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On weak mixing, minimality and weak disjointness of all iterates

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Abstract. This article addresses some open questions about the relations between the topological weak mixing property and the transitivity of the map $f \times f^2 \times \cdots \times f^m$, where $f : X \to X$ is a topological dynamical system on a compact metric space. The theorem stating that a weakly mixing and strongly transitive system is Δ-transitive is extended to a non-invertible case with a simple proof. Two examples are constructed, answering the questions posed by Moothathu [Diagonal points having dense orbit. *Colloq. Math.* **120**(1) (2010), 127–138]. The first one is a multi-transitive non-weakly mixing system, and the second one is a weakly mixing non-multi-transitive system. The examples are special spacing shifts. The latter shows that the assumption of minimality in the multiple recurrence theorem cannot be replaced by weak mixing.

1. *Introduction*

The systematic study of transitivity and recurrence in dynamics dates back (as is often the case in this subject) to Poincaré. In 1967 Furstenberg [**8**] published his seminal paper, which in recent years has become the basis for a broad classification of dynamical systems by their recurrence properties. For an account of these results and their connections with combinatorics, harmonic analysis and number theory, we refer the reader to Glasner's survey [**10**].

Our purpose here is to study recurrence properties of $f \times f^2 \times \cdots \times f^m$. We clarify dependences between some variants of transitivity by solving open problems posed by Moothathu [**22**]. Our interest in recurrence properties of $f \times f^2 \times \cdots \times f^m$ is motivated by the following version of the celebrated topological multiple recurrence theorem. From it one can deduce the famous van der Waerden theorem on the existence of arbitrarily long arithmetical progressions in some element of a partition of the integers (see [**12**, pp. 46–47]).

TOPOLOGICAL MULTIPLE RECURRENCE THEOREM. [**12**, Theorem 1.56] *Let f be a minimal homeomorphism of a compact metric space X. If U is a non-empty open subset of X, then for every positive integer n there exists a positive integer k with*

$$U \cap f^k(U) \cap f^{2k}(U) \cap \cdots \cap f^{(n-1)k}(U) \neq \emptyset.$$

It follows that if f is a minimal homeomorphism, then for every $m \geq 1$ the map $f \times f^2 \times \cdots \times f^m$ has a residual set of recurrent points. This last observation raises some natural questions: what other recurrence properties does $f \times f^2 \times \cdots \times f^m$ have? Can it be minimal? Must it be at least topologically transitive? Can we replace the assumption of minimality of f by some other recurrence assumption such as weak mixing? We discuss some of these problems in §5. Here we note that it is an immediate consequence of the above theorem that for every n the set $N(U, U; f) = \{m > 0 : f^m(U) \cap U \neq \emptyset\}$ contains an arithmetic progression $k, 2k, \ldots, k(n-1)$. Moreover, the same must hold if f is continuous and topologically mixing. Then one can wonder if weak mixing is also enough. Since weak mixing implies that $N(U, U; f)$ contains arbitrarily long intervals of consecutive integers, it is easy to see that in a weak mixing system for any non-empty open subset of $U \subset X$ and every positive integer n there exist positive integers k, m with

$$m + k, m + 2k, \ldots, m + (n-1)k \in N(U, U; f).$$

Now the question is: *can we demand that $m = 0$?* Theorem 9 shows that the answer must be in the negative.

Another formulation and motivation comes from the notion of *disjointness*, introduced to topological dynamics, as well as to the ergodic theory by Furstenberg in [**8**] and its *weak* form developed in [**1, 12–14**]. Let us recall that f and g are *weakly disjoint* if their Cartesian product $f \times g$ is topologically transitive. Weakly disjoint systems are in a way of *independent* form one another. It is independence in a rather weak sense as it may happen that f is weakly disjoint from itself, that is, f is *weakly mixing*. It is well known that f is weakly mixing if and only if for any $n \geq 2$ the Cartesian product of n copies of f, that is, $f \times \cdots \times f$, is topologically transitive. It follows that if f is weakly mixing, then f^n is topologically transitive for any $n \geq 1$.

Now it is natural to ask: *can f be weakly disjoint from some of its iterates, f^m, where* $m \geq 2$, and *how is the weak disjointness of f and f^m related to weak mixing?* These questions can be thought of as a topological dynamics counterpart of problems considered in ergodic theory (see [**11**]). Here we follow [**22**], and we consider two following properties, very similar to weak mixing.

($\star$) For each $m \in \mathbb{N}$, the map $f \times f^2 \times \cdots \times f^m$ is topologically transitive.

($\star\star$) For each $m \in \mathbb{N}$, there is a residual set $Y \subset X$ such that for every point $x \in Y$ the tuple $(x, \ldots, x) \in X^m$ has a dense orbit in X^m under the map $f \times f^2 \times \cdots \times f^m$.

Following [**22**], we will say that f is *multi-transitive* if it satisfies ($\star$) and that f is Δ-*transitive* if ($\star\star$) holds.

It is known that both properties presented above are equivalent to weak mixing if f is a minimal homeomorphism. The proof of that equivalence using only elementary

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notions of topological dynamics is contained in [**22**]. The implication stating that weak mixing implies Δ-transitivity was earlier proved by Glasner (see [**11**]) with the help of the general structure theorem for minimal homeomorphisms. In [**22**] the question whether this implication holds for non-necessarily invertible continuous maps was left open. Here we answer it affirmatively providing a simple proof for the general case; see Theorem 4 below.

Moreover, we solve another open problem stated in [**22**]. We show that in general there is no connection between weak mixing and multi-transitivity by constructing examples of weakly mixing but non-multi-transitive (Theorem 9) and multi-transitive but non-weakly mixing (Theorem 8) systems. Finally, in §5 we offer some remarks regarding the last question of [**22**] in which Moothathu asked if there is a non-trivial minimal system $f : X \to X$ such that $f \times f^2 \times \cdots \times f^m : X^m \to X^m$ is minimal for some $m \geq 2$.

2. *Preliminaries*

Let X be a compact metric space and $f : X \to X$ be a continuous map. For every $m \geq 1$, denote the Cartesian product of m copies of X with itself by X^m and define two maps of X^m to itself: $f^{(\times m)} = f \times \cdots \times f$ and $f^{(*m)} = f \times f^2 \times \cdots \times f^m$.

Given any sets $U, V \subset X$, we denote $N(U, V; f) = \{n > 0 : f^n(U) \cap V \neq \emptyset\}$. If the map f is clear from the context, we simply write $N(U, V)$.

A map f is *minimal* if it has no proper closed invariant set, that is, if $K \subset X$ is non-empty, closed and $f(K) \subset K$, then $K = X$. We say that f is *(topologically) transitive* if $N(U, V) \neq \emptyset$ for any pair of non-empty open sets $U, V \subset X$. A set $S \subset \mathbb{Z}_+$ is *syndetic* if there is a constant $L > 0$ such that for every $n \geq 0$ we have $[n, n + L] \cap S \neq \emptyset$. Then we say that a map f is *syndetically transitive* if $N(U, V)$ is syndetic for any non-empty open sets $U, V \subset X$. If $f \times f$ is transitive, then we say that f is *weakly mixing*. If for any non-empty open set $U \subset X$ there is $M > 0$ such that $\bigcup_{j=1}^{M} f^j(U) = X$, then f is said to be *strongly transitive*. It immediately follows from the definition that any strongly transitive map is syndetically transitive.

Let f and g be two continuous surjective maps acting on compact metric spaces X and Y, respectively. We say that a non-empty closed set $J \subset X \times Y$ is *a joining* of f and g if it is invariant for the product map $f \times g$ and its projections on first and second coordinate are X and Y respectively. If $X \times Y$ is the only joining of f and g, then we say that f and g are *disjoint*.

The notion of disjointness was first introduced by Furstenberg in [**8**]. It is well known that if f and g are disjoint, then at least one of them is minimal. It is also not so hard to verify that if f, g are both minimal, then they are disjoint if and only if $f \times g$ is minimal.

3. *Strong transitivity and Δ-transitivity*

The main result of this section (Theorem 5) is obtained as a corollary from Theorem 4 below. The Theorem 4 was proved in [**22**, Theorem 4] with the additional assumption that f is a homeomorphism. Here we present it with a new proof, which works for any continuous map.

We recall two results from [**22**], first modifying them to a suitable form.

THEOREM 1. [**22**, Proposition 1] *Let X be a compact metric space. A continuous map $f : X \to X$ is Δ-transitive if and only if for each $m \geq 1$ and non-empty open sets $U, V_1, \ldots, V_m \subset X$, there exists $n \geq 1$ such that*

$$U \cap \bigcap_{i=1}^{m} f^{-in}(V_i) \neq \emptyset.$$

THEOREM 2. [**22**, Corollary 2] *Let X be a compact metric space. If $f : X \to X$ is a weakly mixing and syndetically transitive continuous map, then $f^{(*m)}$ is also weakly mixing and syndetically transitive for any $m \geq 1$. In particular, f is multi-transitive.*

The induction step in the proof of Theorem 4 is based on the following lemma.

LEMMA 3. *Let X be a compact metric space. If $f : X \to X$ is a multi-transitive continuous map, then for any $m \geq 1$ and non-empty open sets $V_1, \ldots, V_m \subset X$, there is a sequence of integers $\{k_n\}_{n=0}^{\infty}$ such that for each $n \geq 0$ we have $k_n - n > 0$, and for each $i = 1, \ldots, m$ there is a sequence $\{V_i^{(n)}\}_{n=0}^{\infty}$ of non-empty open subsets of V_i such that*

$$f^{ik_j - j}(V_i^{(n)}) \subset V_i$$

for $i = 1, \ldots, m$, and $j = 0, \ldots, n$.

Proof. Let $V_1, \ldots, V_m$ be non-empty open subsets of X. Set $W = V_1 \times \cdots \times V_m$. We proceed by induction on n. From the multi-transitivity of f, there is $k_0 > 0$ such that $(f^{(*m)})^{k_0}(W) \cap W \neq \emptyset$, or, equivalently, $f^{-ik_0}(V_i) \cap V_i \neq \emptyset$ for $i = 1, \ldots, m$. Put $V_i^{(0)} = f^{-ik_0}(V_i) \cap V_i \subset V_i$ for $i = 1, \ldots, m$ to complete the base step.

For the induction step, suppose that $n \geq 1$ and we have found a sequence $k_0, \ldots, k_{n-1}$, and for each $i = 1, \ldots, m$ we have a non-empty open set $V_i^{(n-1)} \subset V_i$ such that

$$f^{ik_j - j}(V_i^{(n-1)}) \subset V_i \quad \text{and} \quad k_j - j > 0 \tag{1}$$

holds for $j = 0, \ldots, n-1$. For $i = 1, \ldots, m$, let $U_i = f^{-n}(V_i^{(n-1)})$. Put $U = U_1 \times \cdots \times U_m$. By multi-transitivity, we get an integer k_n such that $k_n - n > 0$ and $(f^{(*m)})^{k_n}(U) \cap W \neq \emptyset$, or, equivalently, $f^{-ik_n}(V_i) \cap U_i \neq \emptyset$, for $i = 1, \ldots, m$. Fix $1 \leq i \leq m$. We have

$$f^{ik_n}(U_i) \cap V_i = f^{ik_n}(f^{-n}(V_i^{(n-1)})) \cap V_i = f^{ik_n - n}(V_i^{(n-1)}) \cap V_i.$$

By the above, $V_i^{(n)} = V_i^{(n-1)} \cap f^{-ik_n + n}(V_i)$ is non-empty, open, and clearly $f^{ik_n - n}(V_i^{(n)}) \subset V_i$. Moreover, $V_i^{(n)} \subset V_i^{(n-1)}$. Using (1), we conclude that

$$f^{ik_j - j}(V_i^{(n)}) \subset V_i$$

for $j = 0, \ldots, n$. This completes the proof. □

THEOREM 4. *Let X be a compact metric space. If $f : X \to X$ is a weakly mixing and strongly transitive continuous map, then f is Δ-transitive.*

Proof. First, note that f is multi-transitive by Theorem 2. In particular, it is transitive and surjective.

To prove that f is Δ-transitive, we are going to use the equivalent condition provided by Theorem 1. We will prove by induction on m that for any non-empty open

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sets $U, V_1, \ldots, V_m \subset X$, there exists $n \geq 1$ such that

$$U \cap \bigcap_{i=1}^{m} f^{-in}(V_i) \neq \emptyset.$$

For $m = 1$, this statement simply follows from the transitivity of f. Assume that we established the result for some $m \geq 1$. We fix non-empty open sets U and $V_1, \ldots, V_{m+1}$, and we want to show that there are $n > 0$ and $z \in U$ such that $f^{in}(z) \in V_i$ for $i = 1, \ldots, m+1$. By strong transitivity, $\bigcup_{j=1}^{N} f^j(U) = X$ for some $N > 0$. Lemma 3 gives us non-empty open sets $V_1^{(N)}, \ldots, V_{m+1}^{(N)}$ and integers $k_0, \ldots, k_N$ such that

$$f^{ik_l - l}(V_i^{(N)}) \subset V_i \quad \text{and} \quad k_l > l$$

for $i = 1, \ldots, m+1$ and $l = 0, \ldots, N$. By the induction hypothesis, we can find $x \in V_1^{(N)}$ and $n > 0$ such that $f^{in}(x) \in V_{i+1}^{(N)}$ for $i = 1, \ldots, m$. Clearly, there is $y \in X$ such that $f^n(y) = x$, but strong transitivity gives us $f^j(z) = y$ for some $z \in U$ and $0 \leq j \leq N$. From the above, we get

$$\begin{aligned} f^{i(n+k_j)}(z) &= f^{i(n+k_j)-j}(y) = f^{ik_j - j}(f^{in}(y)) \\ &= f^{ik_j - j}(f^{(i-1)n}(x)) \in f^{ik_j - j}(V_i^{(N)}) \subset V_i \end{aligned}$$

for any $i = 1, 2, \ldots, m+1$. We showed that

$$z \in U \cap f^{-s}(V_1) \cap \cdots \cap f^{-s\cdot(m+1)}(V_{m+1}),$$

where $s = n + k_j$, which completes the proof. □

THEOREM 5. *Let X be a compact metric space. If $f : X \to X$ is a weakly mixing and minimal continuous map, then f is Δ-transitive.*

Proof. It is well known that any minimal map (invertible or not) on a compact metric space is strongly transitive (see [**19**, Theorem 2.5(8)] for a proof). We apply Theorem 4 to finish the proof. □

Now we may formulate a general version of [**22**, Corollary 7], which was stated there for homeomorphisms. Only the implication given by Theorem 5 is new here. The rest of the proof is identical to that in [**22**].

THEOREM 6. *Let $f : X \to X$ be a minimal continuous map on a compact metric space X. Then the following are equivalent.*

(1) *$f \times f^2$ is transitive.*
(2) *f is multi-transitive.*
(3) *f is weakly mixing.*
(4) *f is Δ-transitive.*

4. *Weak mixing and multi-transitivity*

In [**22**, p. 10] Moothathu asked the following question.

Question 1. Are there any implications between weak mixing and multi-transitivity?

The aim of this section is to show that these notions are not related in a general situation, that is, a continuous map can be multi-transitive and not weakly mixing, or weakly mixing

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and not multi-transitive. As this is often the case, to finish our task we will construct a symbolic system.

Consider the set $A = \{0, 1\}$ endowed with the discrete topology. Let Σ denote the set of all infinite sequences of zeros and ones regarded as the product of infinitely many copies of A with the product topology. All sequences $x \in \Sigma$ are indexed by non-negative integers, $x = x_0x_1x_2 \ldots$. Then the *shift* transformation is a continuous map $\sigma : \Sigma \to \Sigma$ given by $\sigma(x) = y$, where $x = (x_i)$, $y = (y_i)$ and $y_i = x_{i+1}$ for $i = 0, 1, \ldots$. Any closed subset $X \subset \Sigma$ invariant for σ is called a *subshift* of Σ. A *word* is a finite sequence of elements of $\{0, 1\}$. The *length* of a word w is just the number of elements of w, and is denoted by $|w|$. We say that a word $w = w_1w_2 \cdots w_l$ appears in $x = (x_i) \in \Sigma$ at position t if $x_{t+j-1} = w_j$ for $j = 1, \ldots, l$. If X is a subshift, then the *language* of X is the set $\mathcal{L}(X)$ of all words which appear at some position in some element $x \in X$. For any word w, let $[w]_t$ denote the element of the sequence w standing at position t, and let $\mathrm{Sp}(w) = \{|i - j| : [w]_i = [w]_j = 1, i \neq j\}$. The set $\mathcal{L}_n(X)$ consists of all elements of $\mathcal{L}(X)$ of length n.

Let P be a set of non-negative integers. We say that a word $w = w_1w_2 \cdots w_l$ is *P-admissible* if $w_i = w_j = 1$ for some $1 \leq i < j \leq l$ implies $|i - j| \in P$, equivalently, if $\mathrm{Sp}(w) \subset P$. Let Σ_P be the subset of Σ consisting of all sequences x such that every word which appears in x is P-admissible. It is easy to see that Σ_P is a subshift, and $\mathcal{L}(\Sigma_P)$ is the set of all P-admissible words. We will write σ_P for σ restricted to Σ_P, and call the dynamical system given by $\sigma_P : \Sigma_P \to \Sigma_P$ a *spacing* shift. The class of spacing shifts was introduced by Lau and Zame in [**20**], and for a detailed exposition of their properties we refer to [**4**].

Let w be a P-admissible word. By $[w]_P$ we denote the set of all $x \in \Sigma_P$ such that the word w appears at position 0 in x. We call the set $[w]_P$ a *P-admissible cylinder* (a *cylinder* for short). The family of P-admissible cylinders is a base of the topology of Σ_P inherited from Σ. It is easy to see that the definition of a spacing shift implies that $N([1]_P, [1]_P; \sigma_P) = P$. Moreover, σ_P is weakly mixing if and only if P is a *thick* set (see [**4, 20**]). A thick set is a subset of integers that contains arbitrarily long intervals (P is thick if and only if, for every n, there is some k such that $\{k, k+1, \ldots, k+n-1\} \subset P$). If w is a word and $n \geq 1$, then by w^n we denote a word which is a concatenation of n copies of w. If $n = 0$, then w^n is the *empty word*.

4.1. *Multi-transitive and not weakly mixing example.* The results of this section generalize construction of the totally transitive not weakly mixing spacing shift presented in [**4**].

We say that a finite set $S \subset \mathbb{N}$ is *q-dispersed*, where $q \geq 2$, if for every $a, b \in S \cup \{0\}$ such that $a \neq b$, we have $|a - b| \geq q$.

LEMMA 7. *Let M, N be positive integers such that $M \geq 3$ and let $A \subset \mathbb{N}$ be an M-dispersed finite set. Then there exists an M-dispersed finite set B containing A and such that for $k = \max(A) + 1$ and any pair of sequences of words $u_1, \ldots, u_N$ and $v_1, \ldots, v_N$ from $\mathcal{L}_k(\Sigma_B)$, there is $n \geq 0$ such that*

$$\sigma^{in}([u_i]_B) \cap [v_i]_B \neq \emptyset \quad \textit{for } i = 1, \ldots, N.$$

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Proof. Let $k = \max(A) + 1$. Let $m = |\mathcal{L}_k(\Sigma_A)|^{2N}$ be the cardinality of the set of all N-element sequences of pairs of words from $\mathcal{L}_k(\Sigma_A)$. We enumerate all members of this set as a list $W^{(1)}, \ldots, W^{(m)}$. Hence, each $W^{(j)}$ is an ordered list of N pairs of words from $\mathcal{L}_k(\Sigma_A)$:

$$W^{(j)} = ((u_1^{(j)}, v_1^{(j)}), \ldots, (u_N^{(j)}, v_N^{(j)})) \quad \text{for each } j = 1, \ldots, m,$$

where $(u_i^{(j)}, v_i^{(j)}) \in \mathcal{L}_k(\Sigma_A) \times \mathcal{L}_k(\Sigma_A)$ for every $i = 1, \ldots, N$. Choose integers $l_1, \ldots, l_m$ fulfilling the following conditions:

$$l_1 \geq 2k + M - 1, \tag{2}$$

$$l_{j+1} \geq (N+1)^j l_j. \tag{3}$$

Given $1 \leq i \leq N$ and $1 \leq j \leq m$, we define

$$w_i^{(j)} = u_i^{(j)} 0^{il_j - k} v_i^{(j)},$$

where $l_1, \ldots, l_m$ are as above. Using (2) and (3), it is easy to see that

$$[il_\alpha - k + 1, il_\alpha + k - 1] \cap [jl_\beta - k + 1, jl_\beta + k - 1] = \emptyset \tag{4}$$

for $1 \leq \alpha, \beta \leq m$, $\alpha \neq \beta$ and $1 \leq i, j \leq N$. Let

$$B = \bigcup_{j=1}^{m} \bigcup_{i=1}^{N} \mathrm{Sp}(w_i^{(j)}).$$

If $n \in A$, then let $u = 10^{n-1}10^{k-n-1}$. Clearly, $n \in \mathrm{Sp}(u)$ and $u \in \mathcal{L}_k(\Sigma_A)$, since $k = \max(A) + 1$. This gives $A \subset B$. The construction of $w_i^{(j)}$ implies that for $1 \leq i \leq N$ and $1 \leq j \leq m$ we have

$$\mathrm{Sp}(w_i^{(j)}) \backslash A \subset [il_j - k + 1, il_j + k - 1]. \tag{5}$$

Therefore,

$$\min(B \backslash A) \geq l_1 - k + 1 \geq M + k. \tag{6}$$

In particular, $\min B = \min A \geq M$. Moreover, we conclude from (4) and (5) that if $r \in B \backslash A$, then there are unique indexes $i(r)$ and $j(r)$ such that $r \in \mathrm{Sp}(w_{i(r)}^{(j(r))})$.

Next, we are going to prove that B is M-dispersed, that is, $|q - p| \geq M$ for each $q, p \in B$, $q \neq p$. We consider three cases.

Case I: Both p and q belong to A.

Case II: Both p and q belong to $B \backslash A$.

Case III: None of the above cases hold.

The first case is clear, since A is M-dispersed. The third case follows from (6). To prove the remaining case, case II, we consider subcases. First note, however, that in the computations below we use (2)–(5) without further reference. Given $p, q \in B \backslash A$, consider the following.

Case IIA: $j(p) \neq j(q)$. Without loss of generality, we assume $j(q) > j(p)$. We have

$$\begin{aligned} q &\geq i(q) l_{j(q)} - k + 1 \geq l_{j(q)} - k + 1 \\ &\geq (N+1) l_{j(p)} - k + 1 \geq N l_{j(p)} - k + 1 + l_1 \\ &\geq i(p) l_{j(p)} + k + M \geq p + M. \end{aligned}$$

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But then

$$q - p \geq M.$$

Case IIB: $j(p) = j(q)$, but $i(p) \neq i(q)$. Without loss of generality, we assume $i(q) > i(p)$. Let $j = j(p) = j(q)$. Then

$$\begin{aligned} q &\geq i(q)l_j - k + 1 \geq (i(p) + 1) \cdot l_j - k + 1 \\ &\geq i(p)l_j - k + 1 + l_1 \geq i(p)l_j + k + M \geq p + M. \end{aligned}$$

Hence,

$$q - p \geq M.$$

Case IIC: $j(p) = j(q)$, and $i(p) = i(q)$. Let $j = j(p) = j(q)$ and $i = i(p) = i(q)$. For $r \in \{p, q\}$, we define

$$s(r) = \min\{s : [w_i^{(j)}]_s = [w_i^{(j)}]_{s+r} = 1\}.$$

Clearly, either $s(p) \neq s(q)$, or $s(p) + p \neq s(q) + q$. We have

$$\begin{aligned} |q - p| &= |(s(q) + q) - s(q) - (s(p) + p - s(p))| \\ &= |(s(q) + q) - (s(p) + p) - (s(q) - s(p))| \\ &\geq ||(s(q) + q) - (s(p) + p)| - |s(q) - s(p)||, \end{aligned}$$

but

$$|(s(q) + q) - (s(p) + p)|, |s(q) - s(p)| \in A \cup \{0\},$$

so either

$$|(s(q) + q) - (s(p) + p)| \neq |s(q) - s(p)|$$

and then

$$|(s(q) + q) - (s(p) + p)| - |s(q) - s(p)| \geq M,$$

or $|(s(q) + q) - (s(p) + p)| = |s(q) - s(p)| \neq 0$, and then

$$|q - p| \geq 2M.$$

It remains to prove that for any pair of sequences of words $u_1, \ldots, u_N$ and $v_1, \ldots, v_N$ from $\mathcal{L}_k(\Sigma_B)$, there is $n \geq 0$ such that

$$\sigma^{in}([u_i]_B) \cap [v_i]_B \neq \emptyset \quad \text{for } i = 1, \ldots, k.$$

Observe that $\mathcal{L}_k(\Sigma_B) = \mathcal{L}_k(\Sigma_A)$, since $\min(B \setminus A) \geq k$, $\max(A) + 1 = k$, and $A \subset B$. Therefore, according to our notation defined at the beginning of the proof, for any two sequences of words $u_1, \ldots, u_N$ and $v_1, \ldots, v_N$ from $\mathcal{L}_k(\Sigma_B)$, there is $j = 1, \ldots, m$ such that

$$W^{(j)} = ((u_1, v_1), \ldots, (u_N, v_N)).$$

Let $w_i^{(j)} = u_i 0^{il_j - k} v_i$ as above. Clearly, $w_1^{(j)}, \ldots, w_N^{(j)} \in \mathcal{L}(\Sigma_B)$, and from the definition of $w_i^{(j)}$ we conclude that

$$\sigma^{in}(w_i^{(j)}) \in \sigma^{in}([u_i]_B) \cap [v_i]_B \quad \text{for } n = l_j.$$

Hence,

$$\sigma^{in}([u_i]_B) \cap [v_i]_B \neq \emptyset \quad \text{for } i = 1, \ldots, N,$$

where $n = l_j$. □

THEOREM 8. *There exists a set $P \subset \mathbb{N}$ such that the spacing shift (Σ_P, σ_P) is multi-transitive but not weakly mixing.*

Proof. Fix any integer $M \geq 3$ and denote $P_0 = \{M\}$. Define a sequence of sets $P_n \subset \mathbb{N}$ $(n \geq 1)$ inductively by putting $P_{n+1} = B$, where B is the set obtained for $A = P_n$, $N = n$, and M as above by Lemma 7. Denote

$$P = \bigcup_{n=0}^{\infty} P_n.$$

Easy induction gives $|p - q| \geq M$ for every distinct $p, q \in P$ and $P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq \cdots$. In particular, P is not thick, so Σ_P is not weakly mixing. We are going to show that $\sigma_P \times \sigma_P^2 \times \cdots \times \sigma_P^m$ is transitive for any $m = 1, 2, \ldots$. Fix any integer $m \geq 1$ and choose any open sets $U_1, \ldots, U_m, V_1, \ldots, V_m \subset \Sigma_P$. Without loss of generality, we may assume that for each $1 \leq i \leq m$ there are words $u_i, v_i \in \mathcal{L}(\Sigma_P)$ such that $[u_i]_P \subset U_i$, and $[v_i]_P \subset V_i$. We may also assume that for each $1 \leq i \leq m$ we have $u_i, v_i \in \mathcal{L}_k(\Sigma_{P_l})$ for some $l \geq m$ and $k = \max(P_l) + 1$. The last equality implies that $\mathcal{L}_k(\Sigma_{P_l}) = \mathcal{L}_k(\Sigma_P)$. If $m < l$, then we put $u_j = v_j = u_m$ for $j = m + 1, \ldots, l$.

Now, by Lemma 7, there is $j > 0$ such that

$$\begin{aligned} \sigma_P^{ij}(U_i) \cap V_i &\supset \sigma^{ij}([u_i]_P) \cap [v_i]_P \\ &\supset \sigma^{ij}([u_i]_{P_l}) \cap [v_i]_{P_l} \neq \emptyset \end{aligned}$$

for $i = 1, \ldots, l$. We have just proved that $\sigma_P \times \sigma_P^2 \times \cdots \times \sigma_P^m$ is transitive for any $m = 1, 2, \ldots$, which in other words means that σ_P is multi-transitive. □

It is clear from the construction of P in Lemma 7, that the spacing shift σ_P from the assertion of Theorem 8 is not syndetically transitive, since the set P, and as a result $N([1]_P, [1]_P)$, have thick complement. Then the following question arises.

Question 2. Does every multi-transitive and syndetically transitive system have to be weakly mixing?

4.2. *Weakly mixing and not multi-transitive example.* Fix $m \geq 2$. Let

$$B(m, k) = \{m^{2k-1}, m^{2k-1} + 1, \ldots, m^{2k} - 1\} \quad \text{and} \quad P(m) = \bigcup_{k=1}^{\infty} B(m, k).$$

Observe that for every $m \geq 2$ the set $P(m)$ has the following property:

$$p \in P(m) \implies m \cdot p \notin P(m). \tag{7}$$

THEOREM 9. *Let $m \geq 2$ and $P = P(m)$ be as defined above. Then $\tau = \sigma_P \times \cdots \times \sigma_P^{m-1}$ is transitive, but $\tau \times \sigma_P^m$ is not transitive. In particular, the spacing shift (Σ_P, σ_P) is weakly mixing, but not multi-transitive.*

Proof. It is easy to see that P is thick, hence σ_P is weakly mixing. To prove that $\tau = \sigma_P \times \cdots \times \sigma_P^{m-1}$ is transitive, we fix open cylinders:

$$[u^{(1)}]_P, \ldots, [u^{(m-1)}]_P, [v^{(1)}]_P, \ldots, [v^{(m-1)}]_P \in \mathcal{L}(\Sigma_P).$$

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Without loss of generality, we may assume that there is $k \geq 1$ such that for any $i = 1, \ldots, m-1$ we have $|u^{(i)}| = |v^{(i)}| = t$, where $t = m^{2k}$. Set $s = m^{2k+1} + m^{2k}$ and define

$$w^{(i)} = u^{(i)} 0^{is-t} v^{(i)} \quad \text{where } i = 1, \ldots, m-1.$$

Clearly,

$$[w^{(i)}]_P \subset (\sigma_P^i)^{-s}([v^{(i)}]_P) \cap [u^{(i)}]_P,$$

and therefore

$$[w^{(1)}]_P \times \cdots \times [w^{(m-1)}]_P \subset \tau^{-s}([v^1]_P \times \cdots \times [v^{(m-1)}]_P) \cap ([u^{(1)}]_P \times \cdots \times [u^{(m-1)}]_P),$$

so it is enough to prove that $[w^{(i)}]_P \neq \emptyset$, that is, $w^{(i)} \in \mathcal{L}(\Sigma_P)$. It follows from the definition of $w^{(i)}$ that

$$\mathrm{Sp}(w^{(i)}) = \mathrm{Sp}(u^{(i)}) \cup \mathrm{Sp}(v^{(i)}) \cup \{l - k : (l, k) \in \Delta\},$$

where Δ is some subset of

$$\{0, \ldots, m^{2k} - 1\} \times \{i \cdot m^{2k+1} + i \cdot m^{2k}, \ldots, i \cdot m^{2k+1} + (i+1) \cdot m^{2k} - 1\}.$$

Hence, we have

$$l - k \in \{m^{2k+1}, \ldots, m^{2k+2} - 1\} \subset B(m, k+1),$$

and $w^{(i)} \in \mathcal{L}(\Sigma_P)$, as desired. We proved that $\tau = \sigma_P \times \cdots \times \sigma_P^{m-1}$ is transitive. To finish the proof, it is enough to show that $\sigma_P \times \sigma_P^m$ is not transitive. Let $U = V = [1]_P \times [1]_P$. It is easy to see from (7) that

$$(\sigma_P \times \sigma_P^m)^n(U) \cap V = \emptyset$$

for every $n \geq 0$, so $\sigma_P \times \sigma_P^m$ cannot be transitive. □

In the literature other recurrence properties stronger than weak mixing are considered, see [**10**] for example. It is natural to ask if we can replace weak mixing by one of these properties in Theorem 9. In the view of the above results, we would like to pose the following problem.

Question 3. Is there any non-trivial characterization of multi-transitive weakly mixing systems?

5. *Minimal self-joinings*

The last question in [**22**] asks: *can $f \times f^2 \times \cdots \times f^m : X^m \to X^m$ be minimal if $m \geq 2$ and X has at least two elements?* Let us call a map $f : X \to X$ providing an affirmative answer to the above question *multi-minimal*. Apparently, Moothathu, when posing his problem, was not aware that examples of multi-minimal homeomorphisms are known. But since their existence is stated in language slightly different to the terminology used in [**22**], we find it necessary to add some explanations. In fact, the construction of multi-minimal systems is related to considerations of multiple disjointness.

The first example of a system disjoint from any of its iterates (we are aware of), is the example of a POD (*proximal orbit dense*) minimal homeomorphism given by Furstenberg *et al* in [**9**]. By [**21**, Theorem 2.6], every POD system has *positive topological minimal*

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self-joinings (see [**21**]). It also follows from [**21**, Proposition 2.1] that every homeomorphism possessing positive topological minimal self-joinings is multi-minimal, and so is the example from [**9**]. Furthermore, del Junco's work [**15**], together with his joint work with Rahe and Swanson [**16**], shows that Chacon's example [**7**] is POD, and hence also multi-minimal. In [**2**], Auslander and Markley introduced the class of *graphic* minimal systems, which generalizes POD homeomorphisms. They also proved that each graphic flow is multi-minimal [**2**, Corollary 22]. Moreover, as reported in [**2**, p. 490], Markley constructed an example of a graphic homeomorphism which is not POD, hence it is another kind of multi-minimal homeomorphism.

More information about minimal subsystems of $f \times f^2 \times \cdots \times f^m$ is to be found in [**3, 5, 6, 17, 18**], to name but a few. There is also the, in some sense, parallel and certainly deep theory of minimal self-joinings (a part of ergodic theory) introduced by Rudolph [**23**]; see Glasner's book [**12**]. We remark that although every weak mixing minimal map is multi-transitive, it is not necessarily multi-minimal. *Discrete horocycle flow* h is an example of a weakly mixing minimal homeomorphism such that h is topologically conjugated to h^2, and hence it is not multi-minimal (see [**12**, pp. 26, 105–110]). The facts gathered above prompt us to raise following questions.

Question 4. Is there any non-trivial characterization of multi-minimality in terms of some dynamical properties?

It is also interesting to ask if it is it possible to characterize multi-minimal systems adding some mild assumptions to Theorem 6. In particular, we do not know the answer to the following question.

Question 5. Assume that f is a weakly mixing map such that $f \times f^2$ is minimal. Is f necessarily multi-minimal?

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