the Slicing Lemma, which takes place in an environment of fixed size.

Lemma 5.1 ('Miniature' Slicing Lemma). *For all $\varsigma > 0$ and all integers $k \geqslant 2$ and $s \geqslant 1$, there exists an integer $S_0 = S_0(\varsigma, k, s)$ so that the following holds.*

Let $K[A_1,\ldots,A_k]$ be the complete k-partite k-graph with vertex partition $A_1 \cup \cdots \cup A_k$, where $|A_1| = \cdots = |A_k| = S_0$. Let $q_1,\ldots,q_s > 0$ be given where $q_0 = 1 - \sum_{i=1}^s q_i \geqslant 0$. Then, there exists a partition $K[A_1,\ldots,A_k] = J_0 \cup J_1 \cup \cdots \cup J_s$ with the following property.

For every $w : \bigcup_{j=1}^k A_j \to [0,1]$ satisfying, for each $1 \leqslant j \leqslant k$,

$$w(A_j) \overset{\text{def}}{=} \sum_{a\in A_j} w(a) \geqslant \varsigma|A_j|,$$

we have, for each $0 \leqslant i \leqslant s$,

$$(q_i - \varsigma)\prod_{j=1}^k w(A_j) \leqslant \sum_{\{a_1,\ldots,a_k\}\in J_i} w(a_1)\cdots w(a_k) \leqslant (q_i + \varsigma)\prod_{j=1}^k w(A_j).$$

Moreover, the partition above can be found, in time depending on S_0, by an exhaustive search.

We proceed to show that Lemma 5.1 implies Lemma 2.3, and then return to prove Lemma 5.1.

5.1. Proof of Lemma 2.3

Let integer $k \geqslant 2$ and $d_0, \varepsilon' > 0$ be given. Set

$$\varsigma = \frac{\varepsilon'}{2}. \qquad (5.1)$$

Now, for an integer (variable) $1 \leqslant s \leqslant \lceil 1/d_0 \rceil$, let $S_0(s) = S_0(\varsigma, k, s)$ be the integer (function) guaranteed by Lemma 5.1. Define

$$S_0^* = \max\{S_0(s) : 1 \leqslant s \leqslant \lceil 1/d_0 \rceil\}. \qquad (5.2)$$

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Define[1]

$$\varepsilon = \varepsilon_{\text{Lem. 2.3}}(k, d_0, \varepsilon') = \frac{\varsigma^{k+1}}{8kS_0^*}. \tag{5.3}$$

With ε in (5.3), let G be an ε-regular k-partite k-graph with vertex partition $V(G) = V_1 \cup \cdots \cup V_k$, where $|V_1| = \cdots = |V_k| = m$ is sufficiently large. Write $D = d_G(V_1, \ldots, V_k)$, for simplicity. Let $p_1, \ldots, p_s \geqslant d_0$ be given satisfying $\sum_{i=1}^{s} p_i \leqslant D$. We say a word about constants. Since s is a fixed integer, $S_0 = S_0(s)$ (described above) is also a fixed integer, where

$$sd_0 \leqslant \sum_{i=1}^{s} p_i \leqslant D \implies s \leqslant D/d_0 \leqslant \lceil 1/d_0 \rceil \overset{(5.2)}{\implies} S_0 \leqslant S_0^*.$$

Thus, by (5.3),

$$\varepsilon \leqslant \frac{\varsigma^{k+1}}{8kS_0}. \tag{5.4}$$

To define the promised partition $G = G_0 \cup G_1 \cup \cdots \cup G_s$, we make two auxiliary considerations. First, consider the complete k-partite k-graph $K[A_1, \ldots, A_k]$, where $A_1, \ldots, A_k$ are arbitrary sets of size $|A_1| = \cdots = |A_k| = S_0$. For each $1 \leqslant i \leqslant s$, set $q_i = p_i/D$, and let

$$K[A_1, \ldots, A_k] = J_0 \cup J_1 \cup \cdots \cup J_s$$

be the partition guaranteed by Lemma 5.1.

Second, refine the vertex classes $V_1, \ldots, V_k$ as follows. For each of the sets A_j above, $1 \leqslant j \leqslant k$, write $A_j = \{a_{j1}, \ldots, a_{jS_0}\}$. Now, for each $a_{j\ell} \in A_j$, $1 \leqslant \ell \leqslant S_0$, choose a subset $V_{j\ell} \subset V_j$ of size

$$|V_{j\ell}| = \left\lfloor \frac{m}{S_0} \right\rfloor \overset{\text{def}}{=} \hat{m} \quad \text{so that} \quad V_j = V_{j0} \cup \bigcup_{a_{j\ell} \in A_j} V_{j\ell} \tag{5.5}$$

is a partition. (The class V_{j0} is the remainder of size at most $S_0 - 1$.)

Now, fix a choice $0 \leqslant \ell_1, \ldots, \ell_k \leqslant S_0$ and consider $G[V_{1\ell_1}, \ldots, V_{k\ell_k}]$. If any $\ell_j = 0$, $1 \leqslant j \leqslant k$, put

$$G[V_{1\ell_1}, \ldots, V_{k\ell_k}] \subset G_0.$$

Otherwise, for each $1 \leqslant i \leqslant s$, put

$$G[V_{1\ell_1}, \ldots, V_{k\ell_k}] \subset G_i \iff \{a_{1\ell_1}, \ldots, a_{k\ell_k}\} \in J_i.$$

This defines the partition $G = G_0 \cup G_1 \cup \cdots \cup G_s$ promised by Lemma 2.3, which is easily constructed in time $O(m^k)$.

[1] It is easy to infer, from the proof of Lemma 5.1, that $S_0(s)$ is monotone increasing in s, and therefore S_0^* is achieved by $s = \lceil 1/d_0 \rceil$. However, for completeness, we avoid using this assumption. (Moreover, it would hardly simplify our presentation.)

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It remains to check that each G_i, $1 \leqslant i \leqslant s$, is (p_i, ε')-regular. To that end, fix $1 \leqslant i \leqslant s$, and for each $1 \leqslant j \leqslant k$, let $V_j' \subseteq V_j$ be given with $|V_j'| > \varepsilon'|V_j| = \varepsilon' m$. We will show that

$$d_{G_i}(V_1', \dots, V_k') = p_i \pm \varepsilon'. \tag{5.6}$$

To that end, we establish a few 'underlying' considerations. First, for each $1 \leqslant j \leqslant k$ and $1 \leqslant \ell \leqslant S_0$, write

$$V_{j\ell}' = V_j' \cap V_{j\ell} \quad \text{and} \quad w(a_{j\ell}) = \frac{|V_{j\ell}'|}{|V_{j\ell}|} = \frac{|V_{j\ell}'|}{\hat{m}}.$$

Then,

$$w(A_j) = \sum_{\ell=1}^{S_0} w(a_{j\ell}) = \frac{1}{\hat{m}} \sum_{\ell=1}^{S_0} |V_{j\ell}'| \implies w(A_j) = \frac{|V_j'|}{\hat{m}}(1 - o(1)) \overset{(5.1)}{\geqslant} \varsigma|A_j| = \varsigma S_0, \tag{5.7}$$

since

$$|V_j'| - S_0 + 1 \leqslant \sum_{\ell=1}^{S_0} |V_{j\ell}'| \leqslant |V_j'|,$$

where $|V_j'| > \varepsilon' m$ and $S_0 = O(1)$. (Thus, $o(1) \to 0$ as $m \to \infty$.) Second, for $1 \leqslant j \leqslant k$ and $1 \leqslant \ell \leqslant S_0$, we say $a_{j\ell}$ is ε-*big* if

$$|V_{j\ell}'| > \varepsilon m \iff w(a_{j\ell}) > \varepsilon \frac{m}{\hat{m}} = \varepsilon S_0(1 - o(1)), \tag{5.8}$$

and ε-*small* otherwise. Let J_i^+ denote the set of all $\{a_{1\ell_1}, \dots, a_{k\ell_k}\} \in J_i$ for which every $a_{j\ell_j}$, $1 \leqslant j \leqslant k$, $1 \leqslant \ell_j \leqslant S_0$, is ε-big, and let $J_i^- = J_i \setminus J_i^+$ denote the set of all $\{a_{1\ell_1}, \dots, a_{k\ell_k}\} \in J_i$ for which some $a_{j\ell_j}$, $1 \leqslant j \leqslant k$, $1 \leqslant \ell_j \leqslant S_0$, is ε-small. Observe then that

$$\begin{aligned}\sum_{\{a_{1\ell_1}, \dots, a_{k\ell_k}\} \in J_i^+} w(a_{1\ell_1}) \cdots w(a_{k\ell_k}) &\overset{(5.8)}{=} \Bigg(\sum_{\{a_{1\ell_1}, \dots, a_{k\ell_k}\} \in J_i} w(a_{1\ell_1}) \cdots w(a_{k\ell_k})\Bigg) \pm 2\varepsilon k S_0^{k+1} \\ &= (q_i \pm \varsigma)\big(w(A_1) \cdots w(A_k)\big) \pm 2\varepsilon k S_0^{k+1},\end{aligned} \tag{5.9}$$

where the last inequalities follow by the application of Lemma 5.1 (*cf.* (5.7)).

Returning to our goal in (5.6), observe that

$$\begin{aligned} &d_{G_i}\big(V_1', \dots, V_k'\big) \\ &= \frac{|G_i[V_1', \dots, V_k']|}{|V_1'| \cdots |V_k'|} = \frac{1}{|V_1'| \cdots |V_k'|} \sum_{\{a_{1\ell_1}, \dots, a_{k\ell_k}\} \in J_i} |G[V_{1\ell_1}', \dots, V_{k\ell_k}']| \\ &= \frac{1}{|V_1'| \cdots |V_k'|} \Bigg[\sum_{\{a_{1\ell_1}, \dots, a_{k\ell_k}\} \in J_i^+} |G[V_{1\ell_1}', \dots, V_{k\ell_k}']| + \sum_{\{a_{1\ell_1}, \dots, a_{k\ell_k}\} \in J_i^-} |G[V_{1\ell_1}', \dots, V_{k\ell_k}']|\Bigg]. \end{aligned}$$

By (5.8), $\sum_{\{a_{1\ell_1}, \dots, a_{k\ell_k}\} \in J_i^-} |G[V_{1\ell_1}', \dots, V_{k\ell_k}']| \leqslant \varepsilon k S_0 m^k$, and with $|V_j'| \geqslant \varepsilon' m$, $1 \leqslant j \leqslant k$, we have

$$\sum_{\{a_{1\ell_1}, \dots, a_{k\ell_k}\} \in J_i^+} \frac{|G[V_{1\ell_1}', \dots, V_{k\ell_k}']|}{|V_1'| \cdots |V_k'|} = d_{G_i}\big(V_1', \dots, V_k'\big) \pm \varepsilon k \frac{S_0}{(\varepsilon')^k}. \tag{5.10}$$

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Observe that

$$\sum_{\{a_{1\ell_1},\dots,a_{k\ell_k}\}\in J_i^+} \frac{|G[V'_{1\ell_1},\dots,V'_{k\ell_k}]|}{|V'_1|\cdots|V'_k|}$$
$$= \sum_{\{a_{1\ell_1},\dots,a_{k\ell_k}\}\in J_i^+} \frac{|G[V'_{1\ell_1},\dots,V'_{k\ell_k}]|}{|V'_{1\ell_1}|\cdots|V'_{k\ell_k}|} w(a_{1\ell_1})\cdots w(a_{k\ell_k}) \frac{|V_{1\ell_1}|\cdots|V_{k\ell_k}|}{|V'_1|\cdots|V'_k|}$$
$$\overset{(5.7)}{=} (1\pm o(1))\frac{1}{w(A_1)\cdots w(A_k)} \sum_{\{a_{1\ell_1},\dots,a_{k\ell_k}\}\in J_i^+} \frac{|G[V'_{1\ell_1},\dots,V'_{k\ell_k}]|}{|V'_{1\ell_1}|\cdots|V'_{k\ell_k}|} w(a_{1\ell_1})\cdots w(a_{k\ell_k}).$$

By the (D,ε)-regularity of G, and the definition of J_i^+ (*cf.* (5.8)), we further infer that

$$\sum_{\{a_{1\ell_1},\dots,a_{k\ell_k}\}\in J_i^+} \frac{|G[V'_{1\ell_1},\dots,V'_{k\ell_k}]|}{|V'_1|\cdots|V'_k|}$$
$$= (1\pm o(1))(D\pm\varepsilon)\frac{1}{w(A_1)\cdots w(A_k)} \sum_{\{a_{1\ell_1},\dots,a_{k\ell_k}\}\in J_i^+} w(a_{1\ell_1})\cdots w(a_{k\ell_k})$$
$$\overset{(5.9)}{=} (1\pm o(1))(D\pm\varepsilon)\frac{1}{w(A_1)\cdots w(A_k)}\left((q_i\pm\varsigma)\big(w(A_1)\cdots w(A_k)\big)\pm 2\varepsilon k S_0^{k+1}\right)$$
$$\overset{(5.7)}{=} (1\pm o(1))(D\pm\varepsilon)\left(q_i\pm\varsigma\pm 2\varepsilon k\frac{S_0}{\varsigma^k}\right). \tag{5.11}$$

Now, comparing (5.10) and (5.11), we infer

$$(1-o(1))(D-\varepsilon)\left(q_i-\varsigma-2\varepsilon k\frac{S_0}{\varsigma^k}\right)-\varepsilon k\frac{S_0}{(\varepsilon')^k} \leqslant d_{G_i}(V'_1,\dots,V'_k)$$
$$\leqslant (1+o(1))(D+\varepsilon)\left(q_i+\varsigma+2\varepsilon k\frac{S_0}{\varsigma^k}\right)+\varepsilon k\frac{S_0}{(\varepsilon')^k}.$$

With $p_i = Dq_i$ and $\varsigma<\varepsilon'$, we further infer that

$$p_i-\varepsilon' \overset{(5.1)}{=} p_i-2\varsigma \overset{(5.4)}{\leqslant} p_i-\varsigma-5\varepsilon k\frac{S_0}{\varsigma^k} \leqslant d_{G_i}(V'_1,\dots,V'_k)$$
$$\leqslant p_i+\varsigma+8\varepsilon k\frac{S_0}{\varsigma^k} \overset{(5.4)}{\leqslant} p_i+2\varsigma \overset{(5.1)}{=} p_i+\varepsilon'.$$

5.2. Proof of Lemma 5.1

Let $\varsigma>0$ and integers $k\geqslant 2$ and $s\geqslant 1$ be given. We take $S_0=S_0(\varsigma,k,s)$ to be sufficiently large (and argue, in context, that this parameter needs only to depend on ς, k and s). Let $K[A_1,\dots,A_k]$ be the k-partite k-graph with vertex partition $A_1\cup\cdots\cup A_k$ with $|A_1|=\cdots=|A_k|=S_0$. Let $q_1,\dots,q_s>0$ be given with $q_0=1-\sum_{i=1}^s q_i\geqslant 0$.

We shall define the promised partition $J_0\cup J_1\cup\cdots\cup J_s$ by a standard random construction. For $0\leqslant i\leqslant s$, let $\mathbb{J}_i$ be defined by, independently for each $\{a_1,\dots,a_k\}\in K[A_1,\dots,A_k]$, $\mathbb{P}[\{a_1,\dots,a_k\}\in\mathbb{J}_i]=q_i$. We seek (exhaustively search for) an instance of $\mathbb{J}_1,\dots,\mathbb{J}_s$ behaving according to the following claim.

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Claim 5.2. *With $S_0 = S_0(\varsigma, k, s)$ sufficiently large, the following holds. For each $0 \leqslant i \leqslant s$,*

(a) *if $q_i \leqslant \frac{\varsigma^{k+1}}{2s}$, then with probability $1 - \frac{1}{2s}$,*

$$|\mathbb{J}_i| \leqslant 2sq_iS_0^k, \tag{5.12}$$

(b) *if $q_i > \frac{\varsigma^{k+1}}{2s}$, then with probability $1 - \frac{1}{2s}$, every choice $A'_j \subseteq A_j$, $1 \leqslant j \leqslant k$, with $|A'_j| \geqslant \frac{1}{2}\varsigma S_0$, satisfies*

$$|\mathbb{J}_i \cap K[A'_1, \ldots, A'_k]| = q_i\left(1 \pm \frac{\varsigma}{2s}\right)|A'_1| \cdots |A'_k|. \tag{5.13}$$

As we show at the end of the section, Claim 5.2 follows by straightforward applications of the Markov and Chernoff inequalities.

Set $J_i = \mathbb{J}_i$, $0 \leqslant i \leqslant s$, to be instances satisfying the properties in (5.12) and (5.13). Let a function $w : \bigcup_{j=1}^k A_j \to [0, 1]$ be given satisfying $w(A_j) = \sum_{a \in A_j} w(a) \geqslant \varsigma S_0$ for all $1 \leqslant j \leqslant k$. For the remainder of the proof, fix $0 \leqslant i \leqslant s$. We show

$$(q_i - \varsigma)\prod_{j=1}^{k} w(A_j) \leqslant \sum_{\{a_1,\ldots,a_k\} \in J_i} w(a_1) \cdots w(a_k) \leqslant (q_i + \varsigma)\prod_{j=1}^{k} w(A_j). \tag{5.14}$$

We proceed by considering two cases, the first of which is nearly trivial. Indeed, assume $q_i \leqslant \varsigma^{k+1}/(2s)$. Then, there is nothing to show for the lower bound of (5.14). For the upper bound, note that

$$\sum_{\{a_1,\ldots,a_k\} \in J_i} w(a_1) \cdots w(a_k) \leqslant |J_i| \overset{(5.12)}{\leqslant} 2sq_iS_0^k.$$

Since $w(A_j) \geqslant \varsigma|A_j| = \varsigma S_0$ for all $1 \leqslant j \leqslant k$, we infer

$$\sum_{\{a_1,\ldots,a_k\} \in J_i} w(a_1) \cdots w(a_k) \leqslant \frac{2sq_i}{\varsigma^k}\prod_{j=1}^{k} w(A_j) \leqslant \varsigma\prod_{j=1}^{k} w(A_j) \leqslant (q_i + \varsigma)\prod_{j=1}^{k} w(A_j),$$

as desired. Thus, for the remainder of the proof, we assume that

$$q_i > \frac{\varsigma^{k+1}}{2s}, \tag{5.15}$$

and proceed with the following claim.

Claim 5.3. *With w given above and $0 \leqslant i \leqslant s$ fixed above, there exists a function*

$$w_0 : \bigcup_{j=1}^{k} A_j \to [0, 1]$$

with the following properties:

(a) *for $1 \leqslant j \leqslant k$, $w_0(A_j) = w(A_j)$,*

(b) *for $1 \leqslant j \leqslant k$, if $M_{A_j}(w_0) \overset{\text{def}}{=} \{a \in A_j : 0 < w_0(a) < 1\}$, then $w_0(M_{A_j}(w_0)) \leqslant 1$,*

(c) *for any $\bar{w} \in \{w, w_0\}$, if $W_i(\bar{w}) \overset{\text{def}}{=} \sum_{\{a_1,\ldots,a_k\} \in J_i} \bar{w}(a_1) \cdots \bar{w}(a_k)$, then $W_i(w) \leqslant W_i(w_0)$.*

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We defer the proof of Claim 5.3 to the end of the section.

To prove the upper bound of (5.14), let the function w_0 guaranteed by Claim 5.3 be given and define, for $1 \leqslant j \leqslant k$, $S_{A_j} \stackrel{\text{def}}{=} \{a \in A_j : w_0(a) = 1\}$. Let us first show that

$$\sum_{\{a_1,\dots,a_k\}\in J_i} w(a_1)\cdots w(a_k) \leqslant |J_i[S_{A_1},\dots,S_{A_k}]| + \frac{k}{\varsigma S_0}\prod_{j=1}^{k} w(A_j). \tag{5.16}$$

Indeed, by Claim 5.3(c), we have

$$\begin{aligned}\sum_{\{a_1,\dots,a_k\}\in J_i} w(a_1)\cdots w(a_k) &= W_i(w) \leqslant W_i(w_0)\\ &\leqslant \sum_{\{a_1,\dots,a_k\}\in J_i[S_{A_1},\dots,S_{A_k}]} 1 + \sum_{h=1}^{k}\sum_{a_h\in M_{A_h}(w_0)} w_0(a_h)\prod_{\substack{j=1\\ j\neq h}}^{k}\prod_{a_j\in A_j} w_0(a_j)\\ &= |J_i[S_{A_1},\dots,S_{A_k}]| + \sum_{h=1}^{k}\left(\prod_{\substack{j=1\\ j\neq h}}^{k}\prod_{a_j\in A_j} w_0(a_j)\right) w_0(M_{A_h}(w_0)).\end{aligned}$$

By Claim 5.3(b), we further conclude

$$\begin{aligned}\sum_{\{a_1,\dots,a_k\}\in J_i} w(a_1)\cdots w(a_k) &\leqslant |J_i[S_{A_1},\dots,S_{A_k}]| + \sum_{h=1}^{k}\prod_{\substack{j=1\\ j\neq h}}^{k}\prod_{a_j\in A_j} w_0(a_j)\\ &= |J_i[S_{A_1},\dots,S_{A_k}]| + \left(\frac{1}{w_0(A_1)}+\cdots+\frac{1}{w_0(A_k)}\right)\prod_{j=1}^{k} w_0(A_j)\\ &= |J_i[S_{A_1},\dots,S_{A_k}]| + \left(\frac{1}{w(A_1)}+\cdots+\frac{1}{w(A_k)}\right)\prod_{j=1}^{k} w_0(A_j),\end{aligned}$$

where we used Claim 5.3(a). Then (5.16) follows from $w(A_j) \geqslant \varsigma S_0$, $1 \leqslant j \leqslant k$.

We may now conclude the upper bound of (5.14). Indeed, by Claim 5.3(a,b),

$$|S_{A_j}| = w_0(A_j) - w_0(M_{A_j}(w_0)) = w(A_j) - w_0(M_{A_j}(w)) \geqslant w(A_j) - 1 \geqslant \frac{1}{2}\varsigma S_0.$$

Thus, from (5.13) from Claim 5.2, we conclude from (5.16) that

$$\begin{aligned}\sum_{\{a_1,\dots,a_k\}\in J_i} w(a_1)\cdots w(a_k) &\leqslant q_i\left(1+\frac{\varsigma}{2s}\right)|S_{A_1}|\cdots|S_{A_k}| + \frac{k}{\varsigma S_0}\prod_{j=1}^{k} w(A_j)\\ &\leqslant \left(q_i\left(1+\frac{\varsigma}{2s}\right)+\frac{k}{\varsigma S_0}\right)\prod_{j=1}^{k} w(A_j) \leqslant \left(q_i+\frac{\varsigma}{s}\right)\prod_{j=1}^{k} w(A_j),\end{aligned} \tag{5.17}$$

where the last inequality follows with $S_0 = S_0(\varsigma, k, s)$ sufficiently large (as a function of k, ς and s alone). Then (5.17) implies the upper bound of (5.14).

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The lower bound of (5.14) is an easy consequence of (5.17), which we may now assume holds for all $0 \leqslant i \leqslant s$. For $0 \leqslant i \leqslant s$ fixed, note that

$$\begin{aligned}\sum_{\{a_1,\dots,a_k\}\in J_i} & w(a_1)\cdots w(a_k) \\ &= \sum_{\{a_1,\dots,a_k\}\in K[A_1,\dots,A_k]} w(a_1)\cdots w(a_k) - \sum_{\substack{h=0\\h\neq i}}^{s} \sum_{\{a_1,\dots,a_k\}\in J_h} w(a_1)\cdots w(a_k) \\ &\geqslant \prod_{j=1}^{k} w(A_j) - \sum_{\substack{h=0\\h\neq i}}^{s} q_h\left(1+\frac{\varsigma}{s}\right)\prod_{j=1}^{k} w(A_j) \geqslant (q_i-\varsigma)\prod_{j=1}^{k} w(A_j),\end{aligned}$$

as promised.

Proof of Claim 5.2. Fix $0 \leqslant i \leqslant s$. The first case follows immediately by Markov's inequality, so assume $q_i \geqslant \frac{\varsigma^{k+1}}{2s}$. Fix $A'_j \subseteq A_j$, $1 \leqslant j \leqslant k$, with $|A'_j| \geqslant \varsigma S_0/2$. By Chernoff's inequality (Fact 2.14),

$$\begin{aligned}\mathbb{P}\left[|\mathbb{J}_i[A'_i,\dots,A'_k]| \neq \left(1\pm\frac{\varsigma}{2s}\right)q_i|A'_1|\cdots|A'_k|\right] &\leqslant 2\exp\left\{-\frac{\varsigma^2}{12s^2}q_i|A'_1|\cdots|A'_k|\right\} \\ &\leqslant 2\exp\left\{-\frac{\varsigma^{2k+3}}{3\cdot 2^{k+3}s^3}S_0^k\right\}.\end{aligned}$$

Over all choices $A'_j \subseteq A_j$, $1 \leqslant j \leqslant k$, we see that Claim 5.2(b) holds with probability

$$1 - 2^{kS_0+1}\exp\left\{-\frac{\varsigma^{2k+3}}{3\cdot 2^{k+3}s^3}S_0^k\right\} \geqslant 1-\frac{1}{2s},$$

where the last inequality holds with $S_0 = S_0(\varsigma,k,s)$ sufficiently large as a function of ς, k and s. □

Proof of Claim 5.3. Recall that $w : \bigcup_{i=1}^{k} \to [0,1]$ and $0 \leqslant i \leqslant s$ are fixed. We determine the promised function w_0 by repeating an iterative procedure. If w (playing the role of w_0) satisfies Claim 5.3(b), set $w_0 = w$ and we are done. Otherwise, there exists some $1 \leqslant j \leqslant k$ so that $w(M_{A_j}(w)) > 1$. Without loss of generality, assume $j = 1$, and write $M_{A_1}(w) = \{\hat{a}_0, \hat{a}_1, \dots, \hat{a}_\ell\}$. We shall define an intermediate function $w' : \bigcup_{j=1}^{k} A_j \to [0,1]$ which will eventually lead us to the promised function w_0.

Since $w(M_{A_1}(w)) > 1$ and every element of $M_{A_1}(w)$ has positive weight, we deduce that there exist $\vartheta_1,\dots,\vartheta_\ell > 0$ such that $w(\hat{a}_h) \geqslant \vartheta_h$ for all $1 \leqslant h \leqslant \ell$ and $w(\hat{a}_0) = 1 - \sum_{h=1}^{\ell}\vartheta_h$. Define $w' : \bigcup_{j=1}^{k} A_j \to [0,1]$ by setting $w'(\hat{a}_0) = 1$, $w'(\hat{a}_h) = w(\hat{a}_h) - \vartheta_h$ for each $1 \leqslant h \leqslant \ell$, and $w'(a) = w(a)$ whenever $a \in A_1 \setminus M_{A_1}(w)$ or $a \in A_2 \cup \cdots \cup A_k$. Note that $M_{A_1}(w') = \{\hat{a}_1,\dots,\hat{a}_\ell\}$.

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We claim that w' (playing the role of w_0) satisfies Claim 5.3(a). In particular, we claim that $w'(A_1) = w(A_1)$. Indeed,

$$\begin{aligned} w'(A_1) &= w'(M_{A_1}(w')) + w'(A_1 \backslash M_{A_1}(w')) \\ &= \sum_{h=1}^{\ell} w'(\hat{a}_h) + w'(A_1 \backslash M_{A_1}(w')) \\ &= \sum_{h=1}^{\ell} w'(\hat{a}_h) + w'(A_1 \backslash M_{A_1}(w)) + w'(\hat{a}_0) \\ &= \sum_{h=1}^{\ell} \big(w(\hat{a}_h) - \vartheta_h\big) + w(A_1 \backslash M_{A_1}(w)) + 1 \\ &= \sum_{h=0}^{\ell} w(\hat{a}_h) + w(A_1 \backslash M_{A_1}(w)) \\ &= w(M_{A_1}(w)) + w(A_1 \backslash M_{A_1}(w)) = w(A_1). \end{aligned}$$

We claim that w' (playing the role of w_0) satisfies Claim 5.3(c). To see this, for $0 \leqslant h \leqslant \ell$ define

$$\hat{W}_i(\hat{a}_h) = \sum_{\{\hat{a}_h, a_2, \ldots, a_k\} \in J_i} w(a_2) \cdots w(a_k).$$

Note that we may assume, without loss of generality, that $\hat{W}_i(\hat{a}_0) = \max_{0 \leqslant h \leqslant \ell} \hat{W}_i(\hat{a}_h)$. Now,

$$\begin{aligned} W_i(w') - W_i(w) &= \sum_{\{a_1, a_2, \ldots, a_k\} \in J_i} \Big(\big(w'(a_1) \cdots w'(a_k)\big) - \big(w(a_1) \cdots w(a_k)\big)\Big) \\ &= \sum_{h=0}^{\ell} \big(w'(\hat{a}_h) - w(\hat{a}_h)\big) \hat{W}_i(\hat{a}_h) \\ &= \big(w'(\hat{a}_0) - w(\hat{a}_0)\big) \hat{W}_i(\hat{a}_0) + \sum_{h=1}^{\ell} \big(w'(\hat{a}_h) - w(\hat{a}_h)\big) \hat{W}_i(\hat{a}_h) \\ &\geqslant \big(w'(\hat{a}_0) - w(\hat{a}_0)\big) \hat{W}_i(\hat{a}_0) + \hat{W}_i(\hat{a}_0) \sum_{h=1}^{\ell} \big(w'(\hat{a}_h) - w(\hat{a}_h)\big) \\ &= \left(\sum_{h=1}^{\ell} \vartheta_h\right) \hat{W}_i(\hat{a}_0) - \hat{W}_i(\hat{a}_0) \left(\sum_{h=1}^{\ell} \vartheta_h\right) = 0, \end{aligned}$$

as desired.

It may not be the case that w' satisfies Claim 5.3(b), *i.e.*, it may be that $w'(M_{A_1}(w')) > 1$. However, in this case, recall that $M_{A_1}(w') = \{\hat{a}_1, \ldots, \hat{a}_\ell\} = M_{A_1}(w) \backslash \{\hat{a}_0\}$, and so

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$$w'(M_{A_1}(w')) = \sum_{h=1}^{\ell} w'(\hat{a}_h) = \sum_{h=1}^{\ell} \big(w(\hat{a}_h) - \vartheta_h\big)$$
$$= w(M_{A_1}(w)) - w(\hat{a}_0) - \sum_{h=1}^{\ell} \vartheta_h = w(M_{A_1}(w)) - 1.$$

We would therefore iterate this process to obtain a function w_1 for which $w_1(M_{A_1}(w_1)) \leqslant 1$. We would then repeat again over all $1 \leqslant j \leqslant k$ for which $w_j(M_{A_j}(w_j)) > 1$, to finally arrive at the promised function w_0. □

6. Proof of the Bounding Lemma

We use the following result of Haxell and Rödl (appearing as Theorem 18 in [12]). As defined in Section 4, a packing of a hypergraph $\mathcal{H}_0$ is a set of pairwise disjoint edges, and so a *fractional packing* of $\mathcal{H}_0$ is a function $\phi : \mathcal{H}_0 \to [0,1]$ satisfying, for each vertex $v \in V(\mathcal{H})$, $\sum\{\phi(e) : v \in e \in \mathcal{H}\} \leqslant 1$. If $\mathcal{H}_0$ is equipped with vertex weights $w : V(\mathcal{H}_0) \to [0,1]$, then $\phi : \mathcal{H} \to [0,1]$ is a *weighted fractional packing* of $\mathcal{H}_0$ if, for each vertex $v \in V(\mathcal{H}_0)$, $\sum\{\phi(e) : v \in e \in \mathcal{H}_0\} \leqslant w(v)$. We say ϕ is *β-bounded* if, for every $e \in \mathcal{H}_0$, $\phi(e) \in \{0\} \cup [\beta, 1]$. Finally, we set $|\phi| = \sum_{e \in \mathcal{H}_0} \phi(e)$.

Lemma 6.1 (Haxell and Rödl [12]). *For every integer $p \geqslant 2$ and for all $\xi > 0$, there exists $B_0 = B_0(p, \xi) > 0$ so that the following holds.*

Let $\mathcal{H}_0$ be a p-graph on R vertices with vertex weights $w : V(\mathcal{H}_0) \to [0,1]$. Suppose ϕ is a weighted fractional packing of $\mathcal{H}_0$ where, for every $e \in \mathcal{H}_0$, $\phi(e) < 1/B_0$. Then, there exists a $(1/B_0)$-bounded weighted fractional packing $\bar{\phi}$ of $\mathcal{H}_0$ so that $|\bar{\phi}| \geqslant |\phi| - \xi R$. Moreover, the function $\bar{\phi}$ can be found, in time depending on R, by an exhaustive search.

We now show that Lemma 6.1 implies the Bounding Lemma (Lemma 2.12). To that end, let F_0 be a given k-graph and let $\xi > 0$ be given. To define the constant $\delta = \delta_{\text{Lem. 2.12}}(F_0, \xi) > 0$, we appeal to Lemma 6.1. Set $p = |F_0|$ to be the number of edges of F_0. With integer p and $\xi > 0$ fixed above, let $B_0 = B_0(p, \xi) > 0$ be the constant guaranteed by Lemma 6.1. Set $\delta = 1/B_0$. Now, let H_0 be a k-graph on r vertices with edge weights $\omega : H_0 \to [0,1]$. We construct the δ-bounded (ω, F_0)-packing of H_0 promised by Lemma 2.12 by appealing to Lemma 6.1.

To that end, define a vertex-weighted p-graph $\mathcal{H}_0$ from H_0 as follows. Set $V(\mathcal{H}_0) = H_0$, *i.e.*, each vertex of $\mathcal{H}_0$ corresponds to an edge of H_0. Let $R = |H_0|$ so that $\mathcal{H}_0$ is on R vertices. Set $\mathcal{H}_0 = \binom{H_0}{F_0}$, *i.e.*, each edge of $\mathcal{H}_0$ corresponds to a copy of F_0 in H_0 (and so $\mathcal{H}_0$ is p-uniform). Define vertex weights $w : V(\mathcal{H}_0) \to [0,1]$ by setting, for each $v_e \in V(\mathcal{H}_0)$ where $e \in H_0$, $w(v_e) = \omega(e)$. Finally, let $\psi^* : \binom{H_0}{F_0} \to [0,1]$ be a maximum fractional (ω, F_0)-packing of H_0. Then ψ^* corresponds to a weighted fractional packing ϕ^* of $\mathcal{H}_0$ with

$$|\psi^*| = |\phi^*| = \nu^*_{F_0}(H_0). \tag{6.1}$$

To apply Lemma 6.1, we delete edges $e \in \mathcal{H}_0$ for which $\phi^*(e) \geqslant \delta$. To that end, set $\mathcal{D}_0 = \{e \in \mathcal{H}_0 : \phi^*(e) \geqslant \delta\}$ and set $\mathcal{H}'_0 = \mathcal{H}_0 \setminus \mathcal{D}_0$. Define vertex weights $w' : V(\mathcal{H}'_0) \to [0,1]$ by

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setting, for each $v \in V(\mathcal{H}'_0) = V(\mathcal{H}_0)$,

$$w'(v) = w(v) - \sum_{v \in e \in \mathcal{D}_0} \phi^*(e). \tag{6.2}$$

(Note that $w'(v) \geqslant 0$ on account that ϕ^* is a weighted fractional packing of $\mathcal{H}_0$.) Write $\phi' = \phi^*|_{\mathcal{H}'_0}$ for the restriction of ϕ^* on $\mathcal{H}'_0$ so that

$$|\phi'| = |\phi^*| - \sum_{e \in \mathcal{D}_0} \phi^*(e). \tag{6.3}$$

Note that, by our definition of w' above, ϕ' is a weighted fractional packing of $\mathcal{H}'_0$. Indeed, for each $v \in V(\mathcal{H}')$ we have

$$\sum_{v \in e \in \mathcal{H}'_0} \phi'(e) = \sum_{v \in e \in \mathcal{H}_0} \phi^*(e) - \sum_{v \in e \in \mathcal{D}_0} \phi^*(e) \leqslant w(v) - \sum_{v \in e \in \mathcal{D}_0} \phi^*(e) \overset{(6.2)}{=} w'(v),$$

where in the inequality above, we used that ϕ^* is a weighted fractional packing of $\mathcal{H}_0$.

We now apply Lemma 6.1 to $\mathcal{H}'_0$, which we may do on account that for every $e \in \mathcal{H}'_0$, we have $\phi'(e) = \phi^*(e) < \delta = 1/B_0$, where $B_0 = B_0(p, \xi) > 0$ is the constant required by Lemma 6.1. In time depending on $R = |H_0| \leqslant r^k$, Lemma 6.1 determines a δ-bounded fractional packing $\bar{\phi}$ of $\mathcal{H}'_0$ so that

$$|\bar{\phi}| \geqslant |\phi'| - \xi R \geqslant |\phi'| - \xi r^k. \tag{6.4}$$

Now, define the function $\hat{\phi} : \mathcal{H}_0 \to [0, 1]$ as follows. For each $e \in \mathcal{H}_0$, set

$$\hat{\phi}(e) = \begin{cases} \phi^*(e) & \text{if } e \in \mathcal{D}_0, \\ \bar{\phi}(e) & \text{if } e \in \mathcal{H}'_0. \end{cases}$$

Then, $\hat{\phi}$ is δ-bounded, by construction. Note also that $\hat{\phi}$ is a weighted fractional packing of $\mathcal{H}_0$ since, for each $v \in V(\mathcal{H}_0)$,

$$\sum_{v \in e \in \mathcal{H}_0} \hat{\phi}(e) = \sum_{v \in e \in \mathcal{H}'_0} \bar{\phi}(e) + \sum_{v \in e \in \mathcal{D}_0} \phi^*(e) \leqslant w'(v) + \sum_{v \in e \in \mathcal{D}_0} \phi^*(e) \overset{(6.2)}{=} w(v).$$

Finally, note that

$$\begin{aligned} |\hat{\phi}| &= \sum_{e \in \mathcal{H}_0} \hat{\phi}(e) = \sum_{e \in \mathcal{H}'_0} \bar{\phi}(e) + \sum_{e \in \mathcal{D}_0} \phi^*(e) \\ &\overset{(6.3)}{=} |\bar{\phi}| + |\phi^*| - |\phi'| \overset{(6.4)}{\geqslant} |\phi^*| - \xi r^k \overset{(6.1)}{=} \nu^*_{F_0}(H_0) - \xi r^k. \end{aligned}$$

Thus, $\hat{\phi}$ corresponds to a fractional (ω, F_0)-packing $\hat{\psi}$ of H_0 of promised size.

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