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BIMINIMAL IMMERSIONS

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Dedicated to Professor Renzo Caddeo on his 60th birthday.

Abstract We study biminimal immersions: that is, immersions which are critical points of the bienergy for normal variations with fixed energy. We give a geometrical description of the Euler–Lagrange equation associated with biminimal immersions for both biminimal curves in a Riemannian manifold, with particular attention given to the case of curves in a space form, and isometric immersions of codimension 1 in a Riemannian manifold, in particular for surfaces of a three-dimensional manifold. We describe two methods of constructing families of biminimal surfaces using both Riemannian and horizontally homothetic submersions.

Keywords: biminimal surfaces; biharmonic maps; Willmore functional; Thurston geometry

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1. Introduction

Many stimulating problems in mathematics owe their existence to variational formulations of physical phenomena. In differential geometry, harmonic maps, candidate minimizers of the Dirichlet energy, can be described as constraining a rubber sheet to fit on a marble manifold in a position of elastic equilibrium, i.e. without tension [**6**]. However, when this scheme falls through, and it can, as corroborated by the case of the 2-torus and the 2-sphere [**7**], a best map will minimize this failure, measured by the total tension, called bienergy. In the more geometrically meaningful context of immersions, the fact that the tension field is normal to the image submanifold suggests that the most effective deformations must be sought in the normal direction.

Two approaches to this optimization are available. The first (the free state) consists in finding (normal) extrema of the bienergy, with complete disregard for the behaviour of the energy. In the second, in order to avoid paying too great a price for a smaller tension, a constancy condition is imposed on the energy level.

In more intuitive terms, and even though we never consider the associated flows, these points of view correspond, at least in the more favourable situations, to reducing the overall tension of a surface, with or without controlling the mean curvature.

However different they appear to be, both approaches are unified into a single mathematical description, amounting to a Lagrange multiplier interpretation. This consideration leads to the following definitions.

Definition 1.1. A map $\phi : (M, g) \to (N, h)$ between Riemannian manifolds is called *biharmonic* if it is a critical point, for all variations, of the bienergy functional

$$E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 v_g,$$

where $\tau(\phi) = \operatorname{tr} \nabla \mathrm{d}\phi$ is the tension field, vanishing for critical points of the Dirichlet energy (i.e. harmonic maps),

$$E(\phi) = \frac{1}{2} \int_M |\mathrm{d}\phi|^2 v_g.$$

In the case of non-compact domains, integration is understood as for all compact subsets.

The Euler–Lagrange operator attached to biharmonicity, called the *bitension field* and computed by Jiang [**9**], is

$$\tau_2(\phi) = -(\Delta^\phi \tau(\phi) - \operatorname{tr} R^N(\mathrm{d}\phi, \tau(\phi))\, \mathrm{d}\phi),$$

and vanishes if and only if the map ϕ is biharmonic. We are now ready to define the main object of this paper.

Definition 1.2. An immersion $\phi : (M^m, g) \to (N^n, h)$, $m \leqslant n$, between Riemannian manifolds, or its image, is called *biminimal* if it is a critical point of the bienergy functional E_2 for variations normal to the image $\phi(M) \subset N$, with fixed energy. Equivalently, there exists a constant $\lambda \in \mathbb{R}$ such that ϕ is a critical point of the λ-bienergy

$$E_{2,\lambda}(\phi) = E_2(\phi) + \lambda E(\phi)$$

for any smooth variation of the map $\phi_t :]-\epsilon, +\epsilon[\,$, $\phi_0 = \phi$, such that $V = \mathrm{d}\phi_t/\mathrm{d}t|_{t=0}$ is normal to $\phi(M)$.

Remark 1.3. The functional $E_{2,\lambda}$ has been on the mathematical scene since the early 1970s (see [**8**], where its critical points, for all possible variations, are studied). In particular, it is shown to satisfy Condition (C) of Palais–Smale when the domain has dimension 2 or 3 and the target is non-positively curved, ensuring the existence of minimizers in each homotopy class. However, Lemaire [**11**] constructed counterexamples when no condition is imposed on the curvature.

Using the Euler–Lagrange equations for harmonic and biharmonic maps, we see that an immersion is biminimal if

$$[\tau_{2,\lambda}]^\perp = [\tau_2]^\perp - \lambda[\tau]^\perp = 0,$$

for some value of $\lambda \in \mathbb{R}$, where $[\cdot]^\perp$ denotes the normal component of $[\cdot]$. We call an immersion *free* biminimal if it is biminimal for $\lambda = 0$. In the instance of an isometric immersion $\phi : M \to N$, the biminimal condition is

$$[\Delta^\phi \boldsymbol{H} - \operatorname{tr} R^N(\mathrm{d}\phi, \boldsymbol{H})\,\mathrm{d}\phi]^\perp + \lambda \boldsymbol{H} = 0. \tag{1.1}$$

Note that this variational principle is close to the Willmore problem, the disparity being that we do not vary through isometric immersions. While it is obvious that biharmonic immersions are biminimal, we will see in the following sections that the two notions are distinct. For example, we construct families of biminimal surfaces in three-dimensional space forms of non-positive constant sectional curvature where biharmonic surfaces do not exist [**3**, **4**]. In the same vein, we construct families of biminimal surfaces in almost all three-dimensional Thurston geometries.

1.1. Notation

We shall work in the C^∞ category, i.e. manifolds, metrics, connections and maps will be assumed to be smooth. By (M^m, g) we shall mean a connected manifold, of dimension m, without boundary, endowed with a Riemannian metric g. We shall denote by ∇ the Levi-Civita connection on (M, g). For vector fields X, Y, Z on M we define the Riemann curvature operator by $R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$. For the Laplacian we shall use $\Delta(f) = \operatorname{div} \operatorname{grad} f$ for functions $f \in C^\infty(M)$ and $\Delta^\phi W = -\operatorname{tr}(\nabla^\phi)^2 W$ for sections along a map $\phi : M \to N$.

2. Biminimal curves

Our quest for examples of biminimal immersions starts with curves. Let $\gamma : I \subset \mathbb{R} \to (M^m, g)$ be a curve parametrized by arc length in a Riemannian manifold (M^m, g), i.e. γ is an isometric immersion. Before computing the bitension field of γ, we recall the definition of Frenet frames.

Definition 2.1 (see, for example, [10]). The Frenet frame $\{B_i\}_{i=1,\dots,m}$ associated with a curve $\gamma : I \subset \mathbb{R} \to (M^m, g)$ is the orthonormalization of the $(m+1)$-tuple

$$\left\{ \nabla^{(k)}_{\partial/\partial t}\, \mathrm{d}\gamma\left(\frac{\partial}{\partial t}\right)\right\}_{k=0,\dots,m}$$

described by

$$\begin{aligned}
B_1 &= \mathrm{d}\gamma\left(\frac{\partial}{\partial t}\right),\\
\nabla^\gamma_{\partial/\partial t} B_1 &= k_1 B_2,\\
\nabla^\gamma_{\partial/\partial t} B_i &= -k_{i-1}B_{i-1} + k_i B_{i+1} \quad \forall i = 2, \dots, m-1,\\
\nabla^\gamma_{\partial/\partial t} B_m &= -k_{m-1}B_{m-1},
\end{aligned}$$

where the functions $\{k_1 = k > 0,\ k_2 = \tau,\ k_3, \dots, k_{m-1}\}$ are called the curvatures of γ. Note that $B_1 = T = \gamma'$ is the unit tangent vector field to the curve.

In the instance of a curve γ on a surface ($m = 2$), the Frenet frame reduces to the couple $\{T, N\}$, with T being the unit tangent vector field along γ and N a normal vector field along γ such that $\{T, N\}$ is a positive basis, while $k_1 = k$ is the signed curvature of γ.

Biminimal curves in a Riemannian manifold are characterized as follows.

Proposition 2.2. *Let $\gamma : I \subset \mathbb{R} \to (M^m, g)$, $m \geqslant 2$, be an isometric curve from an open interval of $\mathbb{R}$ into a Riemannian manifold (M, g). Then γ is biminimal if and only if there exists a real number λ such that*

$$\left.\begin{aligned} k_1'' - k_1^3 - k_1 k_2^2 + k_1 g(R(B_1, B_2)B_1, B_2) - \lambda k_1 &= 0, \\ (k_1^2 k_2)' + k_1^2 g(R(B_1, B_2)B_1, B_3) &= 0, \\ k_1 k_3 + k_1 g(R(B_1, B_2)B_1, B_4) &= 0, \\ k_1 g(R(B_1, B_2)B_1, B_j) &= 0, \quad j = 5, \dots, m, \end{aligned}\right\} \tag{2.1}$$

where R is the curvature tensor of (M, g) and $\{B_i\}_{i=1,\dots,m}$ is the Frenet frame of γ.

Proof. With respect to its Frenet frame, the tension field of γ is

$$\tau(\gamma) = \operatorname{tr} \nabla \mathrm{d}\gamma = \nabla^\gamma_{\partial/\partial t}\left(\mathrm{d}\gamma\left(\frac{\partial}{\partial t}\right)\right) - \mathrm{d}\gamma\left(\nabla_{\partial/\partial t} \frac{\partial}{\partial t}\right) = \nabla^\gamma_{\partial/\partial t} B_1 = k_1 B_2$$

and its bitension field is

$$\begin{aligned} -\tau_2(\gamma) &= -\nabla^\gamma_{\partial/\partial t} \nabla^\gamma_{\partial/\partial t}(\tau(\gamma)) + \nabla^\gamma_{\nabla_{\partial/\partial t}(\partial/\partial t)}(\tau(\gamma)) - R\left(\mathrm{d}\gamma\left(\frac{\partial}{\partial t}\right), \tau(\gamma)\right) \mathrm{d}\gamma\left(\frac{\partial}{\partial t}\right) \\ &= -\nabla^\gamma_{\partial/\partial t} \nabla^\gamma_{\partial/\partial t}(k_1 B_2) - R(B_1, k_1 B_2)B_1 \\ &= -\nabla^\gamma_{\partial/\partial t}(k_1' B_2 - k_1^2 B_1 + k_1 k_2 B_3) - k_1 R(B_1, B_2)B_1 \\ &= -(k_1'' - k_1^3 - k_1 k_2^2)B_2 + 3k_1 k_1' B_1 - (k_1' k_2 + (k_1 k_2)')B_3 \\ &\qquad\qquad\qquad\qquad\qquad - k_1 k_3 B_4 - k_1 R(B_1, B_2)B_1. \end{aligned}$$

The vanishing of the normal components yields the system (2.1). $\square$

Remark 2.3. For a free biminimal curve γ to be biharmonic, we require the supplementary condition $[\tau_2(\gamma)]^{B_1} = 0$, which is equivalent to $k_1 k_1' = 0$: that is, either k_1 is constant or γ is a geodesic ($k_1 = 0$).

If the target manifold is a surface or a three-dimensional Riemannian manifold with constant sectional curvature, equations (2.1) are more manageable as shown in the following corollary.

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Corollary 2.4.

(i) *An isometric curve γ on a surface of Gaussian curvature G is biminimal if and only if its signed curvature k satisfies the ordinary differential equation*

$$k'' - k^3 + kG - \lambda k = 0 \tag{2.2}$$

for some $\lambda \in \mathbb{R}$.

(ii) *An isometric curve γ on a Riemannian 3-manifold of constant sectional curvature c is biminimal if and only if its curvature k and torsion τ satisfy the system*

$$\left.\begin{aligned} k'' - k^3 - k\tau^2 + kc - \lambda k = 0,\\ k^2\tau = \text{const.},\end{aligned}\right\} \tag{2.3}$$

for some $\lambda \in \mathbb{R}$.

Proof. (i) The two-dimensional Frenet frame of γ consists only of T and N, and the curve is biminimal, with respect to λ, if and only if

$$k'' - k^3 + kg(R(T,N)T,N) - \lambda k = 0,$$

but since $g(R(T,N)T,N) = G$ we obtain (2.2).

(ii) In dimension 3, the Frenet frame of γ is $\{T,\ N = B_2,\ B = B_3\}$, and the conditions of Proposition 2.2 become

$$\begin{aligned} k'' - k^3 - k\tau^2 + kg(R(T,N)T,N) - \lambda k &= 0,\\ (k^2\tau)' + k^2 g(R(T,N)T,B) &= 0.\end{aligned}$$

The constant sectional curvature of the target means that $g(R(T,N)T,N) = c$ and $g(R(T,N)T,B) = 0$. □

From Corollary 2.4, if γ is an isometric curve in a Riemannian manifold M^n of constant sectional curvature c and dimension 2 or 3, then the curvature of γ (the signed curvature when $n = 2$) satisfies the equation

$$k'' - k^3 - \frac{\alpha^2}{k^3} + k\beta = 0, \tag{2.4}$$

where $\alpha = k^2\tau$ and $\beta = c - \lambda$. Multiplying (2.4) by $2k'$ and integrating, we obtain

$$(k')^2 - \tfrac{1}{2}k^4 + \frac{\alpha^2}{k^2} + \beta k^2 = A, \quad A \in \mathbb{R},$$

and setting $u = k^2$ yields

$$(u')^2 - 2u^3 + 4\alpha^2 + 4\beta u^2 = 4Au.$$

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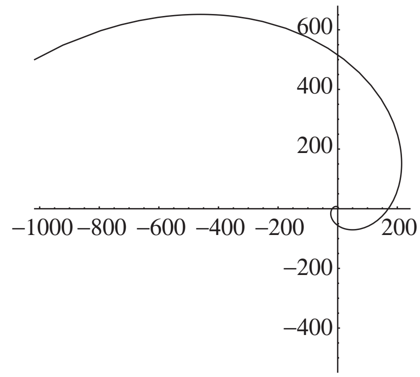

Figure 1. Free biminimal curve in $\mathbb{R}^2$ of logarithmic type.

Since this equation is of the form $(u')^2 = P(u)$, where P is a polynomial of degree 3, it can be solved using standard techniques in terms of elliptic functions (see, for example [**5**]). In a future paper we shall give an accurate description of the solutions of equation (2.4). Here we merely point out that if M is the flat $\mathbb{R}^2$, then equation (2.4), for free biminimal curves, reduces to

$$k'' - k^3 = 0,$$

a solution of which can be expressed in terms of elementary functions: that is, $k(s) = \sqrt{2}/s$, where s is the arc length. Now, using the standard formula to integrate a curve of known signed curvature, we find that, up to isometries of $\mathbb{R}^2$, this free biminimal curve is given by

$$\gamma(s) = \frac{s}{3(\cos(\sqrt{2}\log s) + \sqrt{2}\sin(\sqrt{2}\log s), -\sqrt{2}\cos(\sqrt{2}\log s) + \sin(\sqrt{2}\log s))}.$$

This is the standard parametrization by arc length of the logarithmic spiral plotted in Figure 1.

2.1. Biminimal curves via conformal changes of the metric

On a Riemannian manifold (M, g), any representative of the conformal class $[g]$ can be expressed as $\bar{g} = \mathrm{e}^{2f} g$, $f \in C^{\infty}(M)$, and the Levi-Civita connections are related, for $X, Y \in C(TM)$, by (cf. [**2**])

$$\bar{\nabla}_X Y = \nabla_X Y + X(f)Y + Y(f)X - g(X, Y)\,\mathrm{grad}\, f. \tag{2.5}$$

Observe that a geodesic γ on (M, g) will not remain geodesic after a conformal change of metric, unless the conformal factor is constant since

$$\bar{\nabla}_{\dot{\gamma}}\dot{\gamma} = \nabla_{\dot{\gamma}}\dot{\gamma} + 2\dot{\gamma}(f)\dot{\gamma} - |\dot{\gamma}|^2\,\mathrm{grad}\, f.$$

The following theorem gives a tool with which to construct free biminimal curves.

Theorem 2.5. *Let (M^m, g) be a Riemannian manifold. Fix a point $p \in M^m$ and choose a function f depending only on the geodesic distance from p. Then any geodesic on (M, g) going through p will be a free biminimal curve on $(M^m, \bar{g} = \mathrm{e}^{2f} g)$.*

Proof. Let γ be a geodesic of (M^m, g) and let $\{B_i\}_{i=1,\dots,m}$ be the associated Frenet frame (cf. Definition 2.1). Since the function f depends only on the geodesic distance from p, i.e. the B_1-direction, $B_i f = 0$, for all $i = 2, \dots, m$. Since γ is a geodesic on (M, g), we have

$$\nabla^{\gamma}_{\partial/\partial t} B_1 = 0,$$

and the tension field of γ with respect to the metric $\bar{g} = \mathrm{e}^{2f} g$ is

$$\bar{\tau}(\gamma) = \bar{\nabla}^{\gamma}_{\partial/\partial t} \mathrm{d}\gamma\left(\frac{\partial}{\partial t}\right) = \bar{\nabla}^{\gamma}_{\partial/\partial t} B_1 = \nabla^{\gamma}_{\partial/\partial t} B_1 + 2B_1(f)B_1 - \operatorname{grad} f,$$

where in the above equality we have used (2.5). In addition,

$$\operatorname{grad} f = B_1(f)B_1 + \sum_{i=2}^{m} B_i(f)B_i = B_1(f)B_1;$$

thus $\bar{\tau}(\gamma) = B_1(f)B_1$.

Still working with respect to $\bar{g}$, the bitension field of γ is

$$\begin{aligned}
\bar{\tau}_2(\gamma) &= -\Delta^{\gamma}\bar{\tau}(\gamma) + \operatorname{tr} \bar{R}(\mathrm{d}\gamma, \bar{\tau}(\gamma))\,\mathrm{d}\gamma \\
&= \bar{\nabla}^{\gamma}_{\partial/\partial t}\bar{\nabla}^{\gamma}_{\partial/\partial t}(B_1(f)B_1) - \bar{\nabla}^{\gamma}_{(\nabla_{\partial/\partial t})(\partial/\partial t)}(B_1(f)B_1) + \bar{R}(B_1, B_1(f)B_1)B_1 \\
&= \bar{\nabla}^{\gamma}_{\partial/\partial t}\bar{\nabla}^{\gamma}_{\partial/\partial t}(B_1(f)B_1) \\
&= \bar{\nabla}^{\gamma}_{\partial/\partial t}(B_1B_1(f)B_1 + B_1(f)^2B_1) \\
&= (B_1B_1B_1(f))B_1 + B_1B_1(f)B_1(f)B_1 + 2B_1B_1(f)B_1(f)B_1 + (B_1(f))^3B_1 \\
&= [B_1B_1B_1(f) + 3B_1B_1(f)B_1(f) + (B_1(f))^3]B_1.
\end{aligned}$$

So $\bar{\tau}^2(\gamma)$ has no normal component and γ is free biminimal on $(M^m, \bar{g} = \mathrm{e}^{2f}g)$. □

Corollary 2.6. *Let r be the geodesic distance from a point $p \in (M, g)$ and let $f(r) = \ln(ar^2 + br + c)$, $a, b, c \in \mathbb{R}$. Then a geodesic on (M, g) through p becomes a biharmonic map on $(M, \bar{g} = \mathrm{e}^{2f}g)$.*

Proof. From the proof of Theorem 2.5, a geodesic on (M, g) through p becomes a biharmonic map on $(M, \bar{g})$ if f is a solution of the ordinary differential equation

$$f'''(r) + 3f''(r)f'(r) + f'(r)^3 = 0.$$

To solve this equation we set $y = f'$ to obtain

$$y'' + 3y'y + y^3 = 0. \tag{2.6}$$

Then, using the transformation $y = x'/x$, Equation (2.6) becomes $x'''/x = 0$, which has the solution $x(r) = \bar{a}r^2 + \bar{b}r + \bar{c}$, $\bar{a}, \bar{b}, \bar{c} \in \mathbb{R}$. Finally, from $f(r) = \ln(d\,x(r))$, $d \in \mathbb{R}$, we find the desired f. □

As an example, one can take $(M, g) = (\mathbb{R}^2, g = \mathrm{d}x^2 + \mathrm{d}y^2)$, and $f(r) = \ln(r^2 + 1)$, where $r = \sqrt{x^2 + y^2}$ is the distance from the origin. Thus, any straight line on the flat $\mathbb{R}^2$ turns into a biharmonic curve on $(\mathbb{R}^2, \bar{g} = (r^2 + 1)^2(\mathrm{d}x^2 + \mathrm{d}y^2))$, which is the metric, in local isothermal coordinates, of the Enneper minimal surface. Figure 2 is a plot of the Enneper surface in polar coordinates, so the radial curves on the plot are biharmonic.

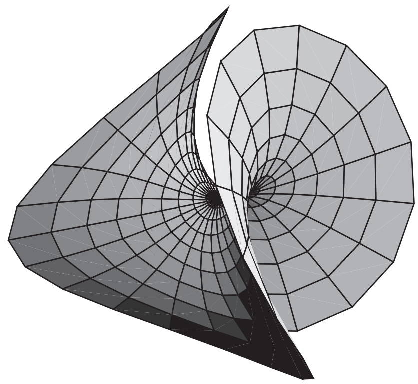

Figure 2. The radial curves from the origin of the Enneper surface are biharmonic.

3. Codimension-1 biminimal submanifolds

Let $\phi : M^n \to N^{n+1}$ be an isometric immersion of codimension 1. We denote by B the second fundamental form of ϕ, by $\boldsymbol{N}$ a unit normal vector field to $\phi(M) \subset N$ and by $\boldsymbol{H} = H\boldsymbol{N}$ the mean curvature vector field of ϕ (H the mean curvature function). Then we have the following result.

Proposition 3.1. *Let $\phi : M^n \to N^{n+1}$ be an isometric immersion of codimension 1 and $\boldsymbol{H} = H\boldsymbol{N}$ its mean curvature vector. Then ϕ is biminimal if and only if*

$$\Delta H = (|B|^2 - \operatorname{Ricci}(\boldsymbol{N}) + \lambda)H \tag{3.1}$$

for some value of λ in $\mathbb{R}$.

Proof. In a local orthonormal frame $\{e_i\}_{i=1,\dots,n}$ on M, the tension field of ϕ is $\tau(\phi) = nH\boldsymbol{N}$ and its bitension field is

$$\begin{aligned}
-\tau_2(\phi) &= -\sum_{i=1}^{n}[\nabla^\phi_{e_i}\nabla^\phi_{e_i}(nH\boldsymbol{N}) + \nabla^\phi_{\nabla_{e_i}e_i}(nH\boldsymbol{N}) - R^{N^{n+1}}(\mathrm{d}\phi(e_i), nH\boldsymbol{N})\,\mathrm{d}\phi(e_i)] \\
&= n\sum_{i=1}^{n}[-\nabla^\phi_{e_i}(e_i(H)\boldsymbol{N} + H\nabla^\phi_{e_i}\boldsymbol{N}) + (\nabla_{e_i}e_i)(H)\boldsymbol{N} + H\nabla^\phi_{\nabla_{e_i}e_i}\boldsymbol{N} \\
&\qquad\qquad - HR^{N^{n+1}}(\mathrm{d}\phi(e_i), \boldsymbol{N})\,\mathrm{d}\phi(e_i)] \\
&= n\sum_{i=1}^{n}[-e_ie_i(H)\boldsymbol{N} - 2e_i(H)\nabla^\phi_{e_i}\boldsymbol{N} - H\nabla^\phi_{e_i}\nabla^\phi_{e_i}\boldsymbol{N} \\
&\qquad + (\nabla_{e_i}e_i)(H)\boldsymbol{N} + H\nabla^\phi_{\nabla_{e_i}e_i}\boldsymbol{N}] - nH\sum_{i=1}^{n}R^{N^{n+1}}(\mathrm{d}\phi(e_i), \boldsymbol{N})\,\mathrm{d}\phi(e_i) \\
&= n(\Delta H)\boldsymbol{N} - 2n\sum_{i=1}^{n}e_i(H)\nabla^\phi_{e_i}\boldsymbol{N} + nH\Delta^\phi\boldsymbol{N} \\
&\qquad\qquad - nH\sum_{i=1}^{n}R^{N^{n+1}}(\mathrm{d}\phi(e_i), \boldsymbol{N})\,\mathrm{d}\phi(e_i).
\end{aligned}$$

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But

(i) $\langle \nabla^\phi_{e_i} \boldsymbol{N}, \boldsymbol{N}\rangle = \frac{1}{2} e_i \langle \boldsymbol{N}, \boldsymbol{N}\rangle = 0$,

(ii) $\langle \sum_{i=1}^n R^{N^{n+1}}(\mathrm{d}\phi(e_i), \boldsymbol{N})\,\mathrm{d}\phi(e_i), \boldsymbol{N}\rangle = \mathrm{Ricci}(\boldsymbol{N})$.

For $\langle \Delta^\phi \boldsymbol{N}, \boldsymbol{N}\rangle$, first we have

$$\langle \Delta^\phi \boldsymbol{N}, \boldsymbol{N}\rangle = \sum_{i=1}^n \langle -\nabla^\phi_{e_i}\nabla^\phi_{e_i}\boldsymbol{N} + \nabla^\phi_{\nabla_{e_i}e_i}\boldsymbol{N}, \boldsymbol{N}\rangle = \sum_{i=1}^n \langle \nabla^\phi_{e_i}\boldsymbol{N}, \nabla^\phi_{e_i}\boldsymbol{N}\rangle.$$

Then, if B is the second fundamental form of ϕ, which, in an orthonormal frame $\{e_1, \dots, e_n, \boldsymbol{N}\}$, is defined by

$$B = (\langle \nabla_{e_i} e_j, \boldsymbol{N}\rangle)_{i,j=1,\dots,n} = -(\langle \nabla_{e_i}\boldsymbol{N}, e_j\rangle)_{i,j=1,\dots,n},$$

we have

$$|\nabla^\phi_{e_i}\boldsymbol{N}|^2 = \langle \nabla^\phi_{e_i}\boldsymbol{N}, \nabla^\phi_{e_i}\boldsymbol{N}\rangle = \sum_{j=1}^n \langle \nabla^\phi_{e_i}\boldsymbol{N}, e_j\rangle^2 \quad \forall i = 1, \dots, n,$$

which implies that

$$\sum_{i=1}^n \langle \nabla^\phi_{e_i}\boldsymbol{N}, \nabla^\phi_{e_i}\boldsymbol{N}\rangle = |B|^2.$$

In conclusion, we obtain

$$-\langle \tau_{2,\lambda}(\phi), \boldsymbol{N}\rangle = n(-\Delta H + H|B|^2 - H\,\mathrm{Ricci}(\boldsymbol{N}) + \lambda H).$$

□

Corollary 3.2. *An isometric immersion $\phi : M^n \to N^{n+1}(c)$ into a space form of constant curvature c is biminimal if and only if there exists a real number λ such that*

$$\Delta H - H(n^2H^2 - s + n(n-2)c + \lambda) = 0,$$

where H is the mean curvature and s the scalar curvature of M^n. Moreover, an isometric immersion $\phi : M^2 \to N^3(c)$ from a surface to a three-dimensional space form is biminimal if and only if

$$\Delta H - 2H(2H^2 - G + \tfrac{1}{2}\lambda) = 0 \tag{3.2}$$

for some λ in $\mathbb{R}$.

Proof. Let $\{e_1, \dots, e_n\}$ be an orthonormal frame of M^n corresponding to the principal curvatures $\{k_1, \dots, k_n\}$ and let B be its second fundamental form. Then

$$\begin{aligned} |B|^2 &= k_1^2 + \cdots + k_n^2 \\ &= n^2H^2 - 2\sum_{i,j=1,\, i<j}^n k_ik_j \\ &= n^2H^2 - 2\sum_{i,j=1,\, i<j}^n (K(e_i, e_j) - c) \end{aligned}$$

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$$= n^2H^2 - \sum_{i,j=1}^{n} K(e_i, e_j) + n(n-1)c$$
$$= n^2H^2 - s + n(n-1)c,$$

where $K(e_i, e_j)$ is the sectional curvature on M^n of the plane spanned by e_i and e_j, and $s = \sum_{i,j=1}^{n} K(e_i, e_j)$ is the scalar curvature of M^n. Since $\mathrm{Ricci}(\boldsymbol{N}) = nc$, the map ϕ is biminimal if and only if

$$\Delta H = (n^2H^2 - s + n(n-2)c + \lambda)H,$$

for some λ in $\mathbb{R}$. □

Remark 3.3. Condition (3.2), for free biminimal immersions, is very similar to the equation of the Willmore problem ($\Delta H + 2H(H^2 - K) = 0$) but the minus sign in (3.2) rules out the existence of compact solutions when $c \leqslant 0$.

We shall now describe some constructions with which to produce examples of biminimal immersions. Recall that a submersion $\phi : (M, g) \to (N, h)$ between two Riemannian manifolds is *horizontally homothetic* if there exists a function $\Lambda : M \to \mathbb{R}$, the *dilation*, such that

(i) at each point $p \in M$ the differential $\mathrm{d}\phi_p : H_p \to T_{\phi(p)}N$ is a conformal map with factor $\Lambda(p)$, i.e. $\Lambda^2(p)g(X, Y)(p) = h(\mathrm{d}\phi_p(X), \mathrm{d}\phi_p(Y))(\phi(p))$ for all $X, Y \in H_p = \ker_p(\mathrm{d}\phi)^{\perp}$,

(ii) $X(\Lambda^2) = 0$ for all horizontal vector fields.

Lemma 3.4. *Let $\phi : (M^n, g) \to (N^2, h)$ be a horizontally homothetic submersion with Λ and minimal fibres and let $\gamma : I \subset \mathbb{R} \to N^2$ be a curve parametrized by arc length, of signed curvature k_γ. Then the codimension-1 submanifold $S = \phi^{-1}(\gamma(I)) \subset M$ has mean curvature $H_S = \Lambda k_\gamma/(n-1)$.*

Proof. Let $\{T, N\}$ be the Frenet frame of γ, i.e. $\nabla^{\gamma}_{\partial/\partial t}T = k_\gamma N$. Choose a local orthogonal frame $\{e_1, e_2, e_3, \dots, e_n\}$ on M^n such that $\mathrm{d}\phi(e_1) = T \circ \phi$, $\mathrm{d}\phi(e_2) = N \circ \phi$ and $\mathrm{d}\phi(e_i) = 0$, for $i = 3, \dots, n$. Since ϕ is a horizontally homothetic submersion, we have $|e_1|^2 = |e_2|^2 = 1/\Lambda^2$ and we can choose $\{e_3, \dots, e_n\}$ of unit length. The restriction to $S = \phi^{-1}(\gamma(I)) \subset M$ of the vector fields e_1 and $\{e_3, \dots, e_n\}$ gives a local frame of vector fields tangent to the submanifold $S = \phi^{-1}(\gamma(I))$, while the restriction of Λe_2 gives a unit vector field normal to S. Therefore, the mean curvature of S is

$$H_S = \frac{1}{n-1}\Lambda^3\langle\nabla_{e_1}e_1, e_2\rangle + \frac{1}{n-1}\Lambda\sum_{i=3}^{n}\langle\nabla_{e_i}e_i, e_2\rangle = \frac{1}{n-1}\Lambda^3\langle\nabla_{e_1}e_1, e_2\rangle + \frac{n-2}{n-1}\Lambda(H_{\mathrm{f}}),$$

where H_{f} is the mean curvature of the fibres. The fibres being minimal ($H_{\mathrm{f}} = 0$), we have

$$H_S = \frac{1}{n-1}\Lambda^3\langle\nabla_{e_1}e_1, e_2\rangle. \tag{3.3}$$

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Moreover,

$$\Lambda^2\langle\nabla_{e_1}e_1, e_2\rangle = \langle \mathrm{d}\phi(\nabla_{e_1}e_1), \mathrm{d}\phi(e_2)\rangle = \langle \mathrm{d}\phi(\nabla_{e_1}e_1), N\circ\phi\rangle$$
$$= \langle \nabla^{\phi}_{e_1}\mathrm{d}\phi(e_1), N\circ\phi\rangle = \langle \nabla^{\phi}_{e_1}(T\circ\phi), N\circ\phi\rangle,$$

since $(\nabla\mathrm{d}\phi)(e_1, e_1) = 0$ for a horizontally homothetic submersion (cf. [**1**]).

Finally,

$$\nabla^{\phi}_{e_1}(T\circ\phi) = (\nabla_{\mathrm{d}\phi(e_1)}T)\circ\phi = (\nabla_T T)\circ\phi = k_\gamma N\circ\phi,$$

and, taking into account (3.3), $(n-1)H_S = \Lambda k_\gamma$. □

Theorem 3.5. *Let $\phi : M^3(c) \to (N^2, h)$ be a horizontally homothetic submersion with dilation Λ, minimal fibres and integrable horizontal distribution, from a space form of constant sectional curvature c to a surface. Let $\gamma : I \subset \mathbb{R} \to N^2$ be a curve parametrized by arc length such that the surface $S = \phi^{-1}(\gamma(I)) \subset M^3$ has constant Gaussian curvature c. Then $S = \phi^{-1}(\gamma(I)) \subset M^3$ is a biminimal surface (with respect to $2c$) if and only if γ is a free biminimal curve.*

Proof. Let $\{T, N\}$ be the Frenet frame of γ, i.e. $\nabla_T T = k_\gamma N$. Let $\tilde\gamma : I \to M^3$ be a horizontal lift of γ, so that $\tilde\gamma'$ is horizontal and $\mathrm{d}\phi(\tilde\gamma') = T\circ\phi$. Let $\psi(t, s) = \eta_s(\tilde\gamma(t))$ be a local parametrization of the surface $S = \phi^{-1}(\gamma(I)) \subset M^3$, where, for a fixed $t_0 \in I$, $\eta_s(\tilde\gamma(t_0))$ is a parametrization by arc length of the fibre of ϕ through $\tilde\gamma(t_0)$. Then ψ induces on the surface S the metric

$$g_S = \frac{1}{\Lambda^2}\,\mathrm{d}t^2 + \mathrm{d}s^2,$$

where Λ is the dilation of ϕ which, when restricted to the surface S, depends only on s. The Laplacian on S is then given by

$$\Delta = \Lambda^2\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial s^2} - \mathrm{grad}(\log\Lambda)\frac{\partial}{\partial s}, \tag{3.4}$$

while the Gaussian curvature of S reduces to

$$G_S = \frac{\Delta\Lambda}{\Lambda} - (\mathrm{grad}(\log\Lambda))^2. \tag{3.5}$$

Now, assuming that S has constant Gaussian curvature $G_S = c$, from (3.2) we see that S is biminimal (with respect to $2c$) in $M^3(c)$ if and only if

$$\Delta H - 2H(2H^2 - c + c) = \Delta H - 4H^3 = 0.$$

By Lemma 3.4, $2H = \Lambda k_\gamma$; thus,

$$\begin{aligned}2(\Delta H - 4H^3) &= \Delta(\Lambda k_\gamma) - (\Lambda k_\gamma)^3\\ &= \Lambda^3\left[k_\gamma'' - k_\gamma^3 + \frac{k_\gamma}{\Lambda^2}(G_S + (\mathrm{grad}(\log\Lambda))^2)\right]\\ &= \Lambda^3\left[k_\gamma'' - k_\gamma^3 + \frac{k_\gamma}{\Lambda^2}(c + (\mathrm{grad}(\log\Lambda))^2)\right].\end{aligned} \tag{3.6}$$

Finally, from the generalized O'Neill formula [**12**] relating the sectional curvatures of the domain and target manifolds for a given horizontally homothetic submersion with integrable horizontal distribution (see, for example [**1**, Corollary 11.2.3]) we get

$$\frac{1}{\Lambda^2}(c + (\operatorname{grad}(\log \Lambda))^2) = G_N,$$

which, together with (3.6), gives

$$2(\Delta H - 4H^3) = \Lambda^3(k_\gamma'' - k_\gamma^3 + k_\gamma G_N).$$

Then the theorem follows from Corollary 2.4. □

When the horizontal space is not integrable, we can reformulate Theorem 3.5 for Riemannian submersions.

Theorem 3.6. *Let $\phi : M^3(c) \to N^2(\bar{c})$ be a Riemannian submersion with minimal fibres from a space of constant sectional curvature c to a surface of constant Gaussian curvature $\bar{c}$. Let $\gamma : I \subset \mathbb{R} \to N^2$ be a curve parametrized by arc length. Then $S = \phi^{-1}(\gamma(I)) \subset M^3$ is a biminimal surface (with respect to λ) if and only if γ is a biminimal curve (with respect to $\lambda + \bar{c}$).*

Proof. First, from (3.4) and (3.5), since $\Lambda = 1$, we have

$$G_S = 0, \quad \Delta = \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial s^2}.$$

Thus, taking into account Lemma 3.4, S is biminimal if and only if

$$\Delta(2H) - (2H)^3 - 2H\lambda = k_\gamma'' - k_\gamma^3 - k\lambda = 0.$$

From Corollary 2.4, the latter equation is clearly biminimal (with respect to $\lambda + \bar{c}$) for a curve $\gamma : I \to N^2(\bar{c})$. □

4. Examples of biminimal surfaces in three-dimensional space forms

4.1. Examples of biminimal surfaces in $\mathbb{R}^3$

We apply Theorem 3.5 to construct examples of biminimal surfaces in $\mathbb{R}^3$ with the flat metric.

(i) First we consider the orthogonal projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$, given by $\pi(x, y, z) = (x, y)$. The projection π is clearly a Riemannian submersion with minimal fibres (vertical straight lines in $\mathbb{R}^3$) and integrable horizontal distribution. Thus, from Theorem 3.5, a vertical cylinder with generatrix a free biminimal curve of $\mathbb{R}^2$ is a free biminimal surface. For example, one can consider the cylinder on the logarithmic spiral.

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(ii) The space $\mathbb{R}^3\setminus\{0\}$ can be described as the warped product $\mathbb{R}^3\setminus\{0\} = \mathbb{R}^+ \times_{t^2} \mathbb{S}^2$ with the warped metric $g = \mathrm{d}t^2 + t^2\,\mathrm{d}\theta^2$, $\mathrm{d}\theta^2$ being the canonical metric on $\mathbb{S}^2$. Then projection onto the second factor $\pi_2 : \mathbb{R}^+ \times_{t^2} \mathbb{S}^2 \to \mathbb{S}^2$ is a horizontally homothetic submersion with dilation $1/t$, integrable horizontal distribution and minimal fibres. Geometrically, π_2 is the radial projection $p \mapsto p/|p|$, $p \in \mathbb{R}^3 \setminus \{0\}$. Again applying Theorem 3.5, we see that the cone on a free biminimal curve on $\mathbb{S}^2$ is a free biminimal surface of $\mathbb{R}^3$. For example, if we take the parallel on $\mathbb{S}^2$ of latitude $\frac{1}{4}\pi$, which is a biharmonic curve, and thus free biminimal, we get the standard cone of revolution in $\mathbb{R}^3$.

(iii) The following example does not seem to enter the picture of Theorem 3.5. Let $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$ be a space curve with curvature k equal to its torsion τ and with $\{T, N, B\}$ its Frenet frame. It is easy to see that the envelope S of γ, parametrized by $X(u, s) = \alpha(s) + u(B + T)$, has mean curvature $H = k$. Thus, S is free biminimal if and only if

$$\Delta H - 4H^3 = k'' - 4k^3 = 0. \tag{4.1}$$

Geometrically, the curve γ is a curve with constant slope, i.e. there exists a vector $u \in \mathbb{R}^3$ such that $\langle T, u\rangle$ is constant. Then γ can be described as a helix of the cylinder on a plane curve β (the orthogonal projection of γ onto a plane orthogonal to u) whose geodesic curvature is a solution of (4.1). For example, we can take β to be the logarithmic spiral of the natural equation $k_\beta = 1/(\sqrt{2}s)$.

4.2. Examples of biminimal surfaces in $\mathbb{H}^3$

(i) Let $\mathbb{H}^3 = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$ be the half-space model for the hyperbolic space endowed with the metric of constant sectional curvature -1 given by $g = (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)/z^2$. Then the projection onto the plane at infinity defines a horizontally homothetic submersion $\pi : \mathbb{H}^3 \to \mathbb{R}^2$ with dilation $\Lambda = z$, integrable horizontal distribution and minimal fibres (vertical lines in $\mathbb{H}^3$). Then, from Theorem 3.5, a vertical cylinder with generatrix a free biminimal curve of $\mathbb{R}^2$ is a biminimal surface (with respect to -2) in the hyperbolic space. For example, the cylinder on the logarithmic spiral is free biminimal in $\mathbb{R}^3$, while it is biminimal (with respect to -2) in $\mathbb{H}^3$.

(ii) Let $\pi : \mathbb{H}^3 \to \mathbb{H}^2$ be defined by $\pi(x, y, z) = (x, 0, \sqrt{y^2 + z^2})$. The fibre of π over $(x, 0, r)$ is the semicircle with centre $(x, 0, r)$ and radius r, and it is parallel to the coordinate yz-plane. Thus, the map π has minimal fibres. Geometrically, this map is a projection along the geodesics of $\mathbb{H}^3$ which are orthogonal to $\mathbb{H}^2$. This is again a horizontally homothetic submersion with dilation, along the fibres, $\Lambda(s) = 1/\cosh(s)$, with s being the arc length parameter of the fibre. An easy computation shows that for any curve γ parametrized by arc length in $\mathbb{H}^2$ the surface $S = \pi^{-1}(\gamma(I))$ is of constant Gaussian curvature -1. Then, applying Theorem 3.5, for any free biminimal curve of $\mathbb{H}^2$, $S = \pi^{-1}(\gamma(I))$ is a biminimal surface (with respect to -2) in the hyperbolic space.

4.3. Examples of biminimal surfaces in $\mathbb{S}^3$

(i) Let $p, q \in \mathbb{S}^3$ be two antipodal points. Then the space $\mathbb{S}^3 \setminus \{p, q\}$ can be described as the warped product $\mathbb{S}^3 \setminus \{p, q\} = (0, \pi) \times_{\sin^2(t)} \mathbb{S}^2$ with the warped metric $g = \mathrm{d}t^2 +$

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$\sin^2(t)\,\mathrm{d}\theta^2$, $\mathrm{d}\theta^2$ being the canonical metric on $\mathbb{S}^2$. Then the projection to the second factor $\pi_2 : \mathbb{R}^+ \times_{t^2} \mathbb{S}^2 \to \mathbb{S}^2$ is a horizontally homothetic submersion with dilation $1/\sin(t)$, integrable horizontal distribution and minimal fibres. Geometrically, π_2 is the projection along the longitudes onto the equatorial sphere. Theorem 3.5 gives a correspondence between free biminimal curves on $\mathbb{S}^2$ and biminimal surfaces (with respect to 2) of $\mathbb{S}^3$ given by $S = \pi_2^{-1}(\gamma(I))$.

(ii) This is the only example for which we use Theorem 3.6. Let $H : \mathbb{S}^3 \to \mathbb{S}^2(\frac{1}{2})$ be the Hopf map defined by $H(z, w) = (2z\bar{w}, |z|^2 - |w|^2)$, where we have identified

$$\mathbb{S}^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\} \quad \text{and} \quad \mathbb{S}^2(\tfrac{1}{2}) = \{(z, t) \in \mathbb{C} \times \mathbb{R} : |z|^2 + t^2 = \tfrac{1}{4}\}.$$

The Hopf map is a Riemannian submersion with minimal fibres (great circles in $\mathbb{S}^3$). Thus, from Theorem 3.6, we see that a Hopf cylinder $H^{-1}(\gamma(I))$ is a biminimal surface (with respect to λ) of $\mathbb{S}^3$ if and only if the curve γ is a biminimal curve (with respect to $\lambda + 4$) of $\mathbb{S}^2(\frac{1}{2})$.

5. Examples of biminimal surfaces in Thurston's three-dimensional geometries

Of Thurston's eight geometries (cf. [**1**, § 10.2]), three have constant sectional curvature ($\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$) and contain biminimal surfaces as described in the previous section, two are Riemannian products ($\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$), and will be our first class of examples, two are line bundles, over $\mathbb{R}^2$ for $\mathcal{H}_3$ and over $\mathbb{R}^2_+$ for $(\mathrm{SL}_2(\mathbb{R}))\tilde{}$, and one, Sol, does not allow Riemannian submersion or horizontally homothetic maps with minimal fibres to a surface, even locally, and therefore does not fit our framework.

5.1. Biminimal surfaces of $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$

In both cases, consider the Riemannian submersion with totally geodesic fibres, given by the projection onto the first factor, $\pi : N^2 \times \mathbb{R} \to N^2$. Given a curve $\gamma : I \subset \mathbb{R} \to N^2$ parametrized by arc length, take its Frenet frame $\{T, N\}$ and consider $\{e_1, e_2\} \in T(N^2 \times \mathbb{R})$ its horizontal lift. The unit vertical vector e_3 completes $\{e_1, e_2\}$ into an orthonormal frame of $T(N^2 \times \mathbb{R})$, such that $\{e_1, e_3\}$ is a basis of TS, for $S = \pi^{-1}(\gamma(I))$, with e_2 the normal to the surface. Then, from Lemma 3.4, the mean curvature of S is $H = \frac{1}{2}k$, where k is the signed curvature of γ and, from Proposition 3.1, S is biminimal (with respect to λ) in $N^2 \times \mathbb{R}$ if

$$\Delta H = (|B|^2 - \mathrm{Ricci}(e_2) + \lambda)H.$$

With respect to the frame $\{e_1, e_3\}$ the matrix associated with the second fundamental form of S is

$$B = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}.$$

In addition,

$$\mathrm{Ricci}^{N^2 \times \mathbb{R}}(e_2) = \mathrm{Ricci}^{N^2}(e_2) = \begin{cases} +1 & \text{if } N^2 = \mathbb{S}^2, \\ -1 & \text{if } N^2 = \mathbb{H}^2. \end{cases}$$

In both cases, using (3.4), $\Delta H = \Delta(\frac{1}{2}k) = \frac{1}{2}k''$, so S is biminimal in $N^2 \times \mathbb{R}$ if

$$k'' = \begin{cases} k^3 - k + \lambda k & \text{if } N^2 = \mathbb{S}^2, \\ k^3 + k + \lambda k & \text{if } N^2 = \mathbb{H}^2. \end{cases}$$

Now comparing these results with (2.2), we have the following proposition.

Proposition 5.1. *The cylinder $S = \pi^{-1}(\gamma(I))$ is a biminimal surface (with respect to λ) in $N^2 \times \mathbb{R}$ if and only if γ is a biminimal curve (with respect to λ) on N^2 ($\mathbb{S}^2$ or $\mathbb{H}^2$).*

5.2. Biminimal surfaces of the Heisenberg space

The three-dimensional Heisenberg space $\mathcal{H}_3$ is the two-step nilpotent Lie group standardly represented in $\mathrm{GL}_3(\mathbb{R})$ by

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$$

with $x, y, z \in \mathbb{R}$. Endowed with the left-invariant metric

$$g = \mathrm{d}x^2 + \mathrm{d}y^2 + (\mathrm{d}z - x\,\mathrm{d}y)^2, \tag{5.1}$$

$(\mathcal{H}_3, g)$ has a rich geometric structure, reflected by the fact that its group of isometries is of dimension 4, the maximal possible dimension for a metric of non-constant curvature on a 3-manifold. Also, from the algebraic point of view, this is a two-step nilpotent Lie group, i.e. 'almost Abelian'. An orthonormal basis of left-invariant vector fields is given, with respect to the coordinates vector fields, by

$$E_1 = \frac{\partial}{\partial x}, \qquad E_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \qquad E_3 = \frac{\partial}{\partial z}. \tag{5.2}$$

Now let $\pi : \mathcal{H}_3 \to \mathbb{R}^2$ be the projection $(x, y, z) \mapsto (x, y)$. At a point $p = (x, y, z) \in \mathcal{H}_3$ the vertical space of the submersion π is $V_p = \ker(\mathrm{d}\pi_p) = \mathrm{span}(E_3)$ and the horizontal space is $H_p = \mathrm{span}(E_1, E_2)$. An easy computation shows that π is a Riemannian submersion with minimal fibres. Take a curve $\gamma(t) = (x(t), y(t))$ in $\mathbb{R}^2$, parametrized by arc length, with signed curvature k, and consider the flat cylinder $S = \pi^{-1}(\gamma(I))$ in $\mathcal{H}_3$. Since the left-invariant vector fields are orthonormal, the vector fields

$$e_1 = x'E_1 + y'E_2 \quad \text{and} \quad e_2 = E_3$$

give an orthonormal frame tangent to S and

$$N = -y'E_1 + x'E_2$$

is a unit normal vector field of S in $\mathcal{H}_3$. The second fundamental form of S is

$$B = \begin{pmatrix} k & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}.$$

Clearly, $H = \operatorname{tr} \frac{1}{2}(B) = \frac{1}{2}k$, $|B|^2 = k^2 + \frac{1}{2}$ and a direct computation shows that $\operatorname{Ricci}(N) = -\frac{1}{2}$. Thus, from (3.1), S is biminimal with respect to λ if and only if

$$\Delta H = (|B|^2 - \operatorname{Ricci}(N) + \lambda)H$$

or, equivalently,

$$k'' = (k^2 + \tfrac{1}{2} + \tfrac{1}{2} + \lambda)k = k^3 + k(1 + \lambda).$$

Finally, taking (2.2) into account, we propose the following.

Proposition 5.2. *The flat cylinder $S = \pi^{-1}(\gamma(I)) \subset \mathcal{H}_3$ is a biminimal surface (with respect to λ) of $\mathcal{H}_3$ if and only if γ is a biminimal curve (with respect to $\lambda + 1$) of $\mathbb{R}^2$.*

5.3. Biminimal surfaces of $(\mathrm{SL}_2(\mathbb{R}))\tilde{}$

Following [**1**, p. 301] we identify $(\mathrm{SL}_2(\mathbb{R}))\tilde{}$ with

$$\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$$

endowed with the metric

$$\mathrm{d}s^2 = \left(\mathrm{d}x + \frac{\mathrm{d}y}{z}\right)^2 + \frac{\mathrm{d}y^2 + \mathrm{d}z^2}{z^2}. \tag{5.3}$$

Then the projection $\pi : (\mathrm{SL}_2(\mathbb{R}))\tilde{} \to \mathbb{R}^2_+$ defined by $(x, y, z) \mapsto (y, z)$ is a submersion, and if we denote, as usual, by $\mathbb{H}^2$ the space $\mathbb{R}^2_+$ with the hyperbolic metric $(\mathrm{d}y^2 + \mathrm{d}z^2)/z^2$, the submersion $\pi : (\mathrm{SL}_2(\mathbb{R}))\tilde{} \to \mathbb{H}^2$ becomes a Riemannian submersion with minimal fibres. The vertical space at a point $p = (x, y, z) \in (\mathrm{SL}_2(\mathbb{R}))\tilde{}$ is $V_p = \ker(\mathrm{d}\pi_p) = \operatorname{span}(E_1)$ and the horizontal space at p is $H_p = \operatorname{span}(E_2, E_3)$, where

$$E_1 = \frac{\partial}{\partial x}, \quad E_2 = z\frac{\partial}{\partial y} - \frac{\partial}{\partial x} \quad \text{and} \quad E_3 = z\frac{\partial}{\partial z} \tag{5.4}$$

give an orthonormal frame on $(\mathrm{SL}_2(\mathbb{R}))\tilde{}$ with respect to the metric (5.3). Now, given a curve $\gamma(t) = (y(t), z(t))$ on $\mathbb{H}^2$, parametrized by arc length, and the flat cylinder $S = \pi^{-1}(\gamma(I))$ in $\mathrm{SL}_2(\mathbb{R})\tilde{}$, as E_1, E_2 and E_3 are orthonormal, the vector fields

$$e_1 = \frac{y'}{z}E_2 + \frac{z'}{z}E_3 \quad \text{and} \quad e_2 = E_1 \tag{5.5}$$

give an orthonormal frame tangent to S and

$$N = -\frac{z'}{z}E_2 + \frac{y'}{z}E_3$$

is a unit normal vector field of S in $(\mathrm{SL}_2(\mathbb{R}))\tilde{}$. With calculations similar to those of the previous example, we find that, with respect to the orthonormal frame (5.5),

$$B = \begin{pmatrix} k & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad \operatorname{Ricci}(N) = -\tfrac{3}{2}.$$

Thus, the following proposition holds.

Proposition 5.3. *The flat cylinder* $S = \pi^{-1}(\gamma(I)) \subset (\mathrm{SL}_2(\mathbb{R}))\tilde{}$ *is a biminimal surface* (*with respect to* λ) *of* $(\mathrm{SL}_2(\mathbb{R}))\tilde{}$ *if and only if* γ *is a biminimal curve* (*with respect to* $\lambda + 1$) *of* $\mathbb{H}^2$.

Remark 5.4. These links between biminimal cylinders and biminimal curves are very similar to those described by Pinkall [**13**] between Willmore Hopf tori of $\mathbb{S}^3$ and elastic curves on $\mathbb{S}^2$.

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