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CONSTANT HIGHER-ORDER MEAN CURVATURE HYPERSURFACES IN RIEMANNIAN SPACES

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Abstract It is still an open question whether a compact embedded hypersurface in the Euclidean space $\mathbb{R}^{n+1}$ with constant mean curvature and spherical boundary is necessarily a hyperplanar ball or a spherical cap, even in the simplest case of surfaces in $\mathbb{R}^3$. In a recent paper, Alías and Malacarne (*Rev. Mat. Iberoamericana* **18** (2002), 431–442) have shown that this is true for the case of hypersurfaces in $\mathbb{R}^{n+1}$ with constant scalar curvature, and more generally, hypersurfaces with constant higher-order r-mean curvature, when $r \geqslant 2$. In this paper we deal with some aspects of the classical problem above, by considering it in a more general context. Specifically, our starting general ambient space is an orientable Riemannian manifold $\bar{M}$, where we will consider a general geometric configuration consisting of an immersed hypersurface into $\bar{M}$ with boundary on an oriented hypersurface P of $\bar{M}$. For such a geometric configuration, we study the relationship between the geometry of the hypersurface along its boundary and the geometry of its boundary as a hypersurface of P, as well as the geometry of P as a hypersurface of $\bar{M}$. Our approach allows us to derive, among others, interesting results for the case where the ambient space has constant curvature (the Euclidean space $\mathbb{R}^{n+1}$, the hyperbolic space $\mathbb{H}^{n+1}$, and the sphere $\mathbb{S}^{n+1}$). In particular, we are able to extend the symmetry results given in the recent paper mentioned above to the case of hypersurfaces with constant higher-order r-mean curvature in the hyperbolic space and in the sphere.

Keywords: higher-order mean curvature; Newton transformations; ellipticity; transversality; flux formula; spherical caps

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1. Introduction

An old problem in classical differential geometry consists on finding all compact surfaces in Euclidean space $\mathbb{R}^3$ with constant mean curvature and circular boundary. As is well known, a circle C in $\mathbb{R}^3$ is the boundary of two spherical caps with constant mean curvature H for any positive number H, less than or equal to the inverse of the radius of the circle C. A natural question to ask [**10**] is whether a compact constant mean curvature surface in $\mathbb{R}^3$ which is bounded by a circle is necessarily a spherical cap or a flat disc.

Actually, a constant mean curvature surface with circular boundary is the mathematical model of a soap bubble which has its boundary on a round hoop, and the surfaces we almost always observe are spherical caps, so that it is natural to ask if these are the only solutions. In [**13**] Kapouleas gave a negative answer to this question by showing that there exist examples of higher genus compact, non-spherical immersed surfaces with constant mean curvature in $\mathbb{R}^3$ bounded by a circle. However, it has been conjectured that there must be a positive answer to this question if one requires in addition that the surface has genus zero or that it is embedded [**9**].

In recent years, several authors have obtained some partial answers to these problems. For instance, Barbosa [**4, 5**] proved that the only compact immersed surfaces with constant mean curvature $H \neq 0$ and circular boundary which are contained either in a sphere or in a cylinder of radius $1/|H|$ are the spherical caps. On the other hand, in the genus zero case the first author, jointly with López and Palmer, has showed that the only stable constant mean curvature immersed surfaces of disc type which are bounded by a circle are spherical caps [**3**] (see also [**7**] for another characterization of spherical caps as the only stable examples, given by Barbosa and Jorge under a stronger idea of stability).

It is clear that this classical question can be stated in a more general context as follows. Let Σ^{n-1} be a compact $(n-1)$-dimensional submanifold contained in a hyperplane $\Pi \subset \mathbb{R}^{n+1}$, and let M^n be an n-dimensional connected orientable manifold with smooth boundary ∂M. As usual, M is said to be a hypersurface of $\mathbb{R}^{n+1}$ with boundary Σ if there exists an immersion $\psi : M^n \to \mathbb{R}^{n+1}$ such that the immersion ψ restricted to the boundary ∂M is a diffeomorphism onto Σ. In this context, the classical question above consists on finding the compact hypersurfaces in $\mathbb{R}^{n+1}$ with constant mean curvature whose boundary Σ is a round $(n-1)$-sphere. At this point, it is interesting to recall that a classical result by Alexandrov [**1**] states that round spheres are the only *closed* hypersurfaces with constant mean curvature which are embedded in Euclidean space $\mathbb{R}^{n+1}$ (here by closed we mean compact and *without* boundary). More recently, Alexandrov theorem was extended by Ros to the case of constant scalar curvature [**24**], and more generally to the case of hypersurfaces with constant higher-order mean curvature [**23**], showing that round spheres are the only closed embedded hypersurfaces with constant r-mean curvature in $\mathbb{R}^{n+1}$ (see also [**19**] for an extension of Alexandrov theorem for higher-order mean curvatures in the hyperbolic space $\mathbb{H}^{n+1}$ and in the sphere $\mathbb{S}^{n+1}$).

As for the case of non-empty boundary, in [**14**] Koiso gave a new interpretation of the problem by studying under what conditions the symmetries of the boundary $\Sigma \subset \Pi$ of a non-zero constant mean curvature hypersurface M in $\mathbb{R}^{n+1}$ are inherited by the whole hypersurface. She showed that this necessarily occurs when the hypersurface M is embedded and it does not intersect the outside of Σ in Π; as a consequence, if the boundary Σ is a round $(n-1)$-sphere, then M is symmetric with respect to every hyperplane through the centre of Σ which is orthogonal to Π, and hence M must be a spherical cap. Related to Koiso's symmetry theorem, Brito *et al*. [**9**] also showed that when Σ is strictly convex and M is embedded and transverse to Π along the boundary ∂M, then M is entirely contained in one of the half-spaces of $\mathbb{R}^{n+1}$ determined by Π and, therefore, the so-called *Alexandrov reflection technique* [**1**] implies that M inherits all the symmetries of Σ. In particular, if Σ is a round sphere, then M must be a spherical cap. Here, transversality means that the hypersurface M is never tangent to the hyperplane Π along its boundary. In what follows, we will use the term *symmetry result* to refer to a result of this type.

The technique introduced in [**9**] makes extensive use of two essential ingredients, the Alexandrov reflection technique mentioned above, and an integral formula first found by Kusner [**16**], which is now known as the *flux formula*. This fact indicates that the symmetry result in [**9**] can be extended from two new viewpoints: by considering constant mean curvature hypersurfaces in other space forms; or by considering the case of hypersurfaces with constant higher-order r-mean curvature. From the first point of view, Nelli and Rosenberg [**20**] studied the case of hypersurfaces in hyperbolic space $\mathbb{H}^{n+1}$, and, more recently, Lira [**11**] considered the case of hypersurfaces in the sphere $\mathbb{S}^{n+1}$, establishing corresponding symmetry results for the case of constant mean curvature. On the other hand, in [**25**] Rosenberg established a version of the flux formula for hypersurfaces with constant higher-order r-mean curvature in Euclidean space $\mathbb{R}^{n+1}$, and applied it to extend the symmetry result given in [**9**] to the case of the higher-order r-mean curvatures.

In this paper, we will deal with some aspects of the classical problem above. Our initial strategy is to study this problem in a more general context. Specifically, our general ambient space will be an $(n+1)$-dimensional connected orientable Riemannian manifold $\bar{M}$, where we will consider the following geometric configuration (for the details, see § 4). Let us fix $P^n \subset \bar{M}$ an orientable connected totally geodesic hypersurface in $\bar{M}$, and let $\Sigma^{n-1} \subset P$ be an orientable $(n-1)$-dimensional compact embedded submanifold contained in P^n. Consider M^n an n-dimensional connected orientable manifold with smooth boundary ∂M. Then, M is said to be a hypersurface of $\bar{M}$ with boundary Σ if there exists an immersion $\psi : M^n \to \bar{M}^{n+1}$ such that the immersion ψ restricted to the boundary ∂M is a diffeomorphism onto Σ.

From this geometric configuration, the following question, closely related to the symmetry problem, naturally arises.

How is the geometry of M along its boundary ∂M related to the geometry of the inclusion $\Sigma \subset P$?

A first partial answer to this question is given by the following expression, which holds along the boundary ∂M and for every $1 \leqslant r \leqslant n-1$ (see Corollary 6.1):

$$\langle T_r \nu, \nu \rangle = (-1)^r s_r \langle \xi, \nu \rangle^r.$$

Here T_r stands for the rth classical Newton transformation associated to the second fundamental form on M (see § 3 for the details), ν is the outward pointing unit conormal vector field along ∂M, ξ is the unitary normal field of $P \subset \bar{M}$, and $s_r = s_r(\tau_1, \dots, \tau_{n-1})$ is the rth elementary symmetric function of $\tau_1, \dots, \tau_{n-1}$, the principal curvatures of $\Sigma \subset P$ with respect to the outward pointing unitary normal. As a first consequence of this expression, we obtain a very strong relationship between the transversality of M with respect to P along the boundary ∂M, and the ellipticity on M of the rth Newton transformation T_r, that is, the positivity of the quadratic form associated to T_r. This fact, along with Theorem 7.3 in [**25**], allows us to state the following symmetry theorem for hypersurfaces in $\mathbb{R}^{n+1}$ (Theorem 7.1).

Let Σ be a strictly convex compact $(n-1)$-dimensional submanifold in a hyperplane $\Pi \subset \mathbb{R}^{n+1}$, and let $\psi : M^n \to \mathbb{R}^{n+1}$ be a compact embedded hypersurface with boundary Σ. Let us assume that for a given $2 \leqslant r \leqslant n$, the r-mean curvature H_r of M is a non-zero constant. Then M has all the symmetries of Σ. In particular, if the boundary Σ is a round $(n-1)$-sphere of $\mathbb{R}^{n+1}$, then M is a spherical cap.

As a consequence, we can conclude that the conjecture of the spherical cap [**9**] is true for the case of embedded hypersurfaces with constant r-mean curvature in $\mathbb{R}^{n+1}$, when $r \geqslant 2$. This includes, in particular, the case of constant scalar curvature, when $r = 2$ [**2**].

In order to extend this symmetry result to the case of hypersurfaces in hyperbolic space and hypersurfaces in the sphere, it is necessary to establish a certain *flux formula*, which is one of the key ingredients of the used techniques. For that reason, § 8 is devoted to deriving a general flux formula for the considered geometric configuration in the case where the Riemannian ambient space $\bar{M}$ is equipped with a conformal vector field (Proposition 8.1). Our general flux formula becomes specially simple when the ambient space has constant sectional curvature, and the conformal vector field is indeed a Killing vector field. In that case, we are able to extend the flux formula given by Rosenberg in [**25**, Theorem 7.2] to the case of the other space forms, as follows (Corollary 8.2).

Let $\psi : M^n \to \bar{M}^{n+1}$ be an immersed compact orientable hypersurface with boundary ∂M, and let D^n be a compact orientable hypersurface with boundary $\partial D = \partial M$. Assume that $M \cup D$ is an oriented n-cycle of $\bar{M}$, and let $\boldsymbol{N}$ and n_D be the unit normal fields which orient M and D, respectively. Assume that $\bar{M}$ has constant sectional curvature. If the r-mean curvature H_r is constant, $1 \leqslant r \leqslant n$, then for every Killing vector field $Y \in \mathcal{X}(\bar{M})$ the following flux formula holds

$$\oint_{\partial M} \langle T_{r-1}\nu, Y\rangle \,\mathrm{d}s = -r\binom{n}{r} H_r \int_D \langle Y, n_D\rangle \,\mathrm{d}D,$$

where ν is the outward pointing conormal to M along ∂M.

As first applications of our general flux formula, we derive some interesting estimates for the volume of minimal hypersurfaces with boundary on a geodesic sphere of the ambient space, in the case where the ambient space is the Euclidean space (Corollary 8.4), the hyperbolic space (Corollary 8.5), or the sphere (Corollary 8.6).

On the other hand, and as another application of our flux formula and the expression for $\langle T_r\nu, \nu\rangle$ given in Corollary 6.1 (see above), we establish in § 9 some interesting estimates for the constant r-mean curvature in terms of the geometry of the boundary. Specifically, when the ambient space is the Euclidean space we obtain the following (Theorem 9.1).

Let Σ be an orientable $(n-1)$-dimensional compact submanifold in a hyperplane $P \subset \mathbb{R}^{n+1}$, and let $\psi : M^n \to \mathbb{R}^{n+1}$ be an orientable immersed compact (connected) hypersurface with boundary $\Sigma = \psi(\partial M)$ and constant r-mean curvature H_r, $1 \leqslant r \leqslant n$.

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Then

$$0 \leqslant |H_r| \leqslant \frac{1}{n\,\mathrm{vol}(D)} \oint_{\partial M} |h_{r-1}|\,\mathrm{d}s,$$

where h_{r-1} *stands for the* $(r-1)$*-mean curvature of* $\Sigma \subset P$*, and* D *is the domain in* P *bounded by* Σ*. In particular, when* Σ *is a round* $(n-1)$*-sphere of radius* ϱ *it follows that*

$$0 \leqslant |H_r| \leqslant \frac{1}{\varrho^r}.$$

This estimate is the natural generalization of an estimate first obtained by Barbosa in the case of constant mean curvature ($r = 1$) [**5**]. On the other hand, when the ambient space is the hyperbolic space, our estimate reads as follows (Theorem 9.2).

Let Σ *be an orientable* $(n-1)$*-dimensional compact submanifold contained in a totally geodesic hyperplane* $P \subset \mathbb{H}^{n+1}$*, and let* $\psi : M^n \to \mathbb{H}^{n+1}$ *be an orientable immersed compact connected hypersurface with boundary* $\Sigma = \psi(\partial M)$ *and constant* r*-mean curvature* H_r*,* $1 \leqslant r \leqslant n$*. Then*

$$0 \leqslant |H_r| \leqslant \frac{C}{n\,\mathrm{vol}(D)} \oint_{\partial M} |h_{r-1}|\,\mathrm{d}s.$$

Here h_{r-1} *stands for the* $(r-1)$*-mean curvature of* $\Sigma \subset P$*,* D *is the domain in* P *bounded by* Σ*, and* $C = \max_\Sigma \cosh(\tilde{\varrho}) \geqslant 1$*, where* $\tilde{\varrho}(p)$ *is the geodesic distance along* P *between a fixed arbitrary point* $a \in \mathrm{int}(D)$ *and* p*. In particular, when* Σ *is a geodesic sphere in* P *of geodesic radius* ϱ*, it follows that*

$$0 \leqslant |H_r| \leqslant \coth^r(\varrho).$$

Similarly, for the case of hypersurfaces in the sphere, our estimate states the following (Theorem 9.3).

Let Σ *be an orientable* $(n-1)$*-dimensional compact submanifold contained in an open totally geodesic hemisphere* $P_+ \subset \mathbb{S}^{n+1}$*, and let* $\psi : M^n \to \mathbb{S}^{n+1}$ *be an orientable immersed compact connected hypersurface with boundary* $\Sigma = \psi(\partial M)$ *and constant* r*-mean curvature* H_r*,* $1 \leqslant r \leqslant n$*. Then*

$$0 \leqslant |H_r| \leqslant \frac{C}{n\,\mathrm{vol}(D)} \oint_{\partial M} |h_{r-1}|\,\mathrm{d}s.$$

Here h_{r-1} *stands for the* $(r-1)$*-mean curvature of* $\Sigma \subset P$*,* D *is the domain in* P_+ *bounded by* Σ*, and* $C = \max_\Sigma \cos(\tilde{\varrho}) / \min_D \cos(\tilde{\varrho})$*, where* $\tilde{\varrho}(p)$ *is the geodesic distance along* P_+ *between a fixed arbitrary point* $a \in \mathrm{int}(D)$ *and* p*. In particular, when* Σ *is a geodesic sphere in* P_+ *of geodesic radius* $\varrho < \frac{1}{2}\pi$*, it follows that*

$$0 \leqslant |H_r| \leqslant \cot^r(\varrho).$$

Finally, the two remaining sections of the paper are devoted to the extension of our symmetry results to the case of hypersurfaces in the hyperbolic space and hypersurfaces in the sphere. Specifically, in § 10 we obtain the following symmetry result for hypersurfaces in hyperbolic space (Theorem 10.1).

Let Σ^{n-1} be a strictly convex compact $(n-1)$-dimensional (connected) submanifold of a totally geodesic hyperplane $P^n \subset \mathbb{H}^{n+1}$, and let $M^n \subset \mathbb{H}^{n+1}$ be a compact (connected) embedded hypersurface with boundary Σ. Let us assume that for a given $2 \leqslant r \leqslant n$, the r-mean curvature H_r of M is a non-zero constant. Then M has all the symmetries of Σ. In particular, when the boundary Σ is a geodesic sphere in $P^n \subset \mathbb{H}^{n+1}$, then M is a spherical cap.

As a consequence, we can conclude, as in the Euclidean case, that the conjecture of the spherical cap is true for the case of embedded hypersurfaces with constant r-mean curvature in hyperbolic space, when $r \geqslant 2$. Finally, in the case of hypersurfaces in the sphere, we state the following symmetry result (Theorem 11.1).

Let Σ^{n-1} be a convex $(n-1)$-dimensional submanifold of a totally geodesic n-sphere $P^n \subset \mathbb{S}^{n+1}$, and let $M^n \subset \mathbb{S}^{n+1}$ be a compact (connected) embedded hypersurface with boundary Σ. Let us assume that M is contained in an open hemisphere $\mathbb{S}^{n+1}_+$, and that the r-mean curvature H_r of M is a non-zero constant, for a given $2 \leqslant r \leqslant n$. Then M has all the symmetries of Σ. In particular, when the boundary Σ is a geodesic sphere in $P^n \subset \mathbb{S}^{n+1}$, then M is a spherical cap.

In particular, the only compact embedded hypersurfaces in $\mathbb{S}^{n+1}_+$ with constant r-mean curvature $H_r \neq 0$ (with $2 \leqslant r \leqslant n$) and spherical boundary are the spherical caps.

2. Preliminaries

Throughout this paper, $\bar{M}^{n+1}$ will denote an $(n+1)$-dimensional connected orientable Riemannian manifold, and $\langle\cdot,\cdot\rangle$ and $\bar{\nabla}$ will stand for its Riemannian metric and its Levi-Civita connection, respectively. Let M^n be an n-dimensional connected orientable manifold with smooth boundary ∂M; M is said to be a hypersurface of $\bar{M}$ if there exists an isometric immersion $\psi : M^n \to \bar{M}^{n+1}$. In that case, since M and $\bar{M}$ are both orientable, we may choose along $\psi(M)$ a globally defined unit normal vector field $\boldsymbol{N}$, and we may assume that M is oriented by $\boldsymbol{N}$. If ∇ denotes the Levi-Civita connection on M, then the Gauss and Weingarten formulae for the immersion are given, respectively, by

$$\bar{\nabla}_V W = \nabla_V W + \langle AV, W\rangle \boldsymbol{N}, \tag{2.1}$$

and

$$A(V) = -\bar{\nabla}_V \boldsymbol{N}, \tag{2.2}$$

for all tangent vector fields $V, W \in \mathcal{X}(M)$.

Here $A : \mathcal{X}(M) \to \mathcal{X}(M)$ defines the shape operator (or the second fundamental form) of the hypersurface with respect to $\boldsymbol{N}$. The curvature tensor R of the hypersurface M is described in terms of A and the curvature tensor $\bar{R}$ of the ambient space $\bar{M}$ by the so-called Gauss equation, which can be written as

$$R(U,V)W = (\bar{R}(U,V)W)^{\top} + \langle AU, W\rangle AV - \langle AV, W\rangle AU \tag{2.3}$$

for all tangent vector fields $U, V, W \in \mathcal{X}(M)$, where the superscript '$\top$' denotes projection on $\mathcal{X}(M)$. Observe that our criterion here for the definition of the curvature tensor is the one in [**21**]. On the other hand, the Codazzi equation of the hypersurface describes the normal component of $\bar{R}(U,V)W$ in terms of the derivative of the shape operator, and it is given by

$$\langle \bar{R}(U,V)W, \boldsymbol{N}\rangle = \langle (\nabla_V A)U - (\nabla_U A)V, W\rangle, \tag{2.4}$$

where $\nabla_U A$ denotes the covariant derivative of A. In particular, when the ambient space has constant sectional curvature, then $\bar{R}(U,V)W$ is tangent to M for every $U,V,W \in \mathcal{X}(M)$, and (2.4) becomes

$$(\nabla_V A)U = (\nabla_U A)V. \tag{2.5}$$

As is well known, A is a self-adjoint linear operator in each tangent plane T_pM, and its eigenvalues $\kappa_1(p),\dots,\kappa_n(p)$ are the principal curvatures of the hypersurface. Associated to the shape operator there are n algebraic invariants given by

$$S_r(p) = \sigma_r(\kappa_1(p),\dots,\kappa_n(p)), \quad 1 \leqslant r \leqslant n,$$

where $\sigma_r : \mathbb{R}^n \to \mathbb{R}$ are the elementary symmetric functions in $\mathbb{R}^n$ given by

$$\sigma_r(x_1,\dots,x_n) = \sum_{i_1<\cdots<i_r} x_{i_1}\cdots x_{i_n}.$$

Observe that the characteristic polynomial of A can be written in terms of the S_r as

$$\det(tI - A) = \sum_{r=0}^{n} (-1)^r S_r t^{n-r}. \tag{2.6}$$

The r-mean curvature H_r of the hypersurface is then defined by

$$\binom{n}{r} H_r = S_r.$$

In particular, when $r=1$, $H_1 = (1/n)\operatorname{tr}(A) = H$ is the mean curvature of M, which is the main extrinsic curvature of the hypersurface. On the other hand, when $r=2$, H_2 defines a geometric quantity which is related to the (intrinsic) scalar curvature of the hypersurface. Indeed, it follows from the Gauss equation (2.3) that the Ricci curvature of M is given by

$$\operatorname{Ric}(U,V) = \overline{\operatorname{Ric}}(U,V) - \langle \bar{R}(U,\boldsymbol{N})V,\boldsymbol{N}\rangle + nH\langle AU,V\rangle - \langle AU,AV\rangle,$$

for $U,V \in \mathcal{X}(M)$, where $\overline{\operatorname{Ric}}$ stands for the Ricci curvature of the ambient space $\bar{M}$. Therefore, the scalar curvature S of the hypersurface M is

$$S = \operatorname{tr}(\operatorname{Ric}) = \bar{S} - 2\overline{\operatorname{Ric}}(\boldsymbol{N},\boldsymbol{N}) + n(n-1)H_2.$$

For instance, if the ambient space has constant sectional curvature $\bar{c}$, then

$$S = n(n-1)(\bar{c} + H_2). \tag{2.7}$$

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3. The Newton transformations

The classical Newton transformations $T_r : \mathcal{X}(M) \to \mathcal{X}(M)$ are defined inductively from A by

$$T_0 = I \qquad \text{and} \qquad T_r = S_r I - A T_{r-1}, \quad 1 \leqslant r \leqslant n,$$

where I denotes the identity in $\mathcal{X}(M)$, or equivalently by

$$T_r = S_r I - S_{r-1} A + \cdots + (-1)^{r-1} S_1 A^{r-1} + (-1)^r A^r.$$

Note that by the Cayley–Hamilton theorem, we have $T_n = 0$.

Let us recall that each T_r is also a self-adjoint linear operator in each tangent plane $T_p M$ which commutes with A. Indeed, A and T_r can be simultaneously diagonalized; if $\{e_1, \dots, e_n\}$ are the eigenvectors of A corresponding to the eigenvalues $\kappa_1(p), \dots, \kappa_n(p)$, respectively, then they are also the eigenvectors of T_r corresponding to the eigenvalues of T_r, and $T_r(e_i) = \mu_{i,r}(p) e_i$ with

$$\mu_{i,r}(p) = \frac{\partial \sigma_{r+1}}{\partial x_i}(\kappa_1(p), \dots, \kappa_n(p)) = \sum_{i_1 < \cdots < i_r,\, i_j \neq i} \kappa_{i_1}(p) \cdots \kappa_{i_r}(p),$$

for every $1 \leqslant i \leqslant n$. From here it can be easily seen that

$$\operatorname{tr}(T_r) = (n-r) S_r = c_r H_r, \tag{3.1}$$

$$\operatorname{tr}(A T_r) = (r+1) S_{r+1} = c_r H_{r+1}, \tag{3.2}$$

where

$$c_r = (n-r)\binom{n}{r} = (r+1)\binom{n}{r+1}.$$

For the details, we refer the reader to the classical paper by Reilly [**22**] (see also [**25**] for a more accessible modern treatment by Rosenberg).

On the other hand, the divergence of T_r is defined by

$$\operatorname{div}_M T_r = \operatorname{tr}(\nabla T_r) = \sum_{i=1}^{n} (\nabla_{e_i} T_r)(e_i),$$

where $\{e_1, \dots, e_n\}$ is a local orthonormal frame on M. Below we will compute $\operatorname{div}_M T_r$, which will be necessary for its later use.

Lemma 3.1. *The divergence of the Newton transformations T_r are given by the following inductive formula:*

$$\left.\begin{aligned} &\operatorname{div}_M T_0 = 0, \\ &\operatorname{div}_M T_r = -A(\operatorname{div}_M T_{r-1}) - \sum_{i=1}^{n} (\bar{R}(\boldsymbol{N}, T_{r-1} e_i) e_i)^{\top}, \end{aligned}\right\} \tag{3.3}$$

where $\bar{R}$ stands for the curvature tensor of $\bar{M}$, and $(\bar{R}(\boldsymbol{N}, V)W)^{\top}$ denotes the tangential component of $\bar{R}(\boldsymbol{N}, V)W$. Equivalently, for every tangent field $V \in \mathcal{X}(M)$ it follows that

$$\langle \operatorname{div}_M T_r, V \rangle = \sum_{j=1}^{r} \sum_{i=1}^{n} \langle \bar{R}(\boldsymbol{N}, T_{r-j} e_i) e_i, A^{j-1} V \rangle. \tag{3.4}$$

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The expression (3.4) has also been recently obtained by Lima in [**17**], using a very different argument to ours.

Proof. It is clear that $\operatorname{div}_M T_0 = \operatorname{div}_M I = 0$. When $r \geqslant 1$, from the inductive definition of T_r we have, for $V, W \in \mathcal{X}(M)$,

$$\begin{aligned}(\nabla_V T_r)W &= \langle \nabla S_r, V\rangle W - \nabla_V(AT_{r-1})W \\ &= \langle \nabla S_r, V\rangle W - (\nabla_V A)(T_{r-1}W) - A((\nabla_V T_{r-1})W),\end{aligned}$$

so that

$$\operatorname{div}_M T_r = \sum_{i=1}^n (\nabla_{e_i} T_r)(e_i) = \nabla S_r - \sum_{i=1}^n (\nabla_{e_i} A)(T_{r-1}e_i) - A(\operatorname{div}_M T_{r-1}).$$

Using now the Codazzi equation (2.4) we get, for $V \in \mathcal{X}(M)$,

$$\begin{aligned}\langle (\nabla_{e_i} A)(T_{r-1}e_i), V\rangle &= \langle (\nabla_{e_i} A)V, T_{r-1}e_i\rangle \\ &= \langle (\nabla_V A)e_i, T_{r-1}e_i\rangle + \langle \bar{R}(V, e_i)T_{r-1}e_i, \boldsymbol{N}\rangle \\ &= \langle T_{r-1}((\nabla_V A)e_i), e_i\rangle + \langle \bar{R}(\boldsymbol{N}, T_{r-1}e_i)e_i, V\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}\langle \operatorname{div}_M T_r, V\rangle = \langle \nabla S_r, V\rangle &- \operatorname{tr}(T_{r-1}\nabla_V A) \\ &- \sum_{i=1}^n \langle \bar{R}(\boldsymbol{N}, T_{r-1}e_i)e_i, V\rangle - \langle A(\operatorname{div}_M T_{r-1}), V\rangle. \quad (3.5)\end{aligned}$$

Using now equation (4.4) in [**25**] we have that

$$\operatorname{tr}(T_{r-1}\nabla_V A) = \langle \nabla S_r, V\rangle,$$

which jointly with (3.5) gives (3.3). Finally, equation (3.4) follows easily from (3.3) by an inductive argument. □

In particular, when the ambient Riemannian space $\bar{M}$ has constant sectional curvature, then $(\bar{R}(\boldsymbol{N}, V)W)^\top = 0$ for every tangent vector fields $V, W \in \mathcal{X}(M)$ and equation (3.4) implies that $\operatorname{div}_M T_r = 0$ for every r.

Corollary 3.2. *When the ambient Riemannian space $\bar{M}$ has constant sectional curvature, then the Newton transformations are divergence-free: $\operatorname{div}_M T_r = 0$ for each r.*

4. A geometric configuration

Throughout this paper, we will be particularly interested in the following geometric configuration, which is suggested by the classical question stated in § 1. Let $P^n \subset \bar{M}$ be an orientable connected hypersurface in $\bar{M}$, and let $\Sigma^{n-1} \subset P$ be an orientable $(n-1)$-dimensional compact embedded submanifold contained in P^n. Let $\psi : M^n \to \bar{M}^{n+1}$ be an orientable compact connected hypersurface in $\bar{M}$ with smooth boundary ∂M. As usual, M is said to be a hypersurface *with boundary* Σ if the immersion ψ restricted to the boundary ∂M is a diffeomorphism onto Σ. The following question naturally arises from this geometric configuration.

How is the geometry of M along its boundary ∂M related to the geometry of the inclusion $\Sigma \subset P$ and the inclusion $P \subset \bar{M}$?

In what follows, we will study this question. Let us start by choosing the orientation of this configuration. Let us consider the hypersurface M oriented by a globally defined unit normal vector field $\boldsymbol{N}$. The orientation of M induces a natural orientation on its boundary as follows: given a point $p \in \partial M$, a basis $\{v_1, \dots, v_{n-1}\}$ for $T_p(\partial M)$ is said to be positively oriented if $\{u, v_1, \dots, v_{n-1}\}$ is a positively oriented basis for T_pM, whenever $u \in T_pM$ is outward pointing. We will denote by ν the outward pointing unit conormal vector field along ∂M. By means of the diffeomorphism $\psi|_{\partial M} : \partial M \to \Sigma$, the orientation of ∂M is induced on each connected component of Σ. On each connected component P_0 of P, we distinguish a connected component $\Sigma_0 \subset P_0$ of Σ. Let η_0 be the unitary vector field normal to Σ_0 in P_0 which points outward with respect to the domain in P_0 bounded by Σ_0. Now, we choose ξ_0 the unique unitary vector field normal to P_0 in $\bar{M}$ which is compatible with η_0 and with the orientation of Σ_0. We note that the chosen orientation of P_0 given by the field ξ_0 determines a unique choice to the unitary vector field η normal to each components of Σ in P_0 such that $\eta|_{\Sigma_0} = \eta_0$. We repeat this process to the others connected components of P and hence we obtain unitary vector fields η normal to Σ in P, and ξ normal to P in $\bar{M}$. With this choice, given a point $p \in \Sigma$, a basis $\{v_1, \dots, v_{n-1}\}$ for $T_p\Sigma$ is positively oriented if and only if $\{\eta(p), v_1, \dots, v_{n-1}\}$ is a positively oriented basis for T_pP.

Let $\{e_1, \dots, e_{n-1}\}$ be a (locally defined) positively oriented frame field along a fixed connected component of ∂M. Using this frame, we can write $\nu = e_1 \times \cdots \times e_{n-1} \times \boldsymbol{N}$, and similarly $\eta = e_1 \times \cdots \times e_{n-1} \times \xi$, since $\det(\nu, e_1, \dots, e_{n-1}, \boldsymbol{N}) = 1 = \det(\eta, e_1, \dots, e_{n-1}, \xi)$.

From these expressions we easily compute

$$\begin{aligned} \eta &= e_1 \times \cdots \times e_{n-1} \times \xi \\ &= e_1 \times \cdots \times e_{n-1} \times (\langle \xi, \boldsymbol{N}\rangle \boldsymbol{N} + \langle \xi, \nu\rangle \nu) \\ &= \langle \xi, \boldsymbol{N}\rangle \nu - \langle \xi, \nu\rangle \boldsymbol{N}, \end{aligned}$$

that is,

$$\langle \eta, \nu\rangle = \langle \xi, \boldsymbol{N}\rangle \quad \text{and} \quad \langle \eta, \boldsymbol{N}\rangle = -\langle \xi, \nu\rangle. \tag{4.1}$$

Let A_Σ (respectively, A_P) denote the shape operator of $\Sigma^{n-1} \subset P^n$ (respectively, $P^n \subset \bar{M}^{n+1}$) with respect to the unit normal vector field η (respectively, ξ). It then follows that

$$\bar{\nabla}_{e_i} e_j = \sum_{k=1}^{n-1} \langle \bar{\nabla}_{e_i} e_j, e_k\rangle e_k + \langle \bar{\nabla}_{e_i} e_j, \nu\rangle \nu + \langle A e_i, e_j\rangle \boldsymbol{N},$$

for every $1 \leqslant i, j \leqslant n-1$, and also

$$\bar{\nabla}_{e_i} e_j = \sum_{k=1}^{n-1} \langle \bar{\nabla}_{e_i} e_j, e_k\rangle e_k + \langle A_\Sigma e_i, e_j\rangle \eta + \langle A_P e_i, e_j\rangle \xi,$$

so that from (4.1) we have that

$$\langle A e_i, e_j\rangle = -\langle A_\Sigma e_i, e_j\rangle \langle \xi, \nu\rangle + \langle A_P e_i, e_j\rangle \langle \xi, \boldsymbol{N}\rangle. \tag{4.2}$$

Equality (4.2) above shows us that it is not possible to go further without any additional geometric hypothesis on the geometry of the inclusion $P \subset \bar{M}$. A hypothesis of relevant geometric nature, and which is also technically quite appropriate for us, consists on assuming the umbilicity of $P \subset \bar{M}$. Then, from now on let us suppose that P is a totally umbilical hypersurface in $\bar{M}$. Therefore, there exists a smooth function $\lambda \in \mathcal{C}^\infty(P)$ such that $A_P = \lambda I$, where I denotes the identity in $\mathcal{X}(P)$, and (4.2) becomes

$$\langle Ae_i, e_j\rangle = -\langle A_\Sigma e_i, e_j\rangle\langle \xi, \nu\rangle + \lambda\langle \xi, \boldsymbol{N}\rangle\delta_{ij}, \quad 1 \leqslant i, j \leqslant n-1. \tag{4.3}$$

We now suppose that the basis $\{e_1, \dots, e_{n-1}\} \subset T_p(\partial M)$ on the boundary is chosen such that it is formed by eigenvectors of A_Σ, and let us denote its corresponding eigenvalues by $\tau_1(p), \dots, \tau_{n-1}(p)$. In other words,

$$A_\Sigma e_i = \tau_i e_i, \quad 1 \leqslant i \leqslant n-1.$$

Hence, by (4.3), $\langle Ae_i, e_j\rangle = 0$ when $i \neq j$, and for each $p \in \partial M$, the matrix of A in the orthonormal basis $\{e_1, \dots, e_{n-1}, \nu\}$ of T_pM is given by

$$A = \begin{pmatrix} \gamma_1 & 0 & \cdots & 0 & \langle A\nu, e_1\rangle \\ 0 & \gamma_2 & \cdots & 0 & \langle A\nu, e_2\rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \gamma_{n-1} & \langle A\nu, e_{n-1}\rangle \\ \langle A\nu, e_1\rangle & \langle A\nu, e_2\rangle & \cdots & \langle A\nu, e_{n-1}\rangle & \langle A\nu, \nu\rangle \end{pmatrix}, \tag{4.4}$$

where $\gamma_i = -\tau_i\langle \xi, \nu\rangle + \lambda\langle \xi, \boldsymbol{N}\rangle$ for $1 \leqslant i \leqslant n-1$.

Now we compute the characteristic polynomial of A. To do that, we begin by observing that

$$\begin{aligned} \det(tI_n - A) = (t - \gamma_{n-1})\det(tI_{n-1} - \Lambda(\gamma_1, \dots, \gamma_{n-2})) \\ - \langle A\nu, e_{n-1}\rangle^2(t - \gamma_1)\cdots(t - \gamma_{n-2}), \end{aligned} \tag{4.5}$$

where

$$\Lambda(\gamma_1, \dots, \gamma_{n-2}) = \begin{pmatrix} \gamma_1 & 0 & \cdots & 0 & \langle A\nu, e_1\rangle \\ 0 & \gamma_2 & \cdots & 0 & \langle A\nu, e_2\rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \gamma_{n-2} & \langle A\nu, e_{n-2}\rangle \\ \langle A\nu, e_1\rangle & \langle A\nu, e_2\rangle & \cdots & \langle A\nu, e_{n-2}\rangle & \langle A\nu, \nu\rangle \end{pmatrix}.$$

Therefore, applying a simple induction argument on n in (4.5), we obtain that the characteristic polynomial of A is given by

$$\det(tI_n - A) = (t - \langle A\nu, \nu\rangle)\sum_{i=0}^{n-1}(-1)^i s_i(\gamma)t^{n-1-i} - \sum_{i=1}^{n-1}\langle A\nu, e_i\rangle^2\sum_{j=0}^{n-2}(-1)^j s_j(\hat{\gamma}_i)t^{n-2-j},$$

where $s_r(\gamma)$ (respectively, $s_r(\hat{\gamma}_i)$) stands for the elementary symmetric functions of $\gamma_1, \dots, \gamma_{n-1}$, (respectively, $\gamma_1, \dots, \hat{\gamma}_i, \dots, \gamma_{n-1}$), and, as usual, $s_0(\gamma) = s_0(\hat{\gamma}_i) = 1$ by definition. Comparing the terms of above polynomials, we conclude from (2.6) that the symmetric function of curvature S_r of the hypersurface M, at a boundary point $p \in \partial M$, is given by

$$S_1 = s_1(\gamma) + \langle A\nu, \nu \rangle, \tag{4.6}$$

$$S_2 = s_2(\gamma) + s_1(\gamma)\langle A\nu, \nu \rangle - \sum_{i=1}^{n-1} \langle A\nu, e_i \rangle^2, \tag{4.7}$$

$$S_r = s_r(\gamma) + s_{r-1}(\gamma)\langle A\nu, \nu \rangle - \sum_{i=1}^{n-1} s_{r-2}(\hat{\gamma}_i)\langle A\nu, e_i \rangle^2, \tag{4.8}$$

for $3 \leqslant r \leqslant n$.

5. The Newton transformations on the boundary

Observe that expressions (4.6)–(4.8) provide us with a partial answer to our initial question, since it relates the geometry of the hypersurface M along its boundary ∂M (given by the r-curvature S_r) to the geometry of $\Sigma \subset P$ and the geometry of $P \subset \bar{M}$ (given by $s_r(\gamma)$). But this expression it is not still satisfactory for our purposes. We need the following essential auxiliary result.

Lemma 5.1. *Let $P^n \subset \bar{M}$ be an orientable totally umbilical hypersurface in $\bar{M}$, and let $\Sigma \subset P$ be an orientable $(n-1)$-dimensional compact submanifold in P^n. Let $\psi : M^n \to \bar{M}^{n+1}$ be an orientable connected hypersurface with boundary $\Sigma = \psi(\partial M)$, and let ν stand for the outward pointing unit conormal vector field along $\partial M \subset M$. Then, along the boundary ∂M and for every $1 \leqslant r \leqslant n-1$, it holds*

$$\langle T_r\nu, \nu \rangle = s_r(\gamma) = s_r(\gamma_1, \dots, \gamma_{n-1}), \tag{5.1}$$

where $\gamma_i = -\tau_i\langle \xi, \nu \rangle + \lambda\langle \xi, \boldsymbol{N} \rangle$ for $1 \leqslant i \leqslant n-1$. Here $\tau_1, \dots, \tau_{n-1}$ are the principal curvatures of $\Sigma \subset P$ with respect to the outward pointing unitary normal, $\boldsymbol{N}$ is the unitary normal field of M, ξ is the unitary normal field of $P \subset \bar{M}$, and λ is the umbilicity factor of $P \subset \bar{M}$ (with respect to ξ).

Proof. We will use induction on r. First, observe that from (4.6) it follows that (5.1) holds for $r = 1$. For a given $2 \leqslant r \leqslant n-1$, suppose that

$$\langle T_j\nu, \nu \rangle = s_j(\gamma) \tag{5.2}$$

holds for all $1 \leqslant j \leqslant r-1$. Observe that

$$A\nu = \sum_{i=1}^{n-1} \langle A\nu, e_i \rangle e_i + \langle A\nu, \nu \rangle \nu,$$

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so that from the inductive definition of T_r and (5.2) we conclude that

$$\begin{aligned}\langle T_r\nu,\nu\rangle &= S_r - \langle T_{r-1}\nu, A\nu\rangle \\ &= S_r - \langle T_{r-1}\nu,\nu\rangle\langle A\nu,\nu\rangle - \sum_{i=1}^{n-1}\langle T_{r-1}\nu,e_i\rangle\langle A\nu,e_i\rangle \\ &= S_r - s_{r-1}(\gamma)\langle A\nu,\nu\rangle - \sum_{i=1}^{n-1}\langle T_{r-1}\nu,e_i\rangle\langle A\nu,e_i\rangle. \end{aligned} \tag{5.3}$$

On the other hand, we also know from (4.4) that

$$Ae_i = \gamma_i e_i + \langle A\nu, e_i\rangle\nu,$$

so that from our induction hypothesis (5.2) we have, for every $1 \leqslant j \leqslant r-1$,

$$\langle T_j\nu, e_i\rangle = -\langle T_{j-1}\nu, Ae_i\rangle = -\gamma_i\langle T_{j-1}\nu, e_i\rangle - s_{j-1}(\gamma)\langle A\nu, e_i\rangle.$$

This implies by a recursive argument that

$$\langle T_{r-1}\nu, e_i\rangle = -\langle A\nu, e_i\rangle\sum_{j=0}^{r-2}(-1)^j s_{r-2-j}(\gamma)\gamma_i^j = -\langle A\nu, e_i\rangle s_{r-2}(\hat{\gamma}_i), \tag{5.4}$$

since it is not difficult to see that

$$s_m(\hat{\gamma}_i) = \sum_{j=0}^{m}(-1)^j s_{m-j}(\gamma)\gamma_i^j$$

for every $1 \leqslant m \leqslant n-1$. Using now (5.4) in (5.3), along with (4.8), we conclude that

$$\langle T_r\nu,\nu\rangle = S_r - s_{r-1}(\gamma)\langle A\nu,\nu\rangle + \sum_{i=1}^{n-1} s_{r-2}(\hat{\gamma}_i)\langle A\nu, e_i\rangle^2 = s_r(\gamma).$$

This finishes the proof of Lemma 5.1. □

Now, it remains to know how the elementary symmetric function $s_r(\gamma)$ can be expressed in terms of the principal curvatures $\tau_1,\dots,\tau_{n-1}$ of the inclusion $\Sigma \subset P$ and the umbilicity factor λ of $P \subset \bar{M}$. To see this, let us write $\gamma_i = \alpha_i + \beta$, where $\alpha_i = -\tau_i\langle \xi, \nu\rangle$ and $\beta = \lambda\langle \xi, \boldsymbol{N}\rangle$, for each $i = 1,\dots,n-1$.

Lemma 5.2.

$$s_r(\gamma) = \sum_{j=0}^{r}\binom{n-1-j}{r-j}\beta^{r-j}s_j(\alpha), \quad 1 \leqslant r \leqslant n-1.$$

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Proof. Recall that $s_r(\gamma)$ can be defined by the following polynomial identity (2.6):

$$\sum_{r=0}^{n-1}(-1)^r s_r(\gamma)t^{n-1-r} = (t-\gamma_1)\cdots(t-\gamma_{n-1}).$$

Since each $\gamma_i = \alpha_i + \beta$, the right-hand side of this equality can be written as follows:

$$((t-\beta)-\alpha_1)\cdots((t-\beta)-\alpha_{n-1}) = \sum_{j=0}^{n-1}(-1)^j s_j(\alpha)(t-\beta)^{n-1-j}.$$

On the other hand, computing the right-hand side of this last equality, we obtain

$$\sum_{j=0}^{n-1}(-1)^j s_j(\alpha)(t-\beta)^{n-1-j} = \sum_{j=0}^{n-1}\sum_{k=0}^{n-1-j}(-1)^{k+j}\binom{n-1-j}{k}\beta^k s_j(\alpha)t^{n-1-k-j},$$

which after a reordering becomes

$$\sum_{r=0}^{n-1}(-1)^r\left(\sum_{j=0}^{r}\binom{n-1-j}{r-j}\beta^{r-j}s_j(\alpha)\right)t^{n-1-r}.$$

Therefore, we have obtained the following equality between polynomials:

$$\sum_{r=0}^{n-1}(-1)^r s_r(\gamma)t^{n-1-r} = \sum_{r=0}^{n-1}(-1)^r\left(\sum_{j=0}^{r}\binom{n-1-j}{r-j}\beta^{r-j}s_j(\alpha)\right)t^{n-1-r},$$

which concludes the proof. □

We summarize the reasoning above in the following result.

Proposition 5.3. *Let $P^n \subset \bar{M}$ be an orientable totally umbilical hypersurface in $\bar{M}$, and let $\Sigma \subset P$ be an orientable $(n-1)$-dimensional compact submanifold in P^n. Let $\psi : M^n \to \bar{M}^{n+1}$ be an orientable hypersurface with boundary $\Sigma = \psi(\partial M)$, and let ν stand for the outward pointing unit conormal vector field along $\partial M \subset M$. Then, along the boundary ∂M and for every $1 \leqslant r \leqslant n-1$, it holds that*

$$\langle T_r\nu,\nu\rangle = \sum_{j=0}^{r}(-1)^j\binom{n-1-j}{r-j}\lambda^{r-j}\langle\xi,\boldsymbol{N}\rangle^{r-j}\langle\xi,\nu\rangle^j s_j. \tag{5.5}$$

Here $s_j = s_j(\tau_1,\dots,\tau_{n-1})$, $0 \leqslant j \leqslant n-1$, are the elementary symmetric functions of $\tau_1,\dots,\tau_{n-1}$, the principal curvatures of $\Sigma \subset P$ with respect to the outward pointing unitary normal, $\boldsymbol{N}$ is the unitary normal field of M, ξ is the unitary normal field of $P \subset \bar{M}$, and λ is the umbilicity factor of $P \subset \bar{M}$ (with respect to ξ).

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6. Transversality versus ellipticity

The relationship between the S_r and the $s_r(\gamma)$ given in (4.6)–(4.8), as well as the expression for $\langle T_r\nu,\nu\rangle$ given in (5.5) becomes specially simple in the case where the inclusion $P \subset \bar{M}$ is *totally geodesic*, that is, when $\lambda = 0$. In that case $\gamma_i = -\tau_i\langle\xi,\nu\rangle$, and we have the following.

Corollary 6.1. *Let Σ be an orientable $(n-1)$-dimensional compact submanifold in an orientable totally geodesic hypersurface $P^n \subset \bar{M}^{n+1}$. Let $\psi : M^n \to \bar{M}^{n+1}$ be an orientable hypersurface with boundary $\Sigma = \psi(\partial M)$, and let ν stand for the outward pointing unit conormal vector field along $\partial M \subset M$. Then, along the boundary ∂M and for every $1 \leqslant r \leqslant n$, it holds that*

$$S_1 = -s_1\langle\xi,\nu\rangle + \langle A\nu,\nu\rangle, \tag{6.1}$$

$$S_2 = s_2\langle\xi,\nu\rangle^2 - s_1\langle\xi,\nu\rangle\langle A\nu,\nu\rangle - \sum_{i=1}^{n-1}\langle A\nu,e_i\rangle^2, \tag{6.2}$$

$$\begin{aligned} S_r = (-1)^r s_r\langle\xi,\nu\rangle^r &+ (-1)^{r-1}s_{r-1}\langle\xi,\nu\rangle^{r-1}\langle A\nu,\nu\rangle \\ &- (-1)^{r-2}\langle\xi,\nu\rangle^{r-2}\sum_{i=1}^{n-1}s_{r-2}(\hat{\tau}_i)\langle A\nu,e_i\rangle^2, \end{aligned} \tag{6.3}$$

for $3 \leqslant r \leqslant n$, and

$$\langle T_r\nu,\nu\rangle = (-1)^r s_r\langle\xi,\nu\rangle^r, \tag{6.4}$$

where $s_n = 0$ and, for every $1 \leqslant r \leqslant n-1$,

$$s_r = s_r(\tau_1,\dots,\tau_{n-1})$$

is the rth elementary symmetric function of $\tau_1,\dots,\tau_{n-1}$, the principal curvatures of $\Sigma \subset P$ with respect to the outward pointing unitary normal, and ξ is the unitary normal field of $P \subset \bar{M}$.

It is not difficult to see that (6.4) establishes a very strong relationship between the *transversality* of M with respect to P along the boundary ∂M, and the *ellipticity* on M of the rth Newton transformation T_r, when $r \geqslant 1$ (recall that $T_0 = I$). That relationship between transversality and ellipticity will actually be one of the keys of the proof of our symmetry results (Theorem 7.1, Theorem 10.1 and Theorem 11.1). In fact, saying that M is not transverse to P along its boundary ∂M means that there exists a point $p \in \partial M$ such that $\langle\xi,\nu\rangle(p) = 0$, which implies from (6.4) that $\langle T_r\nu,\nu\rangle(p) = 0$, $r \geqslant 1$. Therefore, we can conclude that if the Newton transformation T_r is positive definite on M for some $1 \leqslant r \leqslant n-1$, then the hypersurface M is necessarily transverse to P along its boundary.

Observe that in the case where S_n does not vanish on ∂M and $n \geqslant 3$, transversality easily follows from expression (6.3). In fact, by (6.3) we have along the boundary ∂M

$$S_n = (-1)^{n-1}s_{n-1}\langle\xi,\nu\rangle^{n-1}\langle A\nu,\nu\rangle + (-1)^{n-1}\langle\xi,\nu\rangle^{n-2}\sum_{i=1}^{n-1}s_{n-2}(\hat{\tau}_i)\langle A\nu,e_i\rangle^2.$$

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In particular, if there exists a point $p \in \partial M$ where $\langle \xi, \nu \rangle(p) = 0$, then $S_n(p) = 0$ (since $n \geqslant 3$). In the same way, if we assume that $n \geqslant 2$ and S_2 is positive everywhere on ∂M, then (6.2) also implies that M is transverse to P along the boundary.

We summarize the computations above in the following result.

Proposition 6.2. *Let Σ be an orientable $(n-1)$-dimensional compact submanifold in an orientable totally geodesic hypersurface $P^n \subset \bar{M}^{n+1}$ and let $\psi : M^n \to \bar{M}^{n+1}$ be an orientable hypersurface with boundary $\Sigma = \psi(\partial M)$. Then each one of the following hypotheses individually implies that M is transverse to P along the boundary ∂M.*

(i) *For a given $1 \leqslant r \leqslant n-1$, the Newton transformation T_r is definite positive on M.*

(ii) *$n \geqslant 3$ and $S_n \neq 0$ on ∂M.*

(iii) *$S_2 > 0$ on ∂M.*

7. Symmetry for hypersurfaces in Euclidean space

The totally umbilic hypersurfaces of Euclidean space $\mathbb{R}^{n+1}$ are the totally geodesic hyperplanes and the round n-spheres. They trivially have constant r-mean curvature for each $r = 0, \dots, n$. Actually, the hyperplanes have vanishing r-mean curvature $H_r = 0$, and, after an appropriate choice of the unit normal vector field, the round n-spheres of radius $\varrho > 0$ have constant r-mean curvature $H_r = 1/\varrho^r$. Let us fix a hyperplane $\Pi \subset \mathbb{R}^{n+1}$ and an $(n-1)$-sphere $\Sigma \subset \Pi$. Then the hyperplanar round ball bounded by Σ in Π, and the spherical caps bounded by Σ (of radii greater than or equal to the radius of Σ) are examples of compact hypersurfaces embedded into $\mathbb{R}^{n+1}$ with constant r-mean curvature and bounded by Σ. In this context, it was conjectured in [**9**] that these examples are the only compact embedded hypersurfaces in $\mathbb{R}^{n+1}$ with constant mean curvature and spherical boundary. Related to this conjecture we have the following symmetry theorem for hypersurfaces in Euclidean space [**2**].

Theorem 7.1. *Let Σ be a strictly convex compact $(n-1)$-dimensional submanifold in a hyperplane $\Pi \subset \mathbb{R}^{n+1}$, and let $\psi : M^n \to \mathbb{R}^{n+1}$ be a compact embedded hypersurface with boundary Σ. Let us assume that for a given $2 \leqslant r \leqslant n$, the r-mean curvature H_r of M is a non-zero constant. Then M has all the symmetries of Σ. In particular, if the boundary Σ is a round $(n-1)$-sphere of $\mathbb{R}^{n+1}$, then M is a spherical cap.*

Proof. It is not difficult to see that under the hypothesis above there exists at least one interior elliptic point of M, that is, an interior point of M where, after an appropriate orientation of M, all the principal curvatures are positive. In fact, since M is not part of a hyperplane (because of $H_r \neq 0$), then one easily finds a radius $R > 0$ and a point $a \in \mathbb{R}^{n+1}$ such that the closed round ball $\bar{B}(a, R)$ contains M and such that there is a point $p_0 \in \operatorname{int}(M) \cap \partial B(a, R)$ (englobe M with spheres of large radius until such a sphere touches M on one side at an interior point). In particular, in the chosen orientation the constant $H_r = H_r(p_0) > 0$ is positive. The existence of an elliptic point, jointly with the fact that H_r is a positive constant, allows us to conclude that the Newton transformation

T_{r-1} is positive definite on M (see [**6**, Proposition 3.2] and [**25**, p. 232]). Therefore, from Proposition 6.2 it follows that M is transverse to Π along the boundary ∂M. Our result then is a consequence of Theorem 7.3 in [**25**]. □

As a consequence of Theorem 7.1 we can conclude that the conjecture of the spherical cap [**9**] is true for the case of embedded hypersurfaces with constant r-mean curvature in $\mathbb{R}^{n+1}$, when $r \geqslant 2$ [**2**].

Corollary 7.2. *The only compact embedded hypersurfaces in $\mathbb{R}^{n+1}$ with constant r-mean curvature H_r (with $2 \leqslant r \leqslant n$) and spherical boundary are the hyperplanar round balls (with $H_r = 0$) and the spherical caps (with H_r a non-zero constant).*

Indeed, if M is not a hyperplanar round ball, then the constant r-mean curvature must be necessarily non-zero because there exists at least one interior elliptic point of M. In particular, when $r = 2$ saying that H_2 is constant is equivalent to saying that the scalar curvature is constant (see equation (2.7)), so that the result reads as follows.

Corollary 7.3. *The only compact embedded hypersurfaces in $\mathbb{R}^{n+1}$ with constant scalar curvature and spherical boundary are the hyperplanar round balls (with zero scalar curvature) and the spherical caps (with positive constant scalar curvature).*

Our objective in §§ 10 and 11 is to extend the symmetry result given in Theorem 7.1 to the case of hypersurfaces in hyperbolic space and hypersurfaces in sphere, as well as the corresponding solution to the spherical cap conjecture for the case of constant r-mean curvature, $r \geqslant 2$. A result of this type was first given by Nelli and Rosenberg in [**20**, Theorem 3.1] for hypersurfaces with constant mean curvature in hyperbolic space. On the other hand, the corresponding result for the case of hypersurfaces with constant mean curvature in the sphere $\mathbb{S}^{n+1}$ has been recently given by Lira [**11**]. As observed by Nelli and Rosenberg, their result could be extended to the case of constant r-mean curvature as soon as a certain *flux formula* could be established. In the next section, we will derive such a flux formula.

8. A flux formula

In this section we will derive a general flux formula for the geometric configuration considered in § 4 in the case where the Riemannian ambient space $\bar{M}$ is equipped with a conformal vector field $Y \in \mathcal{X}(\bar{M})$. Recall that the fact that Y is conformal means that the Lie derivative of the metric tensor of $\bar{M}$ with respect to Y satisfies

$$\mathcal{L}_Y\langle\cdot\,,\cdot\rangle = 2\phi\langle\cdot\,,\cdot\rangle$$

for a certain smooth function $\phi \in \mathcal{C}^\infty(\bar{M})$. In other words,

$$\langle\bar{\nabla}_V Y, W\rangle + \langle V, \bar{\nabla}_W Y\rangle = 2\phi\langle V, W\rangle, \tag{8.1}$$

for every vector fields $V, W \in \mathcal{X}(\bar{M})$.

In order to derive our general flux formula, let us consider $Y^\top \in \mathcal{X}(M)$ the vector field obtained on the hypersurface M by taking the tangential component of Y, that is,

$Y^\top = Y - f\boldsymbol{N}$, where $f = \langle Y, \boldsymbol{N}\rangle$. Most of the interesting and useful integral formulae in Riemannian geometry are obtained by computing the divergence of certain vector fields and applying the divergence theorem. The interesting integral formulae therefore correspond to vector fields with interesting divergences. Our idea here is to compute the divergence $\operatorname{div}_M(T_rY^\top)$. Using that $\nabla_U T_r$ is self-adjoint for any tangent vector field $U \in \mathcal{X}(M)$, an easy computation shows that

$$\operatorname{div}_M(T_rY^\top) = \langle \operatorname{div}_M T_r, Y\rangle + \sum_{i=1}^n \langle \nabla_{e_i} Y^\top, T_r e_i\rangle, \tag{8.2}$$

where $\{e_1, \dots, e_n\}$ is a local orthonormal frame on M and $\operatorname{div}_M T_r$ is given by (3.3) in Lemma 3.1. From the conformal equation (8.1), we obtain

$$\begin{aligned} 2\phi\langle T_rU, U\rangle &= \langle \bar\nabla_{T_rU} Y, U\rangle + \langle \bar\nabla_U Y, T_rU\rangle \\ &= \langle \bar\nabla_{T_rU} Y^\top, U\rangle + f\langle \bar\nabla_{T_rU}\boldsymbol{N}, U\rangle + \langle \bar\nabla_U Y^\top, T_rU\rangle + f\langle \bar\nabla_U \boldsymbol{N}, T_rU\rangle \\ &= \langle \nabla_{T_rU} Y^\top, U\rangle + \langle \nabla_U Y^\top, T_rU\rangle - f\langle AT_rU, U\rangle - f\langle AU, T_rU\rangle, \end{aligned}$$

that is,

$$\langle \nabla_{T_rU} Y^\top, U\rangle + \langle \nabla_U Y^\top, T_rU\rangle = 2\phi\langle T_rU, U\rangle + 2f\langle AT_rU, U\rangle. \tag{8.3}$$

Choose $\{e_1, \dots, e_n\}$ a local orthonormal frame on M diagonalizing A. We know then that it also diagonalizes T_r with eigenvalues $\mu_{1,r}, \dots, \mu_{n,r}$, and therefore

$$\langle \nabla_{e_i} Y^\top, T_r e_i\rangle = \mu_{i,r}\langle \nabla_{e_i} Y^\top, e_i\rangle = \langle e_i, \nabla_{T_r e_i} Y^\top\rangle,$$

so that from (8.3) we obtain

$$\langle \nabla_{e_i} Y^\top, T_r e_i\rangle = \phi\langle e_i, T_r e_i\rangle + \langle Y, \boldsymbol{N}\rangle\langle AT_r e_i, e_i\rangle.$$

Taking trace here and using (3.1) and (3.2), equation (8.2) becomes

$$\operatorname{div}_M(T_rY^\top) = \langle \operatorname{div}_M T_r, Y\rangle + c_r(\phi H_r + \langle Y, \boldsymbol{N}\rangle H_{r+1}), \tag{8.4}$$

where

$$c_r = (r+1)\binom{n}{r+1}.$$

Integrating now (8.4) on M, the Stokes theorem implies the following integral formula for every $0 \leqslant r \leqslant n-1$:

$$\begin{aligned} \oint_{\partial M} \langle T_r\nu, Y\rangle \, \mathrm{d}s &= \int_M \operatorname{div}_M(T_rY^\top) \, \mathrm{d}M \\ &= \int_M \langle \operatorname{div}_M T_r, Y\rangle \, \mathrm{d}M + c_r \int_M (\phi H_r + \langle Y, \boldsymbol{N}\rangle H_{r+1}) \, \mathrm{d}M. \end{aligned} \tag{8.5}$$

Here $\mathrm{d}M$ denotes the n-dimensional volume element of M with respect to the induced metric and the chosen orientation, and $\mathrm{d}s$ is the $(n-1)$-dimensional volume element induced on ∂M.

On the other hand, let D^n be a compact orientable hypersurface in $\bar{M}$ with smooth boundary that satisfies $\partial D = \partial M$, such that $M \cup D$ is an oriented n-cycle of $\bar{M}$, with D oriented by the unit normal field n_D. We suppose that $M \cup D = \partial \Omega$, where Ω is a compact oriented domain immersed in $\bar{M}$. From the conformal equation (8.1), we easily see that $\operatorname{div}_{\bar{M}} Y = (n+1)\phi$. Therefore, from the Stokes theorem we obtain that

$$\int_M \langle Y, \boldsymbol{N} \rangle \, \mathrm{d}M = -\int_D \langle Y, n_D \rangle \, \mathrm{d}D + (n+1) \int_\Omega \phi \, \mathrm{d}\bar{M}, \tag{8.6}$$

where $\mathrm{d}D$ stands for the n-dimensional volume element of D with respect to the orientation given by n_D, and $\mathrm{d}\bar{M}$ denotes the $(n+1)$-dimensional volume element on $\bar{M}$. Now, from (8.5) and (8.6) we conclude the following general *flux formula*.

Proposition 8.1. *Let $\psi : M^n \to \bar{M}^{n+1}$ be an immersed compact orientable hypersurface with boundary ∂M, and let D^n be a compact orientable hypersurface with boundary $\partial D = \partial M$. Assume that $M \cup D$ is an oriented n-cycle of $\bar{M}$, and let $\boldsymbol{N}$ and n_D be the unit normal fields which orient M and D, respectively. If the r-mean curvature H_r is constant, $1 \leqslant r \leqslant n$, then for every conformal vector field $Y \in \mathcal{X}(\bar{M})$ the following formula holds*

$$\begin{aligned}\oint_{\partial M} \langle T_{r-1}\nu, Y \rangle \, \mathrm{d}s = {}& \int_M \langle \operatorname{div}_M T_{r-1}, Y \rangle \, \mathrm{d}M + r\binom{n}{r} \int_M \phi H_{r-1} \, \mathrm{d}M \\ & - r\binom{n}{r} H_r \int_D \langle Y, n_D \rangle \, \mathrm{d}D + (n+1) r \binom{n}{r} H_r \int_\Omega \phi \, \mathrm{d}\bar{M},\end{aligned} \tag{8.7}$$

where ν is the outward pointing conormal to M along ∂M.

Formula (8.7) becomes especially simple when the ambient space $\bar{M}$ has constant sectional curvature, and the field Y is a Killing vector field, that is, $\phi = 0$. In that case, the Newton transformations are divergence free (Corollary 3.2) and, from formula (8.7), we derive the *balancing formula* given by Rosenberg in [**25**, Theorem 7.2] (see also [**8**, **9**, **16**, **20**] for the case of constant mean curvature).

Corollary 8.2. *If $\bar{M}$ has constant sectional curvature, then for every Killing vector field $Y \in \mathcal{X}(\bar{M})$ the flux formula becomes*

$$\oint_{\partial M} \langle T_{r-1}\nu, Y \rangle \, \mathrm{d}s = -r\binom{n}{r} H_r \int_D \langle Y, n_D \rangle \, \mathrm{d}D, \tag{8.8}$$

where ν is the outward pointing conormal to M along ∂M.

On the other hand, when the ambient space $\bar{M}$ has constant sectional curvature, and the field Y is a homothetic (and non-Killing) vector field, then we may assume without loss of generality that $\phi = 1$ and (8.7) becomes

$$\begin{aligned}\oint_{\partial M} \langle T_{r-1}\nu, Y \rangle \, \mathrm{d}s = {}& -r\binom{n}{r} H_r \int_D \langle Y, n_D \rangle \, \mathrm{d}D \\ & + r\binom{n}{r} \int_M H_{r-1} \, \mathrm{d}M + (n+1) r \binom{n}{r} H_r \operatorname{vol}(\Omega).\end{aligned} \tag{8.9}$$

As a consequence of (8.9) we obtain the following flux formula for r-minimal hypersurfaces.

Proposition 8.3. *Let $\psi : M^n \to \bar{M}^{n+1}$ be a compact orientable hypersurface with boundary ∂M immersed into a Riemannian space of constant sectional curvature. If M is r-minimal in $\bar{M}$, that is, $H_r = 0$, then for every homothetic (non-Killing) vector field $Y \in \mathcal{X}(\bar{M})$ the following formula holds:*

$$\oint_{\partial M} \langle T_{r-1}\nu, Y\rangle \,\mathrm{d}s = r\binom{n}{r}\int_M H_{r-1}\,\mathrm{d}M. \tag{8.10}$$

In particular, for minimal hypersurfaces in Euclidean space with boundary in a round sphere we have the following consequence.

Corollary 8.4. *Let Σ be an orientable $(n-1)$-dimensional compact submanifold in a round sphere $\mathbb{S}^n(\varrho) \subset \mathbb{R}^{n+1}$ of radius ϱ, and let $\psi : M^n \to \mathbb{R}^{n+1}$ be an immersed orientable compact minimal hypersurface with boundary $\Sigma = \psi(\partial M) \subset \mathbb{S}^n(\varrho)$. Then*

$$\mathrm{vol}(M) \leqslant \frac{\varrho}{n}\,\mathrm{vol}(\partial M),$$

and equality holds if and only if M is orthogonal to $\mathbb{S}^n(\varrho)$ along the boundary ∂M.

Proof. Consider the radial vector field $Y(p) = p$ in $\mathbb{R}^{n+1}$, which is a homothetic vector field in $\mathbb{R}^{n+1}$ with $\phi = 1$, and let ξ be the unit vector normal to $\mathbb{S}^n(\varrho)$. Then, along $\mathbb{S}^n(\varrho)$ we have $Y = \varrho\xi$ and (8.10) gives

$$n\int_M \mathrm{d}M = n\,\mathrm{vol}(M) = \oint_{\partial M}\langle \nu, \varrho\xi\rangle\,\mathrm{d}s \leqslant \varrho\oint_{\partial M}\mathrm{d}s = \varrho\,\mathrm{vol}(\partial M).$$

Besides, equality holds if and only if $\xi = \nu$ along the boundary ∂M, or equivalently (see (4.1)) $\langle \boldsymbol{N}, \xi\rangle = 0$ along ∂M. □

Let us now consider the case of a hypersurface immersed into the hyperbolic space $\mathbb{H}^{n+1}$. In that case, it will be appropriate to use the Minkowski space model of hyperbolic space. Write $\mathbb{R}^{n+2}_1$ for $\mathbb{R}^{n+2}$ with the Lorentzian metric

$$\langle\cdot\,,\cdot\rangle_1 = -\mathrm{d}x_0^2 + \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_{n+1}^2.$$

Then

$$\mathbb{H}^{n+1} = \{x \in \mathbb{R}^{n+2}_1 : \langle x, x\rangle_1 = -1,\ x_0 > 0\}$$

is a complete spacelike hypersurface in $\mathbb{R}^{n+2}_1$ with constant sectional curvature -1, which provides the Minkowski space model for the hyperbolic space.

Let $\Sigma \subset \mathbb{H}^{n+1}$ be an orientable $(n-1)$-dimensional compact submanifold in a geodesic sphere $S(a, \varrho)$ of $\mathbb{H}^{n+1}$ of centre $a \in \mathbb{H}^{n+1}$ and geodesic radius ϱ, and let $\psi : M^n \to \mathbb{H}^{n+1}$ be an orientable compact hypersurface with boundary $\Sigma = \psi(\partial M)$.

Consider the vector field in $\mathbb{H}^{n+1}$ represented in this model as $Y(p) = -a - \langle a, p\rangle p$ for every $p \in \mathbb{H}^{n+1}$. Observe that Y is a conformal vector field in $\mathbb{H}^{n+1}$ which is orthogonal to the geodesic spheres centred at the point a, with $\phi(p) = -\langle a, p\rangle = \cosh(\tilde{\varrho}(p))$ and $|Y(p)| = \sinh(\tilde{\varrho}(p))$, where $\tilde{\varrho}(p) = \mathrm{dist}(p, a)$ for every $p \in \mathbb{H}^{n+1}$. Therefore, along $S(a, \varrho)$ we have $Y = \sinh(\varrho)\xi$. Assume now that M is minimal in $\mathbb{H}^{n+1}$. Then, it follows from (8.7) that

$$\oint_{\partial M} \langle \nu, Y\rangle \,\mathrm{d}s = \sinh(\varrho) \oint_{\partial M} \langle \nu, \xi\rangle \,\mathrm{d}s = n \int_M \cosh(\tilde{\varrho}) \,\mathrm{d}M.$$

Thus, since $\cosh(\tilde{\varrho}) \geqslant 1$, we conclude that

$$n \,\mathrm{vol}(M) \leqslant n \int_M \cosh(\tilde{\varrho}) \,\mathrm{d}M = \sinh(\varrho) \oint_{\partial M} \langle \nu, \xi\rangle \,\mathrm{d}s \leqslant \sinh(\varrho) \,\mathrm{vol}(\partial M).$$

Summing up, we have obtained the following result.

Corollary 8.5. *Let Σ be an orientable $(n-1)$-dimensional compact submanifold in a geodesic sphere $S(a, \varrho)$ of $\mathbb{H}^{n+1}$ of centre $a \in \mathbb{H}^{n+1}$ and geodesic radius ϱ, and let $\psi : M^n \to \mathbb{H}^{n+1}$ be an immersed orientable compact minimal hypersurface with boundary $\Sigma = \psi(\partial M) \subset S(a, \varrho)$. Then*

$$\mathrm{vol}(M) \leqslant \frac{\sinh(\varrho)}{n} \mathrm{vol}(\partial M).$$

Finally, let us consider the case of a hypersurface immersed into the sphere $\mathbb{S}^{n+1}$:

$$\mathbb{S}^{n+1} = \{x = (x_0, \dots, x_{n+1}) \in \mathbb{R}^{n+2} : \langle x, x\rangle = 1\}.$$

Let Σ be an orientable $(n-1)$-dimensional compact submanifold in a geodesic sphere $S(a, \varrho)$ of $\mathbb{S}^{n+1}$ of centre $a \in \mathbb{S}^{n+1}$ and radius $\varrho < \frac{1}{2}\pi$, and let $\psi : M^n \to \mathbb{S}^{n+1}$ be an orientable compact hypersurface with boundary $\Sigma = \psi(\partial M) \subset S(a, \varrho)$.

In this case, consider the vector field in $\mathbb{S}^{n+1}$ given by $Y(p) = -a + \langle a, p\rangle p$ for every $p \in \mathbb{S}^{n+1}$, with singularities at the focal points $\{a, -a\}$. Observe that Y is a conformal vector field in $\mathbb{S}^{n+1}$ which is orthogonal to the geodesic spheres centred at the point a, with $\phi(p) = \langle a, p\rangle = \cos(\tilde{\varrho}(p))$ and $|Y(p)| = \sin(\tilde{\varrho}(p))$, where $\tilde{\varrho}(p) = \mathrm{dist}(p, a)$ for every $p \in \mathbb{S}^{n+1}$. Therefore, along $S(a, \varrho)$ we have $Y = \sin(\varrho)\xi$. Assume now that M is minimal in $\mathbb{S}^{n+1}$. Then, it follows from (8.7) that

$$\oint_{\partial M} \langle \nu, Y\rangle \,\mathrm{d}s = \sin(\varrho) \oint_{\partial M} \langle \nu, \xi\rangle \,\mathrm{d}s = n \int_M \cos(\tilde{\varrho}) \,\mathrm{d}M.$$

Let us assume now that M is contained in the open hemisphere centred at a. In that case, it is clear that $\min_M \cos(\tilde{\varrho}) = \cos\varrho_0$, where $\varrho_0 = \max_M \tilde{\varrho} < \frac{1}{2}\pi$, so that

$$n \cos(\varrho_0) \,\mathrm{vol}(M) \leqslant n \int_M \cos(\tilde{\varrho}) \,\mathrm{d}M = \sin(\varrho) \oint_{\partial M} \langle \nu, \xi\rangle \,\mathrm{d}s \leqslant \sin(\varrho) \,\mathrm{vol}(\partial M).$$

This leads to the following result.

Corollary 8.6. *Let Σ be an orientable $(n-1)$-dimensional compact submanifold in a geodesic sphere $S(a,\varrho)$ of $\mathbb{S}^{n+1}$ of centre $a \in \mathbb{S}^{n+1}$ and geodesic radius ϱ, and let $\psi : M^n \to \mathbb{S}^{n+1}$ be an immersed orientable compact minimal hypersurface with boundary $\Sigma = \psi(\partial M) \subset S(a,\varrho)$. Assume that M is contained in the open hemisphere centred at a. Then*

$$\mathrm{vol}(M) \leqslant \frac{\sin(\varrho)}{n\cos(\varrho_0)}\,\mathrm{vol}(\partial M),$$

where $\varrho_0 < \frac{1}{2}\pi$ stands for the maximum over M of the distance to the point a.

9. Estimating the r-mean curvature by the geometry of the boundary

In this section, we will describe an interesting application of our flux formula (8.8) and the formula (6.4). Let us consider the geometric configuration given in Proposition 6.2; that is, let Σ be an orientable $(n-1)$-dimensional compact submanifold in an orientable totally geodesic hypersurface $P^n \subset \bar{M}$, and let $\psi : M^n \to \bar{M}^{n+1}$ be an orientable compact connected hypersurface with boundary $\Sigma = \psi(\partial M)$ and constant r-mean curvature H_r. Our objective here is to estimate H_r by the geometry of the boundary. Assume that there exists a Killing vector field $Y \in \mathcal{X}(\bar{M})$ which is orthogonal to P. Then, we can write Y along the boundary ∂M both as $Y = \langle Y,\xi\rangle\xi$ and also as $Y = \langle Y,\nu\rangle\nu + \langle Y,\boldsymbol{N}\rangle\boldsymbol{N}$, and using (6.4) we obtain

$$\langle T_{r-1}\nu, Y\rangle = \langle Y,\nu\rangle\langle T_{r-1}\nu,\nu\rangle = (-1)^{r-1}s_{r-1}\langle Y,\xi\rangle\langle\xi,\nu\rangle^r$$

along the boundary ∂M.

Let us consider $D \subset P$ the domain in P bounded by Σ, and let us orient D by the unit normal field n_D, so that $M \cup D$ is an oriented n-cycle in $\bar{M}$. Let us denote by h_j the jth mean curvature of $\Sigma \subset P$ with respect to the outward pointing unitary normal η, that is,

$$\binom{n-1}{j}h_j = s_j = s_j(\tau_1,\dots,\tau_{n-1}), \quad 0 \leqslant j \leqslant n-1.$$

In the case where the ambient space $\bar{M}$ has constant sectional curvature, then our flux formula (8.8) allows us to write

$$nH_r\int_D \langle Y,n_D\rangle\,\mathrm{d}D = (-1)^r\oint_{\partial M} h_{r-1}\langle Y,\xi\rangle\langle\xi,\nu\rangle^r\,\mathrm{d}s. \tag{9.1}$$

Let us first apply formula (9.1) to the Euclidean case, $\bar{M} = \mathbb{R}^{n+1}$.

Theorem 9.1. *Let Σ be an orientable $(n-1)$-dimensional compact submanifold in a hyperplane $P \subset \mathbb{R}^{n+1}$, and let $\psi : M^n \to \mathbb{R}^{n+1}$ be an orientable immersed compact (connected) hypersurface with boundary $\Sigma = \psi(\partial M)$ and constant r-mean curvature H_r, $1 \leqslant r \leqslant n$. Then*

$$0 \leqslant |H_r| \leqslant \frac{1}{n\,\mathrm{vol}(D)}\oint_{\partial M}|h_{r-1}|\,\mathrm{d}s, \tag{9.2}$$

where h_{r-1} stands for the $(r-1)$-mean curvature of $\Sigma \subset P$, and D is the domain in P bounded by Σ. In particular, when Σ is a round $(n-1)$-sphere of radius ϱ it follows that

$$0 \leqslant |H_r| \leqslant \frac{1}{\varrho^r}. \tag{9.3}$$

This estimate was first obtained in the case of constant mean curvature ($r = 1$) by Barbosa in [**5**].

Proof. Let ξ be the unit vector normal to P. Then ξ is a constant vector field in $\mathbb{R}^{n+1}$, and therefore $Y = \xi$ is a Killing field in $\mathbb{R}^{n+1}$. On the other hand, we also have that $n_D = \pm\xi$, so that, from (9.1), we obtain

$$n|H_r| \operatorname{vol}(D) = \left| \oint_{\partial M} h_{r-1} \langle \xi, \nu \rangle^r \, \mathrm{d}s \right| \leqslant \oint_{\partial M} |h_{r-1}| \, \mathrm{d}s,$$

which yields (9.2).

In particular, when $\Sigma = \mathbb{S}^{n-1}(\varrho)$ is a round sphere of radius ϱ, then we have that $\tau_i = -1/\varrho$ for every $i = 1, \dots, n-1$, so that $h_{r-1} = (-1)^{r-1}/\varrho^{r-1}$. Besides, the domain D is an n-dimensional round ball of radius ϱ, with volume $n \operatorname{vol}(D) = \varrho \operatorname{vol}(\mathbb{S}^{n-1}(\varrho))$, and the estimate (9.2) becomes (9.3). □

Let us now consider the case of a hypersurface immersed into the hyperbolic space $\mathbb{H}^{n+1}$. As in § 8, it will be appropriate to use the Minkowski space model of $\mathbb{H}^{n+1}$:

$$\mathbb{H}^{n+1} = \{x = (x_0, \dots, x_{n+1}) \in \mathbb{R}^{n+2}_1 : \langle x, x \rangle_1 = -1,\ x_0 > 0\}.$$

We may assume, up to an isometry of $\mathbb{H}^{n+1}$, that the totally geodesic hyperplane P containing Σ is given by

$$P^n = \mathbb{H}^{n+1} \cap \{x \in \mathbb{R}^{n+2}_1 : x_{n+1} = 0\}.$$

In this case, the unit vector normal to P in $\mathbb{H}^{n+1}$ is given by $\xi(p) = e_{n+1} = (0, \dots, 0, 1) \in \mathbb{R}^{n+2}_1$ for every $p \in P$. Observe that, for every arbitrary fixed point $a \in P$, the vector field given by

$$Y(p) = -\langle p, a \rangle e_{n+1} + \langle p, e_{n+1} \rangle a, \quad p \in \mathbb{H}^{n+1},$$

is a Killing vector field on $\mathbb{H}^{n+1}$ which is orthogonal to P, since at every $p \in P$

$$Y(p) = -\langle p, a \rangle e_{n+1} = \cosh(\tilde{\varrho}(p)) \xi(p),$$

where $\tilde{\varrho}(p)$ is the geodesic distance along P between a and p. Let D be the compact domain D bounded by Σ in P, then $n_D = \pm\xi$ and from (9.1) we obtain

$$n|H_r| \int_D \cosh(\tilde{\varrho}) \, \mathrm{d}D = \left| \oint_\Sigma h_{r-1} \cosh(\tilde{\varrho}) \langle \xi, \nu \rangle^r \, \mathrm{d}s \right|. \tag{9.4}$$

Choose $a \in \mathrm{int}(D)$. Then $\min_D \cosh(\tilde{\varrho}) = \cosh(\tilde{\varrho}(a)) = 1$, so that from (9.4) we conclude that

$$n|H_r| \operatorname{vol}(D) \leqslant n|H_r| \int_D \cosh(\tilde{\varrho}) \,\mathrm{d}D \leqslant \oint_{\partial M} |h_{r-1}| \cosh(\tilde{\varrho}) \,\mathrm{d}s$$
$$\leqslant \max_{\Sigma} \cosh(\tilde{\varrho}) \oint_{\partial M} |h_{r-1}| \,\mathrm{d}s. \tag{9.5}$$

In particular, when Σ is a geodesic sphere in P of geodesic radius ϱ and a is chosen to be the geodesic centre of Σ, then $\tilde{\varrho}(p) = \varrho$ at every $p \in \Sigma$, $|h_{r-1}| = \coth^{r-1}(\varrho)$, and (9.5) simply becomes

$$n|H_r| \operatorname{vol}(D) \leqslant \cosh(\varrho) \coth^{r-1}(\varrho) \operatorname{vol}(\Sigma). \tag{9.6}$$

Moreover, in this case D is the geodesic ball in P of radius ϱ centred at a, that is,

$$D = \{p \in P : 1 \leqslant -\langle p, a\rangle < \cosh(\varrho)\},$$

and

$$\Sigma = \partial D = \{p \in P : -\langle p, a\rangle = \cosh(\varrho)\}.$$

Observe then that Σ is in fact a round $(n-1)$-sphere of Euclidean radius $\sinh(\varrho)$, and D is a round n-dimensional ball of Euclidean radius $\sinh(\varrho)$. Therefore, $\operatorname{vol}(\Sigma) = (n/\sinh(\varrho)) \operatorname{vol}(D)$ and (9.6) simplifies to

$$|H_r| \leqslant \coth^r(\varrho).$$

We summarize this as follows.

Theorem 9.2. *Let Σ be an orientable $(n-1)$-dimensional compact submanifold contained in a totally geodesic hyperplane $P \subset \mathbb{H}^{n+1}$, and let $\psi : M^n \to \mathbb{H}^{n+1}$ be an orientable immersed compact connected hypersurface with boundary $\Sigma = \psi(\partial M)$ and constant r-mean curvature H_r, $1 \leqslant r \leqslant n$. Then*

$$0 \leqslant |H_r| \leqslant \frac{C}{n \operatorname{vol}(D)} \oint_{\partial M} |h_{r-1}| \,\mathrm{d}s.$$

Here h_{r-1} stands for the $(r-1)$-mean curvature of $\Sigma \subset P$, D is the domain in P bounded by Σ, and $C = \max_{\Sigma} \cosh(\tilde{\varrho}) \geqslant 1$, where $\tilde{\varrho}(p)$ is the geodesic distance along P between a fixed arbitrary point $a \in \mathrm{int}(D)$ and p. In particular, when Σ is a geodesic sphere in P of geodesic radius ϱ, it follows that

$$0 \leqslant |H_r| \leqslant \coth^r(\varrho).$$

Finally, let us consider the case of a hypersurface immersed into the sphere $\mathbb{S}^{n+1}$:

$$\mathbb{S}^{n+1} = \{x = (x_0, \dots, x_{n+1}) \in \mathbb{R}^{n+2} : \langle x, x\rangle = 1\}.$$

We may assume, up to an isometry of $\mathbb{S}^{n+1}$, that the totally geodesic n-sphere P containing Σ is given by

$$P^n = \mathbb{S}^{n+1} \cap \{x \in \mathbb{R}^{n+2} : x_{n+1} = 0\}.$$

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In this case, the unit vector normal to P in $\mathbb{S}^{n+1}$ is given by $\xi(p) = e_{n+1} = (0, \dots, 0, 1) \in \mathbb{R}^{n+2}$ for every $p \in P$. Observe that, for every arbitrary fixed point $a \in P$, the vector field given by

$$Y(p) = \langle p, a\rangle e_{n+1} - \langle p, e_{n+1}\rangle a, \quad p \in \mathbb{S}^{n+1},$$

is a Killing vector field on $\mathbb{S}^{n+1}$ which is orthogonal to P, since at every $p \in P$

$$Y(p) = \langle p, a\rangle e_{n+1} = \cos(\tilde{\varrho}(p))\xi(p),$$

where $\tilde{\varrho}(p)$ is the geodesic distance along P between a and p. Suppose that Σ is contained in an open hemisphere P_+ of P determined by an equator S of P, and let D be the compact domain D bounded by Σ in P_+. Then $n_D = \pm\xi$ and from (9.1) we obtain

$$n|H_r|\left|\int_D \cos(\tilde{\varrho})\,\mathrm{d}D\right| = \left|\oint_\Sigma h_{r-1}\cos(\tilde{\varrho})\langle\xi,\nu\rangle^r\,\mathrm{d}s\right|. \tag{9.7}$$

Choose $a \in \operatorname{int}(D)$. Since we are assuming that $\Sigma = \partial D$ is contained in the open hemisphere P_+, then $0 \leqslant \tilde{\varrho} < \frac{1}{2}\pi$ on D and $\min_D \cos(\tilde{\varrho}) > 0$, so that from (9.7) we conclude that

$$\begin{aligned} n|H_r|\min_D \cos(\tilde{\varrho})\operatorname{vol}(D) \leqslant n|H_r|\int_D \cos(\tilde{\varrho})\,\mathrm{d}D &\leqslant \oint_{\partial M} |h_{r-1}|\cos(\tilde{\varrho})\,\mathrm{d}s \\ &\leqslant \max_\Sigma \cos(\tilde{\varrho})\oint_{\partial M} |h_{r-1}|\,\mathrm{d}s. \end{aligned} \tag{9.8}$$

In particular, when Σ is a geodesic sphere in P of geodesic radius $\varrho < \frac{1}{2}\pi$ and a is chosen to be the geodesic centre of Σ, then $\tilde{\varrho}(p) = \varrho$ at every $p \in \Sigma$, $|h_{r-1}| = \cot^{r-1}(\varrho)$. Now a computation similar to that in hyperbolic space leads us from (9.8) to

$$|H_r| \leqslant \cot^r(\varrho),$$

because, in this case Σ is in fact a round $(n-1)$-sphere of Euclidean radius $\sin(\varrho)$, and D is a round n-dimensional ball of Euclidean radius $\sin(\varrho)$. Summing up, we can state the following result.

Theorem 9.3. *Let Σ be an orientable $(n-1)$-dimensional compact submanifold contained in an open totally geodesic hemisphere $P_+ \subset \mathbb{S}^{n+1}$, and let $\psi : M^n \to \mathbb{S}^{n+1}$ be an orientable immersed compact connected hypersurface with boundary $\Sigma = \psi(\partial M)$ and constant r-mean curvature H_r, $1 \leqslant r \leqslant n$. Then*

$$0 \leqslant |H_r| \leqslant \frac{C}{n\operatorname{vol}(D)}\oint_{\partial M} |h_{r-1}|\,\mathrm{d}s.$$

Here h_{r-1} stands for the $(r-1)$-mean curvature of $\Sigma \subset P$, D is the domain in P_+ bounded by Σ, and $C = \max_\Sigma \cos(\tilde{\varrho})/\min_D \cos(\tilde{\varrho})$, where $\tilde{\varrho}(p)$ is the geodesic distance along P_+ between a fixed arbitrary point $a \in \operatorname{int}(D)$ and p. In particular, when Σ is a geodesic sphere in P_+ of geodesic radius $\varrho < \frac{1}{2}\pi$, it follows that

$$0 \leqslant |H_r| \leqslant \cot^r(\varrho).$$

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10. Symmetry for hypersurfaces in hyperbolic space

Hyperbolic space is rich in totally umbilic hypersurfaces. Besides the totally geodesic hyperplanes, there are the horospheres, the hyperspheres and the equidistant hypersurfaces. In all of them, the second fundamental form is proportional to the metric by a constant factor, and therefore they all have constant r-mean curvature, for $1 \leqslant r \leqslant n$. After an appropriate choice of the unit normal vector field, hyperspheres have r-mean curvature bigger than 1, horospheres have r-mean curvature 1, and equidistant hypersurfaces have r-mean curvature in the interval $(0, 1)$.

Let us fix a totally geodesic hyperplane $P^n \subset \mathbb{H}^{n+1}$ and a geodesic sphere $\Sigma^{n-1} \subset P^n$ in $\mathbb{H}^{n+1}$. Then each of the totally umbilic hypersurfaces above contains at least a compact domain M^n with boundary being the sphere Σ. Those examples are called the *spherical caps* in hyperbolic space. That terminology is due to the fact that, working in the half-space model of hyperbolic space, after an appropriate isometry of $\mathbb{H}^{n+1}$, the totally umbilic hypersurfaces above are given as intersections of $\mathbb{H}^{n+1}$ with Euclidean spheres in $\mathbb{R}^{n+1}$. Because of the existence of these examples in $\mathbb{H}^{n+1}$, it is natural to consider the *conjecture of the spherical cap* in hyperbolic space.

In this context, the corresponding result analogous to our Theorem 7.1 for the case of hypersurfaces in hyperbolic space can be stated as follows.

Theorem 10.1. *Let Σ^{n-1} be a strictly convex compact $(n-1)$-dimensional (connected) submanifold of a totally geodesic hyperplane $P^n \subset \mathbb{H}^{n+1}$, and let $M^n \subset \mathbb{H}^{n+1}$ be a compact (connected) embedded hypersurface with boundary Σ. Let us assume that for a given $2 \leqslant r \leqslant n$, the r-mean curvature H_r of M is a non-zero constant. Then M has all the symmetries of Σ. In particular, when the boundary Σ is a geodesic sphere in $P^n \subset \mathbb{H}^{n+1}$, then M is a spherical cap.*

As a consequence of Theorem 10.1 we can conclude, as in the Euclidean case, that the conjecture of the spherical cap is true for the case of embedded hypersurfaces with constant r-mean curvature in hyperbolic space, when $r \geqslant 2$.

Corollary 10.2. *The only compact embedded hypersurfaces in $\mathbb{H}^{n+1}$ with constant r-mean curvature H_r (with $2 \leqslant r \leqslant n$) and spherical boundary are*

(i) *the geodesic balls of a totally geodesic hyperplane* (*with* $H_r = 0$);

(ii) *the geodesic balls of an equidistant hypersurface* (*with* $0 < |H_r| < 1$);

(iii) *the geodesic balls of a horosphere* (*with* $|H_r| = 1$);

(iv) *the geodesic balls of a hypersphere* (*with* $|H_r| > 1$).

Proof of Theorem 10.1. Let us work in the half-space model of hyperbolic space. We may assume, up to an isometry of $\mathbb{H}^{n+1}$, that the totally geodesic hyperplane P is given by

$$P = \{x = (x_1, \ldots, x_{n+1}) \in \mathbb{H}^{n+1};\ |x| = 1,\ x_{n+1} > 0\}. \tag{10.1}$$

Let $\mathcal{B}$ be the connected component of $\mathbb{H}^{n+1} \backslash P$ containing the point $(0, \ldots, 0, 2) \in \mathbb{H}^{n+1}$. We will first see that there exists an interior elliptic point, that is, a point $p_0 \in \operatorname{int}(M)$

where all the principal curvatures of M are positive (after an appropriate orientation of M). In fact, since H_r is a non-zero constant, M cannot be entirely contained in P. After an inversion with centre $(0,\dots,0)\in\mathbb{R}^{n+1}$ which fixes P (an isometry of $\mathbb{H}^{n+1}$), if necessary, we may assume that $M\cap\mathcal{B}\neq\emptyset$. Let $C\subset P$ be the geodesic sphere in P given as the boundary of a geodesic ball in P centred at the point $(0,\dots,0,1)$ and containing Σ. Let us consider $\Gamma^\varepsilon\subset\mathbb{H}^{n+1}$ the equidistant sphere with centre on the vertical geodesic through the centre of C such that $\Gamma^\varepsilon\cap P=C$, and such that the exterior angle between Γ^ε and the asymptotic boundary of $\mathbb{H}^{n+1}$ is $\frac{1}{2}\pi-\varepsilon>0$. Since $\Gamma^\varepsilon\to P$ as $\varepsilon\to 0$, and taking into account that $M\cap\mathcal{B}\neq\emptyset$, we may choose $\varepsilon>0$ such that $\Gamma^\varepsilon\cap M\neq\emptyset$. Besides, since Σ is contained in the geodesic ball in P bounded by C, then the points in $\Gamma^\varepsilon\cap M$ are interior points of M. Now, for every $t\geqslant 0$, let us consider $\Gamma^\varepsilon_t\subset\mathbb{H}^{n+1}$ the equidistant sphere in $\mathbb{H}^{n+1}$ obtained from Γ^ε by an homothety centred at $(0,\dots,0)\in\mathbb{R}^{n+1}$ (which is also an isometry of $\mathbb{H}^{n+1}$), and let us define $\Gamma^\varepsilon_0=\Gamma^\varepsilon$. If t is large enough, then M is contained in the interior of the domain enclosed by Γ^ε_t; thus, we may find $t_0>0$ such that M is tangent to $\Gamma^\varepsilon_{t_0}$ at a point p_0, which is necessarily an interior point of M. Finally, it is easy to conclude that the normal curvatures of M at p_0, with respect to the normal direction of the mean curvature vector of $\Gamma^\varepsilon_{t_0}$, are greater than or equal to those of $\Gamma^\varepsilon_{t_0}$, which are positive. In particular, choosing the appropriate orientation of M, all the principal curvatures of M at p_0 are positive.

Therefore, we may assume that $H_r=H_r(p_0)$ is a positive constant. This implies that for every $1\leqslant j\leqslant r-1$, the Newton transformation T_j is positive definite on M (see [**6**, Proposition 3.2]), and in particular the mean curvature is positive on M, so that we may assume that M is oriented by the mean curvature vector field. From Proposition 6.2 we know that M is transverse to P along the boundary ∂M. This implies that, in a neighbourhood of the boundary ∂M, M is contained in one of the two connected components of $\mathbb{H}^{n+1}\backslash P$, which, without loss of generality, can be assumed to be $\mathcal{B}$. Beside, we may also assume that M is globally transverse to P.

In this situation, we will prove that M is above P, that is, $M\subset\bar{\mathcal{B}}$. Let us consider $\tilde M$ the connected component of $M\cap\bar{\mathcal{B}}$ containing Σ. Then, $\tilde M$ is a compact embedded hypersurface in $\mathbb{H}^{n+1}$ with boundary $\partial\tilde M$ contained in P. If the boundary $\partial\tilde M$ were connected, then $\tilde M=M$ and there is nothing to prove. Our objective is to show that actually $\partial\tilde M$ must be connected. We will prove it by showing that assuming that $\partial\tilde M$ is not connected yields a contradiction.

Thus, let us assume that the boundary $\partial\tilde M$ consists of a finite number of disjoint connected compact embedded $(n-1)$-dimensional submanifolds $\Sigma_i\subset P$ $(0\leqslant i\leqslant k)$, with $\Sigma_0=\Sigma$. We orient this configuration as in § 4, with $\tilde M$ oriented by the mean curvature vector of M. Let ν be the outward pointing conormal to $\tilde M$ along each connected component of $\partial\tilde M$. Then, the mean curvature vector of M, together with ν, allows us to orient each Σ_i. Let η be the unitary vector field normal to Σ in P which points outward with respect to the domain D bounded by Σ in P, and let ξ be the unique unitary vector field normal to P which is compatible with η and with the orientation of Σ. Now, there exists a unique choice for the unitary vector field η_i normal to Σ_i in P which is compatible with the orientation of Σ_i and with the orientation of P given by ξ. We remark that we cannot ensure here that, for $i \geqslant 1$, η_i points outward to the domain D_i bounded by Σ_i in P. In this way, we have that formula (6.2) holds at each point $p \in \partial\tilde{M}$ with $r = 1$, giving

$$\langle T_1\nu, \nu\rangle(p) = -s_1(p)\langle \xi, \nu\rangle(p). \tag{10.2}$$

Here s_1 denotes the trace of the shape operator, with respect to η_i, of the inclusion $\Sigma_i \subset P$ which contains the point p.

As Σ is a compact strictly convex submanifold of P and η points outward of D, then $s_1 < 0$ on Σ. On the other hand, as T_1 is positive definite on M, it follows from (10.2) that $\langle \xi, \nu\rangle > 0$ on ∂M. Besides $\tilde{M} \subset \mathcal{B}$ implies that $\langle \xi, \nu\rangle > 0$ on each component of $\partial\tilde{M}$. Hence, along Σ, the mean curvature vector of M points into D. Therefore, if $\partial\tilde{M}$ has a connected component contained in the interior of D, then there exists at least one component Σ_i, for some $i \geqslant 1$, contained in the interior of D on which the mean curvature vector of M points outward to the domain $D_i \subset P$ bounded by Σ_i in P. As $\langle \xi, \nu\rangle > 0$ on Σ_i, then η_i must point into D_i. This contradicts the formula (10.2), because if η_i points into D_i, then we can easily conclude from the compactness of Σ_i that there must be a point $p \in \Sigma_i$ where $s_1(p) > 0$. It then follows that the connected components of $\partial\tilde{M}$ must be all contained in $P\backslash D$.

Now, let us assume that there exists one of them, say Σ_j ($j \geqslant 1$), which is homotopic to Σ in $P\backslash D$. Without loss of generality, we may assume that, between Σ_j and Σ there is no other component of $\partial\tilde{M}$ which is homotopic to Σ in $P\backslash D$. We showed above that, along Σ, the mean curvature vector of M points into D. Therefore, along Σ_j, the mean curvature vector of M must point outward of the domain $D_j \subset P$ bounded by Σ_j in P. Since $\langle \xi, \nu\rangle > 0$ on Σ_j, it then follows that the unitary vector field η_j normal to Σ_j in P points into D_j. This situation gives again a contradiction with formula (10.2), because if η_j points into D_j, then there must be a point $p \in \Sigma_j$ where $s_1(p) > 0$.

Finally, the only remaining case is the one where $\partial\tilde{M}$ has a connected component Σ_l ($l \geqslant 1$) which is contained in $P\backslash D$ and is null homotopic in $P\backslash D$. However, this final possibility is discarded by using the Alexandrov reflection technique [**1**], exactly as in the proof of [**9**, Theorem 1] or [**20**, Theorem 3.1]. For the sake of completeness, we will include the argument here. Let γ be an infinite length geodesic in P starting at a point of D and intersecting Σ_l in at least two points.

Consider a family $Q(t)$, $t < \infty$, of geodesic hyperplanes of $\mathbb{H}^{n+1}$ orthogonal to γ, such that for each $q \in \gamma$ there exists exactly one $Q(t)$ which intersects γ orthogonally at q. Each $Q(t)$ is orthogonal to P, so a hyperbolic symmetry through $Q(t)$ leaves P and $\mathcal{B}$ invariant. Now we apply Alexandrov reflection method to M (observe that this can be done because the equation $H_r = \text{const.} > 0$, under the existence of an elliptic point, is an elliptic equation [**15**]). For t large enough, $Q(t)$ is disjoint from M. As t decreases, there must exist a first point of contact of some $Q(t)$ with M. One continues to decrease t and considers the symmetries of M through the geodesic hyperplanes $Q(t)$. Since γ intersects Σ_l in at least two points, there must exist some hyperplane $Q(t_0)$ such that the symmetry of M through $Q(t_0)$ will touch M at an interior point. This occurs at an interior point since Σ is convex and γ intersects Σ exactly at one point. Thus, M is invariant under symmetry through $Q(t_0)$, which is impossible (for M would then be part of an embedded closed manifold with constant r-mean curvature, hence, a sphere; but a sphere cannot meet P in more that one component).

Summing up, we conclude from the reasoning above that $\partial\tilde{M}$ has no other connected component on P except of Σ, and therefore $M \subset \bar{\mathcal{B}}$. Now that we know that M is above P and transverse to P along ∂M, the proof finishes applying again the Alexandrov reflection method to $M \cup D$, exactly as in the final step of the proof of [**20**, Theorem 2.1]. $\square$

11. Symmetry for hypersurfaces in sphere

The totally umbilic hypersurfaces of $\mathbb{S}^{n+1}$ are given by the intersections of $\mathbb{S}^{n+1}$ with the hyperplanes of Euclidean space $\mathbb{R}^{n+2}$. When the hyperplane passes through the origin of $\mathbb{R}^{n+2}$, they are totally geodesic, and when the hyperplane is an affine hyperplane, they are totally umbilic. We will refer to them as totally geodesic n-spheres and totally umbilic n-spheres of $\mathbb{S}^{n+1}$, respectively. They all have constant r-mean curvature. After an appropriate choice of the unit normal vector field, the totally umbilic n-spheres have r-mean curvature $H_r = \cot^r(\varrho)$, where $\varrho > 0$ denotes the geodesic radius of the convex geodesic ball of $\mathbb{S}^{n+1}$ whose boundary is the totally umbilic n-sphere.

Let us fix a totally geodesic n-sphere $P^n \subset \mathbb{S}^{n+1}$ and a geodesic sphere $\Sigma^{n-1} \subset P^n$ in $\mathbb{S}^{n+1}$. Then, for a given value of H_r, there are two compact domains M_1^n and M_2^n of a totally umbilic n-sphere of $\mathbb{S}^{n+1}$ whose boundaries are the geodesic sphere Σ. These examples are called the *spherical caps* in $\mathbb{S}^{n+1}$. As in hyperbolic space, because of the existence of these examples in $\mathbb{S}^{n+1}$, it is also natural to consider the *conjecture of the spherical cap* in $\mathbb{S}^{n+1}$. In this context, the corresponding result for the case of hypersurfaces in $\mathbb{S}^{n+1}$ can be stated as follows.

Theorem 11.1. *Let Σ^{n-1} be a convex $(n-1)$-dimensional submanifold of a totally geodesic n-sphere $P^n \subset \mathbb{S}^{n+1}$, and let $M^n \subset \mathbb{S}^{n+1}$ be a compact (connected) embedded hypersurface with boundary Σ. Let us assume that M is contained in an open hemisphere $\mathbb{S}_+^{n+1}$, and that the r-mean curvature H_r of M is a non-zero constant, for a given $2 \leqslant r \leqslant n$. Suppose that the convex disc D bounded by Σ in P contains a focal point of $P_1 \cap P$, where $P_1 = \partial\mathbb{S}_+^{n+1}$. Then M has all the symmetries of Σ. In particular, when the boundary Σ is a geodesic sphere in $P^n \subset \mathbb{S}^{n+1}$, then M is a spherical cap.*

Corollary 11.2. *Let M be a compact (connected) embedded hypersurface in $\mathbb{S}_+^{n+1}$ with constant r-mean curvature $H_r \neq 0$ (with $2 \leqslant r \leqslant n$) and spherical boundary contained in a totally geodesic n-sphere $P^n \subset \mathbb{S}^{n+1}$. Suppose that the convex disc D bounded by the spherical boundary of M in P contains a focal point of $P_1 \cap P$, where $P_1 = \partial\mathbb{S}_+^{n+1}$. Then M is a spherical cap.*

As we pointed out before, corresponding results for $r = 1$ can be found in [**11**].

Before we go further, we need to fix a suitable notion of symmetry in the spherical space form. This is done in the definition below.

Definition 11.3. We say that a totally geodesic n-sphere Q is a *n-sphere of symmetry* of a subset S of $\mathbb{S}^{n+1}$ if for each point $p \in S$ and any complete geodesic γ perpendicular to Q

and containing p, we have $\tilde{p} \in S$, where $\tilde{p}$ is the point of γ such that p and $\tilde{p}$ lie in opposite hemispheres of Q at a distance of less than or equal to $\frac{1}{2}\pi$ and $\operatorname{dist}(\tilde{p}, Q) = \operatorname{dist}(p, Q)$.

We observe that the choice of D in this section is compatible with the orientations established in § 4, which allows us to use the calculations made in the earlier parts of the article.

Proof. Let $a \in \mathbb{S}^{n+1}$ and consider $P_1 = \{x \in \mathbb{S}^{n+1} : \langle x, a \rangle = 0\}$ the totally geodesic n-sphere which defines the open hemisphere $\mathbb{S}^{n+1}_+ = \{x \in \mathbb{S}^{n+1} : \langle x, a \rangle > 0\}$ where M is contained. Now, we may assume without loss of generality that the totally geodesic n-sphere containing the boundary of M is

$$P = \{x \in \mathbb{S}^{n+1} : \langle x, e_0 \rangle = 0\}, \quad a \neq e_0.$$

Our first objective is to see that there exists an interior elliptic point of M, that is, a point $p_0 \in \operatorname{int}(M)$ where all the principal curvatures of M have the same sign. To see it, let $B_t(a) \subset \mathbb{S}^{n+1}_+$ be the geodesic ball with centre a and geodesic radius t, where $0 < t < \frac{1}{2}\pi$, and let $S_t(a) = \partial B_t(a)$ be the corresponding geodesic sphere. Since M is compact and $M \subset \mathbb{S}^{n+1}_+$, there exists a minimum value t' such that $M \subset \overline{B_{t'}(a)}$, and a contact point $p_0 \in M \cap S_t(a)$. Observe that the height function $\langle x, a \rangle$ on M attains its minimum value precisely at that contact point. Therefore, if such a contact point is an interior point of M, then it is also a tangency point and all the principal curvatures of M, with respect to the unit normal vector field of $S_{t'}(a)$, are positive at p_0. If the contact point is a boundary point, then we can consider a geodesic ball $B_t(a)$ with $t > t'$ so that $B_t(a) \cap \Sigma = \emptyset$. Now, we can simultaneously move the centre a of the geodesic ball and decrease its radius, keeping always M contained in the interior of this geodesic ball, and we consider the intersection of this geodesic ball with $\mathbb{S}^{n+1}_+$. From this process it follows that either some geodesic ball $B_t(a') \cap \mathbb{S}^{n+1}_+$ is tangent to M at an interior point, or M is entirely contained in the totally geodesic n-sphere P. However, the second possibility cannot happen because H_r is a non-zero constant. Then, reasoning as above, such an interior tangency point is an elliptic point of M. Thus, we may always (including when r is even) assume that the r-mean curvature H_r of M is a positive constant. This implies that T_j is positive definite on M, for each $1 \leqslant j \leqslant r-1$, and, since $H = H_1 > 0$, we may orient M by the mean curvature vector. By Proposition 6.2, we conclude that M is transversal to P along its boundary ∂M. So, there is a neighbourhood $\mathcal{U}$ of Σ in M contained in only one of the hemispheres $\bar{P}^+$ and $\bar{P}^-$ determined by P. We fix $\mathcal{U} \subset \bar{P}^+$.

Let $D \subset P$ be the domain bounded by Σ which does not contain points of $\Sigma_1 := P_1 \cap P$. Denote by $\operatorname{int}(D)$ the interior of D in P and by $\operatorname{ext}(D)$ the subset $P - D$. According to this notation, we have the following claim.

Claim 11.4. *If $M \cap \operatorname{int}(D) \neq \emptyset$, then $M \cap \operatorname{ext}(D) \neq \emptyset$.*

We omit the proof of this claim since it follows the same guidelines as the similar one for the hyperbolic case (see § 10).

To guarantee that $M \cap P = \Sigma$ it suffices then, by using the claim, to prove that $M \cap \operatorname{ext}(D) = \emptyset$. Suppose otherwise; that is, suppose that $M \cap \operatorname{ext}(D) \neq \emptyset$. So, we may assume, without loss of generality, that $M \cap \text{ext}(D)$ consists of a finite number of disjoint connected embedded submanifolds of P.

In order to apply correctly the reflection procedure, we consider now on M the connected component containing Σ of the topological embedded submanifold N obtained after excision of small annuli in M surrounding each one of the domains D_i bounded by the components of $\Sigma \cap \text{int}\, D$ containing no points of Σ and gluing domains homeomorphic to each D_i at the boundary of these annuli (see [**9**] for the similar device in $\mathbb{R}^3$). This construction allows us to consider $M \cup D$ separating $\mathbb{S}^{n+1}$ in two connected components. By Ω we denote the component that contains no points of Σ_1. Note that the set $M \cup \text{ext}(D)$ is not diminished in this process. In fact, the only components of $M \cup \text{ext}(D)$ that could be discarded are the ones that contains points in $\text{int}(D)$ and points in $\text{ext}(D)$, whose existence should oblige the mean curvature vector to point outside Ω, contradicting the Maximum Principle applied to geodesic graphs in $\mathbb{S}^{n+1}$ (see [**12**]).

Case 1. Suppose initially that there exist components Σ_k of $M \cap \text{ext}(D)$ homologous to zero in $P - \text{int}(D)$. For each k, denote by M_k the connected component of M which has boundary Σ_k and contains no points of Σ. We note that M_k contains points of $\bar{P}^-$ in a neighbourhood of Σ_k.

We fix $\Sigma_1 = \{x = (0, 0, x_2, \dots, x_{n+1}) \in \mathbb{S}^{n+1}\}$ and $x_1 > 0$ throughout the hemisphere of P which contains no points of Σ. Define $P_1(t)$, $0 \leqslant t \leqslant \pi$, as the family of totally geodesic n-spheres such that $P_1(t) \cap P = \Sigma_1$, for all t, and $P_1(\alpha) = P_1$, where α is the angle between P_1 and P. The normal vector to $P_1(t)$ is given by $n_t = (\cos t, -\sin t, \dots, 0, 0)$.

For t, $0 \leqslant t < \alpha$, let $M_k^-(t)$ be the set $\{x \in M; \langle x, n_\alpha\rangle > 0 \text{ and } \langle x, n_t\rangle < 0\}$ and let $\tilde{M}_k(t)$ be the reflected image of $M_k^-(t)$ through $P_1(t)$, i.e.

$$\tilde{M}_k(t) = \{\tilde{x} \in \mathbb{S}^{n+1};\ \tilde{x} = x - 2\langle x, n_t\rangle n_t,\ x \in M_k^-(t)\}.$$

By the fact that $P_1(\alpha) \cap M_k = \emptyset$, there exists $t_0, 0 \leqslant t_0 < \alpha$, such that

(i) $P_1(t_0) \cap M_k \neq \emptyset$;

(ii) $P_1(t) \cap M_k = \emptyset$, for all $t > t_0$.

So, M_k is tangent to $P_1(t_0)$ at their common points and there is a neighbourhood of each one of these points in M_k which is a geodesic graph over a domain in $P_1(t_0)$.

Thus, unless $t_0 = 0$, we have that $\tilde{M}_k(t) \subset \bar{P}^-$ for t sufficiently close to t_0. However, for $t_0 = 0$, we consider a rotation of P by a small angle, fixing Σ_1, to return to the previous situation.

We claim that M_k is a geodesic graph over the domain D_k in P bounded by Σ_k which contains no points of Σ_1, with $\tilde{M}_k(0) \subset \text{int}(\Omega)$. In particular, M_k is not perpendicular to P at points of Σ_k. Otherwise, for some k and $t_1 \in [0, t_0)$, one of the following possibilities occurs:

(i) $\tilde{M}_k(t_1) \cap M$ contains interior points of M_k;

(ii) $P_1(t_1)$ is perpendicular to M_k at points of $\partial M_k(t_1)$;

(iii) $\tilde{M}_k(t_1) \cap M$ contains points of $(M - \Sigma) - M_k$;

(iv) $\tilde{M}_k(t_1)$ contains points of Σ.

Cases (i)–(iii) are all ruled out by the Maximum Principle. In fact, otherwise, $P_1(t_1)$ should be a sphere of symmetry and M a compact hypersurface without boundary (see [**14**, pp. 572–573]).

Thus, we conclude that the points of $\tilde{M}_k(t_1)$ away from Σ are contained in $\operatorname{int}(\Omega)$. Since there exists a neighbourhood $\mathcal{U}$ of Σ in M as above, the reflected image of $\tilde{M}_k(t_1)$ through $P = P(\pi)$ is not contained in Ω, if we suppose $\tilde{M}_k(t_1) \cap \Sigma \neq \emptyset$. In particular, the reflection of $\tilde{M}_k(t_1)$ through P is not contained in the open domain bounded by $\tilde{M}_k(t_1)$ in $\operatorname{int}(\Omega)$. Therefore, a sphere $P_1(\tau)$, $\alpha < \tau < \pi$, should exist such that the reflected image of $\tilde{M}_k(t_1)$ through $P_1(\tau)$ is tangent to $\tilde{M}_k(t_1)$ and, in this way, $P_1(\tau)$ is sphere of symmetry of $\tilde{M}_k(t_1)$. Hence, since the portion of $\tilde{M}_k(t_1)$ lying between $P_1(\alpha)$ and $P_1(\tau)$ does not contain points of $\partial\tilde{M}_k(t_1) = \partial M_k(t_1)$ in $P_1(t_1)$, we obtain a contradiction, proving that case (iv) does not occur at any instant $t \in [0, t_0)$.

Notice that it is equally impossible to have $\tilde{M}_k(t_1)$ tangent to M at points of opposite orientation, because if it is the case, then the reflected image of a portion of M_k would have left $\operatorname{int}(\Omega)$ before t_1.

So, we conclude from the impossibility of cases (i)–(iv), for each $t \in [0, t_0)$, that $\overline{M_k(t)}$ is a geodesic graph over the domain in $P_1(t)$ bounded by $\partial M_k^-(t)$ which does not contain points of Σ_1, with $\tilde{M}_k(t) \subset \operatorname{int}(\Omega)$, proving the claim. Besides this, we guarantee that $M_k^-(t)$ is not perpendicular to $P_1(t)$ at any point of $\partial M_k^-(t)$.

Case 2. Now, suppose there exist components of $M \cap \operatorname{ext}(D)$ homologous to Σ. We will prove that whenever exist such components, they are graphs over domains in P.

This case is handled as in [**11**], with minor modifications concerning the use of the flux formula there, which must be changed by the appropriate formula (8.8).

Now, as above, define for $t \in (\alpha, \frac{1}{2}\pi]$ the submanifold of M given by $M^-(t) = \{x \in P^+;\ \langle x, n_\alpha\rangle > 0 \text{ and } \langle x, n_t\rangle < 0\}$ and its reflected image through $P_1(t)$ as

$$\tilde{M}(t) = \{\tilde{x} \in \mathbb{S}^{n+1};\ \tilde{x} = x - 2\langle x, n_t\rangle n_t,\ x \in M^-(t)\}.$$

Since $M \cap P_1 = \emptyset$, either M is contained in the open hemisphere determined by $P_1(\frac{1}{2}\pi)$ containing Σ, or there exists a $t_0 \in (\alpha, \frac{1}{2}\pi]$ such that

(i) $P_1(t_0) \cap M \neq \emptyset$;

(ii) $P_1(t) \cap M = \emptyset$, for all $\alpha < t < t_0$.

For $t_0 = \frac{1}{2}\pi$, there is a neighbourhood of M that is a graph over a domain of $P_1(\frac{1}{2}\pi)$. If $t_0 < \frac{1}{2}\pi$, suppose that there exists $t_1 \in (t_0, \frac{1}{2}\pi]$ for which one of the following statements hold:

(i) $\tilde{M}(t_1)$ is tangent to M at interior points;

(ii) $P_1(t_1)$ is perpendicular to M at some points of $M \cap P_1(t_1)$;

(iii) $\tilde{M}(t_1) \cap \Sigma \neq \emptyset$.

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If (i) or (ii) occurs, then $P_1(t_1)$ is a sphere of symmetry of M. However, Σ is contained in only one of the hemispheres determined by $\Sigma_1 = P_1(t_1) \cap P$ on P.

Suppose (iii) occurs; if there exists $p \in M^-(t_1)$ such that $\tilde{p} \in \Sigma$, then p and $\tilde{p}$ are points at the same distance from $P_1(t_1)$ on a geodesic Σ perpendicular to $P_1(t_1)$. If $t_1 = \frac{1}{2}\pi$, then $p \in P$, since P is totally geodesic. If $t_1 < \frac{1}{2}\pi$, we have $\operatorname{dist}(\tilde{p}, P_1(t_1)) < \frac{1}{2}\pi$; thus, $\operatorname{dist}(p, \tilde{p}) < 2t_1 < \pi$, which implies that $p \in P^-$. Both situations contradict the fact that $M^-(t_1) \subset P^+$.

We conclude that $\tilde{M}(t_1) \subset \operatorname{int}(\Omega)$, for all $t \in (\alpha, \frac{1}{2}\pi]$. Furthermore, $\overline{M^-(\frac{1}{2}\pi)}$ is a geodesic graph over the domain bounded by $\partial M^-(\frac{1}{2}\pi)$ in $P_1(\frac{1}{2}\pi)$ containing points of Ω.

Let $p \in D$ the geodesic centre of Σ_1 and σ an arc of geodesic starting from p passing through Σ and crossing orthogonally Σ_1. We may assume, initially, that the component $\Sigma_{k_0} = \partial M_{k_0}$ of $M \cap \operatorname{ext}(D)$ nearest from Σ_1 in the direction given by σ is homologous to zero. Modifying slightly the direction of σ, if necessary, we may assume that σ crosses Σ_{k_0} at least twice.

For each point $\sigma(t)$, $0 \leqslant t \leqslant d$, we consider the intersection $Q(t)$ of $\mathbb{S}^{n+1}$ and the Euclidean hyperplane containing the origin of $\mathbb{R}^{n+2}$ and perpendicular to $\{x_0 = 0\}$ whose normal vector is $\sigma'(t)$. Denote by $Q^-(t)$ the hemisphere determined by $Q(t)$ containing $\sigma[t, d]$ and by $\mathcal{Q}_t$ the reflection through $Q(t)$.

As we have proved, the portion of M in the hemisphere $Q^-(d)$ determined by $Q(d) = P_1(\frac{1}{2}\pi)$, if it is not empty, is a geodesic graph over a domain in $P_1(\frac{1}{2}\pi) \cap P^+$ at a distance of less than $\frac{1}{2}\pi$ from the sphere $P_1(\frac{1}{2}\pi)$. This remains true, for t sufficiently close to d. By the choice of Σ_{k_0}, we have that the first point of contact, if exists, between the planes $Q(t)$ and $M \cap \bar{P}^-$ must be in Σ_{k_0}. More precisely, there exists $t_0 \in (0, d)$ such that we have:

(i) $M \cap Q^-(t)$ is contained in the portion of the geodesic cylinder over a domain of $Q(t)$ contained in P^+ and $\mathcal{Q}_t(M \cap Q^-(t)) \subset \Omega \cap P^+$, for all $t < t_0$;

(ii) $M \cap \bar{P}^- \cap Q(t_0)$ is a non-empty subset of Σ_{k_0}.

These statements follow from the fact that M_{k_0} is, as proved above, contained in the geodesic cylinder over a domain in P and $Q(t_0)$ is a totally geodesic sphere perpendicular to P. So, since σ crosses Σ_{k_0} at least twice, and M_{k_0} is compact, there exists $t_1 \in (0, t_0]$ so that $M \cap Q^-(t)$ is contained in the geodesic cylinder over a domain of $Q(t)$, in such a way that $\mathcal{Q}_t(M \cap Q^-(t)) \subset \operatorname{int}(\Omega)$, whenever $t > t_1$. Furthermore, one of the following assertions holds:

(i) $\mathcal{Q}_{t_1}(M \cap Q^-(t_1))$ is tangent to M_{k_0} at points not belonging to $Q(t_1)$ with the same orientation;

(ii) $Q(t_1)$ is perpendicular to $\overline{M \cap Q^-(t_1)}$ at points of $Q(t_1)$ or, equivalently,

$$\overline{\mathcal{Q}_{t_1}(M \cap Q^-(t_1))}$$

is tangent to M at points of $Q(t_1)$.

In any case, $Q(t_1)$ should be a sphere of symmetry of M. Let p' be the last point of Σ in $\sigma[0, d)$. The distance between p' and $Q(t_1)$ is less than t_1. Thus, prolonging σ until the point $\mathcal{Q}_{t_1}(p')$, we obtain an arc of geodesic of length strictly less than $2t_1 < \pi$. Since $Q(t_1)$ is a sphere of symmetry of M and, in particular, of Σ, we have that $\mathcal{Q}(p')$ is a point of Σ. However, since that Σ is convex, σ does not return to Σ until it has just crossed all of the hemisphere determined by Σ_1 in P which does not contain points of Σ, that is, just after $t > \pi$. As $2t_1 < \pi$, we have a contradiction. From this contradiction, we conclude that there is no components of $M \cap \operatorname{ext}(D)$ homologous to zero outside the region in $P - \Sigma_1$ bounded by Σ and some component of $M \cap \operatorname{ext}(D)$ homologous to Σ; otherwise, there exists at least a direction σ starting from p so that the component of $M \cap \operatorname{ext}(D)$ nearest from Σ_1 in its direction is homologous to a constant.

Now, suppose Σ_{k_0} is homologous to Σ. By construction, it is clear that σ crosses each component Σ_k of $M \cap \operatorname{ext}(D)$ homologous to Σ at least once. So, proceeding as in [**11**] we find a sphere of symmetry of M before reach Σ, a contradiction.

So far, we have proved that M is contained in $\bar{P}^+$ and that $M \cap P = \Sigma$. Furthermore, $M^-(\frac{1}{2}\pi)$, if it is not empty, is a geodesic graph over a domain in $P_1(\frac{1}{2}\pi)$ having height less than $\frac{1}{2}\pi$ and its reflected image through $P_1(\frac{1}{2}\pi)$ is entirely contained in $\operatorname{int}(\Omega)$.

Let then R be a sphere of symmetry of Σ and $q \in R \cap D$. Let μ be the geodesic perpendicular to R starting from q and reaching a point $q' \in \Sigma_1$. We define $R(t)$, $0 \leqslant t \leqslant 2\pi$, as the intersection of $\mathbb{S}^{n+1}$ and the Euclidean hyperplane containing the origin of $\mathbb{R}^{n+2}$ and perpendicular to $\{x_0 = 0\}$, whose normal is $\mu'(t)_{(0,\dots,0)}$. It is clear that $R(0) = R$. Suppose that $R(d)$ and $P_1(\frac{1}{2}\pi)$ coincide. Then, the facts above imply that we have no touching points until the time $t = d$ on the reflection process through the spheres $R(t)$. However, since M is compact and $M \cap P = \Sigma$, there exists $t_1 \in [0, d)$ such that $R(t_1)$ is a sphere of symmetry of M and, in particular, of Σ. Since R and $R(t_1)$ are both perpendicular to μ, it follows from the convexity of Σ that $R = R(t_1)$, i.e. that R is a sphere of symmetry of M.

If $R(d)$ and $P_1(\frac{1}{2}\pi)$ are distinct spheres, let $\Sigma_2 = P_1(\frac{1}{2}\pi) \cap R(d)$ and consider the totally geodesic spheres $T(t)$, $0 \leqslant t \leqslant \alpha_0$, obtained by rotation, fixing Σ_2, of $P_1(\frac{1}{2}\pi)$ towards $R(d)$, with $T(0) = P(\frac{1}{2}\pi)$ and $T(\alpha_0) = R(d)$. It is clear that $\Sigma_2 = \cap_t T(t)$. Moreover, we have that $T(t) \cap \Sigma = \emptyset$, for all t, since each $T(t)$ is contained in the domain $\mathcal{C}$ of $\mathbb{S}^{n+1}$ bounded by $P_1(\frac{1}{2}\pi)$ and $R(d)$ that does not contain points of Σ. Denote $T^-(t)$ and $\mathcal{T}(t)$ as before.

By continuity, we have that, for t close enough to 0, each component of $M \cap T^-(t)$, when this set is not empty, is still a geodesic graph over a domain $T(t)$ at distance from $T(t)$ strictly less than $\frac{1}{2}\pi$. Furthermore, since M is compact, it is possible to consider t sufficiently small so that $\mathcal{T}(t)(M \cap T^-(t)) \subset \operatorname{int}(\Omega)$. Thus, either this remains true for each $t \in (0, \alpha_0]$, or there exists $t_1 \in (0, \alpha_0]$ such that one of the following situations occurs:

(i) $\mathcal{T}_{t_1}(M \cap T^-(t_1))$ is tangent to M at points not belonging to $T(t_1)$ with the same orientation;

(ii) $T(t_1)$ is perpendicular to $\overline{M \cap T^-(t_1)}$ at points of $T(t_1)$ or, equivalently,
$$\overline{\mathcal{T}_{t_1}(M \cap T^-(t_1))}$$
is tangent to M at points of $T(t_1)$.

In these cases, $T(t_1)$ should be a sphere of symmetry of M and, in particular, of Σ. However, this contradicts the fact that there are no points of Σ in $\mathcal{C}$. Therefore, we conclude from this contradiction that $\mathcal{T}_{\alpha_0}(M \cap T^-(\alpha_0))$ is contained in Ω and that $M \cap T^-(\alpha_0)$ is either empty or a graph over $T(\alpha_0) = R(d)$. In this way, we return to the previous case. The theorem is proved. □

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