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ACTUARIES AND DERIVATIVES

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ABSTRACT

This paper draws analogies between techniques used to reserve for, control and manage derivatives and techniques used by actuaries in other fields. It concentrates on equity derivatives. It also includes a review of the factors which significantly influence the appropriate size of reserves to hold for a derivatives portfolio. These include the likelihood of market jumps, uncertainty in future market volatility and the size of transaction costs, as well as on more obvious factors like position risk.

KEYWORDS

Derivatives; Reserving for Derivatives; Hedging; Principle of No Arbitrage; Risk-Neutral Probability Law

1. INTRODUCTION

1.1 *History of Derivatives Markets*

1.1.1 The derivatives markets have seen a huge explosion in variety and use over the last 20 years or so. Despite the occasional hiccup, derivatives have become an accepted fact of modern investment life.

1.1.2 This is illustrated by the growth of business transacted on the major derivatives exchanges. For example, the London International Financial Futures and Options Exchange (LIFFE) has seen explosive growth in most of its contracts over the last 10 years or so. Figure 1 shows the growth in its FT-SE 100 Index futures contract, which is not atypical.

1.1.3 Indeed, in most major locations the volumes of equity exposures transacted through the derivatives markets now significantly exceed the volumes transacted on the underlying stock markets (see Figure 2).

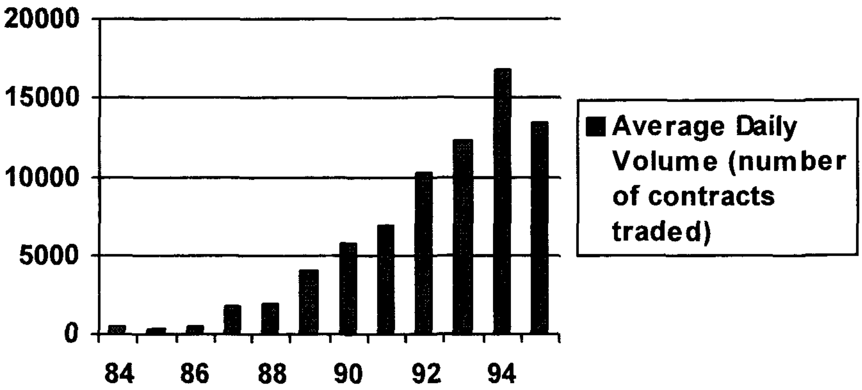
1.2 *What Role is there for Actuaries in Derivatives Markets?*

1.2.1 Derivatives are tools for managing financial risks. They have acquired a reputation for complicated mathematics.

1.2.2 Actuaries also have a reputation for applying mathematics to problems involving financial risks. Indeed, the motto of the Institute of Actuaries is *Certum ex Incertis*, i.e. 'certainty from uncertainty'. There ought, therefore, to be a natural fit, although the involvement to date of actuaries in the derivatives field has been relatively limited.

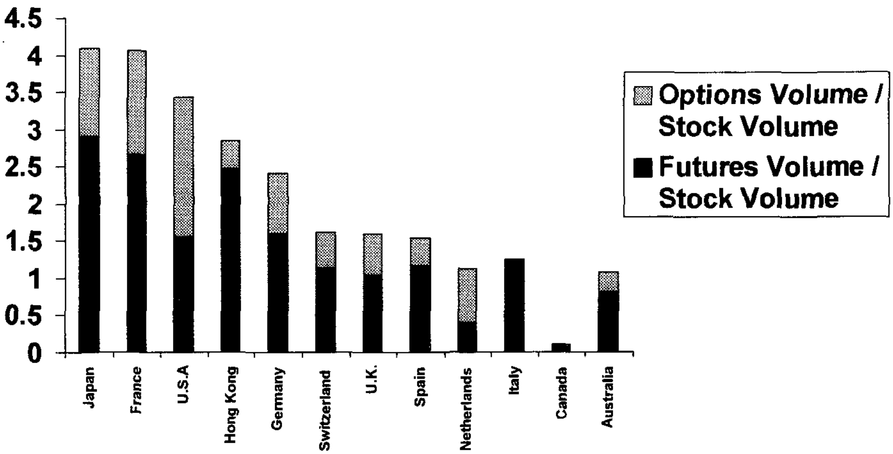
1.3 *The Aim of this Paper*

It is the aim of this paper to develop this fit further, drawing analogies



Source: LIFFE

Figure 1. Growth of FT-SE 100 Index futures volume



Source: Goldman Sachs

Figure 2. Relative amounts traded in futures, options and stocks on various markets

between techniques relevant to (principally equity) derivatives and those used by actuaries in other fields.

1.4 *The Structure of this Paper*

1.4.1 In Sections 2 and 3 the main types of derivatives are summarised, as

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are the main uses to which they are put (by institutions of the sort most typically advised by actuaries).

1.4.2 Sections 4 to 10 discuss the factors influencing the *pricing*, *reserving requirements* and *hedging* of derivatives. Derivatives are risk management tools, which involve the transfer of risk from one party to another. There will, in many instances, be a *price* (paid from one party to the other) for this transfer of risk. The party assuming the risk may need to set aside capital (in actuarial terminology *set up reserves* or *provisions*) to protect its balance sheet against the risks it is acquiring. The size of the capital/reserves, implicit or explicit, will, in general, depend on how the risks are being *hedged*.

1.4.3 A lot can be said about these topics without introducing a large amount of complicated mathematics. However, in my opinion, a fuller understanding does require some more detailed knowledge of the mathematics of option pricing and hedging, and study of this subject repays the effort involved. Some of the key results needed to understand the results in the main text are covered in Appendices A and B. Interest rate derivatives have some special characteristics which are described briefly in Appendix C. Appendix D contains details of the reserving requirements to which banks are now subject, since these may not be familiar to many actuaries.

1.4.4 Losses involving derivatives often seem to have little to do with failures linked to the more complicated mathematics, and are often much more influenced by basic failures in control systems. Sections 11 and 12 consider how to control and manage portfolios that include derivatives. Many of the issues essentially boil down to sound common sense and a good understanding of the way in which derivatives (and derivatives businesses) operate.

1.4.5 Finally, Section 13 draws together the analogies touched on in the rest of the paper and summarises the factors influencing the reserves required for a derivatives book.

2. The Main Types of Derivatives

2.1 *The Basic Nature of Derivatives*

2.1.1 Derivatives are investment instruments whose value or behaviour 'derives' from the value or behaviour of other more basic economic variables.

2.1.2 Typically, the *underlying* variables (often shortened merely to the 'underlying') would be traded securities or corresponding market indices, interest rates, currencies or commodity prices. Derivatives can also 'derive' from the behaviour of more esoteric economic variables, e.g. the credit standing of a specific organisation, a basket of insurance contracts, or the price of a specific fund or unit trust.

2.2 *The Main Sorts of Instruments Typically Categorised as Derivatives*

2.2.1 *Forwards and futures contracts*

2.2.1.1 These are contracts in which two parties agree to carry out a transaction at some future date (the *maturity* date) on terms agreed now. These sorts of derivatives are *symmetric*, in the sense that both parties are obliged to carry out the transaction (unless they mutually agree to cancel it before it matures). Such contracts effectively involve the swapping of economic exposures at the date the contract is effected, without the legal transfer of assets taking place until the contract matures (although the tax treatment of such contracts may assume that the transfer takes places as soon as the contract is entered into).

2.2.1.2 Usually the term *forward* is applied to all such contracts, with the term *futures* limited to contracts traded on a recognised exchange.

2.2.2 *Options*

2.2.2.1 These are contracts, which again relate to a transaction at some future date on terms agreed now, but in which one party (the purchaser of the option) is free to decide whether or not the transaction will go ahead. They are *asymmetric*, in that the purchaser of the option can decide to allow it to lapse, but the seller (otherwise known as the *writer* of the option) is obliged to carry out the transaction if required to do so. A *call option* gives the purchaser the right to *buy* the underlying security at a given price. A *put option* gives the holder the right to *sell* the underlying security at some predefined price. Simple options like puts and calls are called *vanilla* options. Many more complicated sorts of options exist, some of which are described later on in this paper.

2.2.2.2 The price involved in the option transaction is normally fixed in monetary terms. However, it can, in principle, be based on any *numeraire*. For example, pension funds (or life offices) wishing to protect their solvency positions may find it helpful to purchase *relative performance* put options, where the price at which equities can be sold is based on the price of suitable sorts of gilt-edged securities. The numeraire is then not cash, but some gilt index.

2.2.2.3 Selling equities for gilts is the same as buying gilts for equities. Thus, in this example, the option could be described as either a relative performance put option on equities or a relative performance call option on gilts. For many derivatives some clarification of numeraire is needed. For example, a currency option might be referred to as a United States dollar call/United Kingdom sterling put (indicating that the option permits the purchasers to buy a predetermined amount of U.S. dollars by selling a predetermined amount of sterling at a set time in the future).

2.2.3 *Swaps, swaptions, caps, floors and collars*

2.2.3.1 A *swap* is, in effect, a series of forward contracts bundled together. One party agrees to swap a whole series of payments for another series. For example, an interest rate swap would typically involve one party swapping floating rate interest payments in return for receiving a series of payments calculated on a fixed rate of interest. *Swaptions* are options to take out swaps on predetermined terms in the future. The most common sorts of interest rate

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options are *caps*, *floors* and *collars*. With a cap, the floating interest payments are capped at some predefined maximum if the floating rate rises above this maximum. A floor has a predefined minimum below which the floating payments do not fall. A collar involves both a cap and a floor.

2.2.3.2 Another sort of interest rate derivative is a *forward rate agreement*. In such an agreement parties, in effect, lock in the future interest rate for loans or deposits yet to be made.

2.3 *Other Contracts with Derivative-Like Characteristics*

2.3.1 The definition of a derivative in Section 2.1 is the textbook one. If it were followed strictly, then a very large number of investment transactions could be characterised as derivatives transactions.

2.3.2 For example, in England the vast majority of properties are bought using an exchange of contract followed by completion of the contract some time (e.g. two weeks) later. This technically satisfies the definition of a forward contract, as set out above. The buyer and seller of the property commit to carry out the transaction on the date of exchange of contract, but the asset does not legally change hands until completion.

2.3.3 Also, the standard method of carrying out share transactions practically everywhere in the world involves two sides agreeing a deal, but then settling the deal several days later (with legal ownership only passing once the deal has settled). Again, this is strictly a forward transaction.

2.3.4 Option-like characteristics also appear in many guises, e.g. in convertibles or warrants. Indeed, arguably, they even appear with ordinary shares, because of the option-like feature introduced by limited liability. The economic disciplines of pricing options and of pricing corporate contingent claims are very closely allied.

2.3.5 One could even argue that the above definition of a derivative encompasses with-profits contracts. In these contracts, the life office pays sums which depend on the behaviour of some underlying assets. They also have option-like characteristics, since there is a minimum sum payable (the guaranteed sum assured plus reversionary bonuses already accrued). One difference between with-profits contracts and the sorts of contracts typically understood as derivatives is that the behaviour of a derivative contract is normally specified precisely in advance, whereas the behaviour of a with-profits contract is less clearly defined (since bonuses declared are based on actuarial judgement).

2.3.6 On this basis, it could be argued that U.K. life offices are by far the largest writers of equity derivatives in the U.K. via their with-profits contracts. It is, therefore, a little surprising that actuaries do not make more explicit use of techniques borrowed from the derivatives industry within their work. In some quarters of the profession the similarity between with-profits contracts and options has been recognised for some time, see e.g. Wilkie (1987) and Beenstock & Brasse (1986). There has also been some discussion recently within the North American actuarial profession about whether they should specifically incorporate option valuation techniques in the methodologies they use for liability valuations.

2.4 *The Tendency of People to Choose a Definition that Suits Themselves*

2.4.1 It is, therefore, normal practice to narrow the definition of what is, or is not, understood as a derivative. There is a tendency to tailor the definition so that things that have been done 'since time immemorial' are not viewed as derivatives.

2.4.2 For example, forward currency contracts may be excluded, since they are very commonly employed when an overseas transaction is undertaken. Having them classified as a derivative might open them to greater scrutiny by one's superiors.

2.4.3 Stocklending is also rarely viewed as a derivative. This is despite it legally consisting of an agreement to sell the underlying security now, and to repurchase it at some later time. The second leg of the transaction is clearly some form of forward contract!

2.4.4 Sometimes it is advantageous if an instrument *is* classified as some specific form of derivative (perhaps because it can then receive a favourable tax treatment). The definition of what constitutes a derivative may then be widened.

2.5 *Margining and Credit Risk*

2.5.1 When two participants enter into a forward contract, it is usually entered into at nil market value (in the sense that neither party to the transaction pays the other anything, and, in theory, both would at outset be willing to close out the transaction without any payment either). However, in general, its net present value will not remain zero as time progresses. Thus, both participants potentially acquire the credit risk of the other. The credit exposures involved can become substantial if there is a sustained price movement in the underlying.

2.5.2 Derivatives exchanges (which are often called futures exchanges, even if they also trade options) generally try to limit these credit exposures by the use of a *central clearing house* and a *margining system*. Both sides of the transaction pay a returnable good faith deposit, called *initial margin*, to the clearing house, which is normally set up with as high a credit-worthiness as possible. The clearing house typically interposes itself between the two parties by replacing (*novating*) the original contract with two separate equal and opposite contracts, one between itself and the first party and one between itself and the second party. Thus, each party is only exposed to the credit risk of the clearing house and not, potentially, to that of other market participants. The interposition of the clearing house has the further advantage that it enables either party to close out the transaction separately at a future date prior to the maturity of the contract (as long as it can find a third party willing to carry out the necessary transaction).

2.5.3 To avoid additional credit exposures arising as the price of the underlying changes, futures contracts are typically *marked to market*, usually at the close of each day, with any capital gain or loss being paid from or to the clearing house. These payments are called *variation margin*.

2.5.4 Whilst the theory behind margining and other credit risk reduction techniques is fairly clear, the precise mechanics (and who it covers) may be less obvious. For example, on LIFFE, the novation of contracts by the clearing house relates only to *clearing members* of the exchange. External market participants need to deal via a clearing member, known as the investor's *clearing agent*, and therefore remain exposed to the credit risk of their clearing agent.

2.5.5 Credit risk is a very important operational consideration with derivatives (as with other sorts of investment transactions), and we return to it in Section 12.

2.5.6 The control of credit risk with *over-the-counter* (OTC) contracts, i.e. ones that are traded outside a recognised exchange, is usually less strong. However, it may be possible to wrap up the derivative within some form of margining or *collateralisation*, in which, say, purchasers of options have credited to a suitable escrow account or deposited back with them part or all of the market value of the derivative contract. Any such system will typically add complexity to dealing in the derivative, although not necessarily any greater complexity than applies to other sorts of investment transactions.

3. The Main Uses of Derivatives

3.1 *The Potential Uses of Derivatives*

3.1.1 The huge growth in the derivatives business (according to accepted industry definitions of what constitutes a derivative) reflects the large number of potential uses to which derivatives can be put.

3.1.2 Textbooks typically introduce these uses by referring to corporates using currency and interest rate derivatives to hedge income and expenditure components of their profit and loss accounts. They may, for example, make goods in one country, but sell them somewhere else. They can protect themselves against the risk of adverse currency movements by selling forward the receipts they expect to get from the sales of their goods, locking in a known exchange rate at outset. Interest rate swaps may help them fix more precisely their likely borrowing costs.

3.1.3 Financial institutions more usually advised by actuaries, such as pension funds and insurance companies, typically use derivatives for slightly different purposes, e.g.:

(a) to alter the effective asset allocation of the fund, typically using futures or forwards;
(b) to protect the fund against some adverse market movement, e.g. a substantial decline in equity values, typically using options; or
(c) to hedge fairly precisely some specific liability, e.g. the liability incurred by a

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life office when writing a guaranteed equity bond, see e.g. Sheldon & Dodhia (1994).

3.1.4 Pension funds and insurance companies may also use options on individual securities, principally to take a specific investment view relating to that security. Swap contracts are used by insurance companies, but not very frequently.

3.1.5 Derivatives are also often used for tax planning purposes, e.g. to avoid crystallising past capital gains. Tax uses are not just limited to traditional 'net' funds. For example, purchase of overseas market exposure by use of futures leaves the stocks with the underlying investor, who may be a local, and thus able to avoid withholding tax on dividends. Even for a typical 'gross' fund like a pension fund, the return available by investing though a futures contract may, therefore, be greater than is available by investing directly in the underlying equities (and incurring withholding tax).

3.1.6 A large number of new types of derivatives have been developed over the last few years. For example, there are now credit derivatives which pay out if there is a defined credit 'event', like a default, for a specific counterparty. Summarising all the possible types (or uses) of derivatives is practically impossible (and even if undertaken would become rapidly out of date), so the remainder of this section concentrates on the main uses, as set out in ¶3.1.3.

3.2 *Asset Allocation using Futures*

3.2.1 The most important use of derivatives by pension funds and insurance companies, at the present time, is for asset allocation.

3.2.2 Most funds have some sort of benchmark around which they are positioned. The benchmark is often the average asset mix of funds against which the house is competing. However, increasingly it may be set specifically on the basis of the fund's liabilities. A fund management house will be overweight (relative to the benchmark) in those markets it likes and underweight in markets it dislikes.

3.2.3 Usually houses separate out decisions relating to individual stock selection from decisions relating to markets as a whole. Indeed, in all but the smallest houses, individual fund managers will be divided up into teams, or *desks*, on a geographical basis, with asset allocation chosen by a committee of the house's more senior fund managers.

3.2.4 If a large house wishes to shift market exposure rapidly across many funds, it faces some significant problems. Trading large volumes in the relevant stock markets may move the markets against the house. It also takes time for each desk to decide which stocks to buy or sell, but the asset allocation committee may wish to crystallise the change in market exposure much more quickly.

3.2.5 The solution to both of these problems is to use futures. For example, suppose that the asset allocation committee becomes bearish on U.S. equities, and decides to reduce its exposure there and increase its U.K. equity exposure.

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We shall assume that foreign equity exposure is not hedged in the benchmark (as is usual in the U.K.). The desired change in exposures can be implemented very rapidly by buying appropriate amounts of FT-SE futures and selling Standard & Poors (S&P) futures. Because the S&P future is denominated in dollar terms, we would also need to sell dollars forward for sterling if we wished to eliminate currency risk.

3.2.6 The U.S. fund manager can then identify suitable stocks to sell in a more leisurely fashion. Whenever he sells some stock, some of the S&P futures position and some of the currency forward would be unwound. The FT-SE futures position would be unwound as the U.K. fund manager found suitable opportunities for investment, but this does not need to be at the same time as the U.S. stocks are sold.

3.2.7 For a typical balanced fund, even a shift of 2% out of a market like the U.S.A. could involve selling 25% or more of that part of the portfolio. This would be a very stressful activity if the U.S. fund manager has to finish it in an afternoon, to meet some demanding timetable set by the asset allocation committee.

3.2.8 The main advantages of futures, apart from a less stressful life for the fund manager, are:

(a) *Speed and liquidity.* Futures are now usually more liquid than the markets underlying them, so it is easier to deal in size without moving the market. The very largest houses would find it nearly impossible to make substantial asset allocation shifts quickly without using futures. The level of liquidity in major futures markets versus the level of liquidity in the underlying stock markets can, in part, be gauged by the volumes dealt on the each sort of market, as shown in Figure 2. Volumes on derivatives markets now substantially exceed those on the underlying stockmarkets in most major financial centres. In addition, futures contracts are much more standardised than the individual underlying securities. This provides a further boost to the liquidity of futures contracts.

(b) *Cost.* The cost of buying or selling market exposure using futures can be one-tenth, or even less, of the cost of equivalent trades in the underlying assets. This is of particular benefit if there is a reasonable possibility of the asset allocation decision being reversed in the near future. If this is not so, then it will eventually be necessary to buy and sell the underlying stocks, but it may be possible to do these deals on more advantageous terms if there is less need to rush them.

3.2.9 There are many related ways in which futures can be used to gain or shed market exposure. For example, an equity index fund will almost always contain a little cash, both to meet cash flow needs and because it will receive a steady stream of dividend income. To maintain a fully invested stance, it would normally convert this cash element to equity exposure by buying index futures. Another example is when a fund changes its fund managers. The incoming manager can avoid being ‘out of the market’ by judicious use of futures, even if the outgoing manager's stocks take some time to arrive.

3.2.10 Swaps may be used for similar purposes. For example, the fund may need to hold cash or other sorts of physical investments to meet regulatory requirements. If the returns on these are swapped for something else, then the actual underlying economic exposures within the fund may be quite different.

3.2.11 *Stocklending* (and *repo*) agreements can, perhaps, be thought of as the reverse of asset allocation using derivatives. In them the underlying asset allocation remains unaltered, but the credit risk is changed, with the stock temporarily being able to be used by someone else (for a suitable fee!).

3.3 *Basket Trades*

3.3.1 The cost benefits of using futures are particularly pronounced when the individual underlying securities are bought and sold piecemeal. An alternative way of buying or selling a portfolio of securities is to use *programme*, *basket* or *block trades* (the terms are synonymous in this context). In such a trade, the fund manager buys and/or sells a whole list, or basket, of stocks simultaneously. Often it is carried out in conjunction with a corresponding futures deal, in which case it is known as an *exchange for physical* (or EFP). Typically the actual names in the list are not supplied to the market-maker on the other side of the deal until the terms of the deal are agreed, but only some broad characteristics of the portfolio. In return for receiving all of the deal, the market-maker will generally agree much finer terms (e.g. to buy/sell all the stocks at mid-market plus a 'turn' or commission that is much smaller than the usual bid/offer spread on the individual securities).

3.3.2 The sums involved in such basket trades can be very large, and so even small changes in the dealing terms can be worth pursuing. It is particularly important to bear in mind the market impact that a large trade may have, since this can make the actual terms achieved less favourable than might appear at first sight.

3.3.3 However, it may be counter-productive to put such trades out to tender amongst many market-makers. Those who lose the tender will know that there is another market-maker with an unbalanced book. This is valuable market information which the losers can use to their advantage and to the winner's disadvantage. It is known in the trade as the *winner's curse*. All participating market-makers in the tender will thus need to worsen their terms, in case they are 'unfortunate' enough to win the tender!

3.3.4 In some markets it is possible to buy and sell such baskets for forward settlement. Such trades can largely replicate the effect of using futures. Indeed, this sort of trade, or merely buying the underlying and then selling it later, is a key constraint on the price at which futures can trade.

3.4 *Options Contracts for Asset Allocation Purposes*

3.4.1 Options contracts, at least exchange-traded versions, are rather less effective for asset allocation purposes. Suppose, for example, that the house is underweight in U.S. equities, but overweight in U.K. equities. However, it wants to hedge against the possibility that the U.S. market rises against the U.K. market. Ideally the house wants a *relative performance option* which will pay out if the U.S. market rises (in sterling terms) against the U.K. market. These sorts of options are available OTC, but not from exchanges.

3.4.2 If the house is only prepared to use exchange traded options and standard currency options to gain the same sort of protection, it will need to buy three separate options, because there are three ways in which U.S. equities can outperform U.K. equities:

(a) The U.S. equity market can rise in dollar terms. The house can protect against this by buying S&P call options.

(b) The U.S. equity market can stay level in dollar terms whilst the dollar appreciates against sterling. Buying an S&P call option is useless for hedging this. Instead, the house needs to buy a suitable currency option.

(c) The U.S. equity market and the U.S. dollar can both remain unchanged, whilst the U.K. market falls in sterling terms. Neither the S&P call option nor the currency option is any good at hedging this risk; the house needs to buy a FT-SE put option.

3.4.3 The combined price of all three is, perhaps, double the price of the corresponding relative performance option. The reason is that the three option strategy provides too much protection in some circumstances. In some scenarios two or even three of the options may pay out, when only one would be sufficient.

3.5 *Options for Strategic Risk Management*

3.5.1 Index options are much more useful for *strategic risk control* or for hedging some specific *liability feature* of a fund.

3.5.2 For example, many retail investment providers have sold products which give the retail investor equity upside subject to some kind of capital floor. The providers will often hedge the risks involved by investing in some fixed deposit, to guarantee the floor, and buying call options on the equity market, to gain equity upside. Alternatively, the fund may invest directly in equities, gaining downside protection by buying put options. There are many variations on this basic theme. Some involve quite complicated options to mimic all the specific characteristics of the product in question.

3.5.3 Some institutional investors need the same sort of protection. For example, pension funds will shortly have to meet a minimum funding standard. As a consequence they may be exposed to falls in equity values relative to fixed-interest values. A relative performance option, paying out if equities fall too much relative to gilts, may help. Life insurance companies need to establish mismatch reserves, the size of which can be reduced by purchasing suitable sorts of options. Usually the best sort of option for these needs would be a specially tailored OTC option. A package of exchange traded options providing enough cover in all scenarios may, again, provide too much cover in some circumstances.

3.6 *Option Strategies Relating to Specific Securities*

3.6.1 The other main uses of derivatives by fund managers are ones involving options on individual securities. Nearly all mainstream funds advised by actuaries prohibit gearing and require that all sold or written options are 'covered' by holding, at the same time, appropriate underlying assets. For example, insurance companies are specifically required to meet certain cover requirements under the Insurance Companies Regulations for unit-linked funds, and, although they may hold uncovered derivatives in non-linked funds, such derivatives are then inadmissible and may experience a valuation penalty. Insurance companies are also required to have assets which the derivative is 'in connection with'. These introduce constraints somewhat similar to those introduced by cover requirements for unit trusts.

3.6.2 There are four main strategies satisfying cover requirements which involve options on individual shares:

(a) *Hold shares and buy put options.* This protects the fund against a sharp fall in the price of the share.
(b) *Hold shares and sell call options.* This is called *covered call writing*. The investor gives up some possible (and uncertain) future upside in return for receipt now of income from the sale of the options.
(c) *Hold cash and buy call options.* This is essentially the same as buying warrants issued by companies on their own shares, but can be more flexible.
(d) *Hold cash and sell put options.* This is essentially the same as underwriting, but again may be more flexible, as the terms are not necessarily those imposed by the issuer of the security.

The overall impact of (a) is similar to (c), and the overall impact of (b) is similar to (d). Combinations of these basic strategies may also be useful in some circumstances.

3.6.3 Individual share options are relatively fiddly compared to index options. Some fund managers are very keen on them, believing that they offer good ways to tweak their portfolio to closer to a perceived ideal. However, others view individual share options as more like clutter within the portfolio, distracting the fund manager from more weighty matters. The growth in the use of options on individual stocks has been much less marked than the growth in the use of derivatives based on market indices.

4. Market Prices

4.1 *Market Values*

4.1.1 Whatever use a derivative is put to, its price will have a fundamental impact on how it is used. The natural way to price a derivatives contract is the same as for any other investment instrument. It is to obtain its *market value* from some market place within which it is traded.

4.1.2 Market places have changed considerably over the last few centuries.

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Often we think of financial markets in terms of transactions on exchange floors. Buyers and sellers are balanced at some suitable price, which we call the *market price*. In practice, the price charged to external buyers (the *offer* price) may include a premium and the price received by external sellers (the *bid* price) a discount. The resulting bid-offer spread represents, in some sense, the cost (to external participants) of trading in the market. Often market price is equated with the *mid-market* price, which is half-way between the bid and offer prices.

4.1.3 In practice, no formal exchange floor is needed. Dealing can be carried out over the telephone (as now happens for U.K. equities). This market still has a centralised means of disseminating prices (SEAQ). However, even this is not needed, as with the much larger global foreign exchange market (a substantial fraction of which is transacted in London). Transactions can happen very rapidly (as with these two markets) or much more slowly, e.g. the property market.

4.1.4 There is a whole spectrum of market places, in all of which we could identify, with greater or lesser precision, some form of market price and some type of bid-offer spread mechanism. The bid-offer spread may, however, become very large for very illiquid assets, such as venture capital.

4.1.5 There is the same spectrum of markets in the derivatives arena. The markets in some derivatives are highly liquid, e.g. successful exchange traded contracts, such as the FT-SE 100 Index futures contract. Others are much less liquid, with infrequent trades and high (and perhaps difficult to quantify) bid-offer spreads, e.g. 15-year OTC equity options.

4.1.6 The size of the typical bid/offer spread within a market will depend on a variety of factors, including commission levels, taxes and market structure. Investors wishing to deal in significant size also need to bear in mind the concept of *market impact*. A large buyer (seller) may inflate (depress) the price in the market to his own detriment. The market impact will depend on the size of the deal relative to total volumes traded (which will vary according to market circumstances), on the structure of the market (e.g. how easy it is to deal anonymously) and on the skills of the person actually carrying out the trade.

4.2 *Pricing Derivatives*

4.2.1 Pricing daily margined futures contracts, such as the FT-SE futures contract, is trivial, at least if we have defined market value strictly as above. At the end of each day, after the contract has been marked-to-market, the underlying 'value' of a futures contract is zero (if the returnable initial margin is treated separately). This sort of 'market value' is also called the *close-out* or *replacement* value of the derivative.

4.2.2 A different terminology is therefore usually used within equity futures markets. The 'market price' of a futures contract is usually taken to refer to the amount of exposure to the underlying involved with the futures contract, or, to be more precise, the price from which capital gains or losses are computed in

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the margining process. This 'market value' is thus derived after stripping out the impact of any margining processes.

4.2.3 The difference between these two definitions of 'market price' reflects an important characteristic of the margining system. The buyer of a futures contract takes on two separate sorts of exposure. He acquires *market exposure* (to the underlying) and also *credit exposure* (to the seller of the derivative). Margining is designed to reduce credit exposure without altering market exposure. Buyers of the underlying acquire the same two sorts of exposures, but once the deal has settled the exposures are both to the same organisation/market (i.e. the underlying company). If there is any doubt about how 'market value' is defined, the user needs to ensure that he fully understands the characteristics of the contract in question, the operation of any margining system and the conventions used within the relevant market place.

4.2.4 Options contracts are also sometimes margined so that the net replacement value (defined as per Section 4.1) can differ from the price of the (market) risk transference involved. It is, therefore, again important for the user to understand precisely what is meant by 'market value' when it arises in connection with such instruments.

4.3 *Some Economic Theory — the Principle of No Arbitrage*

4.3.1 If the whole economic world consisted of a series of completely discrete commodities or 'goods', each of which was in some completely separate sphere of human endeavour, then any discussion on how to price each might easily end here. In practice, different sort of goods can often be partially or wholly substituted for one another.

4.3.2 For example, in the gilt-edged (i.e. U.K. government fixed-interest) market, there are gilts of varying terms, often quite similar. Usually buyers do not have to buy a specific gilt, but, instead, might wish to buy one with a term which is, say, approximately 10 years. There might be two or three that have terms which are relatively similar. All other things being equal (including, in this context, the coupons on the gilts), the prices of these gilts should trade at similar prices, as otherwise buyers will tend to shun the most expensive gilt and buy the cheapest one.

4.3.3 Exactly the same principle applies in the derivatives market. For example, we would intuitively expect a six-month put option with a strike price of 1000 to have a market price very similar to a six-month put option with a strike of 1001 or a six-month and a day put option with the same strike.

4.3.4 We can extend this to the ultimate limit, where the two goods are economically identical. The two 'ought' to have exactly the same price.

4.3.5 Essentially this is the *principle of no arbitrage*. Strictly speaking, the absence of arbitrage means that it is impossible to carry out a series of transactions which:

(a) involve in aggregate no capital outlay; and
(b) in all circumstances will lead to no loss, and in some circumstances will lead to some profit.

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4.3.6 The principle of no arbitrage is most easily applied to idealised markets which are *frictionless*, i.e. have no transaction costs or other distortions, such as might arise from tax systems, and where there are no restrictions on short sales. In such markets any deviation between the price of two identical goods will be arbitraged away by arbitrageurs, and derivatives would trade at their *fair price* relative to the mid-market price of the underlying. In the presence of transaction costs, etc., we can still apply the concept of no arbitrage, but there will be a range of prices within which the market price might lie. Only when the price reaches one or other limit of this range will arbitrageurs start to be active. In most circumstances we would expect the market to trade within this range. Market participants, by substituting one good for another in the determination of supply and demand, will drive a process similar to arbitrage, even if explicit arbitrage strategies are impractical.

4.4 *Are Markets Arbitrage-Free?*

4.4.1 Identifying instances of 'arbitrage' opportunities, as defined above, is difficult. Many of the ones presented as such confuse the concept of 'arbitrage-free' with *market inefficiencies*, i.e. anomalies in market prices that provide profitable investment opportunities, but which are not risk-free.

4.4.2 For example, central banks intervene to stabilise currency markets. This ought to provide profitable investment opportunities, but they are not risk-free, because the actions of the central bank are not completely predictable in advance.

4.4.3 Another example is described by Lee *et al.* (1990) when analysing the discrepancy between the prices of shares in closed-end mutual funds (such as investment trusts) and the net asset values of their underlying shares. For example, the median U.S. closed-end fund sold at a premium of 47% in the third quarter of 1929, just before the Great Crash (and in real terms the level of issuance far exceeded those seen today). During this wave of enthusiasm, theories explaining why closed-end funds should sell at discounts were not advanced. Even today, closed-end funds sold to retail investors often sell at a premium to net asset value, but then move rapidly to a discount for the sorts of reasons described in Mehta *et al.* (1996). Lee *et al.* (1990) conclude that such discrepancies reflect the existence of a pool of irrational investors, but importantly they also conclude that arbitrageurs cannot operate, because it is unclear for how long the discrepancy might last.

4.4.4 Normally the only sorts of arbitrage strategies seen in practice are ones where the arbitrageur can achieve identical economic exposure in two different ways. If, say, a futures contract deviates by more than a certain amount from the price of the equities underlying it, then arbitrageurs can (and do!) become active. The discrepancy concerned cannot last longer than the time to maturity of the futures contract. Even these strategies may carry some risk (e.g. the arbitrageur's estimate of dividend flows may be wrong) and may not always be classifiable as true arbitrage. Even if they constitute arbitrage, the opportunities are normally fleeting and available only to those particularly well placed to benefit from them (e.g. equity market-makers).

4.4.5 It is thus generally prudent for *reserving* and *hedging* purposes to assume that markets are arbitrage-free, or at least that any arbitrage profits that may exist are ephemeral and only available to someone else. It would be an unusual actuary who was willing to increase the discount rate used to value liabilities merely because the investment managers claimed that they could add value relative to some suitable market index. Prudence dictates that such claims are treated with scepticism, and only taken credit for after they have arisen. The same applies in the field of derivatives.

4.4.6 An exception to this rule might be tax arbitrage, where different ways of achieving the same economic effect may have different tax treatments. It is usual to take into account the tax treatment likely to be received. Even here, however, it is important to bear in mind that retrospective tax changes are not unknown, and that there may be some uncertainty in the way that the transaction will be viewed by the Inland Revenue (especially if the tax arbitrage involved is likely to be viewed as excessive by the Revenue).

4.4.7 A sort of 'arbitrage-like' profit that actuaries will come across is the profit a life office's shareholders receive by the sale of a profitable life insurance policy. The policyholder (or perhaps a group of them acting in concert) could, in theory, replicate the underlying economic consequences of the policy without paying away this profit margin to the shareholders. At least conceptually, the policyholder accepts the payment of the profit margin because the life insurance product is more convenient, better packaged, viewed as more secure, involves less hassle, etc., i.e. that some economic reward is appropriate to the supplier of the product. Much economic endeavour and corporate finance is, of course, about the search for synergy, i.e. how to produce something in which the economic reward attributable to the whole is greater than the sums of the economic costs of its component parts.

4.4.8 The same applies in the derivatives industry (and indeed to practically all financial services). Investment banks selling derivatives apply considerable ingenuity to coming up with some new twist to their stock of OTC derivatives products which have some tax benefit, or meet some client need better than the products of the investment bank next door.

4.5 *Price, Value and Utility*

4.5.1 It is sometimes claimed (e.g. by Daykin in the discussion of Dyson & Exley (1995) in London) that actuaries focus on *value*, whereas financial economists focus on *price*. In this context, value is generally taken to mean some assessment of the underlying 'worth' of an investment (to a given investor), whilst 'price' is used in the more immediate sense of the market price at which the investment can be bought or sold. The implication is that the methods used by financial economists to value financial instruments (including derivatives) are too price orientated and place too little emphasis on the concept of underlying worth to the investor.

4.5.2 I think that this distinction is overstated. It is certainly true that when

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financial economists and derivatives practitioners use the term value, they often use it synonymously with price. However, there is a whole branch of financial economics called *utility theory*. This is based on the concept that different goods and services (and by implication future cash flows) will have different utilities to different people. The actuarial concept of value, as described in ¶4.5.1, seems to me to differ little from this concept.

4.5.3 Nevertheless, I think that there is much that actuaries could contribute in this area. Financial economists often assume that investors have fairly arbitrary utility functions which are convenient from a mathematical perspective. A common one is a log utility function, i.e. utility U of a certain level of wealth W is a logarithmic function $U(W) = \log(W)$.

4.5.4 This sort of utility function is clearly inappropriate for many of the sorts of investors advised by actuaries. For a liability-driven investor such as a pension fund or an insurance company, it would be natural to assume that the investor's utility function depends in some way on the investor's liabilities. For example, an outcome that involves going insolvent (even by a modest amount) might be given a very heavily negative utility, whilst an outcome where surpluses accumulate rapidly might be given a significantly positive utility. An investor with 'real' liabilities might assign a higher utility to RPI-linked cash flows than to nominal cash flows, whereas an investor with nominal liabilities might do the reverse.

4.5.5 A key point to note is that, in some circumstances, the value/price may depend little on the nature of the utility function being assumed. Indeed, in some circumstances, including those consistent with the assumptions underlying the celebrated Black-Scholes option pricing formulae, option prices are completely independent of the utility function. The same price rules regardless. Such valuations are called *preference independent*, since they do not depend on investors' preferences, i.e. utility functions.

4.5.6 However, in general, derivative prices *are* dependent on investors' utility functions (see Section 8 and Appendix B), i.e. they are *preference dependent*. As we shall see, this is particularly important when assessing the level of reserves appropriate for a derivatives portfolio, since the features that introduce these dependencies give rise to a whole slice of the reserves required.

5. Pricing Symmetric Derivatives such as Futures, Forwards and Swaps

5.1 *The Importance of the Principle of No Arbitrage*

5.1.1 The principle of no arbitrage, simple though it looks, is of fundamental importance in derivatives pricing. We can, for example, use it directly to identify fair market prices for symmetric derivatives (such as forward contracts and swaps) in frictionless markets.

5.1.2 Suppose that the annualised risk-free rate of interest (i.e. redemption yield) at time t, with continuous compounding, for a zero coupon bond maturing at time T is r. Suppose that the spot price of one unit of the underlying at time t is $S(t)$, and that the underlying generates an income/interest yield (again annualised with continuous compounding) of q between t and T. Then, in the absence of arbitrage, the *value* $f(t)$, of a *forward contract* with a delivery price of E is given by:

$$f = Se^{-q(T-t)} - Ee^{-r(T-t)}.$$

5.1.3 The *forward price* F, is defined as the delivery price at which a forward contract for that maturity would have zero value. Thus it is the value of E for which $f = 0$. In this instance it would be:

$$F = Se^{(r-q)(T-t)}.$$

5.1.4 We can justify the formula in ¶5.1.2 by considering two portfolios:

(a) *Portfolio A*: consisting of one long forward contract plus $Ee^{-r(T-t)}$ of the risk-free asset, or, to be more precise, E zero coupon bonds each maturing at time T providing 1 at that time (the definition of r is the value which equates $e^{-r(T-t)}$ to the value of the zero coupon bond).

(b) *Portfolio B*: consisting of $e^{-q(T-t)}$ units of the underlying with all income being reinvested in the underlying (or, to be more precise, an instrument consisting of the underlying, but stripped of all income/interest payments prior to time T, but again the definition of q makes these equivalent).

5.1.5 Both portfolios provide exactly one unit of the underlying at time T, and hence, by the principle of no-arbitrage, must be of identical value.

5.1.6 Even if r and q are time dependent, the above formula is still correct (if r and q are defined as the averages of instantaneous forces of interest/dividend income), as can be determined by considering carefully the more precise definitions of Portfolios A and B.

5.1.7 In many instances it may be more appropriate to assume that the income generated by a share is fixed in monetary terms rather than as a percentage yield. If the income is I (paid at the start of the life of the forward), then the values of the forward contract and the forward price become:

$$f = S - I - Ee^{-r(T-t)} \quad \text{and} \quad F = (S-I)e^{r(T-t)}.$$

5.2 *Hedge Portfolios for Forward Contracts*

5.2.1 Another way of looking at the approach adopted in Section 5.1 is that if we have sold a forward contract, we can hedge it by investing in a *hedge portfolio* consisting (if the dividend yield is fixed) of:

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(a) going short (i.e. borrowing) $Ee^{-r(T-t)}$ of the risk-free asset; and
(b) going long (i.e. buying) $e^{-q(T-t)}$ units of the underlying (reinvesting all income generated on the units in the underlying).

5.2.2 As the hedge portfolio mimics the effect of the forward contract, we can introduce the concept of the *associated economic exposure* of the contract. This is the equivalent amount of the underlying that an investor would need to hold to have the same economic effect as holding the forward. It is the value of the underlying within the hedge portfolio, i.e. $Se^{-q(T-t)}$. More generally, it may be found (for equity derivatives) as:

$$S\frac{\partial V}{\partial S}.$$

This is also known as the *delta* of the contract (although often delta refers to this expression, but without the leading term in S). For interest rate derivatives, the delta is usually calculated with respect to changes in the interest rate, rather than the value of the investment equivalent to S, i.e. a zero coupon bond, and therefore has somewhat different characteristics. In certain special cases delta hedging of interest rate derivatives is equivalent to Redington's duration matching, see e.g. Jarrow & Turnbull (1994).

5.2.3 If income is fixed in monetary terms, then the hedge portfolio changes. It would be short $(I+E)e^{-r(T-t)}$ of the risk-free asset and long one unit in the underlying, so its associated economic exposure or delta would be S.

5.2.4 In either case, this sort of hedge could be described as a *static hedge*, since we do not need to alter the hedge portfolio if the price of the underlying moves, except to reinvest dividend income (and to disinvest borrowing costs) in a suitable fashion. It may be contrasted with *dynamic hedging*, in which the structure of hedge portfolio is altered in some fashion, depending on the movement in the price of the underlying.

5.2.5 Static hedging ought to be widely understood in actuarial circles, since it is equivalent to the concept of *matching*, e.g. as described by Wise (1987). Dynamic hedging is less widely written about in actuarial circles. However, it too is not far away from the actuarial mainstream; it is implicit in the actuarial principle that the smaller the surplus the more a fund should adopt a position that matches its liabilities.

5.2.6 Astute readers will have spotted that there may be an element of circularity in the hedge portfolio described above. For currency forwards, it is possible to buy suitable investments in the underlying stripped of income prior to maturity of the forward. However, for equity markets such investments do not exist. Indeed, the usual way of estimating the value of the stripped component is to work backwards from the price at which suitable futures, forward contracts or swaps trade! In practice, it may be possible to estimate future dividend income and hence q (or I), but if maturity is a long time into the future, then

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this alternative becomes less accurate. There is also a problem deciding what tax rate to apply to the estimated dividend income. Usually the implied market rate is somewhere between a gross and a net rate (which, incidentally, can introduce possibilities for tax arbitrage, since forwards and futures contracts can be used to convert income into capital gains or vice versa).

5.3 *Pricing Futures Contracts*

5.3.1 The fair price of a futures contract (subject to daily margining) would appear to correspond to the forward price of a contract with the same maturity date. A complication is that futures contracts are generally marked to market daily, and therefore the quantity corresponding to the exercise price keeps being reset to the current futures price. However, if the risk-free rate of return r, is constant and the same for all maturities, the fair market price of the future would be the same as the forward price (see Hull, 1992), i.e.:

$$\begin{aligned} &\text{known dividend yield, } q: && Se^{(r-q)(T-t)} \\ &\text{known dividend income, } I: && (S-I)e^{r(T-t)}. \end{aligned}$$

5.3.2 The associated economic exposure of the future is also more complicated, since any gains or losses are credited/debited immediately. If the price of the underlying doubled instantaneously or fell to zero instantaneously (and futures remained priced at their fair value), then a portfolio holding one futures contract would rise or fall in value by $Se^{(r-q)(T-t)}$ (or $Se^{r(T-t)}$ if dividend income is constant in monetary terms). These may therefore be used to identify the associated economic exposure of a futures contract.

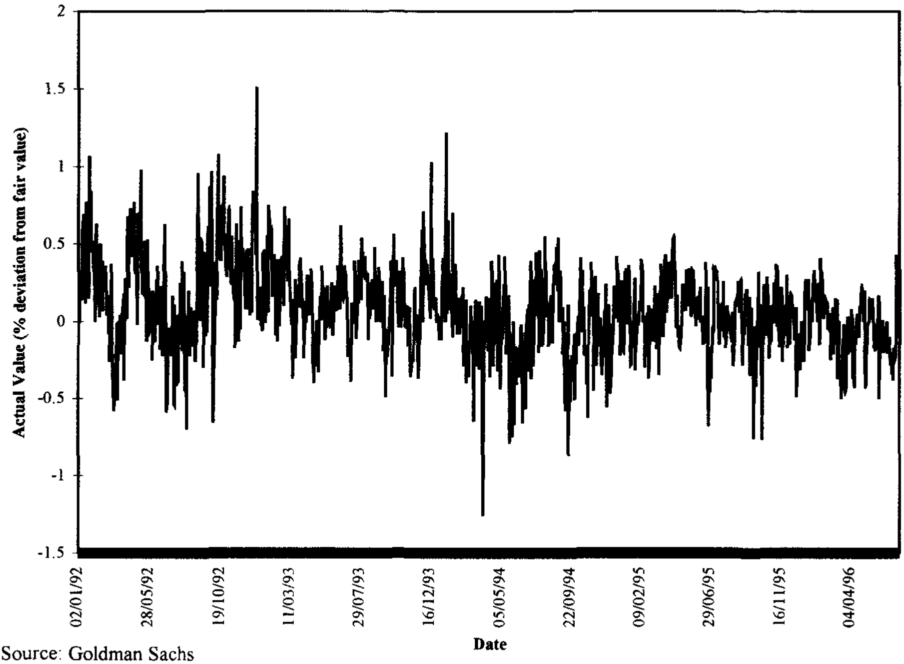
5.3.3 In practice, of course, the risk-free rate is unlikely to be constant, and the price of a futures contract may then vary from the forward price. The reason is hinted at in Section 3.6 of Hull (1992). In essence, the hedge portfolio described above is no longer perfect for a futures contract, since capital gains or losses credited daily must be reinvested at a rate which is not known in advance.

5.3.4 For contracts lasting just a few months, the theoretical differences between equity forwards and futures are usually small enough to be ignored, unless they differ in other ways (e.g. tax treatment, transaction costs or the initial margin deposited with the clearing house is not credited with as high a rate of interest as other deposits).

5.3.5 Users of the LIFFE long gilt future (and some other gilt futures) should also note that these futures are settled by delivery of stock from a range of alternatives (with delivery being possible on a range of dates). This element of optionality can influence its price.

5.4 *Deviations from Fair Value*

5.4.1 Neither forwards nor futures are actually guaranteed to trade at the



Source: Goldman Sachs

Figure 3. FT-SE Index futures: percentage deviations from fair values

no-arbitrage fair prices described above. If the difference between the actual price and the fair price is less than the bid/offer spread in the underlying market, or if short-selling is difficult or impossible, then arbitrageurs may not be able to profit from the difference. Usually the deviation is modest (see Figure 3), although still potentially important in some instances.

5.4.2 Very occasionally the difference between the actual price and the fair price derived from no arbitrage criteria can become much larger. This should only happen if a major shock prohibits the process of arbitrage from drawing the two prices closer together again. An example was the October 1987 Crash, when the underlying stock markets moved so rapidly and so far, that for much of the day the two markets appeared to be out of synchronisation (in the case of the U.S.A., the futures market was closed altogether).

5.4.3 The impact of differences between the actual price and the fair price of futures or forwards is known as *basis risk* (or, if it refers to the impact when the contract is rolled forward, as *roll-over risk*).

5.5 *Index Arbitrage*

5.5.1 Arbitrageurs of equity derivatives are most likely to be players who are market-makers on both the underlying physical stock market and on the corresponding futures exchange (because their transaction costs are then generally lower than for other market participants). Each such organisation will have a *book* of positions, usually both *long* and *short*, in both individual stocks and in instruments such as futures relating to index exposures. They will have purchased their long positions directly in the market. To cover their short positions, they will often have borrowed stock from a stock lender.

5.5.2 To minimise the likelihood of shortfalls in their overall book, they will normally try to minimise the *net* exposures that they have to any one stock or any one type of stock. Of course, the nature of market making means that they will always be taking temporary positions (sometimes quite large), but they will try to avoid permanent large long or short positions, as it ties up scarce capital.

5.5.3 Assessing the level of capital required to back such a book (if it contains no options) ought to be conceptually straightforward for most actuaries already involved with investments. The market-maker's book can be thought of as a portfolio of many different individual equities (long and short). Index exposures gained through futures can be included by decomposing them into the various index constituents, weighted as per the index (plus a further basis risk element as per ¶5.4.3). The riskiness of this portfolio can then be assessed by calculating its *tracking error*, which is usually defined as the expected standard deviation of the return on a portfolio (relative to some suitable benchmark, perhaps here being cash). The concept of tracking error is explained in Rains & Gardner (1995). The capital required may then be set by applying some suitable multiplier to the tracking error to provide an acceptably low estimated risk of ruin.

5.5.4 There are commercially available packages that can estimate such tracking errors from an investment management perspective, e.g. ones supplied by BARRA and Quantec. These generally start with a predefined series of factors that are assumed to influence the returns on individual stocks (e.g. market capitalisation, industry type, book to price ratios), as well as assumptions on the volatility of individual stocks not explained by these factors. The total tracking error is then dependent on how diversified the portfolio is, as well as whether it has any factor biases. In both of these packages the contribution to risk from individual stocks (and their correlations) and the residual *non-systematic* risk from individual stock holdings are estimated from past history, which may not necessarily be a good guide to the future.

5.5.5 Market-makers often develop their own proprietary methods of calculating tracking error. For example, factors influencing the behaviour of stocks can be found by *multiple linear regression* and *principal component analysis* without regard to whether the factors thereby derived bear any obvious relationships to how investors might view stocks. Trading volume and liquidity are also, arguably, more important from a market-maker's perspective than from an investment manager's point of view.

5.6 *Valuing Interest Rate Swaps and Bond Futures*

5.6.1 Although this paper concentrates on equity derivatives, it is also worth noting that exactly the same sorts of arguments can be used to value interest rate swaps and futures or forwards on bonds. However these normally require, not just point parameters, but information relating to the entire yield curve.

5.6.2 Take, for example, an interest rate swap involving a principal amount P, maturing in n years' time. For simplicity, we assume the swap involves Party A paying floating-rate interest payments to Party B in return for fixed-interest payments of aP, and that both the fixed and the floating-rate payments are continuous (with the floating-rate payments reset continuously). We also assume that the value of a zero-coupon bond paying 1 at time t from now is worth $v(t)=1/(1+i(t))^t$, where $i(t)$ is the relevant (annualised) gross redemption yield of such a zero-coupon bond. We also, in this simplified example, ignore the impact of credit risk.

5.6.3 If we buy a £1 zero-coupon bond maturing at time x, sell a £1 zero-coupon bond maturing at time y ($>x$), and put the £1 we receive from the first bond on deposit (at floating rates), then we will receive floating-rate interest on £1 between x and y. Thus the present value *now* of such floating-rate payments is, by the principle of no arbitrage, $v(x) - v(y)$. The value *now* of (continuous) fixed rate payments of a between x and y is:

$$a\int_x^y v(t)dt.$$

5.6.4 Thus the value of the swap *now* to Party B is:

$$P\left(a\int_0^n v(t)dt + v(n) - 1\right).$$

5.6.5 In practice interest rate swaps have discrete reset dates, at which the floating payments are recalculated and on which the fixed and floating-rate payments are made. The value of the fixed-rate payments would thus be discrete rather than continuous annuities, and there would be an adjustment for the period between the date of valuation and the date of the next reset date.

5.7 *Predicting Future Price Movements from Forward Prices*

5.7.1 The 'fair' forward price depends on the current spot price and differentials in expected income streams. If a future/forward is trading cheap or dear relative to its fair value, then it will move back to its fair value by maturity (either by moving itself or by the underlying moving). It thus provides some information on how the underlying market might move in the future, but the amount is quite limited. In an arbitrage-free world there should be no deviation

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from fair value, and the forward price should provide no information at all on whether the market was fundamentally cheap or dear.

5.7.2 This is merely a specific example of it being impossible, in an arbitrage-free world, to predict future price movements from past price movements in the sorts of fashion often used by chartists. Most actuaries are deeply sceptical about chartist analysis, perhaps because they have an in-built acceptance of the prudence of assuming that markets are arbitrage-free.

5.7.3 One often hears of a movement in the futures markets driving a movement in the underlying stock market. What this actually means is that there is an imbalance between supply and demand at the current price, and it influences the futures market more quickly than the underlying. Since futures markets are now often more liquid than their underlying stock markets, it is not too surprising that the imbalance is often spotted and acted upon first in the futures market.

5.7.4 Again, a slight word of caution is required with interest rate derivatives. The short sterling contract on LIFFE *does* provide information on market expectations for sterling interest rates over the next few months or years. However, if the market is arbitrage free, it still does not indicate if zero-coupon bonds are fundamentally 'cheap' or 'dear'.

5.7.5 Of course, investment managers often argue that specific markets are not arbitrage-free or may have systematic inefficiencies (e.g. foreign currency markets, because of the presence of players like the Central Banks who seek to smooth exchange rate movements). In such circumstances, forward prices may have some predictive capability. Forward prices (and forward interest rates) also give an indication of what the market is expecting to happen, and thus can help a fund manager to identify whether his views (and thus the position he will want to take) differ from the prevailing market views.

6. The Greater Complexity of Pricing Asymmetric Derivatives Such as Options

6.1 *Why Options are more Complex than Forwards*

6.1.1 Pricing asymmetric contracts, such as options, is much more complicated than pricing symmetric contracts. The reason is that the principle of no arbitrage, *in isolation*, no longer produces unique fair values for such contracts that all market participants will agree on, although it does introduce certain upper and lower limits.

6.1.2 For vanilla options, such as puts and calls, which never generate a liability for the purchaser, the lower limit must be at least equal to zero. However we can do better than this. For American style options which are potentially exercisable immediately, the option must be worth at least its *intrinsic value*, which for a call option is $\max(S - E,0)$, and for a put option is $\max(E - S,0)$. For a European option these limits become $\max(Se^{-q(T-t)} - Ee^{-r(T-t)},0)$ and

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$\max(Ee^{-r(T-t)} - Se^{-q(T-t)}, 0)$, even though it is still usual to retain the American style formulae as the definitions of 'intrinsic value'.

6.1.3 The upper limits are also relatively easy to establish. In the absence of transaction costs, a European call option giving the holder the right to buy a share for the price of E cannot be worth more than $Se^{-q(T-t)}$, whilst a put option giving the holder the right to sell the underlying for a price E cannot be worth more than $Ee^{-r(T-t)}$. For American put and call options, these limits become S and E respectively, since the option might possibly be exercised immediately. To achieve these limits, the underlying must rise in price infinitely much (for the call option) or fall to zero (for the put option).

6.1.4 It is relatively easy to show that the price of a vanilla option can, in principle, fall anywhere between these bounds. For example, suppose that the price of the underlying S remains fixed until just before the maturity date of the option and then jumps in the instant before maturity to equal x with probability $P(x)$. Given an arbitrary probability density function $P(x)$, any option price between the lower and upper bounds stated above could be consistent with no arbitrage.

6.2 *Decomposing Option Pay-Offs into their (Possibly Infinitesimal) Parts*

6.2.1 So to price options we need something more than just the principle of no arbitrage. One way of tackling the pricing of European options might be to decompose the pay-off at maturity into lots of individual parts, depending on the level the underlying reaches at maturity. Each of these parts pays out 1 if at maturity the underlying lies between $S(x)$ and $S(x) + dx$. Such contracts are technically known as *digital call spreads* (in this instance with infinitesimal spreads of width dx).

6.2.2 The no arbitrage criterion places constraints on the values of these digital call spreads. Each must have a non-negative value. Also, if we purchase Sdx digital call spreads (relating to the spread $S(x)$ to $S(x) + dx$) for each possible x, then their combination is economically identical to a contract which provides one share at maturity, which we know how to value from Section 5.

6.2.3 Thus, in principle, we must determine the value of an infinite number of digital call spreads if we want to be able to value European options, i.e. we need to identify a function $D(x)$ defined such that:

$$D(x)\,dx = \text{value of a digital call spread between } x \text{ and } x + dx.$$

6.2.4 A common way of restating this is to identify a probability distribution $p(x)$, called the *risk-neutral probability distribution*, which is defined so that:

$$D(x) = Zp(x)$$

where Z is the present value of a payment of 1 at the maturity of the option, i.e. the value of a corresponding zero-coupon bond.

6.2.5 The price of a European option can then be defined as the expected value of the pay-off of the option with respect to this risk-neutral probability measure. We consider risk-neutral probabilities in more detail in Section 7 and in the Appendices.

6.3 *Calibration of Option Prices*

6.3.1 In practice we will only have a relatively limited number of actually observable option prices (which may be subject to significant bid-offer spreads). Finding a suitable $D(x)$, or equivalently $p(x)$, which closely fits actually observed option prices is a process which, in the actuarial world, would be known as *calibration*. It has strong similarities to *graduation*, which actuaries have long used to calculate smoothed mortality rates, etc. and which involves fitting some suitable smooth mathematical function to observed mortality rates.

6.3.2 A derivatives market-maker with small net positions generally cares little about what the price of an option 'ought' to be, provided he can find a good way of calibrating the market, and hence valuing consistently the options he is buying and selling. Of course, if he holds significant net derivatives positions on his book (e.g. if he also carries out *proprietary trading*, i.e. trading on his own account), then he becomes more interested in the 'right' value of each option in isolation. He is then, in effect, operating more like an own-account position-taking fund manager than a market maker.

6.3.3 The calibration process becomes rather more complicated if options with different numeraires are to be considered, and much more complicated still when applied to *path dependent* options such as barrier options. These are options whose pay-off structure depends in part on how the underlying gets to a position as well as its final value at maturity. We then, in principle, need to identify an infinite dimensional function $F(\underline{\mathbf{x}},\underline{\mathbf{t}}))$ representing the value of an option that pays out 1 if the path of the underlying S remains within the following band at all times:

$$x(t) < S(t) < x(t) + dx(t).$$

6.3.4 The value of a given path dependent option could then, in principle, be calculated by summing across all possible paths that the price of the underlying might follow. Monte Carlo simulation techniques would often, in practice, need to be used, given the large number of possible paths. The techniques involved are conceptually similar to those used by actuaries in stochastic asset/liability modelling.

6.4 *Hedging Algorithms*

A potentially more useful way of attempting to price options is to identify some model describing how the price of the underlying securities might move, and then to derive from it a pricing algorithm (often by identifying its associ-

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ated risk neutral probability distribution). The reason is that such a model can help us identify ways of *hedging* or otherwise controlling the risks incurred when writing such options. It can also help us to understand the circumstances in which controlling these risks may be problematic. It, too, will normally need calibrating to match closely prices actually observed in the market place.

6.5 *Views on the Black-Scholes Formulae within the Actuarial Profession*

6.5.1 A common assumption to make is that the price of the underlying follows a *diffusion* process. The usual sort of diffusion process adopted is that set out in Appendix A, which, as shown there, results in the celebrated Black-Scholes formula for European put and call options, and to its generalisation, the Garman-Kohlhagen formula for interest/dividend bearing securities.

6.5.2 The Black-Scholes formulae seem to cause controversy amongst some parts of the actuarial profession.

6.5.3 There are some (usually ones who do *not* work for banks) who flatly refuse to believe that equity prices follow anything like a diffusion process, and specifically that such an approach vastly understates the likelihood of extreme outcomes, especially for long-term options. These people then infer that the Black-Scholes formulae will understate dramatically the true value of, say, far out-of-the-money put options, raising the spectre that writers of such options (i.e. principally banks) are putting their capital at serious risk (and may be unable to honour contracts they enter into with insurance companies).

6.5.4 There are others (usually from a banking background) who seem much more comfortable with the Black-Scholes formulae (or some suitable modification), arguing that to-date the application by banks of the formulae seems to have been quite successful and very far from the doomsday scenario painted by the other camp.

6.5.5 The issue seems to be rather more about philosophy (or perhaps a reluctance to accept that a non-actuarial profession can have useful insights in an area where actuaries feel that they ought to be the experts) than about cost or size of reserves. Even within the actuarial literature, see e.g. Beenstock & Brasse (1986), it has been noted that the Black-Scholes formulae produce uncomfortably expensive values for the sorts of maturity guarantees that life offices might want to include within their products. Perhaps, also, this controversy mirrors the sorts of options different members of the profession may come across. Those principally concerned with, say, valuing maturity guarantees will often be focusing on the costs of very long-term far out-of-the-money options. These are, perhaps, less well catered for by the Black-Scholes formulae than the less 'extreme' sorts of options more typically traded by banks.

6.5.6 To some extent the Report of the Institute of Actuaries and Faculty of Actuaries Maturity Guarantees Working Party (1980) set the scene for actuarial opinion on this topic, just as the huge boom in derivatives since the early 1980s started. Section 6 of the Report considered briefly the possibility of following an immunisation strategy to hedge maturity guarantees effectively along the lines of dynamic hedging of option prices using the Black-Scholes formulae. Section 6 of the Report concludes:

> "Although the Working Party did not pursue immunization theory, it concluded that it is a subject which merits further investigation. Also it is likely that a paper will be presented to the Institute on the subject in the near future. In the meantime, the Working Party considers that there is no basis for reducing maturity guarantee reserves because a company follows some form of immunization strategy and, in fact, a company that follows such a strategy without fully appreciating the difficulties could well require greater maturity guarantee reserves than would otherwise be the case."

The paper referred to in the second sentence of this quotation appears never to have been published.

6.5.7 Some argue that this view is bolstered by the experience of the October 1987 Crash, in which share prices on some markets moved by 20% to 25% in a single day. The argument goes that writers of put options (or 'portfolio insurers' trying to replicate the effects of such options) found it impossible to trade in the manner necessary to hedge the risks involved (i.e. to sell enough stock rapidly enough).

6.5.8 In fact, some of the fund managers with the fastest and best hedging programs claim to have been able to largely, or wholly, insulate their funds against the effect of the Crash. Thus, although dynamic hedging does not work perfectly when markets 'gap', it does seem to offer some protection, despite the suggestion of the Maturity Guarantees Working Party to the contrary.

6.5.9 There is, at this point in such discussions, a tendency to digress into issues such as whether portfolio insurance, derivatives markets and indeed all computer driven share trading systems are intrinsically a 'bad thing', guaranteed at times to disrupt the orderly functioning of the underlying stock market. It is not the intention of this paper to digress in this fashion (although I suspect that the widespread use in the U.S.A. of portfolio insurance immediately prior to the October 1987 Crash did contribute to market volatility over that period). Instead I have concentrated on two related questions:

(a) what are the different sorts of risks arising from price movements in the underlying; and
(b) how much of these risks can be mitigated if we have sold an option contract?

6.5.10 I have set out, in Section 9, an analysis of how effective daily dynamic hedging might have been if applied to the U.K. equity market since 1984 (a period including the October 1987 Crash). Although it is always appropriate to treat such simulations with some caution, it seems that pure dynamic hedging can reduce risks, but it is not the panacea that some, perhaps, suggest. There are still significant risks which need to be controlled in other ways.

7. Binomial Lattices and Diffusion Processes

7.1 *A One-Period Binomial Lattice*

7.1.1 However, before we can carry out this analysis, we need to delve more deeply into the mathematics of dynamic hedging.

7.1.2 Suppose we knew for certain that between time $t-h$ and t the price of the underlying could move from S to either Su or to Sd (as in Figure 4), that cash (or more precisely the appropriate risk-free asset) invested over that period would earn a force of interest of r and that the underlying generates a force of dividend income of q.

7.1.3 Suppose we also have a derivative (or indeed any other sort of security) which (at time t) is worth $A=V(Su,t)$ if the share price has moved to Su, and worth $B=V(Sd,t)$ if it has moved to Sd.

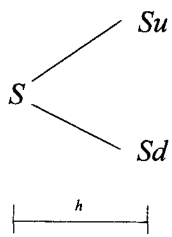

Figure 4. Binomial price movement

7.1.4 Starting at S at time $t-h$, we can (in the absence of transaction costs and in an arbitrage-free world) construct a hedge portfolio at time $t-h$ which is guaranteed to have the same value as the derivative at time t whichever outcome materialises. We do this by investing (at time $t-h$) fS in f units of the underlying and investing gS in the risk-free security, where f and g satisfy the following two simultaneous equations:

$$fSue^{qh} + gSe^{rh} = A = V(Su,t)$$
$$fSde^{qh} + gSe^{rh} = B = V(Sd,t).$$

7.1.5 Thus fS and gS are given by:

$$fS = \frac{e^{-qh}(V(Su,t) - V(Sd,t))}{u-d}$$
$$gS = \frac{e^{-rh}(-dV(Su,t) + uV(Sd,t))}{u-d}.$$

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7.1.6 The value of the hedge portfolio and hence, by the principle of no arbitrage, the value of the derivative at time $t-h$ are thus given by the *backward equation*:

$$V(S,t-h)=fS+gS=\frac{e^{(r-q)h}-d}{u-d}\,e^{-rh}V(Su,t)+\frac{u-e^{(r-Q)h}}{u-d}\,e^{-rh}V(Sd,t).$$

7.2 *Risk-Neutral Probabilities*

7.2.1 We can also write this equation in the form:

$$V(S,t-h)=p_u e^{-rh}V(Su,t)+p_d e^{-rh}V(Sd,t)$$

where :

$$p_u=\frac{e^{(r-q)h}-d}{u-d} \quad \text{and} \quad p_d=\frac{u-e^{(r-q)h}}{u-d}$$

and thus:

$$p_u+p_d=1.$$

7.2.2 p_u and p_d behave like probabilities of jumping up or down, especially if they are both non-negative, i.e. if $u>e^{(r-q)h}>d$ (assuming $u>d$). This is a fairly reasonable assumption to make, since $e^{(r-q)h}$ is the forward price of the security, and it would be an odd sort of binomial tree that did not straddle the expected movement in the underlying. Failure to satisfy these inequalities also introduces arbitrage opportunities.

7.2.3 However, it is important to realise that p_u and p_d do *not* need to correspond to the probabilities that any given investor believes are the actual probabilities of up or down movements. In order to avoid arbitrage, we do not need to know what people actually believe will be the case. Instead, we only need to know the probabilities that would be assigned to up and down movements by a notional *risk-neutral investor*, who is an investor who assumes that the expected return on the underlying is the same as the risk-free rate. p_u and p_d are therefore called *risk-neutral* probabilities. It is relatively easy to show that, if the price movement is as implied by such a binomial arrangement, then this definition of risk-neutral probability is compatible with the more general definition given in Section 6.

7.3 *Multi-Period Binomial Tree*

7.3.1 One way of extending the one time period model, described in Section 7.1, is to build up a *binomial tree*, or *binomial lattice*, as in Figure 5. The price of the underlying is assumed to be able to move in the first period either up or down by a factor u or d, and in second and subsequent periods up or down by u or d from where it had reached at the end of the preceding period.

7.3.2 More generally, u or d could vary depending on the time period, but it would be usual to require u_i/d_i to be fixed, to make the lattice *recombining*.

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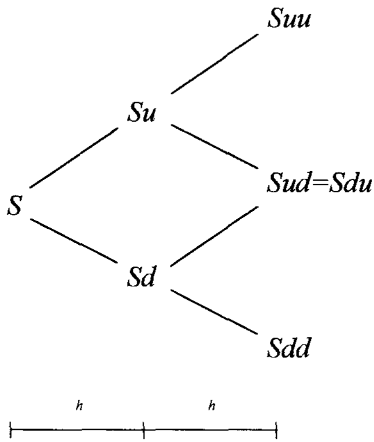

Figure 5. Binomial lattice/tree

In such a lattice an up movement in one time period followed by a down movement in the next leaves the price of the underlying at the same value as a down followed by an up. Non-recombining lattices are possible, but are much more complicated to handle. It would also be common, but not essential, to have each time period of the same length h.

7.3.3 By repeated application of the backward equation in Section 7.1, we can derive the price n periods back, i.e. at $t = T - nh$, of a derivative with an arbitrary payoff at time T. If u, d, p_u and p_d, r and y are the same for each period then:

$$V(S,T-nh) = e^{-rnh} \sum_{m=0}^{n} \binom{n}{m} p_u^m p_d^{n-m} V(Su^m d^{n-m},T) \quad \text{where} \binom{n}{m} = \frac{n!}{m!(n-m)!}.$$

7.3.4 We can also express this in probabilistic terms, placing a value on the derivative by calculating the expected value of the pay-off *by reference to the risk-neutral probability measure, rather than the actual likelihood of up or down movements occurring (as perceived by any specific investor).* Thus we can also express V as:

$$V(S,t) = \mathrm{E}(e^{-r(T-t)}V(S,T) \mid S_t) \quad \text{where} \quad t = T - nh.$$

$\mathrm{E}(X \mid I)$ means the expected value of X given some probability measure conditional on being in state I when the expectation is carried out.

7.3.5 As we saw in Section 6, calculating derivative prices using expectations referring to risk neutral probability measures is a quite general method that can be applied to all European style options, provided that the appropriate risk-neutral probability measure can be identified. In the derivatives literature, the use of a risk-neutral approach is also sometimes referred to as using an *equivalent martingale measure*, since the discounted price process (under the risk-neutral probability measure) is a mathematical artefact called a *martingale*.

7.3.6 Using expected values ought also to be instantly recognisable to actuaries. The 'standard' actuarial approach, if one might be said to exist, to valuing a cash flow dependent on some factor S is:

(a) to estimate the probability $p(S)dS$ of that factor falling in the range S to $S + dS$;
(b) to determine the size of the cash flow arising in these circumstances; and
(c) to discount the cash flow at some suitable discount rate, say r (which can be expressed as a 'force' of interest).

7.3.7 For a European call option with exercise price E, the 'standard' actuarial approach thus results in the use of the following formula:

$$V(S,t)\int_{E}^{\infty} p(S)(S-E)e^{-r(T-t)}dS.$$

7.3.8 At first sight there appears to be an inconsistency between the two approaches, since the derivative pricing formula in ¶7.3.4 requires the use of risk-neutral probabilities, whereas actuaries would tend to use their own perceived estimates of the actual likelihoods of different outcomes occurring. The difference is reconciled through the discount rate. It is accepted actuarial practice, at least for life office appraisal valuations, to use some sort of *risk discount rate* for r, reflecting the risk characteristics of the investment concerned. The natural discount rates to use are ones that ensure that the value of a call option with zero exercise price is equal to the value of a share deliverable at time T (which can be valued as per Section 5). This is because the two have identical economic effects. Smith (1996) shows that use of actual (investor estimated) likelihoods and risk discount rates set in this fashion is mathematically identical to using risk-neutral probability measures and a risk-free discount rate.

7.3.9 In practice, it is often nearly impossible to estimate the appropriate risk discount rate applicable to a given 'real world' probability distribution except by backing it out from the results of a calculation using the risk-neutral probability distribution. Risk-neutral valuation approaches are therefore almost universally preferred within the derivatives industry.

7.4 *Preference Independent Pricing Formulae*

7.4.1 Astute readers will have noticed that nowhere in Sections 7.1 to 7.3 is there any mention of the utility functions of individual investors (or even investors in aggregate). A pricing formula like this, which does not depend on investors' views regarding the likelihood of price movements, is called *preference independent.* Such a lack of dependence on investors' views regarding risk and return is actually very rare. As Neuberger (1992) notes, essentially the only examples are ones where the price movements in the underlying follow *a binomial tree known in advance* or *the limit of such a tree* in which the time interval becomes arbitrarily small (which include the sorts of processes underlying the Black-Scholes formula).

7.4.2 The reason is that, if there are just two possible outcomes at any one time (known precisely in advance), then we have just enough flexibility to replicate the option behaviour exactly using dynamic hedging, by altering the mix of the hedge portfolio. If there are more than two possible outcomes, or if the two are not known with certainty, then we do not have sufficient flexibility, and dynamic hedging becomes intrinsically 'risky'. The presence of transaction costs also introduces preference dependence (see Appendix B.8.3). As we shall see, this has important consequences when we are trying to hedge the risks involved and in assessing the level of reserves needed for positions which are dynamically hedged.

7.4.3 Risk-neutral probability measures are much more widely applicable (as can be seen from ¶6.2.3), since they depend only on the absence of arbitrage. Practically all option pricing formulae seem to have equivalent risk-neutral probability measures, provided we define the concept widely enough. They also, in principle, have wide application in other actuarial areas, see e.g. Smith (1996). If the price movement model is preference independent, then the risk-neutral probability measure can be found without reference to the attitude of investors to risk and return, i.e. to their utility functions, as described in Section 4.5. However, *if the price movement model is preference dependent then, in general, the risk-neutral probability measure will depend on investors' utilities (or some average of them).*

7.5 *Valuing Put and Call Options*

7.5.1 Applying the formula in ¶7.4.3 to a put option, with strike E (assumed to be at a node of the lattice) maturing at time T, we get the price of the put option as:

$$P(S,T) = \max(E - S, 0) \quad \text{where } E = S_0 u^{m_0} d^{n-m_0} \quad \text{say.}$$

Thus:

$$\begin{aligned} P(S,T-nh) &= e^{-rnh} \sum_{m=0}^{m_0} \binom{n}{m} p_u^m p_d^{n-m} (E - S_0 u^m d^{n-m}) \\ &= e^{-rnh} E.\mathrm{B}(m_0, n, p) - e^{-qnh} S_0 \mathrm{B}\left(m_0, n, \frac{u.p_u}{u.p_u + d.p_d} \right) \end{aligned}$$

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where:

$$\mathrm{B}(x,n,p) = \text{binomial probability distribution} = \sum_{m=0}^{x} \binom{n}{m} p^m (1-p)^{n-m}.$$

7.5.2 Suppose we define the *volatility* of the lattice as σ where:

$$\sigma = \frac{\log(u/d)}{2\sqrt{h}}.$$

Then, if we allow h to tend to zero, keeping σ and t fixed, with u/d tending to 1 by, say, setting $\log(u)=\sigma\sqrt{h}$ and $\log(d)=-\sigma\sqrt{h}$, we find that the above formula, and hence the price of the put option, tend to:

$$P(S,t) = Ee^{-r(T-t)}\mathrm{N}(-d_2) - Se^{-q(T-t)}\mathrm{N}(-d_1)$$

where:

$$d_1 = \frac{\log(S/E) + (r - q + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

and $\mathrm{N}(x) = \text{cumulative normal probability function} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} dz.$

7.5.3 The equivalent price (in the limit) of a European call option is:

$$C(S,t) = Se^{-q(T-t)}\mathrm{N}(d_1) - Ee^{-r(T-t)}\mathrm{N}(d_2) \quad \text{where } d_1 \text{ and } d_2 \text{ are as above.}$$

This formula can be justified along the same sorts of lines as used above for a put option. Alternatively, it can be justified on the grounds that the values of a European put option and a European call option satisfy *put-call parity* (as noted in ¶3.6.2), i.e.:

stock + put = cash + call (allowing for dividends and interest).

Mathematically, this means that C and P satisfy:

$$P = C + Ee^{-r(T-t)} - Se^{-q(T-t)}.$$

7.5.4 These formulae for European put and call options are the *Garman-Kohlhagen* formulae for dividend bearing securities. If q is set to zero, then they become the celebrated *Black-Scholes* option pricing formulae. For simplicity, in the remainder of this paper I refer to both generically as the BS

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formulae, and call a world satisfying the assumptions underlying these formulae as a 'Black-Scholes' world.

7.5.5 In this limiting situation, the volatility σ corresponds with the natural meaning of the word 'volatility' as applied to the share price.

7.5.6 Similar formulae can also be derived if we assume that dividends rather than dividend yields are fixed, see e.g. Hull (1992).

7.6 *Derivation of the Black-Scholes Formulae using Stochastic Calculus and Partial Differential Equations*

7.6.1 It is probably more usual to develop the BS formulae using stochastic calculus and partial differential equations.

7.6.2 This approach involves deriving a partial differential equation satisfied by *any* derivative on the security (including one that paid the total return on the underlying security!). The one caveat is that the derivative must depend only on the underlying security and the risk-free asset. The relevant partial differential equation for the Black-Scholes world is described in Appendix A.

7.6.3 Different derivatives then differ only in that they satisfy different *boundary conditions*. For example, a European-style call option with exercise price E maturing at time T will satisfy the partial differential equation developed in Appendix A, subject to the boundary condition that at time T the price must equal $\max(S - E,0)$ for all possible prices of the underlying security S.

7.6.4 It is relatively easy to demonstrate that the formulae set out in ¶7.5.2 and ¶7.5.3 satisfy this partial differential equation and the relevant boundary conditions. However, a better understanding of the formulae can be gained by deriving them from first principles using mathematical techniques appropriate to solving partial differential equations, as is also done in Appendix A.

7.7 *Factors Influencing the Price of an Option*

7.7.1 The price of a call option for a given strike price will vary as the price of the underlying varies. Figure 6 shows the form of this dependency for various times to maturity for a European call option, assuming a constant volatility $\sigma = 15\%$, and constant risk-free interest rate $r = 0\%$ and dividend rate $q = 0\%$.

7.7.2 The delta of an option, or indeed of any type of (equity) derivative (see ¶4.2.2) is the slope of such a curve, i.e. the rate of change of the price of the option with respect to the price of the underlying (multiplied by S if the aim is to express the answer in units of market value rather than units of stock). Deltas equivalent to the prices in Figure 6 are shown in Figure 7. Equivalent graphs for European put options (making the same assumptions as in ¶7.7.1) are shown in Figures 8 and 9.

7.7.3 Another important characteristic of the option is its *gamma*, which is the slope of the curve in Figures 7 or 9, i.e. the rate of change of delta with respect to the price of the underlying. Gamma is important because:

(a) It is directly related to the level of turnover within the sort of portfolio that

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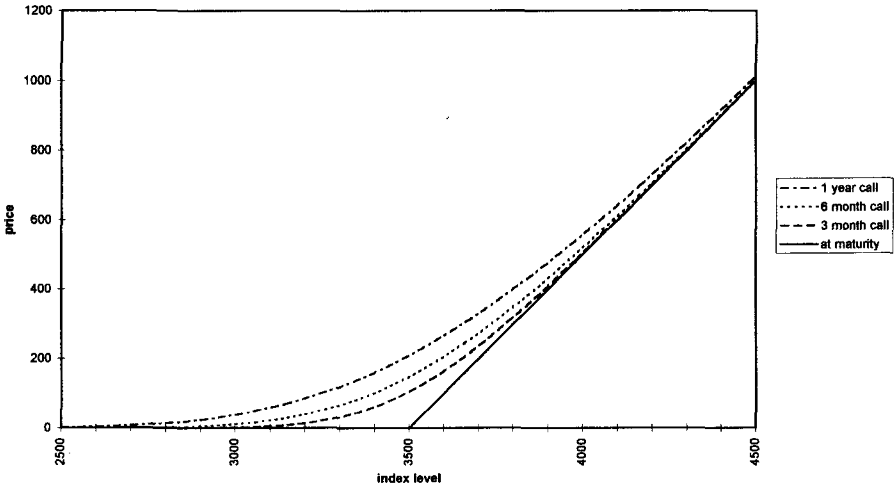

Figure 6. Price of a call option

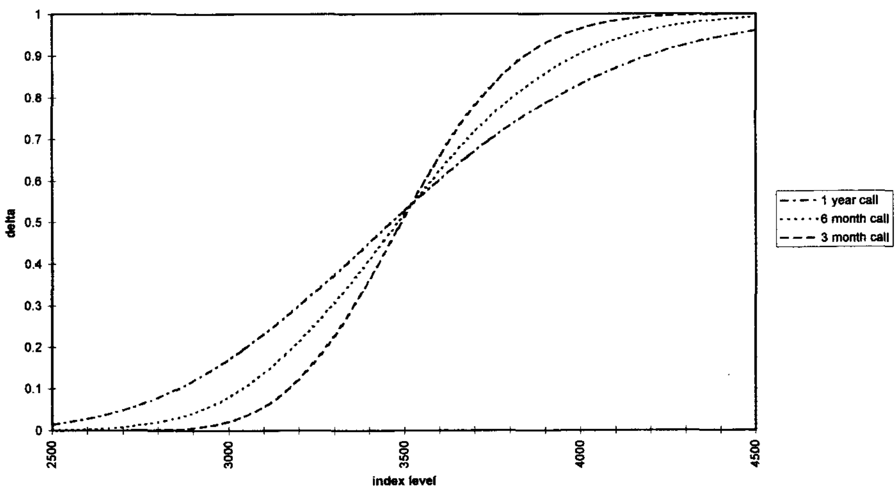

Figure 7. Delta of a call option

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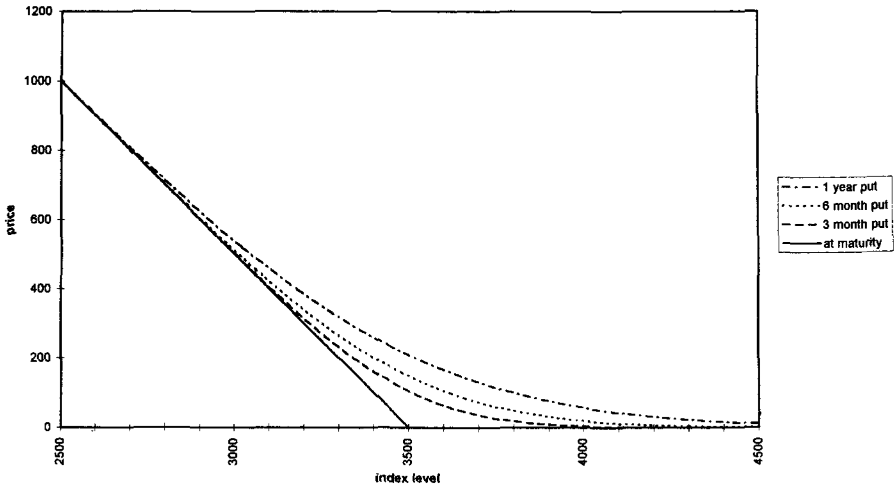

Figure 8. Price of a put option

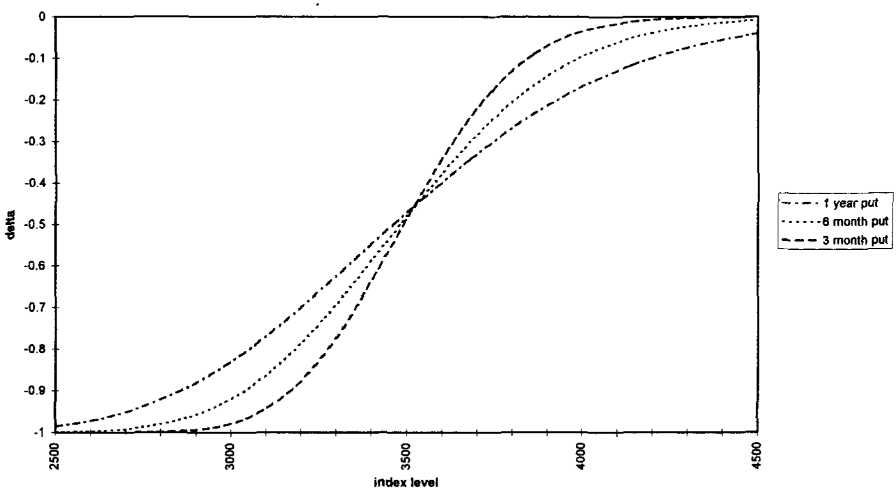

Figure 9. Delta of a put option

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might be used to hedge dynamically the characteristics of the option.

(b) The smaller the absolute size of the gamma, the less, in effect, are the option-like characteristics of the derivative. If gamma (defined as above) is zero in all circumstances, then the derivative will have no asymmetric characteristics, and may be priced using merely the sorts of techniques described in Section 5.

7.7.4 As mentioned in ¶5.2.2, somewhat different definitions of delta and gamma are normally used for interest rate derivatives.

7.7.5 The price of a call option also depends on σ, r, q and the time to maturity (and, for pricing formulae which are more complicated than the BS formulae, on other relevant parameters). For example, the price of vanilla puts and calls will, in general, rise if σ rises. These sensitivities, together with the delta and gamma of the option, are generically known as the option *greeks*, and are described further in Appendix A.

7.8 *Implied Volatility*

7.8.1 It is relatively easy to demonstrate that there exists some σ which equates the observed price of any European put or call option to the price derived from the appropriate BS formula (provided the price of the option is within the constraints imposed by no arbitrage, see Section 5.1).

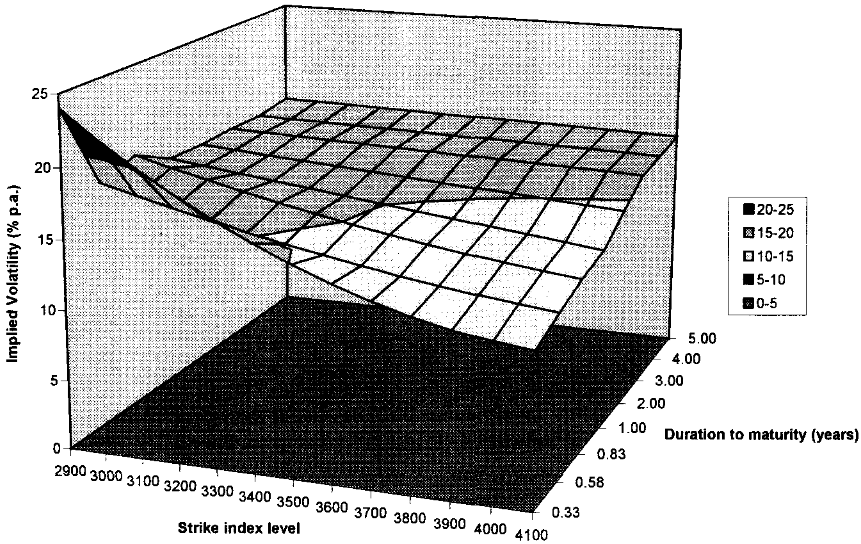
7.8.2 This σ is known as the *implied volatility* of the option, and is widely used within the derivatives markets.

7.8.3 In this artificial sense, the BS equation will be satisfied by all European style options. However, the implied volatility may differ for options with different maturities or for options with the same maturities, but with different exercise prices.

7.8.4 In Section 5.4 we introduced the concept of calibrating option prices, concentrating directly on the price of appropriate digital call spreads. We could equally well restate the calibration problem into one of finding the appropriate implied volatilities for each exercise price. Indeed, this is the usual way in which the problem is framed within derivatives houses.

7.8.5 If the assumptions underlying the BS formulae were correct, then the implied volatility would be constant across all exercise prices. In practice it is not. Market prices exhibit *smile* and *skew* effects. The smile effect is so named because the implied volatility is generally higher for far in-the-money or far out-of-the-money options than for options with exercise prices closer to current levels. An *in-the-money* option is one whose intrinsic value is greater than zero (e.g. for a call, $S>E$) whilst an *out-of-the-money* option is one whose intrinsic value is zero (and which is not at the money, i.e. on the boundary between the two, with $S=E$). Only one side of the smile may be noticeable, in which case the structure is called a skew.

7.8.6 Figure 10 shows how implied volatilities (for FT-SE options as at 7 March 1996) varied according to the exercise price of the option. It also shows how the smile/skew varied by duration at that date.



Source: Paribas

Figure 10. FT-SE smile /skew by strike and duration on 7 March 1996

7.8.7 It is important to note that the implied volatility of the option can change during the lifetime of the option, even if the actual volatility does not. If the option is marked-to-market, this can generate a gain or loss, even if the dynamic hedger believes his position is perfectly dynamically hedged. Thus, there is an element of preference dependence in the way that the BS formulae are applied in practice.

8. Further Comments on the Black-Scholes Formulae

8.1 *Common Misconceptions regarding the Black-Scholes Formulae*

8.1.1 There are many common misconceptions regarding the BS formulae, which can be demonstrated as erroneous by considering carefully the binomial tree derivation set out in Section 7.5.

8.1.2 For example, it is sometimes claimed that the BS formulae require interest rates to be constant. This is not so. If the price at time t of a zero coupon bond delivering E and maturing at time T is $V_E(t)$, and the price at time t of a contract delivering 1 share at time T without dividend income in the

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meantime is $V_S(t)$, then we can rework the derivation and show that the price of, say, a put option will be given by:

$$P(S,t) = V_E(t)\mathrm{N}(-d_2) - V_S(t)\mathrm{N}(-d_1)$$

where:

$$d_1 = \frac{\log(V_S(t)/V_E(t)) + (\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T-t}.$$

Strictly speaking, in this formula σ needs to be defined as the volatility of $\log(V_S(t)/V_E(t))$.

8.1.3 Thus, we may still use the BS formulae even if interest rates and dividend yields vary stochastically, provided we use the appropriate rates applicable at the time of valuation for securities maturing at the same time as the option.

8.1.4 Indeed, it is not even necessary for the variability of the share price to be constant. If it is continuous and a function of time only (i.e. deterministic, not stochastic), then the formulae remain correct, provided σ is taken as a suitable average volatility for the period between now and maturity. This result is usually attributed to Merton (1973).

8.1.5 If σ is continuous and a function of both time and the price of the underlying, then the BS formulae will, in general, give the wrong answers, but the option can still be found as the limit of an appropriate binomial tree or the solution to a partial differential equation which is preference independent (see Appendix A). We might call this *the generalised Black-Scholes framework*, or perhaps better would be the *generalised Brownian framework*, for the sorts of reasons outlined below.

8.1.6 It is also claimed that the BS formula requires the share price to follow a Brownian motion, and thus for the price of the underlying to be log-normally distributed in the future (and, in particular, at maturity). In fact, the derivation in Section 7.5 only requires the stochastic process followed by the price of the underlying to be the limit of a process involving just up and down jumps (with u/d tending to one). It would therefore appear that more general processes are possible. Rather remarkably, however, it is possible to prove that any continuous stochastic process (or to be more precise every continuous martingale) is a sort of time-shifted Brownian motion, see Rogers & Williams (1994).

8.2 *Experience Rated Options*

8.2.1 Neuberger (1990a) provides further insight into this. He uses a concept that he calls the *cumulative quadratic variation* of a process (and what Rogers & Williams call merely *quadratic variation*). This is the cumulative total of the squared changes in the log price (less the mean risk-neutral drift of the price process). If it is continuous, finite and reaches a fixed level at the time the option matures, then he shows that the BS formulae are still correct. He defines what he calls *mileage* options, which are options which mature when the cumulative quadratic variation first reaches a certain level. These options do not actually trade in practice. Given some underlying share price movement, the option will eventually mature, but it is not possible at outset to say when. He shows that in a no arbitrage world a mileage option will be priced according to the BS formula as long as it matures at a fixed point in time, i.e. that the cumulative quadratic variation is fixed at the maturity of the option, and as long as the cumulative quadratic variation is continuous.

8.2.2 Such mileage options have parallels with the actuarial principle of *experience rating*, in which part of any profit or loss arising from an insurance contract may be rebated to, or recovered from, the policyholder. If we could alter the characteristics of the option as time progresses, to reflect actual volatility experienced, then we can dramatically improve the ability of dynamic hedging to replicate the pay-off of the option, as shown in Section 9. As we might expect, the quality of replication is best if we optimise the hedging to be consistent with these mileage options.

8.2.3 In effect, Neuberger shows that the price of such an option at time t, in the absence of transaction costs, if the cumulative quadratic variation of the option $C(t)$ is continuous and $C(T)-C(t)$ is fixed whatever the path taken by the underlying stochastic process, is given by the following formula:

$$P(S,t) = V_E(t)\mathrm{N}(-d_2) - V_S(t)\mathrm{N}(-d_1)$$

where:

$$d_1 = \frac{\log(V_S(t)/V_E(t)) + (C(T)-C(t))/2}{\sqrt{C(T)-C(t)}} \quad \text{and } d_2 = d_1 - \sqrt{C(T)-C(t)}.$$

8.2.4 The cumulative quadratic variation is defined (in discrete time), if $V_E(t)$ is constant and q can be ignored, as the sum of the squares of the log price movements, i.e.:

$$C(T)-C(0) = \sum_{t=0}^{T} x_t = \sum_{t=0}^{T}\left(\log\left(\frac{S_{t+1}}{S_t}\right)\right)^2.$$

8.2.5 In practice $V_E(t)$ will not normally be constant, and, even with a vanilla equity index option, we can conceptually think of the option as having two sorts of underlying, one being a forward contract on the index, and one being a zero coupon bond maturing at time T. This is, in turn, a special case of a more general sort of option, a *relative performance option*, where the exercise price is based on two arbitrary assets. If the two 'underlyings' that now exist in this more generalised formulation of an option are S and E, the cumulative quadratic variation becomes:

$$C(T)-C(0) = \sum_{t=0}^{T} x_t = \sum_{t=0}^{T}\left(\log\left(\frac{V_S(t+1)}{V_S(t)}\right) - \log\left(\frac{V_E(t+1)}{V_E(t)}\right)\right)^2.$$

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8.2.6 We can thus price such relative performance options in a Black-Scholes world using a standard deviation of:

$$\sigma^2 = \sigma_S^2 - 2c_{S,E}\sigma_S\sigma_E + \sigma_E^2$$

where the terms σ_S and σ_E refer to the volatility of S and E in isolation, and the term $c_{S,E}$ represents the correlation between movements in S and E.

8.3 *When will the Black-Scholes Formulae be Wrong?*

8.3.1 Neuberger's approach can also be used to identify when the BS formulae will break down. These are:

(a) if markets are not arbitrage-free;
(b) if markets jump (since the cumulative quadratic variation is then discontinuous);
(c) if the future volatility of the market is uncertain; or
(d) if markets are subject to 'friction', e.g. in the form of transaction costs.

8.3.2 In some fundamental sense we do not worry about (a). If we follow a hedging algorithm based on the assumption that markets are arbitrage-free, then we will forego possible arbitrage profits, but these probably cannot be relied upon in any case. We also eliminate the possibility of arbitrage losses, so adopting a no arbitrage assumption can be thought of as a worst case or prudent approach.

8.3.3 The other three possible sources of deviation from the BS formulae are considered further in Appendix B. All three of them introduce preference dependence (although the uncertainty in volatility of markets only does so if the volatility is itself stochastic rather than depending merely on time and the price of the underlying), and, as we shall see later, thus influence the size of reserves we may need for a derivatives portfolio.

8.3.4 In addition to these, there are some other practical reasons why the BS formulae may not provide a complete picture, including:

(a) basis risk (and roll-over risk), as described in ¶5.4.3, and other uncertainties in the forward prices V_S and V_E; and
(b) miscellaneous practical matters such as whether to strip out weekends when measuring the time between two dates.

8.4 *Pricing Path Dependent Options*

8.4.1 The BS formulae relate solely to European put and call options. These are *path independent*, in the sense that their pay-offs depend only on the price of the underlying at maturity and not on how the price reaches its final level.

8.4.2 Options may also be *path dependent*, with the pay-off also depending on how the price behaved prior to maturity.

8.4.3 For example, a *barrier option* has a pay-off which depends on whether the underlying has crossed some prespecified barrier level prior to maturity. Barrier options may *knock-in* or *knock-out*, depending on whether the holder becomes entitled to a pay-off or loses that entitlement if the price of the underlying crosses the barrier.

8.4.4 Provided the barrier has certain characteristics, a closed form analytic solution (i.e. a relatively straightforward mathematical formula) can be found for the price of such options in the basic Black-Scholes world. The approach uses the *reflection principle*, and involves showing that the boundary conditions satisfied by the option are exactly those generated by another option with a suitable shadow pay-off at maturity. The price of the barrier option prior to maturity (as long as the barrier has not been reached) is therefore the same as the price of a non-barrier option with this shadow pay-off. Such an approach works (in the basic BS framework of constant r, q and σ) as long as the barrier $B(t)$ can be described by a formula along the lines of (where η is constant):

$$B(t) = B_0 e^{-\eta(T-t)}.$$

8.4.5 If the barrier does not have this form, then it may still be possible to decompose it piecewise into parts that do. We can then apply the same approach repeatedly to each part of the barrier (working backwards from final maturity). However, the two cumulative normal functions within the BS formulae become multiple nested cumulative normal functions and the overall formulae become horrendously complicated.

8.4.6 Another sort of path dependent option that actuaries may come across in practice (because it is the basis of some retail products) is a *cliquet* option. In this sort of option, the period to maturity is split into several sub-periods. Typically, the pay-off is like an at-the-money call option, but instead of looking at the rise in underlying over the whole period, the cumulative pay-off at maturity is equal to the sum of the gains over each of these sub-periods. Like barrier options, this sort of option can be valued analytically in the basic Black-Scholes world, although the resulting formula is quite complicated.

8.5 *American Style Options*

8.5.1 A more common, and certainly simpler to describe, sort of path dependent option is an *American* style option. It is the same as a European style option, except that it is exercisable at any time prior to maturity rather than just at maturity. Both European and American style options are traded on both sides of the Atlantic, and so the words no longer have any geographical relevance when used to describe derivatives. Exchange traded options are often, but not always, American style.

8.5.2 The value of an American option is, in a no arbitrage world, at least as much as the value of an equivalent European option. The American option provides the holder with the right (but not the obligation) to exercise the option before maturity if he so wishes. This extra optionality must have a non-negative value. Its value is often called the *early exercise premium*.

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8.5.3 The 'path dependence' of American options arises because the optimal early exercise strategy on the part of the option purchaser depends on the price of the underlying during the lifetime of the option. However, the way in which such options are path dependent is different to that for barrier options, and different valuation techniques are generally needed.

8.5.4 Unfortunately, producing mathematically tractable valuation formulae akin to the BS formulae for American options (or equivalently valuing the early exercise premium) has proved impossible except in special cases. One special case where such a formula does exist is if the dividend yield is zero (for a call option) or the risk-free interest rate is zero (for a put option). In these circumstances it should never be optimal to exercise early, so the early exercise premium should be zero. Another special case is if the term of the option is infinite and r, q and σ are constant, when the option can be priced by:

(a) identifying the time independent solutions to the underlying partial differential equation; and

(b) identifying the level of the boundary B (as a function of E) by noting that on the boundary what is known as the *smooth pasting condition* applies, i.e.:

$$\frac{\partial V}{\partial S} = \frac{\partial I}{\partial S}$$

where $V(S,t)$ is the value of the option and $I(S,t)$ is its intrinsic value, i.e. $S-E$ for an in-the-money call option.

8.6 *Valuing American Options using Binomial Lattices*

8.6.1 The most common way of valuing American options is to use a binomial lattice. We merely place a minimum on the value of the option at each lattice point equal to what the holder would get if he exercised the option immediately, i.e. its intrinsic value. It is possible to show that the value calculated in this fashion will converge to the 'true' value as the spacing between different lattice points tends to zero.

8.6.2 This would appear to solve, at least from a practical perspective, the problem of how to value American options. Unfortunately, the binomial lattice converges only relatively slowly to the true value as the lattice spacing is reduced, although for straightforward options the speed of convergence is usually adequate.

8.7 *Trinomial Lattices*

8.7.1 If binomial lattices converge too slowly, a potentially much quicker way of valuing options is to use a trinomial lattice, as in Figure 11.

8.7.2 In such a lattice greater flexibility is introduced by a third possible price movement. This means that we can get much faster convergence as the lattice spacing becomes smaller, as noted in Kemp (1995).

8.7.3 Careful choice of lattice structure is necessary to get the best conver-

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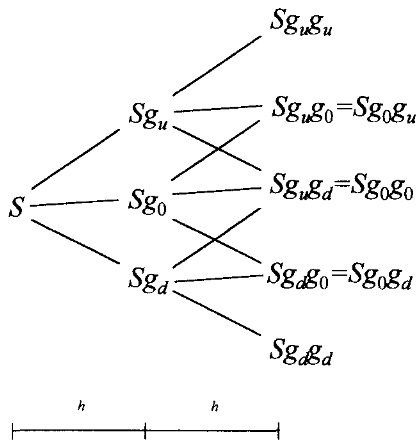

Figure 11. Trinomial lattice/tree

gence characteristics. The mathematics is discussed in more detail in Appendix A.5.

8.8 *Using Binomial Lattices to Replicate Arbitrary European Style Risk-Neutral Probabilities*

8.8.1 There is one further point worth noting on binomial lattices. Although it is usual to adopt constant spacing between lattice points, we do not need to do so. As soon as we permit arbitrary spacing, it becomes possible to fit a much wider range of price processes.

8.8.2 Indeed, the flexibility is so wide that we can replicate exactly the price of any finite number of European option prices satisfying no arbitrage using a suitably chosen binomial tree, see e.g. Rubinstein (1996), although replicating options with different maturities is a non-trivial exercise. In a continuous time framework, and as long as the European option prices satisfy suitable regularity conditions (e.g. they are continuous functions of the price of the underlying and of time), then it seems to be possible to price an entire term and strike structure of European option prices using a suitable $\sigma(S,t)$ which is a deterministic function of S and t, i.e. within the generalised Brownian framework introduced in ¶8.1.5.

8.8.3 And yet we know that not all price processes can be modelled using a binomial tree or limiting versions of it. Thus, the price of European options *in the future* will not necessarily be modelled correctly by this binomial tree. The price will fail to be correct in precisely those circumstances where binomial lattices and their limits break down.

8.8.4 This has potentially important consequences for the 'standard'

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actuarial approach to valuation, described in Section 7.4, and to how we should calculate actuarial reserves for future liabilities:

(a) The fair price of an option/guarantee can depend, *not just on the likelihood of the outcomes at maturity, but also on how the price of the underlying moves before maturity.*

(b) If price movements follow a known binomial tree (or diffusion process) perfectly, then we can always hedge any contingent claims perfectly using dynamic hedging techniques. As long as we actually invest in line with the relevant dynamic hedging algorithm, there would, in principle, be no need to hold any reserves in excess of the value of the option. Thus *the part of the reserves arising because the position is dynamically hedged ought to be directly linked to how far the actual price process followed by the underlying differs from predeterminable binomial trees/diffusion processes.* We return to the issue of how to reserve for derivatives in Sections 9 and 10.

8.9 *'Risk Neutral' and 'Real World' Probability Distributions and Autoregressive Market Behaviour*

8.9.1 In a world characterised by binomial trees or diffusion processes, the only information that an investor needs to derive the price of options is the current price of the underlying and the risk-neutral probabilities. It is not that the 'real world' probability distribution that the investor actually expects to arise is unimportant. It is just that the impact on derivative prices due to differences between it and the risk-neutral probability distribution all fortuitously collapse into the price of the underlying.

8.9.2 The risk-neutral probability distribution is often loosely described as the 'real world' distribution of an investor who believes that all assets will supply the same expected return.

8.9.3 It turns out that this shorthand is misleading if the 'real world' distributions involve autoregressive characteristics, like the Wilkie model as described in Wilkie (1995), which is probably the most written about stochastic model in the U.K. actuarial literature. In such circumstances, the risk-neutral probability distribution may be materially more spread out than the 'real world' distribution.

8.9.4 A random walk model has the characteristic that the volatility $v(n)$ of log price movements over an n-year period satisfies the following formula:

$$v(n) = n\text{-year volatility} = v(1)\sqrt{n}.$$

8.9.5 Autoregressive models, by contrast, are characterised by the volatility of outcomes in n years time being less than the square root of n times the volatility in each given year.

8.9.6 However, Neuberger's mileage options show that the price of an option depends on the cumulative quadratic variation, which, in this context,

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can be thought of as the accumulation of one-year volatilities, ignoring any dampening effect from the autoregressive characteristics of the model. For the Wilkie model (which has log-normal error terms and is therefore modellable using diffusion processes, albeit ones with autoregressive characteristics), the risk-neutral probability distribution is more widely spread out than the 'real world' distribution *even for an investor who assumes all asset categories will have the same expected return.*

8.9.7 Thus, it seems necessary to strip out the autoregressive characteristics of the Wilkie model when pricing guarantees and other option-like characteristics, particularly long-term ones, such as maturity guarantees. Looking merely at the spread of outcomes arising from the Wilkie model will materially understate the actual value of the guarantee. Approximate valuation formulae for options in a Wilkie world are given in Kemp (1996).

8.9.8 Another way of understanding this apparently counterintuitive result is to note that the Wilkie model implies that it is possible to make abnormally high profits by switching away from assets which have recently done well into assets which have recently done poorly. If I buy a put or a call option and then hedge away its option characteristics by dynamic hedging, this involves selling equities as they rise and buying them as they fall, i.e. precisely the approach that leads to anomalously high profits under the model. This places a premium on purchased options, which should bid up their price, fortuitously to precisely the price that arises if the autoregressive characteristics of the model are stripped out.

8.9.9 An interesting corollary for actuaries is in the area of *asset/liability studies* and *dynamic solvency testing*. The idea behind such tests is to identify how robust an insurance company is to different ways in which the future might evolve. Typically the tests involve projections of assets and liabilities under many scenarios, often chosen at random from a suitably chosen probabilistic model of how markets might evolve. These projections are then used to give an indication of the range of likely outcomes the insurer might face and the probability of the insurer running into trouble. However, if an autoregressive model, such as the Wilkie model, is being used, the spread of outcomes revealed by the projections is not necessarily a helpful guide to the (open market) cost of protecting against such risks. The cost should, it seems, be based on a wider potential spread of outcomes than revealed by the projections, with the potential error larger the further into the future the projections go. An extreme view some might take is that all that really matters for shareholders and investors in an insurance company is an estimate of the *value* of the contingent claim they have on the company, rather than some internally estimated likelihood of the claim being of a given size. Proponents of such a view would presumably wish to ignore projections based on the Wilkie model altogether, concentrating exclusively on projections using models without any autoregressive characteristics.

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9. Testing the Effectiveness of Dynamic Hedging

9.1 *What Precisely do we Mean by Dynamic Hedging?*

9.1.1 We are now in a position to test the effectiveness of dynamic hedging; but first it is helpful to restate what we mean by the term.

9.1.2 If an organisation has sold a derivative, then it is usual for it to set aside assets (reserves/provisions) to meet the potential liability represented by the option. *Dynamic hedging* is, strictly speaking, the process of:

(a) investing this portfolio in a mixture of the assets underlying the derivative or futures/forwards on these assets (for a vanilla equity put option the portfolio would consist of cash and equities, or more probably equity futures contracts); and

(b) altering the mix of this portfolio in a manner that mimics the behaviour of the option.

9.1.3 Another term that is often used for the same sort of process is *portfolio insurance*.

9.1.4 The way in which the price of a put option (of the sort described in ¶9.2.3) varies as the index level moves up or down is shown in Figure 8, as long as the price of the option is exactly replicated by the Black-Scholes formula (in that example volatility is 15% p.a. and the exercise price is 3,500). To replicate this behaviour (given the assumptions underlying the Black-Scholes formula), the hedge portfolio needs to have exposure to the equity market that varies in line with the slope of the curve in Figure 8. The required amount invested in equities is the option delta:

$$\Delta(S,t) \equiv S\frac{\partial V}{\partial S}.$$

For a put option delta is negative, and so the hedge portfolio needs to short-sell equities (which would be most easily achieved by selling futures contracts). Figure 9 shows the delta (excluding the multiplier of S) corresponding to the prices in Figure 8.

9.1.5 Portfolio insurance acquired a poor reputation at around the time of the October 1987 Crash. The market movements that then occurred were so extreme as to cause some portfolios run using dynamic hedging techniques to differ significantly in behaviour from the options they were trying to mimic. As there are risks that cannot be hedged using merely pure dynamic hedging (of which jump risk is one), portfolio insurance techniques are nowadays normally deemed to encompass hedging using a variety of instruments, including other options. The meaning of the term 'dynamic hedging' can also be extended in this fashion, but for the purposes of this paper I have treated it as synonymous with 'pure' dynamic hedging, involving merely investment in the securities underlying the derivative.

9.1.6 In the limit, of course, it would be possible to buy an option which was identical to the liability being hedged. This is much like reinsurance, and involves no 'dynamic' characteristics at all, since it is a pure buy-and-hold form of hedging. The term 'portfolio insurance' would normally be limited to the use of relatively liquid (e.g. exchange-traded) instruments. There is a spectrum of portfolio insurance techniques with a greater or lesser degree of reliance on pure dynamic hedging.

9.2 *Testing the Effectiveness of Dynamic Hedging*

9.2.1 The most important market from the perspective of U.K. actuaries is probably the U.K. equity market. This section, therefore, concentrates on the effectiveness or otherwise of dynamic hedging of the U.K. equity derivatives, concentrating on daily movements in the FT-SE 100 Index over the period 1 January 1984 to 26 March 1996 (excluding weekends and bank holidays), as supplied by Datastream. This period includes the largest daily falls on record for FT-SE, which, according to Datastream, were 11% and 12% on 19 and 20 October 1987 respectively.

9.2.2 The spread of daily log price movements over this period is shown in Figure 12. It appears to be remarkably like a normal distribution. However, Figure 12 hides the fact that the spread of movements (i.e. market volatility) is greater over some sub-periods than over others. It also masks a small number of very extreme events, which, although rare, happen far more often than they should do if movements were actually normally distributed. This can be seen by referring to Figures 13 to 17, which show the spread of (overlapping) weekly, two-weekly, monthly, quarterly and yearly log price movements. For comparative purposes these are expressed as deviations from the mean, in units of $0.0094\sqrt{n}$ (where n is the number of working days in the relevant period), since, if the daily log price movements were independent normally distributed, then the graphs would then all have the same shape. In Figure 16 (quarterly log price movements) there are sufficient extreme outcomes (many including the 19 and 20 of October 1987) for the spread to have a more noticeable tail. Figure 17 (annual log price movements) shows a noticeable skew.

9.2.3 For simplicity, I have concentrated on a special sort of European put option. This option gives the holder the right to sell the index (with gross income reinvested) for a given price defined by a certain exercise price at outset, rolled up in line with the risk-free rate. Suppose that S_t is the index level at time t, Y_t is the equivalent total return index with gross dividends reinvested and C_t is a total return cash index with gross income reinvested. Suppose also that the exercise price of the option is E. Then the option provides a pay-off in relation to Y_0 $(=S_0)$ of opening market exposure of:

$$\text{pay-off} = \max\left(E.\frac{C_T}{C_0} - Y_T, 0\right).$$

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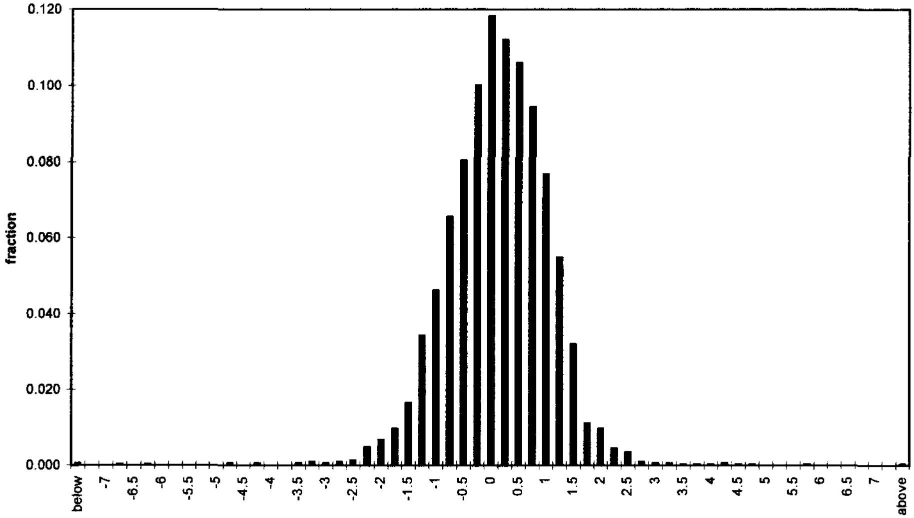

Source: Datastream

Figure 12. Spread of daily log price movements on FT-SE

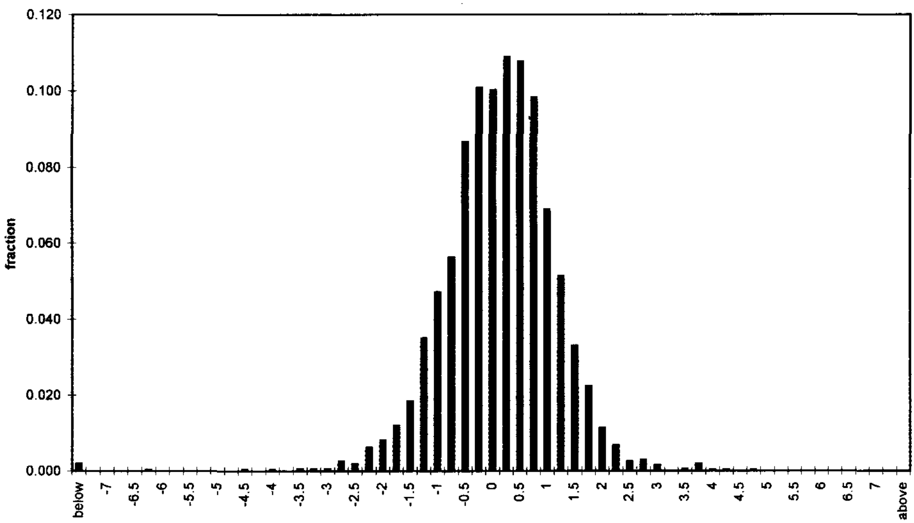

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Figure 13. Spread of weekly log price movements on FT-SE

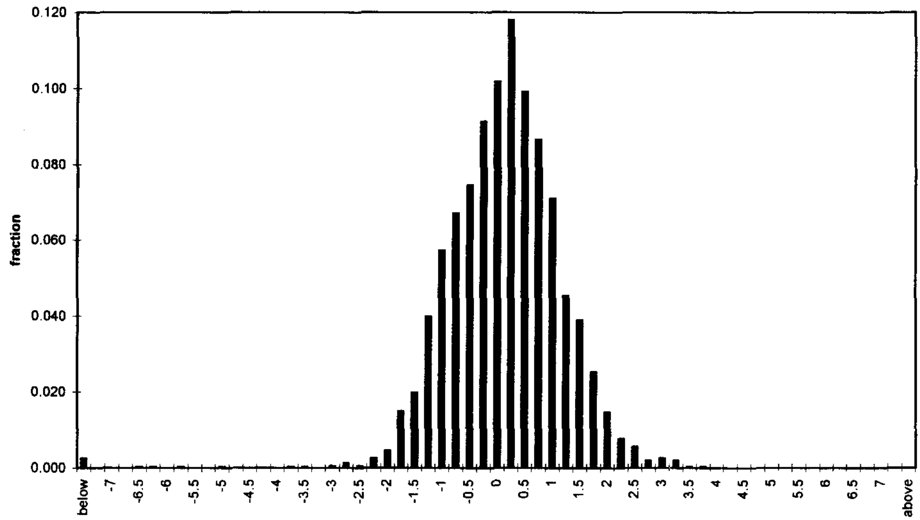

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Figure 14. Spread of two-weekly log price movements on FT-SE

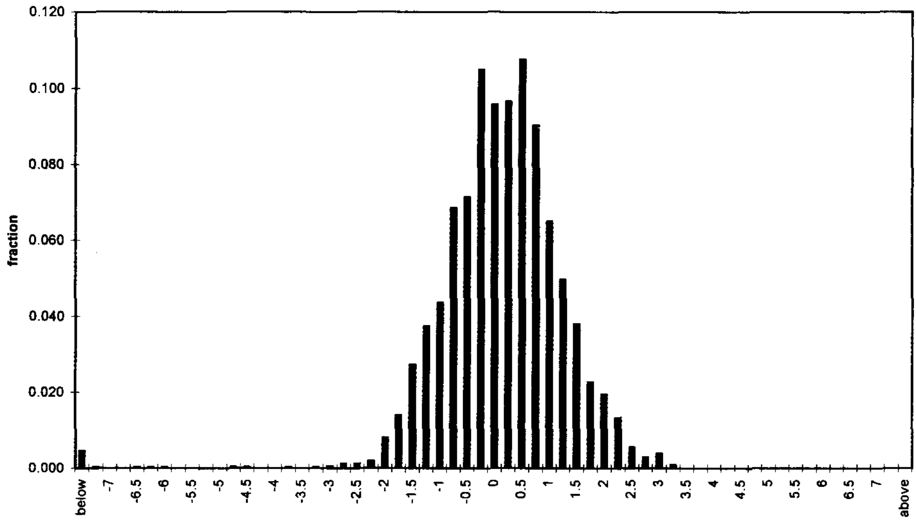

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Figure 15. Spread of four-weekly log price movements on FT-SE

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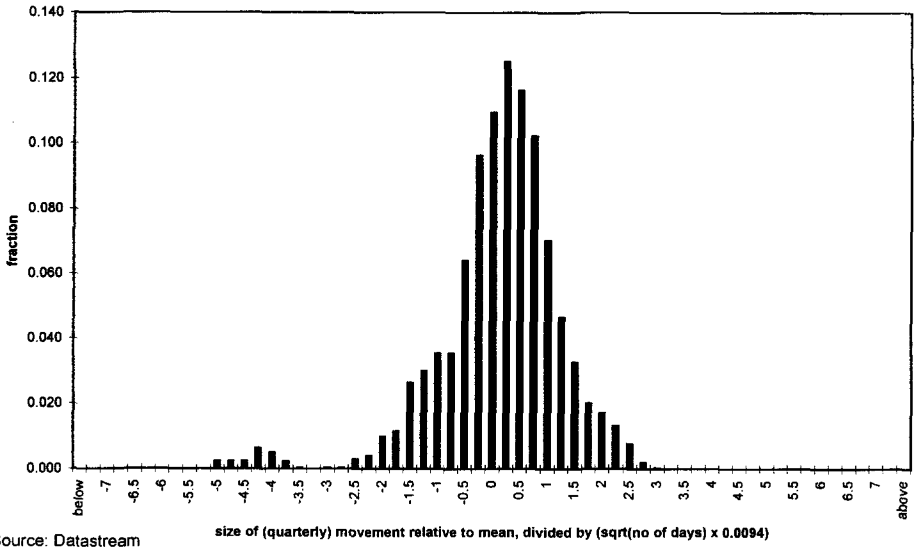

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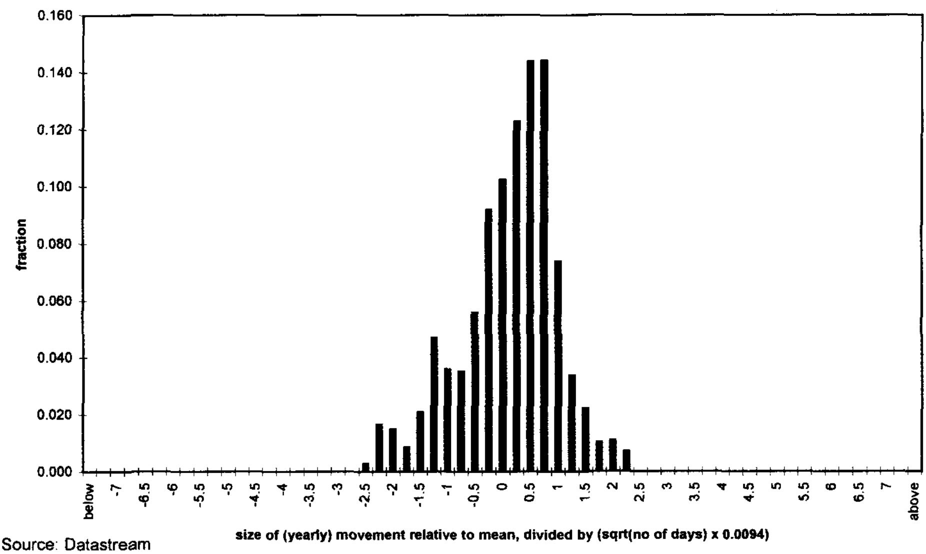
Figure 16. Spread of quarterly log price movements on FT-SE



Source: Datastream

Figure 17. Spread of yearly log price movements on FT-SE

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9.2.4 There are two reasons for considering this apparently rather complicated type of option, despite the potential practical problems a bank would face issuing such a contract (e.g. possible difficulties in defining how to construct the relevant indices and in deciding on what appropriate reinvestment and tax assumptions to make):

(a) This pay-off is appropriate if we are considering a liability that pays the better of 'equities' or 'cash', since what we are then really interested in is the *total return* achieved on these investment classes. The use of options linked only to the *capital value* of an index is common within the life insurance industry, but this is partly so that the income component can be spent to provide the capital guarantee (which helps with marketing).

(b) The pay-off also makes the Black-Scholes option pricing formula simpler, since it means that we can set both the assumed interest rate and any allowance for dividends equal to zero. Thus, the value of this option according to the Black-Scholes formula simplifies to:

$$P(S,t) = E.\mathrm{N}(-d_2) - S.\mathrm{N}(-d_1)$$

where:

$$d_1 = \frac{\log(S/E) + (\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

and

$$\mathrm{N}(x) = \text{cumulative normal probability function} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2} dz.$$

9.2.5 The 'delta' of the put option (in a Black-Scholes world) is also simplified by adopting this pay-off. It is:

$$\Delta(S,t) = -S.\mathrm{N}(-d_1).$$

9.2.6 We can assess how successful dynamic hedging (in the absence of transaction costs) of this option might be by determining the cumulative surplus Z, a dynamic hedging programme would generate, where Z is defined as:

$$Z = \sum_{t=0}^{T-1} z_t \quad \text{where } z_t = \left(\frac{S_{t+1}}{S_t} \Delta(S_t,t) + (P(S,t) - \Delta(S_t,t)) \right) - P(S_{t+1}, t+1).$$

The first bracket in the formula defining z_t is the sum available from the dynamic hedging programme after the index has moved from S_t to S_{t+1}, whilst the last term is the value of the option at time $t+1$, i.e. the liability being hedged. The difference is thus the surplus arising between t and $t+1$ (or rather, given the way that the option is defined, the present value of this surplus at $t=0$, discounted at the risk-free rate). In this formula $t=0$ defines the start of the option.

9.2.7 Suppose we consider put options which are 5% out of the money (i.e. $E=0.95Y_0$), and which are 240 working days (i.e. nearly one year) in length.

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9.2.8 Over the period 1 January 1984 to 26 March 1996 (excluding weekends and bank holidays), the standard deviation of daily log price movements in the index between the close of business on consecutive working days was 0.009407 and the mean of the daily log movements was 0.000419. Thus, one way of hedging the put option would be to assume that σ = 0.009407 per day. The price of the option at outset (and hence the initial value of the hedge portfolio) would then be 3.51% of the opening index level.

9.2.9 To assess how effective this strategy might be, we could calculate the cumulative surplus (always as a percentage of the index level ruling at the start of the contract) for each 240 working day period encompassed by 1 January 1984 to 26 March 1996, order the results and calculate percentiles and minimum and maximum values.

9.2.10 Table 1 shows the results of this exercise. In a material number of cases the cumulative deficit is more than the total value of the option at outset. This is true even if we apply a loading of 10% or even 20% to the volatility (i.e. σ = 0.0103 or 0.0113 per day).

9.2.11 However the approach is better than merely adopting a completely static approach, which is also shown for comparative purposes in Table 1. Over the period under analysis, equities performed better than cash, so, on average, a strategy of taking the option premium and holding it in cash would have outperformed the dynamically hedged portfolios described above (since they have option deltas somewhere between $-S$ and 0). This, in effect, reflects a strategic asset allocation stance which we should strip out to compare like with like. We can do this by considering a static investment mix involving going short –21.45% of the starting index level in equities and investing 21.45+3.51 = 24.96% of the starting index level in cash, since the average cumulative surplus is then zero.

Table 1. Cumulative surplus, daily rebalancing, fixed volatility, no special treatment of jump costs

	No loading	10% loading	20% loading	c.f. static approach
Minimum value	–8.51	–7.80	–7.05	–12.38
99th percentile	–7.37	–6.80	–6.21	–8.98
95th percentile	–4.29	–3.98	–3.64	–5.78
90th percentile	–1.18	–0.59	–0.10	–3.19
70th percentile	–0.02	0.33	0.66	–0.82
50th percentile	0.30	0.67	1.03	0.20
30th percentile	0.62	1.08	1.57	1.46
10th percentile	1.03	1.51	2.01	3.14
5th percentile	1.18	1.69	2.23	3.81
1st percentile	1.42	2.07	2.81	4.38
Maximum value	2.06	2.91	3.71	4.58
c.f. average	–0.12	0.31	0.75	0.00
Standard deviation	1.67	1.67	1.67	2.75
Option premium	3.51	4.04	4.58	3.51

9.3 *Possible Control Mechanisms*

9.3.1 Table 1 might seem to vindicate some of the traditional 'actuarial' mistrust of pure dynamic hedging. It is better than 'doing nothing', but not hugely so. We can, of course, improve the quality of replication by rebalancing more frequently than daily (although, of course, ever more frequent rebalancing becomes more onerous on systems, etc., and transaction costs can become increasingly problematic, see Section 9.7). However, in the sorts of extreme market movements causing the problem, there is no guarantee that the markets will be trading actively during the day. The usual means of gaining and shedding market exposure is via futures (because they are so much cheaper to deal in than the underlying stock). During the October 1987 Crash, the U.S. futures exchange trading in S&P was unable to keep up with the volume traded in the underlying market, and did, indeed, close (although the U.K. equivalent remained open throughout the day).

9.3.2 However, there are several other ways in which we could seek to improve the characteristics of the hedging.

9.4 *Jump Costs*

9.4.1 We could, for example, postulate that we could buy (or charge separately for) some sort of protection which made good any deficit caused by a downward or upward jump of more than, say, *four* times the standard deviation of daily movements. If the probabilities of movements were normally distributed, then the likelihood of any such movements occurring throughout the entire period under analysis is less than 25%, so these events could justifiably be thought of as 'catastrophes'. They are, in practice, only insurable in a manner akin to other 'catastrophe insurance' such as earthquake risks, e.g. by spreading the risk over many different sorts of exposures and/or by stripping out such costs from a day-to-day revenue account and charging for them through some pooling through time. This approach might be thought of as equivalent to purchasing catastrophe excess of loss insurance, or running a separate catastrophe risks fund.

9.4.2 The worst outcomes in the analysis (excluding the cost of jumps) are very considerably reduced if jump risk is paid for separately, as shown in Table 2. Jump costs can arise both with large downwards market movements and with large upwards market movements, since either can lead to a deficit in the hedge portfolio. I have categorised them separately, since this highlights the relative magnitude of the risks involved.

9.4.3 Interestingly, the jump costs (as calculated above) are much less sensitive to the strike price than the overall option premium. If we test equivalent options, but with different strikes, then the down and up jump costs and the average initial option premium are as per Table 3.

9.4.4 This analogy with catastrophe risk also highlights the advantages that come from diversification. If the jumps to which a derivatives house is exposed arise from different sorts of economic exposures, then they are unlikely all to

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Table 2. Cumulative surplus, daily rebalancing, fixed volatility, jump costs itemised separately

	Non-jump cost	Down jump cost	Up jump cost
No loading			
Minimum value	−2.91	−5.43	−1.98
99th percentile	−1.32	−5.12	−1.80
95th percentile	−0.39	−3.55	−0.90
90th percentile	−0.23	−0.31	−0.66
70th percentile	0.08	—	—
50th percentile	0.34	—	—
30th percentile	0.68	—	—
10th percentile	1.06	—	—
5th percentile	1.19	—	—
1st percentile	1.42	—	—
Maximum value	2.06	—	—
c.f. average	0.37	−0.33	−0.16
Standard deviation	0.56	1.07	0.37
Option premium	3.51	n/a	n/a
10% loading			
Minimum value	−2.26	−5.19	−1.92
99th percentile	−0.90	−4.90	−1.73
95th percentile	−0.02	−3.41	−0.88
90th percentile	0.14	−0.29	−0.63
70th percentile	0.42	—	—
50th percentile	0.71	—	—
30th percentile	1.14	—	—
10th percentile	1.57	—	—
5th percentile	1.73	—	—
1st percentile	2.10	—	—
Maximum value	2.91	—	—
c.f. average	0.77	−0.31	−0.15
Standard deviation	0.61	1.03	0.35
Option premium	4.04	n/a	n/a
20% loading			
Minimum value	−1.70	−4.97	−1.85
99th percentile	−0.46	−4.66	−1.65
95th percentile	0.23	−3.27	−0.85
90th percentile	0.42	−0.26	−0.58
70th percentile	0.77	—	—
50th percentile	1.09	—	—
30th percentile	1.62	—	—
10th percentile	2.09	—	—
5th percentile	2.35	—	—
1st percentile	2.84	—	—
Maximum value	3.71	—	—
c.f. average	1.19	−0.30	−0.15
Standard deviation	0.69	0.99	0.33
Option premium	4.58	n/a	n/a

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Table 3. Jump costs for various strikes

	15% out of the money		5% out of the money		5% in the money	
	Down jump cost	Up jump cost	Down jump cost	Up jump cost	Down jump cost	Up jump cost
Minimum value	–4.38	–1.60	–5.43	–1.98	–6.57	–2.56
99th percentile	–4.21	–1.47	–5.12	–1.80	–5.94	–2.06
95th percentile	–2.56	–0.81	–3.55	–0.90	–2.78	–0.69
90th percentile	–0.16	–0.30	–0.31	–0.66	–0.38	–0.41
70th percentile	—	—	—	—	—	—
c.f. average	–0.23	–0.11	–0.33	–0.16	–0.35	–0.12
Standard deviation	0.87	0.29	1.07	0.37	1.13	0.33
Average initial option premium	0.89		3.51		8.78	

happen simultaneously, and the costs when they do will be mitigated by the non-occurrence of jumps in other parts of their business. It would be a fairly unwise catastrophe insurer to, say, have all his exposure linked to Tokyo earthquake risk. Instead, he would try to spread the risk across different geographical regions and different types of exposures. Interestingly, U.K. life offices are peculiarly exposed to the risk of large falls in the U.K. equity market, and thus should be particularly interested in diversifying away from such a preponderance of exposure to this single risk.

9.5 *Experience Rating*

9.5.1 Another way of improving the characteristics of hedging would be to assume that we knew in advance the level of volatility actually occurring during the lifetime of the option or we could alter the pay-off depending on actual experienced volatility. The latter corresponds to the concept of 'experience rating' or 'profit sharing'. This is a common principle within the insurance industry. The most obvious examples within the *life* insurance industry are with-profits contracts.

9.5.2 Suppose we started the hedging exercise with the price of the option based on the actual out-turn volatility over the following 240-day period, running the hedging exercise as if that were then the correct (constant) volatility to use. The range of cumulative surpluses would then be as set out in Table 4. Table 4 also shows the range of cumulative surpluses that would arise if we deliberately overstated the out-turn volatility by 10% or 20%. Jump costs are no longer itemised separately.

9.5.3 Whilst the hedging characteristics do improve very substantially using this sort of experience rating, the improvement is not quite as dramatic as we might have expected. A major reason for this is that index volatility may vary during the life of the option, but the form of the experience rating described

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above does not take this possibility into account. A better approach is to optimise the experience rating to reflect the concept of 'mileage' options, discussed in Section 8.2 and to operate as follows:

(a) We assume that we have been able to identify (or experience rate on the basis of) the out-turn cumulative quadratic variation.

(b) As time progresses, we reduce the cumulative quadratic variation remaining by the actual quadratic variation experienced in each time interval. We determine the option delta, and identify any surpluses or deficits using a remaining life of the option $T-t$ and a volatility σ so that $\sigma^2(T-t)$ equals the remaining cumulative quadratic variation.

Table 4. Cumulative surplus, daily rebalancing, 'crude' experience rating

	No loading	10% loading	20% loading
Minimum value	−1.86	−0.95	−0.37
99th percentile	−1.20	−0.51	−0.26
95th percentile	−0.79	−0.14	0.13
90th percentile	−0.67	−0.03	0.20
70th percentile	−0.20	0.09	0.43
50th percentile	−0.09	0.29	0.63
30th percentile	0.05	0.43	0.90
10th percentile	0.24	0.65	1.37
5th percentile	0.28	0.79	1.71
1st percentile	0.58	1.73	2.60
Maximum value	1.77	3.20	4.80
c.f. average	−0.13	0.30	0.73
Standard deviation	0.37	0.35	0.54
Average option premium	3.33	3.84	4.36

Table 5. Cumulative surplus, daily rebalancing, 'mileage optimised' experience rating

	No loading	10% loading	20% loading
Minimum value	−0.59	0.01	0.12
99th percentile	−0.38	0.06	0.17
95th percentile	−0.14	0.09	0.21
90th percentile	−0.06	0.11	0.26
70th percentile	−0.02	0.19	0.43
50th percentile	−0.01	0.32	0.68
30th percentile	0.00	0.52	1.07
10th percentile	0.05	0.75	1.49
5th percentile	0.14	1.05	1.90
1st percentile	0.71	2.15	3.66
Maximum value	1.42	2.77	4.19
c.f. average	0.00	0.42	0.86
Standard deviation	0.15	0.38	0.65
Average option premium	3.33	3.84	4.36

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9.5.4 The range of cumulative surpluses that would arise using this more accurate form of experience rating (together with the corresponding surpluses that would arise if we deliberately overstated the out-turn volatility at outset by 10% and 20%) are shown in Table 5.

9.5.5 Replication error is much smaller than with the cruder experience rating described previously. Although the small possibility of very substantial jumps does influence the minimum and maximum values significantly, in nearly all circumstances the cumulative surplus (if out-turn volatility is not deliberately overstated at outset) is very close to zero. If out-turn volatility were overstated by 10% at outset, then such a hedging programme would have always ended up in surplus (ignoring transaction costs)!

9.5.6 One reason that the experience rating, described in ¶¶9.5.3 to 9.5.6, is so resilient is that, to some extent, it captures jump risks directly, since the larger the jump, the greater is the quadratic variation during the period encompassing the jump. If the option is at-the-money, and the loading applied to the cumulative quadratic variation is at least 25%, then the surplus arising in the period z_t appears to be greater than or equal to zero, whatever the size of the jump (within a range –50% to +50%). If the option is not at-the-money, then there is no loading that seems to guarantee a non-negative surplus in each period, but the maximum possible loss can be kept very low with a sufficiently large loading (and can be more than compensated for by surpluses in other periods as demonstrated in ¶9.5.4).

9.5.7 Of course, fund managers seeking to run funds in this fashion could find daily rebalancing onerous. Table 6 shows the impact of rebalancing at less frequent intervals. Although the spread of results is significantly wider, such hedging is still surprisingly resilient, as long as it is possible to 'experience rate' the contract in a suitable fashion and to include an adequate loading at outset.

9.6 *'Experience Rating' in the Form of Market Derived Implied Volatilities*

9.6.1 Although I described experience rating as relatively rare within the derivatives industry, there is a sort of way in which something similar happens in practice. Option providers will sell options at the market rates prevailing at the time of sale, i.e. using the implied volatilities at that time. If implied volatility is a good predictor of actual out-turn volatility, then a large part of the uncertainty in future volatility would be discounted for in the actual price at which the derivatives traded.

9.6.2 This does, indeed, seem to be the case, at least over the recent past. Figure 18 shows the three-month implied volatility over the period January 1992 to June 1996 and the corresponding actual out-turn volatility over the following 66 days, based on data supplied by Goldman Sachs. Figure 19 shows a scatter plot of these implied volatilities (minus their average for the period as a whole) against the actual out-turn volatilities (minus their average for the period as a whole). There is a clear correlation (the correlation coefficient is around 0.8).

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Table 6. Cumulative surplus, different rebalancing periods, 'mileage optimised' experience rating

Frequency of rebalancing	Daily			Every three days			Weekly			2-weekly		
Volatility loading	0%	10%	20%	0%	10%	20%	0%	10%	20%	0%	10%	20%
minimum	–0.59	0.01	0.12	–0.83	–0.18	0.10	–1.12	–0.44	0.05	–1.73	–0.53	–0.08
99th percentile	–0.38	0.06	0.17	–0.63	–0.01	0.16	–0.82	–0.13	0.11	–1.01	–0.29	0.01
95th percentile	–0.14	0.09	0.21	–0.33	0.06	0.21	–0.49	0.01	0.17	–0.60	–0.13	0.09
90th percentile	–0.06	0.11	0.26	–0.18	0.08	0.25	–0.27	0.04	0.22	–0.42	–0.07	0.14
70th percentile	–0.02	0.19	0.43	–0.06	0.16	0.43	–0.11	0.11	0.38	–0.24	0.03	0.30
50th percentile	–0.01	0.32	0.68	–0.04	0.29	0.69	–0.07	0.25	0.63	–0.16	0.16	0.53
30th percentile	0.00	0.52	1.07	–0.01	0.51	1.07	–0.04	0.48	1.07	–0.09	0.41	0.98
10th percentile	0.05	0.75	1.49	0.08	0.79	1.58	0.13	0.87	1.69	0.22	0.97	1.82
5th percentile	0.14	1.05	1.90	0.26	1.22	2.14	0.44	1.47	2.48	0.69	1.68	2.67
1st percentile	0.71	2.15	3.66	1.87	3.17	4.51	2.55	3.92	5.32	3.85	5.66	7.44
Maximum	1.42	2.77	4.19	3.36	4.99	6.63	4.11	5.48	6.90	6.31	8.01	9.79
c.f.average	0.00	0.42	0.86	–0.01	0.44	0.90	–0.02	0.45	0.92	–0.04	0.45	0.94
Standard deviation	0.15	0.38	0.65	0.34	0.55	0.81	0.47	0.69	0.97	0.73	0.99	1.29

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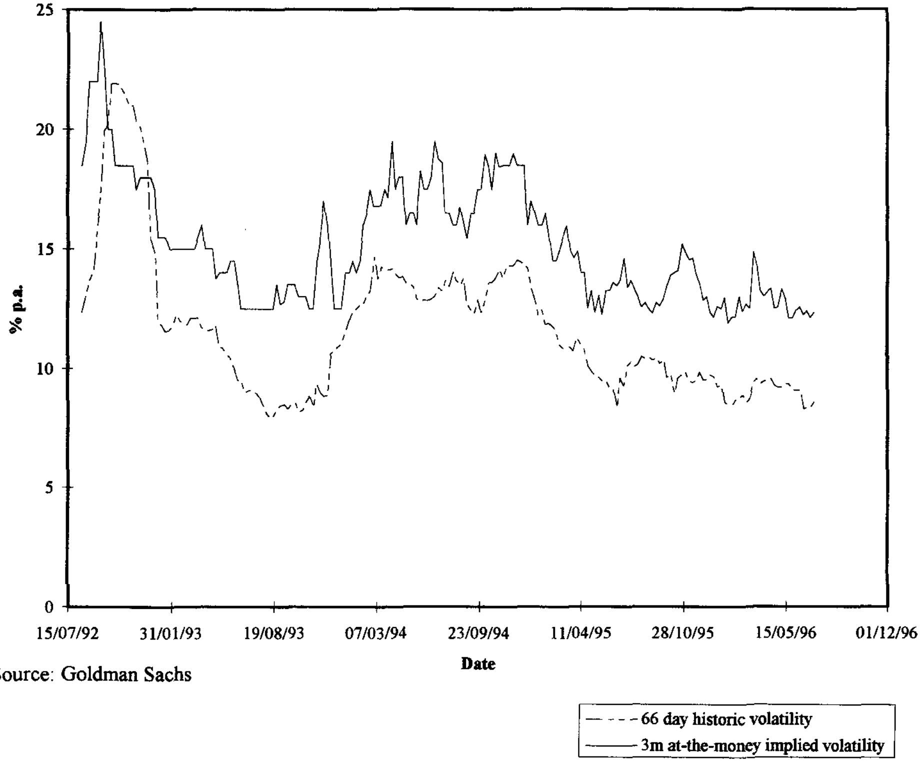

Source: Goldman Sachs

Figure 18. Historic and implied (at-the-money) FT-SE volatilities

9.7 *Transaction Costs*

9.7.1 It might be thought that we can extrapolate the results in ¶9.5.6 (i.e. Table 6) to rebalancing periods shorter than one day. Unfortunately, increasing the frequency of rebalancing increases transaction costs (potentially without limit, see Appendix B.8). Indeed, transaction costs even potentially have an influence on the efficacy of daily rebalancing. Table 7 shows the spread of results arising from following daily rebalancing along the lines above, if transaction costs were 0.2% per trade (perhaps a little cautious if market exposure were being gained through futures) and jump costs were costed separately (but not the rebalancing costs required once a jump had occurred). Daily rebalancing can be costly!

9.7.2 The transaction costs are heavily loaded towards the time that the option matures and towards those options that mature at close to at-the-money. This is when the level of transactions (and the option gamma) is typically greatest.

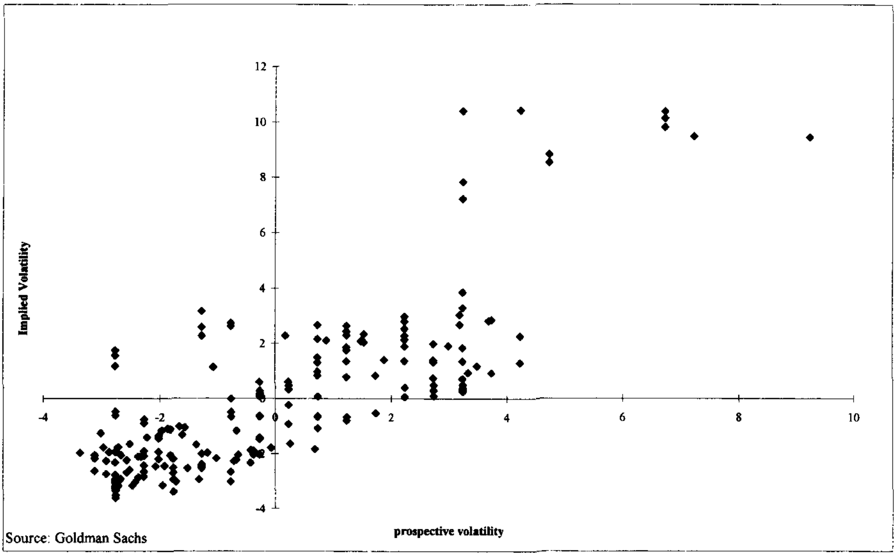

Figure 19. Predictability of future FT-SE volatility from current implied (ATM) volatility

Table 7. Daily rebalancing with transaction costs; cumulative surplus, different rebalancing periods, 'mileage optimised' experience rating

	No transaction costs			0.2% transaction costs (buy or sell)		
	Non-jump costs	Down jump costs	Up jump costs	Non-jump costs	Down jump costs	Up jump costs
Minimum	–2.91	–5.43	–1.98	–4.51	–5.43	–1.98
99th percentile	–1.32	–5.12	–1.80	–2.58	–5.12	–1.80
95th percentile	–0.39	–3.55	–0.90	–1.58	–3.55	–0.90
90th percentile	–0.23	–0.31	–0.66	–1.22	–0.31	–0.66
70th percentile	0.08	—	—	–0.55	—	—
50th percentile	0.34	—	—	–0.19	—	—
30th percentile	0.68	—	—	0.04	—	—
10th percentile	1.06	—	—	0.39	—	—
5th percentile	1.19	—	—	0.51	—	—
1st percentile	1.42	—	—	0.78	—	—
Maximum	2.06	—	—	0.95	—	—
c.f. average	0.37	–0.33	–0.16	–0.34	–0.33	–0.16
Standard deviation	0.56	1.07	0.37	0.69	1.07	0.37
Initial option premium	3.51	n/a	n/a	3.51	n/a	n/a

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Table 8. Cumulative surplus, non jump costs when rebalancing optimised using Whalley & Wilmott approach, transaction costs assumed to be 0.2% (buy or sell)

	No transaction costs	λ=0.5	λ=5	λ=50	λ=500
Minimum	–2.91	–3.90	–4.20	–4.32	–4.40
99th percentile	–1.32	–2.62	–2.41	–2.43	–2.50
95th percentile	–0.39	–1.54	–1.41	–1.44	–1.50
90th percentile	–0.23	–1.23	–1.01	–1.10	–1.16
70th percentile	0.08	–0.41	–0.38	–0.44	–0.48
50th percentile	0.34	–0.16	–0.08	–0.10	–0.13
30th percentile	0.68	0.21	0.18	0.15	0.09
10th percentile	1.06	0.60	0.55	0.49	0.45
5th percentile	1.19	0.83	0.72	0.65	0.59
1st percentile	1.42	1.02	0.95	0.88	0.82
Maximum	2.06	1.39	1.06	1.01	1.00
c.f. average	0.37	–0.21	–0.18	–0.23	–0.28
Standard deviation	0.56	0.72	0.68	0.68	0.68
Initial option premium	3.51	3.51	3.51	3.51	3.51

9.7.3 It is therefore necessary to be selective in the frequency of rebalancing. The optimal strategy for rebalancing appears to be that given in Appendix B.8. This strategy is preference dependent, since it depends explicitly on the utility function of the investor seeking to dynamically hedge the option. For various values of the risk aversion factor λ, described there, the range of cumulative surpluses become as shown in Table 8.

9.7.4 These seem to be fairly independent of the size of λ. This is principally because there are other sources of variation which mask any increase in variability of outcomes by changing the rebalancing approach (within the limits shown).

9.7.5 The impact of rebalancing costs generates substantial economies of scale if the organisation involved is buying options from one party and selling similar ones to another. Rebalancing trades for the different options may then largely cancel out. It is also a further reason for hedging written options with appropriate bought options, since again the required gross rebalancing is reduced (to zero if the options exactly match each other).

9.8 *Other Factors Potentially able to Disrupt Dynamic Hedging*

9.8.1 The above analysis has concentrated on:

(a) *jump costs*;
(b) *uncertainty in future volatility*; and
(c) *transaction costs.*

There are several other factors that can potentially disrupt the operation of dynamic hedging. These are:

(d) *Basis risk and roll-over risk* (as mentioned in ¶5.4.3). This is also significantly influenced by the frequency of rebalancing, since this increases the level of transactions that can be subject to basis risk. Like the transaction costs in

Section 9.7, it is significantly reduced if opposite option positions are held within the book. The potential difference between fair value and actual market price of futures contracts is material (the standard deviation of the difference is around 0.6% for FT-SE). It therefore seems that the impact of basis/roll-over risk could be significantly greater for the option under analysis than the underlying transaction costs (as long as these are controlled appropriately as in Section 9.7). Indeed, basis risk may also be larger than the risk arising from uncertainty in future volatility, although it does not seem to be as large as jump risk (for the option analysed above).

(e) *Position risk*, i.e. taking deliberate positions away from those required for pure dynamic hedging.
(f) *Credit risk*, which could mean that the assets the dynamic hedger hoped would meet his liabilities do not actually do so, because of default (see Section 12).
(g) *Legal risk and taxation risk*, which might cause the proceeds of the hedging instruments to be less than (or the payments on the written option to be more than) expected (see Section 12).
(h) *Other operational risks*, e.g. inadvertent position-taking and fraud (see Section 11).

9.8.2 Ignoring the final three, which we have yet to consider, and (e), which we assume does not take place within a pure dynamic hedging book, the analysis in this section suggests (at least for the option considered above in isolation) that the most important risks a dynamic hedger faces are (a) and (d), i.e. jump risk and basis/roll-over risk.

9.9 *Setting Reserving Levels*

9.9.1 The above sort of analysis can theoretically be used to identify suitable reserving levels for derivative books. We could estimate, using simulations, the potential level of loss that might arise from each sort of risk factor mentioned above, taking into account correlations between the factors, and set the reserves at a level which produces an acceptably low probability of ruin.

9.9.2 Indeed, this is conceptually the sort of approach used by some of the largest players within the derivatives industry to monitor their own capital requirements. In the next section we see how it compares with approaches actuaries will be familiar with in other fields.

10. Reserving for Derivatives

10.1 *The Underlying Framework*

10.1.1 If a financial organisation is run solely on the basis of meeting future commitments from income received when these commitments become payable, it would differ little from the 'pyramid' schemes that appear sporadically in developed economies and more frequently in less developed ones. Such a scheme can offer fantastically attractive rewards as long as there is a continuing stream of people gullible enough to join, but the people left in the scheme once the stream of new entrants dries up can expect to lose most or all of their investment.

10.1.2 The need to set aside suitable reserves or provisions to cover future commitments has been widely recognised amongst responsible financial professionals for many years. It has been central to actuarial thought since the profession came into existence several centuries ago.

10.1.3 Accountants often distinguish between *provisions* for commitments the size of which can be quantified reasonably accurately, and *reserves* for commitments which are less easily quantifiable. The two may be treated differently for tax purposes. Actuaries tend to lump both together, since the underlying rationale for both is the same, i.e. to provide on a prudent basis in advance for future financial commitments. Indeed, for insurance companies the term *mathematical reserves* is explicitly enshrined in European Community (E.C.) legislation, encompassing both.

10.1.4 Nowadays, most financial organisations are regulated by a *regulator* appointed by the government of the country in which they are domiciled or operate. For example, U.K. banks are regulated by the Bank of England and U.K. insurance companies are regulated by the Department of Trade and Industry (DTI). It is generally accepted that governments have a legitimate interest in the orderly functioning of financial markets, and will wish to make it unlikely that financial organisations operating within their borders will default. This is especially so if there are statutory safety nets triggered by such defaults (as exists for both banks and insurance companies in the U.K. and most other developed countries).

10.1.5 Part of the regulatory process will involve setting *statutory minimum capital requirements* and defining what happens if a financial organisation does not meet them. These requirements are normally applied in a fairly uniform fashion across all the organisations that a given regulator regulates. The calculations involved may, therefore, be relatively broad-brush and will usually be supplemented by liaison between the regulator and the organisation concerned. The larger organisations may also spend considerable effort devising their own measures of capital requirements, which they believe more accurately reflect the risks inherent in the businesses that they are carrying out. They may use these internal measures to allocate capital between different lines of business or to determine how much capital they actually think that they need (and to plan the best way to raise it).

10.1.6 There is a strong trend towards harmonisation of reserving standards across countries, especially if organisations can compete extensively internationally. For example, banks domiciled or operating in most of the major financial markets are all subject to similar capital requirements, roughly in line with the requirements of the Capital Adequacy Directive (see Appendix D), which will, in due course, be superseded by other standards being developed by the Basle

Committee. E.C. direct-writing insurance companies are subject to a common (but different) regulatory framework. U.S. insurance companies are regulated at a state level, but similar standards are applied country-wide. Without relatively standard regulatory frameworks, there is a risk that organisations will redomicile to the country/state with the lowest regulatory standards.

10.1.7 There is also a trend, but at a much slower rate, towards harmonisation across different types of financial institutions. This reflects the observation that distinctions between different financial market places are often blurred.

10.1.8 There are two main ways in which the reserving calculation can be presented:

(a) The first is to use a *risk-based capital framework*. Assets and liabilities would typically be valued on a suitable market related basis, and then a separate capital requirement is calculated, reflecting in some way the riskiness of the businesses in which the organisation is involved, including the way in which the assets and liabilities might move relative to each other. Banks in most developed countries use such an approach (at least for their trading book).

(b) Alternatively, *the assets and liabilities may be valued on a prudent basis, including margins*, perhaps with a further solvency margin (i.e. required excess of assets over liabilities) superimposed, calculated on a simplified basis. E.C. insurers currently use this sort of approach. Their statutory minimum solvency margin is based on a relatively simple calculation which depends mainly on premium income and claims incurred for non-life insurers and aggregate sums at risk and mathematical reserves for life insurers. U.K. insurers are also required to establish a mismatch or resilience reserve based on a small number of possible movements in aggregate market levels.

10.1.9 At least conceptually, both of these alternatives ought to be similar, since they can both be re-expressed as:

(a) assets minus liabilities on a market value basis; plus

(b) a balancing item, which can be thought of as some additional capital requirement dependent on the risks being run by the organisation.

10.1.10 In either case, the basic aim of holding additional capital (implicit or explicit) is to provide reasonable certainty that future commitments will be met. Logically, prudence dictates that the balancing item should be positive (i.e. should increase the capital resources of the business), but this may not always be the case in practice.

10.2 *The 'Idealised' Reserving Framework — Two Different Models*

10.2.1 The 'idealised' actuarial framework, at least in the actuarial research literature, can probably be taken as the one prevalent in the general insurance (i.e. non-life) area. We might call this the *Model A* framework. It involves determining some estimate of the risk of ruin for the organisation (i.e. the likelihood of default). The capital requirement is then set so that the risk of ruin is

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acceptably low. It is thus similar conceptually to the analysis in Section 9, although that was based on historic data, whilst an idealised framework might rely on other estimates of prospective risk if the past was considered unlikely to be a good guide to the future.

10.2.2 Arguably, the current solvency capital requirements for U.K. general insurers is a long way from this 'ideal', but this is principally because the current rules were imposed by the E.C. Possible amendments are under discussion which would move closer to the 'ideal', see e.g. Hooker *et al.* (1996). In the U.S.A., property/casualty insurers (their name for general insurers) are subject to risk-based capital requirements much closer to this 'ideal'.

10.2.3 The Model A framework would seem most naturally fitted to an organisation acting as a *principal*. The framework needs modification where the shareholders carry only a small fraction of the risks involved in the business because they are acting as *agents* for some other party. A different sort of framework, which we might call *Model B*, then becomes more appropriate.

10.2.4 Activities falling into this category are conceptually little different to fund management. The Model B framework should thus relate principally to the way in which the underlying assets are managed, discouraging activities deemed to be likely to be incompatible with the sorts of risks the underlying investors would be comfortable with. However, some minimum level of capital would still be needed, to ensure that the business has sufficient resources to meet minimum systems requirements, compliance requirements, etc.

10.2.5 Model B is a better fit than Model A to many sorts of life insurance business. Pure unit-linked business (if it is 'property-linked' as opposed to 'index-linked') is, in effect, fund management of the underlying unit-linked assets and very definitely in the Model B category. With-profits business typically fits somewhere in between, since the shareholders carry some of the risks relating to with-profits assets and liabilities. However, most of the risk is normally carried by policyholders via changes to reversionary and terminal bonuses. Non-profit business, if it is owned by the with-profits fund, would also come into this 'in-between' category, although, if it were owned directly by the shareholders, then it would be more akin to Model A.

10.3 *The Banking Approach*

10.3.1 Exactly the same sort of subdivision exists for businesses in which banks (and securities firms) are involved. Their proprietary and market making businesses are now subject to a risk-based capital framework which is, in many places, remarkably similar in concept to the actuarial 'ideal' reflected in Model A. In banking terminology, Model A could be described as *market values with stress testing*, since the term 'stress test' is usually applied to a scenario analysis concentrating on extreme circumstances. Banks' advisory and fund management businesses have much lower capital requirements, more akin to the those reflected in Model B.

10.3.2 Further details of the risk-based capital requirements to which banks are subject are set out in Appendix D. Arguably banks come closer to the actuarial 'ideal' Model A framework for their proprietary businesses than most organisations more typically advised by actuaries.

10.4 *Current Life Office Reserving for Derivatives*

10.4.1 The division between the two model frameworks can, perhaps, be best seen in the different statutory reserving requirements arising from derivatives currently applicable to insurance companies and banks.

10.4.2 The impact derivatives have on the balance sheet of a U.K. insurance company is described in the DTI Prudential Guidance Note 1995/3. It may be summarised as follows:

(a) If the derivative is an asset, is *covered* with appropriate holdings in the underlying assets, and meets certain other requirements, such as being readily realisable, being 'in connection with' admissible assets, etc., then the derivative is included in the balance sheet at market value, with no risk-based capital requirement being incurred by the insurance company. Potentially, the derivative can also reduce the resilience reserve a life office might need to maintain.

(b) If the derivative is an asset and is covered by, but does not meet, these additional requirements, then it is normally inadmissible, which means that zero value is ascribed to it in solvency calculations. In effect, there is an admissibility penalty for investing in undesirable derivatives. Unit-linked funds are prohibited from holding such derivatives.

(c) If the derivative is a liability and is 'covered' (and certain other requirements are met), then the value of the underlying assets is reduced by the value of the derivative, i.e. again the derivative is included in the balance sheet at market value, without generating a further risk-based capital requirement.

(d) If the derivative is not covered or fails one of the other tests for being acceptable, then the derivative is included at market value, but there is a further *provision for adverse deviations*, taking into account the possibility of, *inter alia*, changes in the volatility of the underlying assets. The provision for adverse deviations is thus, in effect, a risk-based capital charge. Unit-linked funds are prohibited from holding such derivatives.

10.4.3 Thus, insurance companies using derivatives face no capital penalties as long as the derivatives satisfy suitable tests of acceptability. However, additional capital requirements are imposed when the derivative fails these tests. The tests are similar to those that unit trusts need to meet to be permitted to invest in a similar fashion. The regulatory framework follows the Model B approach except if the derivative fails the tests for acceptability, when it reverts to a Model A.

10.4.4 There is much less of a history of risk sharing by banks, and banks generally try to take either all the risks or none of them. Therefore, most businesses that banks are involved with fit very definitely into one or other of the Model A or the Model B categories. If the business involves derivatives, then these are reserved for accordingly. Derivatives held by investment management components or other purely advisory relationships carry no particular capital requirement for the bank business (unless the bank has transgressed its authority limits), but might generate capital requirements for the underlying investor. However, if the derivatives arise because of proprietary trading or the bank's market making activities, then they fall into the bank's trading book and are subject to the Model A type framework set out in the CAD.

10.5 *Efficient Portfolio Management*

10.5.1 One of the interesting, and possibly contentious, requirements a derivative must satisfy to be acceptable in ¶10.4.2(a) is that it must be used for *efficient portfolio management* (or EPM for short) or for the 'reduction of investment risks'. This wording is included with the relevant E.C. Directives, but is not defined there. Indeed, it does not seem to be defined anywhere in legislation. Instead, the DTI has given guidance as to what it thinks the words mean, drawn heavily from the corresponding meaning given to EPM by the Securities and Investments Board (SIB) within the unit trust industry. Although the interpretation can be broad, DTI guidance states that:

(a) If the assets are 'earmarked' to match specific policyholder benefits where the policyholder bears an investment risk (notably unit-linked liabilities), then the use of derivatives must involve a reduction in the risks to either the company or the policyholder, whilst still having a broadly neutral or beneficial impact on the investment risks to the other.
(b) If the assets are not so earmarked, then the derivatives must reduce the risk to the company.
(c) In either case, 'risk' from the company's perspective is to be understood as the risk of mismatching between its assets and liabilities.

10.5.2 The rationale behind this approach can best be understood by remembering the Model B framework that is driving the approach to reserving for derivatives within insurance companies. The Regulations and DTI guidance are essentially starting from the premise that an investor cannot reasonably complain if his assets are invested in sensible sorts of underlying assets, and that the sorts of derivatives activities that would be inappropriate for an insurance company are ones that increase 'risk' from this vantage point.

10.5.3 Identifying precisely what increases or decreases investor 'risk' can be difficult in this context. For example, there has recently been a spate of contracts issued which, over five years, promise income higher than the risk-free rate together with a full return of capital, provided the equity market has not fallen. However, if the equity market has fallen by, say, more than 10%, only 50% of the initial capital is returned to the policyholder when the policy matures. The capital component of the investor's return is thus highly geared to the overall movement in the equity market over 5 years, at least if the movement is in a small band close to 0%. It is not obvious to me that this is easily classifiable as 'efficient portfolio management', at least when the policy is close to maturity. However, the DTI appears to take the view that for such policies:

(a) it is reasonable to consider the position only at outset (since policyholders will, in nearly all cases, remain invested in the policy throughout its five-year life); and

(b) if at outset the delta of the option backing the liability is between 0 and 1, then it does not involve 'gearing' and is compatible with EPM.

10.5.4 The DTI does, however, object to derivatives within linked funds which introduce gearing more directly. For example, portfolios involving cash plus a down-and-in call option can be constructed which fall in value as the price of the underlying asset rises. They would thus appear to provide the opportunity to create 'bear' funds, which give policyholder inverse market exposure. The options involved can also be made to satisfy the 'in connection with' and 'cover' requirements, but the DTI takes the view that they do not satisfy EPM, and therefore cannot normally be used within an insurance company context.

10.5.5 This position contrasts with the relevant unit trust regulatory framework imposed by SIB which does, in some circumstances, permit such combinations of investments to be included within authorised unit trusts (but not the standard type of unit trust which is a 'securities fund', rather special sorts called 'futures and options funds' or 'geared futures and options funds'). 'Bear' unit trusts are therefore permissible in some circumstances.

10.5.6 Regulators in other European countries seem to find it no easier to interpret the concept of EPM in a consistent fashion. For example, in France there are funds which return either 0% or 100% depending on some specific investment event, and there are also 'reverse floaters', giving the investor inverse exposure to interest rate movements.

10.6 *Weaknesses in the Current Approach to Life Office Reserving*

10.6.1 The Model A/Model B framework helps to explain how the current life office reserving requirements for derivatives have arisen. It also highlights a weakness in the current approach. The Model B (fund management) approach is applied to all life office assets, even when the business to which the derivatives relate can, in some instances, be closer to Model A.

10.6.2 This could be used to finesse the regulations. For example, a life office with a large diversified fund which sells sufficiently modest amounts of covered options, but then dynamically hedges away the effects of such options, would suffer essentially no change to its capital requirements, even though it would have acquired exposure to jump risk and to the risk of unforeseen changes to market volatility.

10.6.3 However, of course, as we noted in ¶2.3.5, life offices sell a lot of 'options' in the form of contracts with guarantees and other option-like characteristics, e.g. with-profits contracts. If we wanted a completely justifiable reserving methodology for derivatives, we would also need to revise the reserving

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framework for insurance liabilities. The experience of Scott *et al.* (1996) suggests that it is easier to point out possible flaws in a reserving framework than it is to come up with one that everyone agrees is better! At least the current methodology for reserving for derivatives is generally consistent with the methodology used for reserving for insurance liabilities.

10.7 *Value at Risk*

10.7.1 How might we determine the reserves an organisation needs if the appropriate framework is Model A, i.e. the organisation is acting as principal? We have already discussed in Section 9 scenario/stress testing to estimate risks of ruin. An alternative method used within the banking community, which is less onerous to calculate, is the concept of *value at risk* (VAR). This is the amount by which the net value of a bank's trading book (see Appendix D for a definition) might change over a set period of time (e.g. 1 day, 2 weeks, 1 month, 1 year, etc.), with a set probability (5%, 1%, etc.), based on the sensitivity of the portfolio to small movements in the parameters underlying the pricing model. For a large complex derivatives book, there are many factors that will influence the value of the trading book, and banks can spend considerable sums designing suitable systems to prepare estimated values at risk. A few banks, like J.P.Morgan, make available services (in their case called RiskMetrics®) which provide estimates of the parameters needed for these calculations.

10.7.2 The approach is perhaps most easily recognised by actuaries when it focuses only on market risk, since many portfolio managers use similar sorts of risk measures, in the form of *tracking errors*, see e.g. Rains & Gardner (1995).

10.7.3 In practice, a fairly arbitrary multiplier is applied by the Bank of England to convert VAR model outputs into levels of capital required, and it is also checked periodically against capital requirements derived from simpler calculations.

10.7.4 One question that arises when constructing VAR models is whether they should be based on likely levels of market variabilities and jump probabilities 'implied' from the market or whether these should be based on history. Perhaps both should be analysed, with the one giving the higher reserve adopted for prudence. If parameters based on history are used, then it is necessary to identify some way of extrapolating from the past to the present (and to the future). Research has been carried out into the use of GARCH forecasting techniques, but the results seem mixed. J.P.Morgan's RiskMetrics uses exponentially weighted averages to extrapolate the past into the future.

10.7.5 The main disadvantage of VAR is that it fails to take account of the non-linearities arising from options contracts, and the Bank of England prefers scenario/stress testing in such circumstances.

10.8 *Provisions for Adverse Deviations for Insurance Companies*

10.8.1 The only formal guidance on how insurance companies should set

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such provisions is contained in the DTI Prudential Guidance Note 1995/3. This indicates that the DTI expects companies to set provisions on bases which, in the case of a derivative based on a broadly-based equity index, are at least as prudent as:

(a) assuming that a 25% adverse movement in the index occurs in the near future; and
(b) the possibility of a greater adverse movement should be allowed for, to the extent that this is consistent with the historical record or otherwise where it would be imprudent to ignore the possibility.

10.8.2 This level of provision seems to have been set by reference to the guidance provided by the Government Actuary's Department on how to determine a suitable mismatch or 'resilience' reserve for insurance liabilities, which also talks about a 25% fall in equity values. Such a mismatch reserve is similar conceptually to the stress testing approach used within the banking community, although much cruder.

10.8.3 In normal circumstances this would seem prudent, at least for vanilla put and call options, indeed overly prudent for the sort of option analysed in Section 9 (as long it is considered practical to take account of some mitigation of market movements using dynamic hedging). However, if the term of the option is very long (e.g. 15 years plus) and the option is far out-of-the-money, then upwards shifts in levels of implied volatility of, say, 5% have about the same order of magnitude impact on option values as a 25% adverse market movements. Most derivatives held by insurers will be for much shorter terms, but interestingly, these sorts of characteristics can correspond with the guarantees within with-profits contracts.

10.8.4 For some exotic options, provisions set merely on the basis of ¶10.8.1 will not be prudent. For example, take an insurer that has sold a double barrier option which knocks out if the price of the underlying falls below one barrier or rises above another. Large market movements and increased volatility make the barrier more likely to knock out, and therefore reduce the liability to the insurer. The insurer is most exposed to volatility falling, when the option is more likely to run through to maturity. Again, it is probably more likely that the insurer has such liabilities because of unusual contract features within insurance policies it has written than because it has sold a specific derivative contract.

10.9 *Pension Schemes*

10.9.1 The methods used to establish reserves for U.K. final salary pension schemes do not, it seems to me, fit neatly into either a Model A or a Model B framework.

10.9.2 In principle, there are four different sorts of actuarial valuations applied to pension schemes:

(a) Perhaps the most important is the *ongoing* valuation designed to determine an

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appropriate contribution rate for the sponsoring company to pay into the scheme. In such a valuation, liabilities are typically valued using assumptions regarding future rates of return, rates of inflation, etc. which change only slowly as market conditions alter. Assets would be valued by assuming that their current market value is reinvested into a notional portfolio deemed appropriate in relation to the liabilities, and then valuing the notional portfolio using assumptions consistent with those used to value the liabilities.

(b) Schemes also value their assets and liabilities on a wind-up or *discontinuance* basis. Until recently this was unlikely to be a constraint for most schemes, since a significant proportion of the future benefits were on a discretionary basis, and therefore could be ignored in such calculations. There was also, in many cases, no specific sanction if the liabilities exceeded the assets in such a calculation, on the assumption that over time such deficits could be eradicated by future contributions.

(c) The lack of such a sanction has led to the introduction of the *Minimum Funding Requirement* (MFR). This imposes sanctions on schemes with MFR funding levels that fall below certain trigger points. The MFR funding level (or ratio of assets to liabilities) is calculated in a manner that has some of the characteristics of both (a) and (b). In the calculation, assets are effectively valued at market value, with the liabilities valued in a manner that is independent of the investment strategy being followed.

(d) Schemes approved for tax purposes are also required to carry out a valuation for the Inland Revenue designed to discourage them from being excessively well funded (and thus benefitting too much from the advantageous tax treatment afforded to assets in such schemes). The calculation used is similar to that in (a).

10.9.3 The key point to note is that, with the possible exception of (b), there is no equivalent to the mismatch test or risk-based capital approach applicable to insurance companies and banks. As far as assets are concerned, actual holdings of different sorts of assets with the same market value (at the date of the valuation) are treated equally in the valuations (except possibly if the asset is illiquid). It is the overall market value that is important, not how the assets are distributed.

10.9.4 In practice, of course, the distribution of assets has an important impact on the overall financial health of the fund, but this only appears through time. Actuaries within the pensions arena have developed techniques, principally in the form of asset/liability studies, which help to identify how features such as future MFR funding levels might be influenced if different investment strategies are followed.

10.9.5 As far as derivatives are concerned, the *initial* impact of any such transaction will be minimal (except perhaps in the discontinuance valuation), even if the derivatives being purchased limit down-side risk. Although purchase of an option may involve payment of a premium, the option itself would then become an asset of the scheme (worth the premium paid less some bid/offer adjustment), leaving the overall market value of the scheme's assets largely unchanged. It is *only over time* that the derivative holding will have a material impact on funding levels, and this effect will normally only be captured via the asset/liability modelling process (if the scheme undertakes such exercises).

10.9.6 When carrying out ongoing actuarial valuations, pension scheme actuaries are under a professional duty to advise the scheme's trustees if they believe that the investment strategy being adopted is inappropriate for the scheme's liabilities. In principle, therefore, the actuary may need to comment on the use of derivatives, if any, by the scheme. If the scheme is using merely futures for asset allocation purposes, then the effect is similar to a realignment in the underlying asset distribution, to take account of the change in economic exposure created by the futures. In normal circumstances, this is unlikely to justify a comment. Only large scale usage of derivatives, particularly ones that involve the scheme acquiring exposure to jump risks or volatility risk, are likely to require comments in this context. Usually the scheme's liabilities contain option-like characteristics, and one could argue that, if these are matched by corresponding option characteristics in the assets, then the use of derivatives may be more appropriate for a scheme than no use at all.

11. CONTROL PROCEDURES

11.1 *The Need for Control Procedures*

11.1.1 I have concentrated up to this point on the mathematics and economic theory underlying derivatives. As important, probably, is the need for adequate systems and controls. These often require much less mathematical knowledge, and instead often rely much more on practical common sense.

11.1.2 Suitable control procedures are a fundamental part of all business processes, not just for derivatives. However, there are some specific reasons why it is particularly important to have good systems and control procedures when using derivatives. These include:

(a) *The association many people have between derivatives and risk*, where 'risk' here probably means something going seriously wrong enough to lose your job, to get adverse press comment, etc. Individuals want to avoid the sorts of disasters that have afflicted Barings (see Section 11.5) and others. Of course, there is a converse risk that undue attention may be paid to derivatives at the expense of forgetting about other potentially risky aspects of the business. Losses from irregularities in government bond trading in the U.S. branch of Daiwa Bank were similar in size to the losses sustained by Barings through derivatives trading. One solution may be to set up a specific risk management function (e.g. an asset-liability committee) reviewing all risks (including derivatives risk). However, such a committee will only function properly if it is operating within an appropriate business culture/ethos (and if the information supplied to it is accurate).

(b) *Efficiency and the need to delegate effectively.* Derivatives tend to be more complicated than some other areas of investment, or at least are viewed as such by senior management. To avoid employing too many highly paid 'rocket scientists' with the necessary derivatives expertise, it helps to make the processes involved as simple and as well specified as possible.

11.2 *Formulating Suitable Systems and Control Procedures*

11.2.1 It therefore seems to me that the key to designing suitable control procedures is to apply a healthy dose of common sense, whilst, at the same time, having a good general understanding of derivatives and their mechanics (which I hope that this paper will help to provide!).

11.2.2 Specific guidance on how to frame control procedures is available from many sources. These include:

(a) recommendations produced by the Group of Thirty in 1993 entitled *Derivatives: Practices and Principles*;
(b) the DTI Prudential Guidance Note 1994/6 entitled *Guidance on systems of control over the investments (and counterparty exposure) of insurance companies with particular reference to the use of derivatives* and DTI Prudential Guidance Note 1995/3 entitled *Use of Derivatives Contracts in Insurance Funds*;
(c) for actuaries, Guidance Note 25 entitled *Investments — Derivative Instruments*; and
(d) material produced by trade bodies, such as the Futures and Options Association's (FOA) *Managing Derivatives Risk: guidelines for end-users of derivatives*, produced in December 1995.

11.2.3 The FOA document is particularly helpful as a source of checklists which can benchmark control procedures appropriate to organisations transacting relatively low volumes of derivatives.

11.2.4 Most of the above documents borrow from preceding ones, e.g. GN25 was written taking into account the contents of the DTI Prudential Guidance Note 1994/6, which itself borrowed extensively from the concepts introduced in the Group of 30 Report.

11.2.5 Regulated entities, such as insurers and banks, are often required to follow such guidelines (e.g. insurance companies are required to follow the DTI guidance to prove that they satisfy the principles of *sound and prudent management*).

11.2.6 All these documents and guidance agree that suitable control procedures need to:

(a) involve principles agreed at the highest level within the organisation (e.g. the board) which are laid down in the form of written guidelines approved by the board;
(b) be implemented by people who understand the derivatives business and the issues involved;
(c) involve proper assessment of all risks;

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(d) be regularly reviewed in the light of changing market conditions and experience;
(e) be consistent with the company's overall investment strategy; and
(f) involve suitable reports back to the board.

11.2.7 There is also wide acceptance that implementing such controls requires adequate resources, both of staff and of systems. The greater the usage of derivatives, the greater the resources that are likely to need to be allocated to derivatives systems. For example, an organisation which carries out one or two small derivatives deals a year may be able to keep track of all its positions with a simple spreadsheet, but this would be quite inappropriate for an organisation that carried out thousands of derivatives transactions in lots of different markets.

11.2.8 Another key requirement is for there to be adequate independence between the people responsible for settlements and producing monitoring reports (the 'back office') and those responsible for initiating deals (the 'front office'). Otherwise there is a danger that reports fail to highlight risks that the front office do not wish to be put under a spotlight.

11.3 *Guidelines*

11.3.1 The sorts of material that should be included in the written guidelines agreed by the board are:
(a) the purposes for which derivatives would be used;
(b) the sorts of derivatives the organisation will use, including, probably, which ones will be used for which purposes;
(c) the constraints within which derivatives will be used;
(d) procedures for seeking approval for the usage of new types of derivatives;
(e) the restrictions on the counterparties and brokers through whom derivative deals can occur (e.g. the minimum acceptable credit rating);
(f) details of who is authorised to enter into derivative transactions (and the limits placed on their authorisation); and
(g) procedures for how management will monitor derivative activity (and individual position limits), and who will be responsible for the monitoring process.

11.3.2 It would also be good practice to identify how derivatives will be valued and how credit exposures arising from derivatives are to be calculated (and by whom). These questions are non-trivial for OTC derivatives. It may be helpful for the procedure for seeking approval for the usage of new types of derivatives to cover how these calculations are to be carried out.

11.4 *Management Reporting*

11.4.1 Whatever the level of usage, organisations are likely to want the following information to be able to monitor their derivatives positions:
(a) a summary of derivative transactions that have occurred since the last reporting date, partly for audit purposes, but also so that senior management can spot

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unusual transactions or use of new types of derivatives;

(b) a summary of positions likely to expire in the near future (since these may require particular actions to take place); this summary could include estimates of the likely amounts of cash or stocks that might need to be delivered at that time, and estimates of other funding requirements;

(c) a summary of open derivative positions, sorted by the fund to which they relate; for a derivatives house, many of the deals may be proprietary and thus relate to the same fund (i.e. the firm's own capital), although the summary might be split by the trader/product area involved, and for life offices, the deals may be scattered across different sub-funds (e.g. separate unit-linked funds), which may be managed by separate individuals, or be legally separate entities; and

(d) a summary of the counterparty exposures generated by the derivative positions, sorted by counterparty and the fund/entity to which the derivative relates.

11.4.2 The contents of (a) and (b) are relatively straightforward. Report (a) should contain all purchases and sales separately. It should show some measure of deal size (in a format understood by senior management), as well as the book/market value information needed for accounting reconciliations. Report (b) is principally to flag up positions that need actions regarding renewal/roll-over.

11.4.3 Conceptually the contents of (d) are also relatively straightforward. However, in practice, estimating the actual size of counterparty exposures may be quite complex, see Section 12. The report should be produced in conjunction with estimates of counterparty exposures generated by other aspects of investment activity, e.g. holdings in equity and bond issues, cash deposits, settlement of outstanding equity and bond transactions and stock lending to the relevant counterparty. Connected counterparties, e.g. organisations within the same group, would typically be aggregated, but, in some circumstances, there may be offsetting factors, depending on the nature of the derivatives involved.

11.4.4 The report which is likely to vary the most by type of institution is the open position summary, i.e. (c). For life offices or unit trust management firms carrying out few trades and not actively hedging their derivatives book, a summary showing the current market value and/or associated economic of each open derivative position would be helpful, probably together with the deal slip number, type of contract, contract details (e.g. for options, the strike price, maturity date, whether a put or call, a long or a short position, American or European, etc.), number of contracts, multiplier per contract, currency, current exchange rate and current price.

11.4.5 However, reports that show open derivatives positions in isolation will fail to show the total picture, since derivatives are nearly always used by life offices to modify exposures arising from investments in physical securities. They should, therefore, be supplemented by consolidated summaries of portfolio structure for each fund, including allowance for derivatives, say along the lines recommended by LIFFE (1992a) and as summarised in LIFFE (1992b). Such a summary should also show whether derivatives are meeting relevant 'cover' and 'in connection with' regulatory requirements.

11.4.6 For firms actively managing a derivatives trading book, especially with a significant option component, reports merely along the lines of the above will supply insufficient information to meet traders' and risk managers' needs. They should be extended to show the 'greeks' or sensitivities to various factors of each derivative (and for the book as a whole), as well as consolidated stress tests giving the likelihood of specific severities of outcomes occurring.

11.5 *The Importance of Basics*

11.5.1 The importance of following even basic common sense should not be underestimated. They are highlighted by the collapse of Barings during February 1995. Barings' Singapore exchange traded derivatives operation exceeded its authority and lost large sums of money, having built up a very large exposure to the Japanese equity market by buying exposure on two different exchanges when senior management thought that the positions offset each other.

11.5.2 The collapse was the subject of an inquiry by the Board of Banking Supervision of the Bank of England and by authorities in Singapore. Five key lessons are highlighted by the Bank of England's Board of Banking Supervision Report. All, in retrospect, are obvious. The lessons are:

(a) Management teams have a duty to understand fully the businesses they manage.
(b) Responsibility for each business activity has to be clearly established and communicated.
(c) Clear segregation of duties is fundamental to any effective control system.
(d) Relevant internal controls, including independent risk management, have to be established for all business activities.
(e) Top management, especially the board's audit committee, must ensure that significant weaknesses, identified to them by internal audit or otherwise, are resolved quickly.

11.5.3 A key control highlighted in the report was the need for proper monitoring of margin payments. Another lesson highlighted in the report, again in retrospect obvious common sense, is that anomalously high profits ought to be investigated vigilantly to ensure that something is not going wrong. The very high apparent profitability of Barings Singapore futures operations should have sounded warning bells when it was supposed to be carrying out only low risk activities. As the proverb says: 'there are no free lunches'.

12. Credit Risk and Other Operational Issues

12.1 *Counterparty Risk*

12.1.1 The most important area that we have not yet covered is that of credit, i.e. counterparty risk. Credit risk arises with all investment transactions,

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but usually the risk is directly associated with the underlying economic exposure. With a derivative transaction, however, the two are usually distinct.

12.1.2 Thus, with an equity investment the investor would suffer a loss (once the deal has settled) if the underlying company defaults; the counterparty and economic exposure are identical. However, with an OTC forward, loss can arise because either:

(a) the underlying investment fails; or
(b) the firm from whom the derivative has been purchased defaults.

12.2 *The Importance of Market Values*

12.2.1 If a counterparty fails, the loss that an investor suffers is the replacement cost of the derivative, i.e. an amount related to *market value*. Book costs are of very little help in assessing counterparty risk.

12.2.2 This is why margining systems are so important for exchange traded derivatives. They reduce the replacement cost to zero each time the position is marked-to-market and variation margin is paid (although there may still be some credit risk arising from the initial margin). Even if derivatives are not margined, it is still important to mark them to market in valuation terms, to measure the exposure were a counterparty to default immediately.

12.3 *Potential Future Losses*

12.3.1 However, the current replacement cost is not normally a good measure of total *potential* counterparty exposure. If the underlying moves in such a way as to increase the replacement cost and the counterparty then defaults, a larger loss will occur.

12.3.2 In theory, credit risk can itself be thought of as akin to a derivative (indeed explicit credit derivatives are now traded). Counterparties can be expected to default from time to time. We could estimate the potential loss along the same sorts of lines as general insurers use to calculate premiums for credit guarantee insurance. For complex derivatives, some form of scenario testing might be carried out to assess the likely 'exposure to risk', and the likely rate of default might be estimated using the counterparty's credit rating and tables of default rates for similarly rated entities, as produced by the rating agency. Default rates are also influenced by the general state of the economy, which might also be correlated with movements in the value of the derivative. Appropriate adjustments for this might be incorporated into the calculations.

12.3.3 A suitably conservative estimate of the average potential loss across the entire book could be used as a credit reserve, in a manner similar to that used by actuaries to reserve for mortality or other similar risks.

12.3.4 In practice, accurately estimating potential credit exposure is quite difficult, and more broad brush approaches would normally be used (especially if the size of the derivatives book is quite small). For example, we might set the credit risk attaching to a futures contract as, say, 5% of its market value, on the grounds that some loss might potentially occur with the initial margin (even though these are normally supposed to be in segregated accounts), perhaps because of intra-day exposure, and a conservative estimate of how much a market might move in most circumstances on the day the organisation defaulted might be nearly 5%, and hence this amount of variation margin might be lost.

12.4 *Controlling Counterparty Exposure*

12.4.1 Identifying the sizes of counterparty exposures (and potentially the reserves to hold against these risks) is only one part of the process. Suitable controls should be placed on the total exposure to a single counterparty, based partly on its credit rating. Aggregate limits might also be placed on the combined exposure to counterparties of the same type, e.g. banks domiciled in the same country, since there may be some systemic risks affecting all such organisations simultaneously.

12.4.2 New transactions should be approved only after taking into account the incremental impact they have on counterparty exposure. Some thought should be given as to how positions might be unwound, or credit risks mitigated, if circumstances made this necessary (e.g. a credit downgrade, or if an extreme market movement took the credit exposure above the credit limit assigned to that organisation).

12.4.3 If an organisation's credit rating is unacceptable or its credit limit is exhausted, credit enhancement techniques might be used. For example, a third party could guarantee the contract, or the contract could be collateralised, reducing the replacement cost and hence credit exposure (but of course creating new exposures with wherever the collateral is placed). A credit derivative could be purchased, in which a third party agrees to pay a set sum if the first counterparty defaults in a defined manner.

12.5 *Documentation and other Operational Issues*

12.5.1 Although legal documentation may seem a dry subject, a significant fraction of default losses that have actually occurred over the last decade or so in the derivatives markets have been a result of a lack of legal enforceability. In the decade to 1993, one-half of all losses from worldwide derivative defaults were the result of the Hammersmith and Fulham swaps case in which the U.K. House of Lords ruled that English local authorities did not have the capacity to enter into swap or other derivative transactions.

12.5.2 The Group of Thirty report identified five main areas of legal risk:

(a) *contract formation*, i.e. whether transactions are actually properly documented within appropriate time scales;
(b) *capacity*, i.e. whether the counterparty is legally able to enter into the transaction;
(c) *early termination*, where the problem is that a counterparty that becomes insolvent may be able to suspend its payments whilst at the same time demanding performance by its counterparty; the normal way to try to minimise this risk is by *close-out* netting agreements, including ones that allow netting both of

derivative and non-derivative transactions; however, not all jurisdictions recognise such agreements;

(d) *multi-branch netting arrangements*, where a further complication arises if banks have 'booked' individual transactions in various locations through a multi-branch master agreement; close-out netting may not be possible in such circumstances, since there may be conflicts between regulators in different countries in the event of default (e.g. as happened with BCCI); and

(e) *enforceability*; where in some jurisdictions a counterparty may be able to enter into a derivative transaction, but it may not be enforceable, e.g. it may be viewed as a gambling contract.

12.5.3 Exchange traded contracts are normally dealt over the telephone. Investors need to have a contract in place with their clearing agents. There also needs to be a 'give up' agreement in place with the broker executing the transaction (if the broker is different to the clearing agent, then the broker is said to 'give up' the deal to the clearing agent).

12.5.4 Over-the-counter contracts will typically each have their own agreement (although they may be encompassed within a master netting arrangement). The industry standard is the International Swap Dealers Association (ISDA) agreement.

12.5.5 There are several other ways in which contracts can fail to deliver what is expected of them, e.g. retrospective tax changes or changes in the rules governing the construction of the underlying indices (for index derivatives). The key requirement is to be aware of as many as possible of the factors potentially influencing the business in which the firm is operating and to ensure that the firm has adequate capital (or some other means of fall-back) if these factors actually materialise. Derivatives, of course, are by no means unique in this regard.

12.5.6 The North American actuarial profession specifically includes under its 'C-4' risk category the concept of operational risk (including those relating to failures in control systems and unexpected changes in the regulatory framework). The risk-based capital framework used for U.S. property/casualty insurers specifically incorporates an allowance for C-4 risk (although admittedly on a fairly arbitrary basis).

13. Concluding Remarks

13.1 *Analogies between Derivatives Techniques and Techniques in Other Fields*

The main purpose of this paper is to draw analogies between techniques relevant to derivatives and those used by actuaries in other fields. Quite a few were identified as the paper progressed, including:

Derivative concept	Actuarial concept	Section paragraph
'Static' delta-hedging of forwards	Matching	5.2
'Dynamic' delta hedging of equity positions	No clear equivalent, although implicit in some concepts.	Throughout paper
'Dynamic' delta hedging of bond positions	In some instances equivalent to Redington duration matching	5.2
Calibration of option prices from sparse observed market data	Graduation and other curve fitting techniques	6.3
Risk neutral probability distributions	Formally equivalent to using risk adjusted discount rate, but existing actuarial norm is usually unhelpful as an analogue	7.3
Hedging by reference to cumulative quadratic variation, and implied volatility	Experience rating, but the analogy is by no means perfect	8.2, 9
Monte Carlo simulation	Asset/liability studies, but the analogy is not always helpful if it excludes an adjustment to risk-neutrality	6.3.4, 8.9
Utility theory	The actuarial concept of 'value', but often this focuses too little on market value.	4.5
No arbitrage	Actuarial prudence and dislike of chartism	4
Jump risk	Catastrophe insurance	9
Basis risk/roll-over risk, and value at risk	Tracking error	5.5 10.7.2
Stress testing	Reserving using risk of ruin	9, 10
Hedging options by buying other options	Reinsurance	9.4
Operational risks, such as fraud	C-4 risk, in the terminology of the North American actuarial profession	11, 12.5.6
Multiple jump size pricing models	General insurance pricing approaches	Appendix B.4
Credit reserves	Exposure to risk calculations and standard actuarial reserving techniques	12.4

13.2 *Factors influencing Reserving Requirements*

13.2.1 The other main aim of this paper is to identify the factors influencing the appropriate level of reserves to hold for a derivatives portfolio. We saw, in Section 10, that this depended on the nature of the business being run, i.e. whether the organisation was acting as an agent or as a principal. If the organi-

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sation is merely managing a portfolio containing derivatives for someone else, then the role is that of fund management, and reserves materially in excess of those required by the underlying investor are probably not needed.

13.2.2 If the derivatives are part of the organisation's own book, then the idealised actuarial framework (which is quite similar to approaches actually adopted by some of the large banks extensively involved in derivatives markets) includes components for each of the following:

(a) basis/roll-over risk;
(b) position risk;
(c) the impact of jumps;
(d) uncertainty in future volatility;
(e) transaction costs, if the position is being dynamically hedged;
(f) credit risk; and
(g) other operational risks, including fraud.

13.3 *The Role of Actuaries*

13.3.1 As this paper tries to demonstrate, the mix of skills needed within the derivatives industry is one that involves both the practical and the mathematical. This mix is one that the actuarial profession has been offering in other spheres for many generations.

13.3.2 The derivatives industry is a young industry and one that actuaries have not been very successful at penetrating. I hope that this paper will help more actuaries to be involved extensively in it.

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APPENDIX A

OPTION PRICING FORMULAE

A.1 *The Merits of being Familiar with Option Pricing Formulae*

A.1.1 Although the market price of a derivative can usually be obtained merely by approaching a suitable broker, there are many instances in which it is advantageous to be able to calculate approximate prices oneself. Such expertise can, for example, help in the process of negotiating on the price of OTC derivatives, and will give the prospective purchaser a better understanding of the factors to which the price is most sensitive. It is a useful management discipline, and also helps in the assessment of credit risk.

A.1.2 There are a significant number of suppliers of commercial pricing software, but, in my opinion, it is nearly always helpful to understand the underlying methodologies of the software.

A.1.3 For some sorts of derivatives or with some models, no closed form analytic solution is available, but for simpler sorts of derivatives it is often possible to find such solutions. This appendix concentrates on such solutions, since if they exist they are much easier to code up and understand than more complex numerical methods, and are therefore more suitable for carrying out reasonableness checks. The solutions developed in this appendix are principally for call options; the development of formulae for put options is generally very similar (because of put-call parity). The development of the formulae also provides some insights into the comments in the main body of the text. However, readers should note that this appendix provides no more than a very limited introduction to what is a very complicated subject.

A.2 *Deriving the Black-Scholes Formulae using Stochastic Calculus*

A.2.1 Stochastic calculus is the most important mathematical technique used to price options. I have here developed it from a partial differential equation perspective. Within the derivatives industry an alternative that is often used is based on martingales, but the two are formally equivalent in a mathematical sense.

A.2.2 An example of its use is in the derivation of the Black-Scholes formulae, which can be done as follows. We assume that:

(a) the markets on which the underlying securities and derivatives trade have no transactions costs;
(b) participants can take out long and short positions without constraint;
(c) tax can be ignored; and
(d) markets are arbitrage-free.

A.2.3 We also assume that the movement in the stock price follows a *Gauss-Weiner* or *Brownian* stochastic process. This means that:

$$\frac{dS}{S} = \mu dt + \sigma dz$$

where S_t = stock price at time t, σ = volatility of stock price, μ = mean drift of stock prices and dz are normal random variables with zero mean and variance dt. Thus in each consecutive infinitesimal time period length dt the share price movement is an independent identically distributed normal random variable with mean μdt and standard deviation $\sigma\sqrt{dt}$.

A.2.4 We also assume that a cash holding of H provides a *risk-free* income of $rHdt$ in time dt (i.e. r is the interest rate on a *risk-free* asset) and a stock holding of S provides a dividend of $qSdt$ in time dt (i.e. q is the dividend yield).

A.2.5 Suppose that the price of the derivative is $u(S,t)$. Suppose we also construct a hedge portfolio which will, at any instant in time, rise or fall by the same amount as the value of the option. We therefore need the hedge portfolio to consist of:

$$A = \text{number of units of stock } = \frac{\partial u}{\partial S}$$

$$B = \text{balance} = \text{amount of cash} = u - S\frac{\partial u}{\partial S}.$$

A.2.6 For simplicity, we will use subscripts to refer to partial derivatives, i.e. u_t refers to the partial derivative of u with respect to t. Where two subscripts are used we refer to second partial derivatives, i.e.:

$$u_{SS} = \frac{\partial^2 u}{\partial S^2} \quad \text{etc.}$$

A.2.7 The key to stochastic calculus is *Ito's formula*, which is effectively a Taylor Series expansion, but allowing for the stochastic nature of S. It implies that:

$$du = u_S dS + u_t dt + \frac{\sigma^2 S^2}{2} u_{SS} dt. \qquad (1$$

A.2.8 It also follows, from the hedge portfolio that we have constructed, if du = movement in option price and dS = movement in stock price, that:

$$du = A(dS + qSdt) + Br.dt = u_S dS + qSu_S dt + r(u - Su_S)dt. \qquad (2)$$

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A.2.9 Subtracting (1) from (2) we derive the partial differential equation satisfied by *any* derivative, given the assumptions set out above:

$$-ru + u_t + (r-q)Su_S + \frac{\sigma^2 S^2}{2} u_{SS} = 0. \tag{3}$$

A.3 *Deriving The Black-Scholes Formulae for European Puts and Calls using Stochastic Calculus*

A.3.1 The partial differential equation set out in ¶A.2.9 is a second-order, linear partial differential equation of the *parabolic* type. This sort of equation is the same sort as is used by physicists to describe diffusion of heat. For this reason, a Gauss-Weiner process is also known as a *diffusion* process.

A.3.2 To solve the equation, we first transform it into a standard form, namely (with c constant):

$$\frac{1}{c^2}\frac{\partial w}{\partial y} = \frac{\partial^2 w}{\partial x^2} \quad \text{i.e. } w_y = c^2 w_{xx}.$$

A.3.3 Suppose we replace u by w, where $w = ue^{r(T-t)}$ (assuming r is constant). This transformation removes one of the terms in the partial differential equation:

$$w_t = (-ru + u_t)e^{r(T-t)} \qquad w_S = u_S e^{r(T-t)} \qquad w_{SS} = u_{SS} e^{r(T-t)}$$

$$\Rightarrow w_t + (r-q)Sw_S + \frac{\sigma^2 S^2}{2} w_{SS} = 0.$$

A.3.4 Suppose we also make the following double transformation (assuming r, q and σ are constant):

$$y = T - t \quad \text{and} \quad x = \frac{\log(S)}{\sigma} + \frac{r - q - \sigma^2/2}{\sigma}(T-t) = \frac{\log(S)}{\sigma} + \frac{r - q - \sigma^2/2}{\sigma} y.$$

The partial differential equation then simplifies to $w_y = c^2 w_{xx}$, with $c = 1/\sqrt{2}$.

A.3.5 If r or q are time dependent (or even stochastic in their own right), then more complicated substitutions are needed, but it is still possible to convert the equation in ¶A.2.9 into a standard parabolic form (as is necessary for the logic set out in Section 8.1 to be correct). If σ is dependent only on t, it is also still possible to do this, but not if σ also depends on S. If σ depends on S and t

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only, then equation (3) is still valid, but must be solved in a different way. This corresponds to the 'generalised Brownian framework' mentioned in ¶8.1.5.

A.3.6 The prices of different derivatives will then satisfy the above equation, but subject to different boundary conditions. A common method of identifying solutions to partial differential equations subject to such boundary conditions is by the use of *Green's functions*. This expresses the solution to a partial differential equation, given a general boundary condition applicable at some boundary $B(z)$ formed by the curves $x=\bar{x}(z)$ and $y=\bar{y}(z)$, as an expression of the form:

$$\mathrm{v}(x,y)=\int_B \mathrm{v}_0(z)\mathrm{G}(x,y,\bar{x}(z),\bar{y}(z))dz.$$

A.3.7 G is then known as the Green's function for the partial differential equation. In the case of the equation in ¶A.3.2, the Green's function is:

$$\mathrm{G}(x,y,\bar{x},\bar{y})=\frac{1}{2c\sqrt{\pi}}\frac{e^{-(x-\bar{x})^2/(4c^2(\bar{y}-y))}}{\sqrt{y-\bar{y}}}.$$

A.3.8 For European style options, the boundary problem simplifies to what is sometimes called an *initial value problem* or a *Cauchy problem*, and the Green's function approach becomes the *Poisson* equation, see e.g. Bronshtein & Semendyayev (1978). If the boundary condition is $w(x,0)=w_0(x)$ at $y=0$, where $w_0(x)$ is continuous and bounded for all x, then:

$$w(x,y)=\frac{1}{2a\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{w_0(z)}{\sqrt{y}}e^{-(z-x)^2/(4c^2y)}dz.$$

A.3.9 For a European call option, after making the substitutions described above:

$$u(S,T)=\max(S-E,0)\Rightarrow w_0(x)=\max(e^{\sigma x}-E,0)\ \text{ and } c=\frac{1}{\sqrt{2}}c^2=\frac{1}{2}$$

$$\Rightarrow u(x,y)=e^{-ry}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi y}}\max(e^{\sigma z}-E,0)e^{-(z-x)^2/(2y)}dz$$

$$\Rightarrow u=e^{-ry}\int_{\log\frac{(E)}{\sigma}}^{\infty}\frac{1}{\sqrt{2\pi y}}e^{\sigma z-(z-x)^2/(2y)}dz-e^{-ry}\int_{\log\frac{(E)}{\sigma}}^{\infty}\frac{E}{\sqrt{2\pi y}}e^{-(z-x)^2/(2y)}dz$$

$$\Rightarrow u=\mathrm{I}(1,\log(E),S,y)-E.\mathrm{I}(0,\log(E),S,y)\ \text{ say}$$

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where:

$$\mathrm{I}(k,H,S,y)=e^{-ry}\int_{H/\sigma}^{\infty}\frac{1}{\sqrt{2\pi y}}e^{k\sigma z-(z-x)^2/(2y)}dz.$$

A.3.10 We can simplify this formula as follows:

let:

$$z_1=\frac{z-p-k\sigma y}{\sqrt{y}}\Rightarrow dz_1=\frac{dz}{\sqrt{y}}$$

and

$$k\sigma z-\frac{(z-x)^2}{2y}=-\frac{z_1^2}{2}+xk\sigma+\frac{k^2\sigma^2}{2}$$

$$\begin{aligned}\Rightarrow \mathrm{I}(k,H,S,y)\ &=e^{-ry}e^{xk\sigma+k^2\sigma^2y/2}\int_{-H_1}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz_1\\ &=e^{-ry}e^{(\log(S)+(r-q-\sigma^2/2)y)k+k^2\sigma^2y/2}\mathrm{N}(H_1)\\ &=S^ke^{-kqy}e^{-r(1-k)y}e^{\sigma^2y(k^2-k)/2}\mathrm{N}(H_1)\end{aligned}$$

where:

$$H_1=\frac{-H+p\sigma+k\sigma^2y}{\sigma\sqrt{y}}=\frac{\log(S)-H+(r-q-\sigma^2/2+k\sigma^2)y}{\sigma\sqrt{y}}$$

$$\Rightarrow u(x,y)=Se^{-qy}\mathrm{N}\left(\frac{\log(S/E)+(r-y+\sigma^2/2)y}{\sigma\sqrt{y}}\right)$$

$$-Ee^{-ry}\mathrm{N}\left(\frac{\log(S/E)+(r-y-\sigma^2/2)y}{\sigma\sqrt{y}}\right).$$

If we substitute $y=T-t$, then we recover the BS formula set out in Section 7.

A.3.11 Most closed form option pricing formulae consist of terms that look vaguely like the expression derived for $I(k,H,S,y)$ above.

A.3.12 There is a strong element of symmetry between the terms in S and the terms in E, since:

$$\text{if } m_0=Ee^{-r(T-t)}\ m_1=Se^{-y(T-t)}\ \text{ and } \alpha=\sigma\sqrt{T-t}\ \text{ then:}$$

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$$\text{value of call option} = m_1.\mathrm{N}\left(\frac{\ln(m_1/m_0)}{\alpha}+\frac{\alpha}{2}\right)-m_0.\mathrm{N}\left(\frac{\ln(m_1/m_0)}{\alpha}-\frac{\alpha}{2}\right)$$

$$\text{value of put option} = m_0.\mathrm{N}\left(\frac{\ln(m_0/m_1)}{\alpha}+\frac{\alpha}{2}\right)-m_1.\mathrm{N}\left(\frac{\ln(m_0/m_1)}{\alpha}-\frac{\alpha}{2}\right).$$

A.3.13 The reason for this is that a 'call' option giving the right to buy equities at a certain predetermined rate for cash can also be expressed as a 'put' option giving the holder the right to 'sell' cash for equities at a predetermined rate. This symmetry is often called *spot-strike symmetry*.

A.3.14 For hedging purposes, it is also necessary to know the *greeks* or partial differentials of V. The most important are probably the option *delta* and *gamma*. These are usually defined as follows:

$$\Delta \equiv \frac{\partial V}{\partial S} \equiv V_S \quad \text{and } \Gamma \equiv \frac{\partial^2 V}{\partial S^2} \equiv V_{SS}.$$

A.3.15 Formulae for these (for a call option), given the BS model, are:

$\Delta = e^{-q(T-t)}\ \mathrm{N}(d_1)$ where d_1 is as in ¶7.5.2;

$\Gamma = \dfrac{e^{-q(T-t)}\mathrm{f}(d_1)}{S\sigma\sqrt{T-t}}$; and

$\mathrm{f}(x)$ is the normal probability density function.

A.3.16 Other 'greeks' include:

$$\kappa \equiv \text{vega} \equiv \frac{\partial V}{\partial \sigma} = Se^{-q(T-t)}\mathrm{f}(d_1)\sqrt{T-t}$$

$$\rho \equiv \text{rho} \equiv \frac{\partial V}{\partial r} = (T-t)Ee^{-r(T-t)}\mathrm{N}(d_2)$$

$$\lambda \equiv \text{lambda} \equiv \frac{\partial T}{\partial q} = -(T-t)Se^{-q(T-t)}\mathrm{N}(d_1)$$

$$\Theta \equiv \text{theta} \equiv \frac{\partial V}{\partial t} = -Se^{-q(T-t)}\mathrm{f}(d_1)+qSe^{-q(T-t)}\mathrm{N}(d_1)-rEe^{-r(T-t)}\mathrm{N}(d_2).$$

A.3.17 Given the nature of the hedging equation, theta can be determined from the other greeks.

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A.3.18 For hedging purposes, it is often helpful to restate the formulae for the delta and gamma to relate to the equivalent market exposure involved. This involves multiplying the delta and gamma in ¶A.3.15 by S and S^2 respectively. The formulae then become:

$$\Delta = Se^{-q(T-t)}\mathrm{N}(d_1)$$

$$\Gamma = S\frac{e^{-q(T-t)}\mathrm{f}(d_1)}{\sigma\sqrt{T-t}}.$$

A.4 *Options involving Multiple Underlying Assets*

A.4.1 For more complicated options depending on several variables, there is an equivalent equation to that given in ¶A.2.9. It can be developed using martingale theory, in which case the equation is known as the Feynman-Kac equation, see e.g. Duffie (1992). Equivalently, the equation can be developed in a partial differential equation framework, using no arbitrage arguments, along the lines summarised in Vetzal (1994). In this sort of formulation we assume that the state of the economy (or at least that part we are interested in) may be summarised by a k-dimensional vector of state variables $X \in R^k$, where the movements of X over time are described by a system of stochastic differential equations:

$$dX = \mu(X,t)dt + \sigma(X,t)dZ(t)$$

(where $Z(t)$ is a k-dimensional Brownian motion).

A.4.2 We might call this a multi-dimensional generalised Brownian framework. In an interest rate context and if X relates to zero coupon bonds, but we model the development of forward rates, then this approach would generally be known as the Heath, Jarrow and Morton framework, see Heath *et al.* (1992).

A.4.3 It is then possible to show, subject to suitable regularity conditions, and assuming markets are 'complete', that if the value of any security or derivative on the security is given by $V(X,t)$, the instantaneous pay-out of the security is $C(X,t)$ and the risk-free interest rate is $r(X,t)$, then standard no-arbitrage arguments imply that V satisfies the following partial differential equation:

$$\tfrac{1}{2}\mathrm{tr}[\sigma(X,t)' V_{XX}\sigma(X,t)] + V_X v(X,t) + V_t + C(X,t) - r(X,t)V(X,t) = 0$$

where:

$$V_t = \frac{\partial V}{\partial t} \qquad V_X = \left[\frac{\partial V}{\partial X_1},\ldots,\frac{\partial V}{\partial X_n}\right] \qquad V_{XX} = \left[\frac{\partial^2 V}{\partial X \partial X'}\right]_{k\times k}$$

$$v(X,t) = \mu(X,t) - \sigma(X,t)\phi(X,t)$$

and $\phi(X,t)$ is a column vector called the 'market price of risk' vector, the existence of which is a necessary condition for the absence of arbitrage.

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A.4.4 The price of any specific derivative in such a world is then given by the solution to this equation, subject to an appropriate boundary condition.

A.4.5 It is usually easiest to express the results of this process in the manner equivalent to that described in ¶7.3.4, i.e. as:

$$V(X,t) = \mathrm{E}\left(e^{-\int_t^T r(X,s)ds} V(X,T) \middle| X_t \right)$$

where E(Y) is the expected value of Y under some suitable risk-neutral probability distribution.

A.4.6 It is possible to use this approach to confirm the formula for relative performance options given in ¶8.2.6.

A.4.7 This formulation is also normally essential when trying to value interest rate derivatives, since the state variables for such derivatives are not normally one dimensional, but will depend on the yield curve, which, in general, needs several factors to describe it fully.

A.4.8 For the purposes of this paper, it is helpful to realise that the regularity conditions required to justify the equation in ¶A.4.3 include:

(a) no transaction costs;
(b) σ to be continuous (which precludes jumps); and
(c) σ to be a function of X and t, i.e. σ is not stochastic in its own right.

Otherwise the markets are 'incomplete' and can only be completed by including 'additional' underlyings relating to volatility, jump costs and transaction costs which cannot be priced merely by reference to the price of the 'main' underlyings in X. In this sense the model becomes preference dependent, since derivatives can only, in general, be priced after making assumptions about the utilities of different investors. These are the same circumstances highlighted in Section 8.3 which cause the BS formulae and the use of binomial trees/Brownian motions to break-down.

A.5 *Trinomial Lattices*

A.5.1 A necessary requirement of being able to derive a backwards equation of the sort described in ¶7.1.6 solely from no-arbitrage arguments is that at each node in the lattice there can only be two possible movements (since there are only two assets that exist, in such a framework, to construct the hedge portfolio). More complex lattices can be constructed, e.g. trinomial lattices, at which the share price can go 'up', 'down' or 'sideways' at each node. The backwards equations that these lattices satisfy can no longer be deduced directly from arbitrage principles, but instead have greater degrees of freedom (and are consequently more complicated). This greater flexibility means that it is possible to optimise the numerical properties of the lattice.

A.5.2 In most circumstances a trinomial lattice along the lines of the following

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is likely to be close to optimal. $S(n,m)$ should be allowed to move to $S(n+1,m-1)$, $S(n+1,m)$ or $S(n+1,m+1)$, where the nodes of the lattice are given by $S(n,m)=S_0 b^m k^n$ (so in the notation in Figure 11, $g_u=bk$, $g_0=k$ and $g_d=b^{-1}k$). The backwards equation is defined as follows:

$$V(n,m)=g_u V(n+1,m+1)+g_0 V(n+1,m)+g_d V(n+1,m-1)$$

where:

$$k,b,g_u,g_0,g_d \text{ are defined so that } k=e^{(r-y)h}$$

$$\zeta=e^{\sigma^2 h/32}+e^{2\sigma^2 h/32}+e^{3\sigma^2 h/32}-3$$

$$b=\left(1+\zeta/2+\sqrt{\zeta(\zeta+4)/4}\right)^4$$

$$e^{rh}(g_u+g_0+g_d)=1$$

$$e^{rh}(g_u+g_d)=\frac{\zeta+1+e^{-\sigma^2 h/8}-(\zeta+2)e^{-3\sigma^2 h/32}}{\zeta}$$

and

$$e^{rh}(g_u-g_d)=\frac{(\zeta+4)e^{-3\sigma^2 h/32}-e^{-\sigma^2 h/8}-(\zeta+3)}{\sqrt{\zeta(\zeta+4)}}.$$

A.5.3 This complicated lattice structure is designed so that if one were to price a derivative paying S^p for $p=0$, 0.25, 0.5, 0.75 or 1, then the lattice gives exactly the right answer. For standard types of European and American options, this calibrated trinomial lattice can converge much more rapidly than a binomial lattice.

A.5.4 One might expect to be able to improve convergence still further by using quadrinomial or even more complicated lattices, but the effort involved in programming them becomes progressively more complicated, and it becomes less easy to ensure that the optimal lattice structure is recombining, so, for fast numerical computation of options using lattices, trinomial lattices are likely to be preferred in most circumstances.

A.5.5 It is worth noting that there are two main sorts of errors arising with lattice pricing methods. The first relates to propagation errors from the backwards equation, which can be much reduced by using trinomial lattices rather than binomial ones. The second sort of error involves the approximation of a continuous pay-off at maturity with one involving discrete amounts at each maturity node. It can be minimised by setting the pay-off at each maturity node equal to the average of maturity pay-off for prices of the underlying closest to the node. This second sort of error is not improved merely by use of a trinomial lattice.

APPENDIX B

JUMP PROCESSES, STOCHASTIC VOLATILITY AND TRANSACTION COSTS

B.1 *Circumstances in which the Black-Scholes Formulae break down*

As noted in Section 8 and Appendix A, there are essentially three possible sources of deviation from the BS framework (other than inappropriate management of the hedging process), namely:

(a) market jumping;
(b) market volatility differing from that originally expected; and
(c) the existence of transaction costs.

B.2 *Jump Processes*

B.2.1 If markets jump, it is impossible to move the hedge portfolio fast enough in the fashion required to replicate the behaviour of the option by dynamic hedging. Section 9 suggests that jump risk is the most important of the three possible departures mentioned above for the sort of option considered there.

B.2.2 Perhaps the simplest way of incorporating jumps into option pricing formulae is the *cost of capital* model developed by M.H.D. Kemp and A.D. Smith (see, e.g., Smith, 1995).

B.2.3 This model assumes that in any small instant the underlying may jump in price either infinitely upwards or down to zero. An option writer is putting his capital at risk from such jumps, as they are not hedgeable by investing merely in the underlying and risk-free assets. It is reasonable to assume that the option writer will demand an excess return on the 'risk capital' he thus needs, to reflect this risk. Of course, such risks can be hedged by buying suitable options, but this has merely transferred the jump risk to someone else. Ultimately, someone must carry this risk.

B.2.4 The model assumes that writers of derivatives require some additional return on the cash/shares that they need as risk capital to compensate them for putting their capital at risk. It therefore includes two extra parameters:

r_a = the rate of interest required on the cash the writer would need to hold to make good the loss L_C that he would incur if there were an extreme downward jump in the price of the underlying shares; and

q_a = the enhanced income yield required on the shares the writer would need to hold to make good the loss L_S that he would incur if there were an extreme upward movement in the price of the underlying shares.

B.2.5 The values under this model of European put options $P(S,t)$ and call options $C(S,t)$ with exercise price E are:

$$C(S,t) = Se^{(q_a-q)(T-t)} + Se^{-q_a(T-t)}\mathrm{N}(d_1) - Ee^{-r_a(T-t)}\mathrm{N}(d_2)$$

$$P(S,t) = Ee^{(r_a-r)(T-t)} + Ee^{-r_a(T-t)}\mathrm{N}(-d_2) - Se^{-q_a(T-t)}\mathrm{N}(d_2)$$

where:

$$d_1 = \frac{\log(S/E) + (r_a - q_a + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \text{ and } d_2 = d_1 - \sigma\sqrt{t-t}.$$

B.2.6 The derivation of these formulae is as follows. Suppose we have written a derivative with value $V(S,t)$ and we hedge it using a portfolio containing a suitable mix of risk-free assets and the underlying. We will have:

Assets: A in the risk-free asset + B shares, where:

$$A = V - S\frac{\partial V}{\partial S} \quad \text{and} \quad B = \frac{\partial V}{\partial S}$$

Liabilities: $V(S,t)$, i.e. the value of the option.

B.2.7 Our exposure (i.e. the loss incurred), were the underlying to crash instantaneously to zero, is:

$$L_C = \lim_{\varepsilon \to 0} V(\varepsilon,t) - \left(V(S,t) - S\frac{\partial V}{\partial S} \right).$$

B.2.8 Our exposure, were the underlying to rise instantaneously to infinity, is:

$$\lim_{M\to\infty} V(M,t) - A - B.\lim_{M\to\infty} M.$$

This quantity may grow arbitrarily large for large M, and so we cannot cover all possible losses merely by holding cash. However, we can (at least for usual sorts of derivatives) cover all losses by holding a suitable amount of shares. For very large M, the number of shares required is:

$$L_S = \lim_{M\to\infty} \frac{V(M,t) - A - BM}{M} = \lim_{M\to\infty} \frac{V(M,t)}{M} - \frac{\partial V}{\partial S}.$$

Exceptions include *power* options (which have pay-offs expressed in terms of S^n, where $n > 1$, so that they grow asyptotically faster than S as S becomes large) and *quanto* options (in which, say, the pay-off is, say, \$1 for every one point rise in FT-SE, even though FT-SE is denominated in £ sterling). In the presence of very large jumps, their values become unstable, and this could lead to systemic instability in derivatives markets.

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B.2.9 We assume that shareholders of the ultimate carriers of such jump risks require some additional return on the cash/shares they need as risk capital to compensate them for putting their capital at risk, i.e.:

r_a = rate of interest required on cash backing downward jump risk (i.e. L_C); and
q_a = enhanced income yield required on shares backing the upward jump risk (i.e. L_S).

B.2.10 The writer of the option, therefore, needs to include allowance for the extra costs of capital, i.e. $(r_a - r)L_C$ and $(q_a - q)L_S$, when pricing the option. Thus, the price of the option will no longer be described by the BS partial differential equation, but instead by the following partial differential equation.:

$$-ru + u_t + (r-q)Su_S + \frac{\sigma^2 S^2}{2}u_{SS} + (r_a - r)\left(Su_S - u + \lim_{\varepsilon \to 0} u(\varepsilon, t)\right)$$
$$+(q_a - q)S\left(\lim_{M \to \infty} \frac{u(M,t)}{M} - u_S\right) = 0.$$

B.2.11 This partial differential equation simplifies to:

$$-r_a u + u_t + (r_a - q_a)Su_S + \frac{\sigma^2 S^2}{2}u_{SS} + (r_a - r)P(t) + (q_a - q)Q(t) = 0$$

where:

$$P(t) = \lim_{\varepsilon \to 0} u(\varepsilon, t) \quad \text{and} \quad Q(t) = S \lim_{M \to \infty} \frac{u(M,t)}{M}.$$

B.2.12 The above partial differential equation collapses to the Black-Scholes partial differential equation when $r_a = r$ and $q_a = q$. Even when r_a differs from r and q_a from q, the formula is very similar to the BS partial differential equation apart from the two terms involving $P(t)$ and $Q(t)$.

B.3 *The Cost of Capital Adjustments in Practice*

B.3.1 The cost of capital model can easily be criticised for assuming unrealistic market behaviour, since markets do not, in practice, leap by the sorts of amounts assumed in the model. However, it is relatively easy to demonstrate that, as long as the option is not far in or out of the money (or of very long duration), the losses incurred by more modest sized jumps rapidly approach L_C and L_S as the size of the jumps become significant. The model can thus be thought of as a good approximation to a more accurate one that merely assumes that jumps that need to be protected against are 'significant'.

B.3.2 There are two ways that we can test whether the cost of capital model provides greater explanatory power than the Black-Scholes model:

(a) we can analyse whether it seems to have captured aspects of actual market behaviour in the past; and

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(b) we can analyse whether it seems to explain current market prices.

B.3.3 The cost of capital model seems to do quite well in relation to past market behaviour. The formula in ¶B.2.5 for, say, a put option involves two terms representing contributions from:

(a) jump components: $Ee^{(r_a-r)(T-t)}$

(b) diffusion components: $Ee^{-r_a(T-t)}\mathrm{N}(-d_2) - Se^{-q_a(T-t)}\mathrm{N}(-d_1)$.

The first term is independent of the strike price. Thus, one way of testing whether the cost of capital model is a helpful model is to see whether the jump related costs of hedging an options position are relatively independent of the option strike price. Paragraph 9.4.4 demonstrated that this is indeed a reasonable approximation. We could also conclude from this that, based on the period analysed there, an appropriate value for $r_a - r$ might be around 0.4% p.a. to 0.5% p.a. (this being roughly the sum of the up and down jump costs shown there).

B.3.4 In some respects, the cost of capital model is also helpful when trying to explain current market prices. If the model underlying the BS formulae is correct, then the implied volatility of options will be independent of strike price. The effect of introducing the cost of capital adjustments is to alter the shape of the curve we would get by plotting implied volatility versus strike. The cost of capital model has two extra parameters, and will, therefore, in general permit a curve with three degrees of freedom (i.e. something like a quadratic curve), rather than just a curve with just one degree of freedom (a horizontal line).

B.3.5 These extra parameters mean that the cost of capital model will always be a better fit to the implied volatility curve. The implied volatilities of FT-SE options for different strikes and different terms were shown in Figure 10. Table B.1 contains corresponding cost of capital parameters which closely fit these implied volatility curves. The fit is much better than using a single implied volatility for all strikes.

Table B.1. 'Cost of capital' parameters fitting FT-SE smile/skew at 7 March 1996

Duration (years)	Sigma, σ (% p.a.)	$r_a - r$ (% p.a.)	$q_a - q$ (% p.a.)
0.33	15.0	0.6	−3.4
0.58	14.8	0.2	−2.0
0.83	15.5	0.4	−2.1
1.00	16.3	0.2	−2.1
2.00	16.8	0.2	−1.4
3.00	17.1	0.1	−0.6
4.00	17.1	0.1	−0.5
5.00	17.6	0.1	−0.5

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B.4 *Finite Jumps*

B.4.1 Unfortunately Table B.1 also shows a flaw in the cost of capital model. Logically both $r_a - r$ and $q_a - q$ should be non-negative, since both upward and downward jumps potentially involve losses to the option writer. Although $r_a - r$ is positive, $q_a - q$ is not (at least not based on these exercise prices, which are not far from being at-the-money).

B.4.2 One reason is that markets will not, in practice, jump infinitely up or down (or at least market practitioners expect this to be rare). A more realistic model would involve finite jumps as well, in order to explain any *skewness* and 'fat-tailedness' (usually referred to as *kurtosis*) away from a log normal distribution.

B.4.3 One possible model with these characteristics would be the jump model described in Smith (1996). Another is the generalised beta distribution of the second kind, as described in Bookstaber & McDonald (1987). This distribution contains as special cases a large number of well-known distributions, such as the log-normal, log-*t* and log-Cauchy distributions.

B.4.4 Both of these models are also characterised by four parameters, and thus, like the cost of capital model, can fit any arbitrary mean, variance, skewness and kurtosis (or equivalent measures of dispersion, etc. if the various moments of the relevant distribution are infinite). The mean of the distribution is, in some sense, redundant in this sort of analysis, since it is necessary to set it equal to the risk-free rate in a risk-neutral world.

B.4.5 Interestingly, Bookstaber & McDonald conclude in their paper that the longer the time period, the less justification there is for adopting a model different to the log-normal one underlying the BS formulation. This seems to be consistent with the fall in the absolute values of $r_a - r$ and $q_a - q$ in Table B.1 as the period to maturity rises.

B.4.6 Another set of distributions that some practitioners have considered involves Levy stable distributions (otherwise known as Stable Paretian distributions). These play the same sort of role as the normal distribution does in the central limit theory when we accumulate random returns with infinite variances. Indeed, the normal distribution is a special case of the more generalised Levy stable distribution. They, too, have four parameters which relate to the position of the 'middle' of the distribution, its dispersion, its skewness and how fat tailed it is. However, they have the practical disadvantage that they have infinite variances and are not particularly easy to manipulate mathematically. Longuin (1993), when analysing the distribution of U.S. equity returns, concludes that it is not sufficiently fat-tailed to be adequately modelled by Levy stable distributions, even if it is fatter tailed than implied by the normal distribution.

B.4.7 A final way that is sometimes used by derivatives practitioners to handle possible market jumps is similar to that used to price general insurance contracts. Jumps are assumed to be of a specific size (or to come from a specific size distribution, such as an exponential distribution) and to occur at a rate which is a Poisson process, with parameter λ. Thus the probability that n jumps occur within a short time interval t is $(\lambda t)^n$ (although it is not necessary to assume that λ is constant over time). It is then possible to develop a partial differential equation similar to that in ¶A.2.9, but with extra terms incorporating a risk adjusted version of λ, as well as parameters describing the size distribution and the loss incurred were a jump of the given size to occur. Whilst this is arguably closer to reality than the approaches described above, it is also a lot more complicated mathematically. The cost of capital model is a limiting case of this approach with two jump sizes ($+\infty$ and $-\infty$, with the results normally closely approximating to a model concentrating merely on 'significant' positive or negative jump sizes) and with the risk adjusted λ equal to $r_a - r$ or $q_a - q$ (depending on the direction of the jump).

B.4.8 The link with risk adjusted parameters also reminds us that the pricing of jump risk is preference dependent, and will, therefore, be influenced by the utility functions of the various market participants. It is reasonable to postulate that market participants dislike large downward market jumps more than they dislike large upward jumps. This may drag down the observed $q_a - q$ and increase the observed $r_a - r$.

B.5 *Options with a Moving Average Strike Price*

B.5.1 The original purpose behind developing the cost of capital model was to permit a rather unusual sort of option to be priced. This was a 'catastrophe put' option, with an exercise price based on a moving average of the price of the underlying. The idea was that insurers would be particularly worried about rapid falls in equity markets, but less concerned about slow declines that could, perhaps, be managed by changing business strategy.

B.5.2 In the limit, as the period over which the moving average is calculated falls to zero, such an option would only have a pay-off if the market had jumped downwards by at least a specified amount. In a Black-Scholes world, the appropriate price to charge for such jump risk is zero.

B.5.3 Thus one (theoretical) way of assessing how far markets deviate from the Black-Scholes world in terms of their jump characteristics is to determine the price differential between such a catastrophe put option and the price of the corresponding vanilla option, and to identify its limiting behaviour as the averaging period tended to zero. From this we could identify the market price attaching to jump risk arising from jumps of any specific size. Unfortunately, such options do not trade in practice.

B.6 *Convexity*

B.6.1 The jump characteristics of price movements of the underlying are irrelevant to the pricing of symmetric derivatives such as futures and forwards; the formulae developed in Section 4 remain valid. The key difference with options is their non-linearity. This is more normally referred to within the equity derivatives industry by the term *convexity*. However, readers should note that the way interest rate derivatives are usually described means that convexity has a different meaning for them.

B.6.2 For example, the price u of a call option will depend on the price of the underlying in the manner similar to that shown in Figure 7. It is convex upwards, so a purchaser of the option is said to have positive convexity, whilst the seller/writer of the option is said to have negative convexity.

B.6.3 If we attempt to replicate the effect of the option using dynamic hedging, and the price of the underlying was S, then we would be investing in a portfolio consisting of Su_S in the underlying and $u-Su_S$ in cash. This would change in value as per the tangent line to the graph. The behaviour of this portfolio deviates from the behaviour of the option precisely because of the convexity of the option.

B.6.4 We would expect there to be some price attaching to convexity. We could thus price derivatives in the following manner (since the left hand side is the rate of change in the value of the option with respect to time and the first two terms on the right hand side are the interest/dividends we are receiving from the hedge portfolio):

$$u_t = \frac{\partial u}{\partial t} = r(u - Su_S) + qSu_S$$
$$+ \text{ price of a unit of convexity per unit time} \times \text{amount of convexity.}$$

B.6.5 A common measure of convexity used by mathematicians is how rapidly the tangent angle changes, which we can measure by u_{SS}, the second partial derivative with respect to S. To convert this to monetary quantity we need to multiply by S^2. If we then define the price of a unit of convexity per unit time as $\sigma^2/2$, we recover the BS partial differential equation and the BS formulae. If our measure of convexity includes the asymptotic behaviour of the option as S tends to $\pm\infty$, then we recover the cost of capital model. If our measure includes other aspects of convexity (e.g. finite jumps), then more complicated pricing equations result.

B.7 *Stochastic Volatility*

B.7.1 The second source of discrepancy from the Black-Scholes world is the possibility that the volatility of the price movement of the underlying might change as time progresses, in a way that is not predictable in advance.

B.7.2 Figure 17 shows how the implied volatility of options can vary, and how it also can differ from the historic volatility of the underlying price movements. Volatilities (both market and implied) often seem to rise when there is a downwards market shock, returning only over time to their previous levels.

B.7.3 Such characteristics may be modelled using GARCH models, i.e. models exhibiting Generalised Autoregressive Conditional Heteroscedasticity. These models are often conceptually similar in structure to the Wilkie model,

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often used by actuaries for stochastic asset/liability modelling, except that the Wilkie model concentrates its autoregressive characteristics on the *mean* of the relevant distribution, whereas GARCH models concentrate their autoregressive characteristics on its *volatility*.

B.7.4 Just as the limiting 'catastrophe put' option, described in Appendix B.4, can be thought of as highlighting the jump characteristics of a price movement distribution, we can, in theory, construct a type of option which would characterise the degree to which volatility is stochastic. The basic conceptual building block is the 'Log Contract' described in Neuberger (1990b), whose value is directly related to *out-turn* volatility, i.e. the future volatility actually experienced in practice. If a market maker was prepared to trade options extensively on such contracts (they trade only rarely in practice), then it would probably be possible to mimic any arbitrary volatility structure.

B.8 *Non-Zero Dealing Costs*

B.8.1 The final potential source of discrepancy from the BS formulation is the existence of transaction costs. Suppose we rebalanced our portfolio in accordance with a 'perfect' dynamic hedging programme at intervals separated by a short period h. Then as h tends to zero, it is possible to show that the total volume of transactions between now and maturity would tend to infinity. This is because Brownian motion has the characteristic that, as the time interval becomes shorter and shorter, the observed variability of the price process reduces only by the square root of the time interval. In the presence of non-zero transaction costs, sufficiently frequent rebalancing will always completely extinguish the hedge portfolio.

B.8.2 One possibility would be to 'over-hedge', i.e. always hold more than enough to meet any level of transaction costs. Unfortunately, the characteristics of Brownian motion mean that, to avoid completely the possibility of extinguishing the hedge portfolio, we would need to hold the upper limit on the possible value of the option set out in ¶6.1.3, and then carry out no dynamic hedging whatsoever. This is arguably overkill!

B.8.3 Thus, we will need to accept some possibility of being unable to hedge fully an option pay-off if there are transaction costs. There is a trade-off between the degree to which we rebalance (especially in terms of the frequency of rebalancing), thus incurring transaction costs, and the degree to which we replicate accurately the final option pay-off. There is inherent uncertainty in the quality of replication, as is explained in the seminal work on this subject, Davis, Panas & Zariphopoulou (1993):

> "There is a paradoxical element to the Black-Scholes approach, which has been called the 'Catch-22 of option pricing theory'; the claims that can be priced are just those that are redundant in that the investor could, in principle, simply take a position in the replicating portfolio rather than actually buy the option. Thus, apparently such options have no reason to exist. The fallacy is that we do not live in a Black-Scholes world. In particular, the replicating portfolio cannot be implemented exactly, since it involves incessant rebalancing,

which is impractical in the face of any form of market friction such as transaction costs. In this paper, we develop a theory of option pricing in which transaction costs are explicitly taken into account. Perfect hedging is no longer possible, and therefore buying or writing options involves an unavoidable element of risk. For this reason, a preference-independent valuation is no longer possible, and the investor's or writer's attitude towards risk must be considered."

B.8.4 Davis, Panas & Zariphopoulou (1993) borrow ideas from Hodges & Neuberger (1989). They use a utility maximisation approach and show that the mathematics involved reduce to two stochastic optimal control problems, i.e. partial differential equations involving inequalities. Unfortunately, the mathematics are rather difficult even by the standards of derivative pricing literature. However, a simple asymptotic approximation to the solution to these equations has been found by Whalley & Wilmott (1993). It is also described, in passing, in Smith (1996) and in more detail below. The optimal dynamic hedging strategy involves three sorts of actions characterised by 'buy', 'hold' or 'sell' regions. The 'hold' region, not surprisingly, becomes larger as the transaction costs become larger.

B.8.5 For market makers, especially those who make markets in the underlying physical markets and in corresponding futures contracts, transaction costs may be significantly lower than for external participants. This will generally give market makers an edge in pricing derivatives, since it will reduce their hedging costs and improve the potential accuracy of their hedging programmes. Usually futures contracts would be the main route to obtaining market exposure in dynamic hedging programmes, given the lower transaction costs associated with them. However, if futures are being used, it is important to bear in mind factors such as market impact (see ¶4.1.6) and basis and roll-over risk (see Section 5.4). These factors may increase the effective level of transaction costs that need to be allowed for.

B.8.6 Another practical problem associated with attempting to follow 'perfect' dynamic hedging, as per ¶B.8.1, is that rebalancing of the hedge portfolio becomes very frequent. This could require excessively large staff or systems commitments. A derivatives house could take this into account by a suitable increase in the transaction costs involved. In the extreme hypothetical case where transaction costs are unlimited, the optimal dynamic hedging strategy becomes that set out in ¶B.8.2, i.e. no rebalancing takes place at all. Practical cases will fall somewhere between these two extremes. The frequency of rebalancing can thus, to a substantial degree, be controlled by altering the transaction costs allowed for in the hedging algorithm.

B.8.7 Mohamed (1994) reviewed various ways proposed in the academic literature to minimise transaction costs whilst still hedging derivatives dynamically. He carried out simulations of four approaches and concluded that the most effective was one involving the analytic approximation to a utility maximisation approach as set out in Whalley & Wilmott (1993).

B.8.8 This strategy involves rebalancing the portfolio if it moves outside a certain band (back to the nearest edge of that band). The band is defined in terms of a factor Δ defining how much is invested in the underlying. It is like an option delta, but given by:

$$\Delta = \frac{\partial V}{\partial S} \mp \left(\frac{3kSe^{-r(T-t)}}{2\lambda} \right)^{1/3} \left| \frac{\partial^2 V}{\partial S^2} \right|^{2/3}.$$

The terms on the right hand side relate to the actual delta and gamma of the option, ignoring transaction costs. The assumed utility function is an exponential one, with an index of risk aversion λ. k is the size of the (proportional) transaction cost. In the limit of zero transaction costs, the band collapses to rebalancing in line with the standard BS case, i.e. using:

$$\Delta = \frac{\partial V}{\partial S}.$$

B.8.9 The explicit link with the option gamma (convexity), i.e. the second order partial differential with respect to S, has intuitive appeal. The band tightens when the option becomes deep in-the-money, or deep out-of-the-money (when the rate of change of delta in the zero transaction cost becomes small), but increases when the delta is liable to fluctuate more.

B.9 *Applications to Modern Portfolio Theory*

B.9.1 The earlier parts of this appendix also provide insight into Modern Portfolio Theory, especially the single factor *Capital Asset Pricing Model* (CAPM) and its multi-factor analogue, the *Arbitrage Pricing Theory* (APT). The APT assumes that the log returns on any given investment can be decomposed into various factor components plus a residual (stock-specific) risk. The stock-specific risk is typically assumed to be a normal error term. The CAPM effectively produces the same answers, but assuming that there is just one factor, the 'market' (with the exposure component to the market being the stock's 'beta').

B.9.2 Thus, the APT and CAPM assume that returns can be decomposed in the following fashion:

$$r_i = \alpha_i + \beta_{i,j} x_j + \varepsilon_i.$$

B.9.3 The rest of this appendix shows that a more complete decomposition is along the lines of the following:

$$r_i(t) = \alpha_i(t) + \beta_{i,j}(t) x_j(t) + B_i(t) + \int V_{i,k}(t) d\sigma_k + \int J_{i,l} d\lambda_l$$

+ plus further cross correlation terms and transaction cost terms

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where $B(t)$ are Brownian motions, $V(t)$ represents the contribution from unpredictable changes to volatility and $J(\lambda,t)$ the contribution from market jumps.

B.9.4 In particular, we can think of economic quantities called 'volatility' and jump/'gap' risk, which one can buy or sell via the derivatives markets. Indeed, derivatives practitioners often talk about trading volatility when they mean taking a position that will benefit or suffer if volatility changes. Just as there is a term structure to interest rates, there can also be a term structure to volatility.

B.9.5 Modern portfolio theory teaches us that we can diversify non-systematic risk (i.e. the ε) by holding a diversified portfolio. Exactly the same principle means that we can diversify volatility and jump risk by holding a diversified derivatives book, as long as the types of volatility risk and jump risk are not 100% correlated.

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APPENDIX C

INTEREST RATE DERIVATIVES

C.1 *The Importance of Interest Rate Derivatives*

C.1.1 Interest rate derivatives, including swaps, form by far the world's largest market involving long-term derivatives. The majority of these swaps involve exchanging fixed-interest rates for floating rates. Such swaps can be for very long terms, e.g. 25 years or more. Some effectively include guaranteed reinvestment rates for new money.

C.1.2 Although U.K. life insurance companies are often thought of as principally interested in equity derivatives, many of the larger companies may have executed some swaps to match guaranteed income bonds or for traditional corporate treasury purposes. They may also be attracted by relative performance options linked to both equity and fixed-interest returns, as may pension funds. Interest rate derivatives are much more important for life insurance companies in some other markets, e.g. the U.S.A., where such companies often hold a large proportion of their assets in bonds. Many of these bonds have complicated option-like characteristics (e.g. mortgage backed securities).

C.1.3 General insurance companies, both in the U.K. and elsewhere, tend to invest less in equities and more in bonds than life insurance companies, and again may make more use of interest rate derivatives.

C.1.4 Banks, of course, make very extensive use of swaps. Indeed, they probably place more attention on the interbank swaps market when attempting to analyse future interest rate movements and their required costs of capital than on the government debt markets that insurance companies and investment managers would usually focus on.

C.2 *The Applicability of the Rest of this Paper*

C.2.1 Although the material in the rest of the paper principally focuses on equity derivatives, much remains valid for interest rate derivatives. For example, standard forms of swaps can be decomposed into a series of forward transactions, and priced according to the principles set out in Section 5. The calibration concepts described in Section 6 are also applicable. Indeed they are arguably more important, since it is usually necessary to fit an entire yield curve at outset.

C.2.2 However, there are four key differences between interest rate derivatives and those on other securities:

(a) In terms of control procedures, nearly all options on any underlying have some exposure to interest rates (e.g. a one-year vanilla European equity option is, strictly speaking, dependent on the prices of zero coupon bonds maturing at the same time as the option). However, interest rate derivatives are often self contained. Organisations need to be careful to arrange some bridge between

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reserving and risk management procedures in different derivatives specialities if they wish to capture all their interest rate exposures.

(b) We need to take explicit account of the stochastic nature of the rate of interest r, since this is, of course, the key determinant of the value of interest rate options. We also need to allow for the correlation between different points along the yield curve, since the curve usually moves in a similar direction along its length.

(c) As time progresses, the volatility of the underlying will generally fall, if the derivatives relate to bond instruments, rather than interest rates themselves. This is because the derivative is then linked to the duration of the underlying bond. For example, suppose we have a European option on a zero coupon bond which will be redeemed at par shortly after the option matures. The price of the bond at the maturity of the option is then almost certain to be close to par, unless it is subject to significant credit risk. In the derivatives industry this is called the *pull to parity*.

(d) Indeed, as time progresses, the duration/time to payment of every single fixed cash flow involved in the derivative will steadily fall. Thus, what we really need to tackle interest rate derivatives are models of how the whole yield curve might develop over time, rather than just how some single underlying price might vary.

C.3 *Interest Rate Models*

C.3.1 Interest rate models currently in use are usually based on special cases of the Heath, Jarrow and Morton framework as set out in Section A.4. The main academic focus in this area has been to devise probabilistic models describing how interest rates might evolve which:

(a) produce analytically tractable solutions to this equation, at least for simpler sorts of derivatives; and

(b) bear some resemblance to reality.

C.3.2 Vetzal (1994) provides a useful summary of the main models. The first series of models to reach prominence were *single factor* models, e.g. the *Vasicek* model and the original version of the *Cox, Ingersoll and Ross* model. These invariably assumed that the single factor was the instantaneous risk-free interest rate r. The problem with such models is that they imply the behaviour of the entire yield curve can be perfectly modelled by just one parameter. Experience teaches that this is not the case. For example, investment managers will often split the yield curve into two, three or more areas in order to understand its dynamics more completely. More complex *multiple factor* models have thus been devised, e.g. the two factor *Brennan and Schwartz, Longstaff and Schwartz* and *Vetzal* models.

C.3.3 Early models also had the disadvantage that they were unable to fit the term structure of interest rates perfectly at outset. The argument in favour of models which perfectly replicate the opening term structure is that, if a model cannot even price a straightforward bond correctly at outset, then little confi-

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dence may be placed on its ability to value other more complicated financial instruments. More importantly, calibration of a model becomes much easier if it fits the opening term structure exactly. Models with this characteristic include the *Black, Derman and Toy* and the *Hull and White* models, which are single factor models.

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APPENDIX D

THE CAPITAL ADEQUACY DIRECTIVE

D.1 *The Capital Adequacy Directive (CAD)*

D.1.1 The full title of the CAD is Council Directive 93/6/E.E.C. of 15 March 1993. It prescribes minimum capital requirements for the trading books of banks and other investment firms domiciled in the E.C.

D.1.2 Details of the implementation of the CAD in the U.K. are set out (for organisations like banks which are regulated by the Bank of England) in the Bank of England Supervision and Surveillance Notice Numbered S&S/1995/2. Securities houses are regulated by the Securities and Futures Authority (SFA), and their CAD definition rules are in SFA Board Notice 249, dated 15 May 1995, but these are very similar in approach to the Bank of England's rules.

D.1.3 The CAD subdivides a bank's asset and liabilities into two components — its trading book and its banking book. The former consists of its holdings of financial instruments which it trades in (and other instruments held to hedge tradable financial instruments), whilst the latter refers to deposits, long-term loans and the like. The division between the two is not always clear cut (especially given the trend towards securitisation of loan portfolios!). However, market-making activities, including derivatives activities, will almost always fall within the trading book, and thus within the CAD rules.

D.1.4 Banks are permitted to calculate their capital requirements under CAD in three ways. Two are specified in some detail in the CAD. These are the 'simplified' approach and the 'standard' approach. The former is simpler to calculate, but would normally lead to higher capital requirements. Its main purpose is to avoid smaller banks incurring excessive systems costs in implementing the CAD.

D.1.5 The third approach involves the banks using their own risk management models, as long as these have been approved by the Bank of England. The structure of these models is not laid down in S&S/1995/2, although the requirements they need to satisfy are (see Section D.4).

D.1.6 Exposures above a certain size in relation to the bank's 'own funds' (i.e. after netting off assets and liabilities) receive special treatment. For example, any exposure above 25% of own funds must be reported and incurs extra capital requirements (if permitted at all). There are limits on the total that all exposures above 10% can add up to.

D.1.7 Over time, banks' capital reserving requirements are likely to change. The principal forum in which international harmonisation of reserving requirements is discussed is known as the Basle Committee.

D.2 *Use of Market Values*

D.2.1 Banks are required to be able to produce valuations of their trading book on a daily basis. Thus, their trading book positions need to be marked to market daily. Changes in the values of banking book positions are usually amortised over much longer periods.

D.2.2 Close-out valuations rather than mid-market values are used, i.e. a long position is valued at its current bid price and a short position at its current offer price. If a bank only has access to indicative prices, then the CAD requires that these are adjusted, if necessary, to achieve a prudent valuation. The same applies if the bank is the only market-maker in the instrument being valued.

D.3 *The 'Standard' Approach*

D.3.1 The 'standard' approach operates roughly as follows:

(a) Positions in identical securities with identical counterparties are calculated (netted, if suitable netting arrangements are in place). Positions in derivatives can be netted off against positions in the underlying securities. Interest rate positions are grouped together in pre-specified maturity bands.
(b) These positions are converted into the bank's own reporting currency, although positions in different countries and currencies and with different economic exposures must still be reported separately.
(c) Futures, forwards, etc. are treated as suitable combinations of long and short positions, in line with their fundamental economic effect.
(d) Options, warrants and covered options are converted to equivalent positions in the underlying, based on relatively simple 'carve-out' calculations based on the 'money-ness' of the option and whether the option is naked (i.e. not covered by suitable holdings in the underlying).
(e) Swaps are decomposed into (c) and (d), depending on their nature.
(f) The position risk, i.e. the risk inherent in the above exposures, is split into two components, namely:
    — specific risk (i.e. risks affecting the issuer of the security or derivative); and
    — general risk (i.e. risks arising from changes in general market levels unrelated to any specific attribute of the individual security).

D.3.2 The capital required to back specific risk is:

0% × net positions in qualifying central government items
+ 0.25 to 1.6% × net positions in other qualifying items, depending on maturity
+ 8% × net position in other items.

Essentially, qualifying items are instruments issued either by appropriate investment firms (including banks) or liquid instruments listed on a suitable exchange not subject to undue solvency risk.

D.3.3 The capital required to back general risk is calculated as:

(a) *Interest rate risk.* All positions are weighted according to the maturity band (and coupon band) in which the instrument falls. Adjustments are made to allow partial netting off of positions within the same maturity band and in nearby maturity bands. Floating rate instruments have a maturity which is defined by reference to the time to the next interest rate reset. Alternatively, a duration based approach can be applied, provided it is done so consistently.

(b) *Equities.* The bank calculates its gross position in each individual equity (netting the same counterparty only if suitable netting arrangements are in place). The capital for specific risks is 4% × the gross position (but this can be reduced to 2% if the equity is highly liquid, the counterparty is approved, as in ¶D.4.2, and the position does not account for too large a percentage of the overall trading book).

(c) *FX contracts.* these are generally calculated using a back-test approach relating to how far different currencies might move relative to each other. The approach is similar to the 'own model' approach described below.

D.3.4 Various rules are laid down for other trading book risks, e.g. underwriting, settlement/delivery risk, securities lending and repo activities.

D.4 *Own Models*

D.4.1 Perhaps the most interesting feature of the CAD is that it permits banks to determine their level of risk capital for certain parts of their business on the basis of their own internal risk management models, provided these models have been agreed by the regulator. However, the Bank of England still lays down rules on how the output of such models is converted into the amount of risk capital that the bank needs.

D.4.2 This is very close to the concept of the Appointed Actuary of a life office, who is normally an employee of company, but who is ultimately responsible for setting reserves for insurance liabilities, and for distributing bonuses equitably between different policyholders. The main difference is that there is no equivalent level of professional responsibility imposed on the relevant banking personnel akin to that required of Appointed Actuaries.

D.4.3 These models can be of two main types:

(a) *Pricing models.* These produce hedging parameters, as per ¶¶A.3.15 to A.3.19, along the lines of the BS or the cost of capital model, which can then be used to identify appropriate levels of reserves.

(b) *Risk aggregation models*; including value at risk (VAR) models.

D.4.4 Banks can also use scenario testing approaches. Indeed, they are expected to when carrying out *stress tests*, i.e. simulations (or scenario tests) of what might happen in extreme circumstances. For large and complicated option books, it may be impractical to use any other technique, if the model is to be approved by the Bank of England. The Bank of England generally prefer this sort of analysis to the use of 'greeks' (sensitivities), due to the non-linear nature of the risks involved with a portfolio containing options.

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D.5 *The Model Review Process*

D.5.1 Whatever the model, before it can be used for regulatory capital purposes it must be agreed with the Bank of England. This will include discussions between the firm and the Bank of England on:

(a) the mathematics of the model (and its underlying assumptions);
(b) what systems and controls are in place;
(c) the internal risk management and reporting procedures, including position limits;
(d) staffing issues;
(e) reconciliation and valuation procedures; and
(f) the capital requirements arising from the model.

D.5.2 The discussion of the mathematics of the model and its underlying assumptions is likely to form a relatively small part of the overall review process. The model needs to be fully specified and to be appropriate for the derivative in question. The firm must have sufficient expertise to understand the technical aspects of the model and its weaknesses/limitations. Option models that generate delta values will only be accepted if they "also address the full range of market risks posed by the use of options, including gamma and sensitivity to implied volatility, time decay and interest rates".

D.5.3 The model must also form part of the day-to-day risk management mechanisms used by the firm. The firm must "have the ability to control and monitor its positions, through a timely risk management system and access to a liquid market in hedging instruments." If liquid hedging instruments are not available, then the risk management techniques being used (and the calculated capital requirements) need to reflect this.

D.5.4 Systems and controls are deemed by the Bank of England to be at least as important as the model itself. This reflects the observation by the Bank of England that the rules on capital requirements do not attempt to cover all the market shocks or extraordinary situations the firm might face, yet these are just the times when the firm is most at risk.

D.5.5 Controls are expected in areas such as strategy, staff, risk limits, procedures for dealing with excesses to limits, systems, settlements, revaluation procedures, dealing manuals, disaster recovery and internal audit checks.

ABSTRACT OF THE DISCUSSION

HELD BY THE INSTITUTE OF ACTUARIES

Mr M. H. D. Kemp, F.I.A. (introducing the paper): The basic aim of the paper is to illustrate some similarities between derivative concepts and actuarial concepts. I have concentrated on equity derivatives and on the sorts of uses that would be made by pension funds and insurance companies.

Section 9 covers a particular part of the mathematics underlying derivative pricing, namely 'dynamic hedging'. The concept has become widely accepted within the banking world whilst actuaries have generally been sceptical. Whatever the correctness of this approach, it does mean that there are a lot of good mathematicians with a great deal of financial acumen in the banking world who are not actuaries. It is a shame that actuaries have not been able to make more use of their expertise in this area to the same extent. There are, of course, some problems with dynamic hedging, and perhaps actuaries have been wise to exercise some caution. I have highlighted some of the problems in the paper, in particular the issue of jump risk, that is the risk that markets do not behave in a steady, smooth fashion.

There are lessons to be learned from the collapse of Barings Bank and other so-called derivatives disasters. The control issues are the same for derivatives as for any sort of investment. They include: proper segregation of duties; proper expertise; proper management reporting; and proper organisation of the business.

Section 13.2 summarises reserving for derivatives. Based on the analyses in Section 9, the most important risk that a derivatives house runs, particularly if it is using pure dynamic hedging, is usually that of jump risk. I suspect that the magnitude of the risk is not always fully appreciated within the derivatives community.

Finally, I would note that the largest single writers of equity derivatives within the United Kingdom are the life insurance companies via their with-profits books. It is, therefore, a shame that there is not the same level of expertise and enthusiasm to get to grips with the intricacies of derivatives within the life insurance industry as there is in banking. I hope that this paper will redress some of this imbalance.

Mr M. J. W. Barge, F.I.A (opening the discussion): I first came across derivatives, and the fact that they could be used by actuaries to assist in their work, about six years ago. At that time it was virtually impossible to obtain equity derivatives for durations in excess of three years at prices which could be considered even remotely practical. Since then the market for equity derivatives in the U.K. has seen rapid expansion, and it is encouraging that this expansion has been due, to a large extent, to the activities of actuaries. Actuaries' involvement with derivatives has mushroomed to the extent that derivatives now form an integral part of virtually all investment product development, and no insurance managed fund or pension fund can ignore them, if only for the purpose of tactical asset allocation. The implementation of the Insurance Third Life Directive, and the consequential amendment and upgrading of U.K. legislation, perhaps removes the final obstacle to the practical involvement of life office actuaries with derivatives. Therefore, in adding to the sparse supply of papers dealing specifically with this subject, this paper is not only welcome, but overdue.

The author sets out to cover the now very wide spectrum of actuarial techniques and their analogies with techniques used in the field of derivatives. It should not surprise any reader, therefore, if their favourite topic — be it experience rating, reserving, formula derivation or dynamic hedging — is not given the prominence they would like. The paper, nevertheless, includes something for everyone, and provides further evidence (if any were needed) of the need, at this time, for actuaries to embrace new risk management techniques borne of the technical advances made in the derivatives market. Derivatives are, after all, instruments of risk or risk control involving the management of uncertain outcomes — the actuary's speciality.

In the opening sections of the paper the author sets out a comprehensive description of the various

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forms of derivatives contracts available from both the exchange traded markets and the over-the-counter (OTC) markets. At first glance it appears that derivatives should provide the perfect tool to hedge many of our investment risks. The traditional with-profits contract, when described in terms of derivatives, actually means that insurance companies are the largest writers of equity derivatives in the U.K. Yet derivatives markets are not generally used as a principal tool in the management of such insurance funds. Possible reasons why this might be so are covered in the paper. First among them are what I would call technical reasons. In a perfect world these would not be a problem, but in the real world they introduce high levels of risk. Insurance liabilities are more complex than conventional derivatives. They contain options on the part of both policyholder and insurance company which are impossible to hedge precisely. Added to this are counterparty risks and dealing costs, and if the office decides to delta hedge, then it must take jump risk into consideration. One might thus be tempted to conclude that derivatives are fine in theory, but in practice are not suited for with-profits funds.

The second set of reasons which may explain why derivatives are not widely used to manage with-profits funds is to do with performance. Any attempt to match with-profits liabilities using OTC derivatives is likely to result over the long term in a fund with inferior performance by comparison to a conventionally managed with-profits fund. Given the strong tendency of with-profits funds to invest heavily in equities, any move into 'less risky' put options would have the unwelcome effect of reducing both risk and return. Even if the effect of this were mitigated using geared options, the pattern of performance of such a fund would be likely to be out of line with the rest of the market. Although it is beneficial not to have to reduce bonus rates when everyone else is forced to, this is typically not worth the cost of being the only company unable to raise bonus rates when everyone else can.

However, derivatives are now widely used in other areas of new product design — for example lump sum investment contracts. The subject of complex product development raises the issue of scenario testing and pricing guarantees, including pseudo-guarantees. Option prices or, if you like, values, equate to their discounted value using risk-neutral probabilities and risk-free discount rates. Option prices can also be shown to be equal to the discounted expected payouts using a risk discount rate and real world expectations. It is for the individual actuary to decide which approach to use. The natural actuarial reaction, especially when so many factors are uncertain, is to use real world projections and risk discount rates. Unfortunately there may be some problems with this traditional approach. The first is that autoregressive models, such as those generally used by actuaries, will not produce answers consistent with the market unless the elements of mean reversion are removed. The second problem is that, without knowing how risky the instruments are in terms of volatility, there is no easy way to determine the correct risk discount rate. The only way to do this is to examine the price of the equivalent derivative instrument, as calculated using risk-neutral probabilities, and then to solve for the discount rate by discounting real world projections.

Calculating the value of a guarantee or investment derivative is only half the battle. Determining the underlying price is only useful if you intend to buy the instrument from a supplier (such as a bank) or to delta hedge the risks within your own company. In the case of non-linked insurance funds, most companies do not delta hedge their guarantees nor do they purchase options. This being the case, risk-neutral calculations are of little benefit, and such companies need to consider the use of real world projections to help manage and control their liabilities. Those companies who do wish to use delta hedging to match their investment guarantees must first examine the regulatory framework to determine whether or not such activities are allowed, and what effect they will have on reserves. In the case of index-linked funds, delta hedging is usually not an option, owing to the rules on 'close matching'. Whereas it is possible for banks and securities houses to go to their regulators and seek approval of mathematical models to be used in connection with hedging derivatives, such models are not recognised by the DTI for use by insurance companies. It appears, therefore, that for index-linked funds, insurance companies are technically forced to use the banks and securities houses to match their liabilities. The regulations which have emerged preclude insurance companies from 'hedging' their own index-linked risks with anything other than an exactly matching OTC option from a bank. This situation probably arises from concern for the customer. Nevertheless, those risks are passed on to banks who are doing exactly the thing that the insurance companies are prohibited from doing — that is, creating mathematical models and then delta hedging the risks.

Fortunately, perhaps, for insurance companies, there are instances where the tables are turned, and it is advantageous to the insurer to sell an investment guarantee to a bank and delta hedge the risk. These primarily exist in non-linked insurance funds.

The final two sections of the paper deal with control and credit risk issues. It is tempting to suggest that, given the problems investment banks have had in this area, insurance companies would be wiser to ignore financial developments and stick to traditional investment methods. Only five years ago there were six or seven banks world-wide with an AAA credit rating. There are now only two left, and one of them is on credit watch. The overwhelming cause seems to be the increased level of volatility arising from investment banking activities. Perhaps the best strategy for insurance companies is to pass their index-linked and volatile investment risks on to the banks (ignoring the lure of higher profits and self-sufficiency), and to continue to manage their less volatile positions with traditional techniques, rather than using derivatives or delta hedging.

Even as a practitioner in this field I find the mathematics difficult to follow. However, it is the practice that I am most interested in, and I was very pleased to read a paper which covered such a broad spectrum of actuarial techniques and their analogies within the derivatives markets. I hope that this paper, and papers like it, will not only act as a signal to actuaries that they can make a significant contribution to this field, but will also serve to remind the current investment practitioners of the added value that actuaries can bring to this topic.

Mr S. J. Green, F.I.A.: The author recognises that, in the real world, such things as transaction costs, counterparty risk, roll-over risk and taxation do exist. He has shown that, if properly used, derivatives are powerful tools which can help to minimise risk, or improve return, in many of the fields in which members of our profession operate. I hope that this paper becomes required reading for all aspiring members of the profession — not just those who are entering the investment field.

The author describes derivatives as 'risk management tools'. He must be aware that they are also widely used as gambling tokens. He refers to the losses incurred on derivatives trading at Barings Bank, but implies that derivatives were not involved in the losses incurred at Daiwa. My understanding was that derivatives were also involved there, as they were at Sumitomo, Credit Suisse, Lloyds Bank-Lugano and Rowntrees, to name just a few others, so they can be extremely dangerous instruments.

It is not only rogue traders who gamble with derivatives. A few years ago a major fund management house was known to be frequently more than 100% invested for its pension fund clients, and another was reputed to have an exposure of over 140% of market value. Where one can gain ten times, or more, exposure through buying derivatives than through investing the same amount in the underlying instruments, the temptation for fund managers to improve performance by gearing up will always exist.

The author goes into some detail on control and regulation. Although I agree that, in both of these areas, there is no substitute for common sense, he has not stressed sufficiently some of the elementary in-house controls which can be imposed. For example, as with the old jobbing system, trading limits can be imposed on individual traders, with their 'trading books' examined automatically once or twice a day. All open positions — both gross and net, and both individually and in total — should always be examined, daily, against previously set control limits. Insufficient distinction is often made between covered and uncovered trades, between the writers and the takers of options and between options and futures. Every control report, at each level, should contain a 'what if?', or calamity, scenario.

Most of what the author has written would apply even if the Black-Scholes formula did not exist. Since he has devoted a number of pages to it, I should like to explain why many actuaries, such as myself, have doubts about its applicability in the real world. As an investment manager, I was actively trading in derivatives for at least ten years before Black & Scholes (1973) was published. Because the discount models then in use for valuing derivatives were rather primitive, I was excited when I first read about their formulae; but as I went into it more thoroughly, I was put off by the assumptions which were an essential platform for their calculations:

(1) *Transaction costs.* Except, possibly, for market-makers, transaction costs for users of derivatives are not insignificant, and the idea that trading is frictionless is ludicrous.
(2) *Brownian stochastic process.* Practitioners have always known that prices did not follow a Brownian stochastic process. By the time that Black-Scholes was published, Granger had shown that, in the U.K. equity market, there exists a small, but positive, correlation between any two successive price movements of an individual share. The author believes that this assumption is not a necessary condition for Black-Scholes, but, hitherto, it has been held to be a necessary condition.
(3) *Markets are arbitrage free.* They are not. It is true that most arbitrage positions are traded away fairly rapidly, but as an investment manager, who has, actively and profitably, arbitraged over a number of years in U.K. and overseas equity and derivatives markets and in gilts, this assumption is untenable. Indeed, some arbitrage positions, which are due to different tax regimes, can exist for quite a long time, and are only inhibited by the liquidity of the particular market. This is because the internal revenue authorities become a third player in what the academics always assume is a two-player game.
(4) *Taxation,* too, exists, even if it frequently changes — occasionally with retrospective effect. In all markets it is far too significant to be ignored.
(5) *Continuous rebalancing.* The author has dealt with the problems that continuous rebalancing would throw up, even if it were possible.

I have not mentioned the problem of dividends on the underlying, as apparently Black-Scholes can be modified to make adequate allowance for these.

When I read their paper, I found that the assumptions which had been made by Black and Scholes were so unrealistic that I went back to the beginning and looked for the words 'Once upon a time...'. Frankly, I would be surprised if any experienced investment professionals of that era had wasted their time studying the arcane mathematics by which the formula was derived. At the time, we were rather more aware of the old computer saying, 'Rubbish in; rubbish out', and were too preoccupied in making money for our clients out of the very inefficiencies which Black-Scholes ignored.

The author observes that Black-Scholes is more acceptable to those in banking than those in long-term investment. If we go back to our school days, there is a parallel with elementary Euclidian geometry. If the curve of a circle is short in relation to its diameter, the tangents at any point on the curve all have very similar gradients and each makes a good first proxy for the curve itself. It is only when the curve is longer in relation to the diameter that approximation fails; thus with Black-Scholes. In the short term the unrealistic assumptions do not matter too greatly — except when the historic volatility of the market is tested — and this itself is less likely to happen the shorter the period. In the longer term its inadequacies are shown up.

At the first AFIR colloquium, Nisbet showed that there were sufficient inefficiencies in the London Traded Options Market to question both the efficient market hypothesis and the general equilibrium assumptions which are the basis of the Black-Scholes option pricing model. At the same colloquium, Walter showed that price movements in the MATIF were chaotic. For the fifth AFIR colloquium, Walter produced a paper containing even more convincing evidence that financial markets are chaotic in nature rather than following random Brownian motion, as assumed in the Black-Scholes model. Also, at the first AFIR colloquium, Berg, who, for two years, while he was Professor of Mathematics at Toronto University, worked three days a week as a derivatives trader in Chicago, and thus was (and probably still is) the only academic with any real-life experience of derivative trading as opposed to derivatives, demonstrated a system for making money by exploiting the inefficiencies in Black-Scholes.

Since then there have been numerous studies demonstrating that Black-Scholes does not provide a sufficiently accurate picture of the real world. The French atomic physicists, Bouchaud and Sornette, have revealed that, in a study over a number of years, Black-Scholes consistently undervalued real option prices on the MATIF. They also demonstrated that, in the long run, any bank basing its book on Black-Scholes faced ruin, since Black-Scholes did not, as claimed by others, diversify away 'risk'. The physicists and economists at the Santa Fe Institute — several of them Nobel Prize winners — have proved that price fluctuations in financial markets are not random, but pseudo-random, as in deterministic chaos, and therefore that the mathematics of Black-Scholes is inadequate.

Despite all the evidence, the author finds it necessary to devote inordinate space to a model which is based upon some of the totally discredited theories of MPT.

Very little, if any, of the author's ideas and recommendations are dependent on the pricing formulae used. From studying some of his tables and his comments, it seems that he has his own reservations about the use of Black-Scholes in some circumstances. Has the time not come for those actuaries who aspire to be academics to discard Black-Scholes and search for a better model based on proven actuarial techniques?

Mr S. P. Deighton, F.I.A.: I shall speak on the issues of counterparty risk, efficient portfolio management (EPM) and reduction in risk (RIR), and the interaction between them.

As the author points out, EPM and RIR are not defined in legislation. This results in a distinctly 'grey' area in the regulatory framework, where DTI guidance effectively establishes what can and cannot be done. The DTI has worked extensively with the industry practitioners in arriving at workable interpretations, but still has the final say in drafting its own guidance notes!

The interpretation chosen has an impact on the retail products which can be designed. To some extent, therefore, we have moved away from the traditional 'freedom with disclosure' approach to regulation. The DTI regards the EPM/RIR tests as independent tests which a derivative must pass to be admissible, or to be a permitted link. Whether or not the test is passed is not dependent on whether the policyholder understands the features of the product he or she is buying. We cannot allow the policyholder to decide on his or her own preferred risk/reward profile, at least not if we want to back the product with a derivative.

The EPM and RIR tests, although separate in the legislation and separately defined by the guidance, are often difficult to separate in practice, and the DTI recognises this. It is therefore just as well that the asset only has to pass one of them, not both.

The essence of the tests is that an insurer wishing to use a derivative must ensure that such use benefits either itself or the policyholder under certain circumstances. If it is using the derivative to match liabilities under a guaranteed equity product, it is fairly easy to argue this part for the company. Second, any adverse consequences of using the derivative must be 'unforeseeable' or 'insignificant'. Deciding on whether either of these is the case is entirely a matter of judgement.

The DTI is prepared to accept that derivative-based contracts often provide significant guarantees to the policyholder, and that there is a cost associated with that. The significance of an adverse consequence can be judged, making allowance for the cost of such guarantees.

In the High Income Bond example given by the author, the insurer must carry out a statistical analysis of the product to determine when it would underperform against a similar non-derivative investment. The DTI will then accept significant underperformance, which can be demonstrated to be due to the guarantee, and only occurs in scenarios where returns are exceptionally high.

I believe that we will see increasing demands on the actuarial profession to carry out this type of analysis. It is essential that we watch for any inconsistencies between the tools we use and those used by the banks to price the assets, to prevent any systematic bias creeping into the results.

Quite rightly, the DTI includes counterparty risk as part of investment risk when applying these tests. There has been an increasing tendency over the last couple of years to write guaranteed products as property-linked contracts. This passes any counterparty risk to the policyholder in addition to the investment risk, and thereby reduces the solvency margin requirement. As the author points out, both types of risk occur in the usual investments of an internal linked fund. However, the use of derivatives concentrates the risk, and it is the duty of the profession to ensure that the sales approach and marketing literature are designed to ensure that policyholders understand the risk that they are accepting.

In some cases the derivative assets were not margined, so the policyholder was subject to significant risk. The DTI now conclude that this is unlikely to be consistent with EPM/RIR. Any additional counterparty risk, which can be identified as being due to the use of derivatives, must now be mitigated in some way, for example by margining. The company must also be careful to avoid the re-introduction of counterparty risk in the way it invests the margin. Again we see some 'unfairness' here in the regulatory treatment of derivatives, since unlimited counterparty risk is acceptable in, for example, a deposit-based internal linked fund. However, this unfairness arises from the drafting of the legislation rather than the DTI's interpretation of it.

A major attraction of guaranteed equity products is the relative simplicity of the marketing message. The remarkable increase in sales volumes confirms that the consumer does understand them. More complicated offerings have not had the same success, confirming that the consumer will let us know when he or she does not understand.

Currently the regulations make design of the product more complex from the insurer's perspective than for traditional alternatives, although we have progressed a long way from the position before 1994. It is hoped, as all parties become more familiar and more comfortable with the use of derivatives to provide innovative retail products, that we may see further changes in legislation to support this.

Mr A. D. Smith: I have a few things to say about short-term derivative pricing models and long-term actuarial models, because I believe that a greater integration between the two would be of considerable benefit to those offering and receiving long-term guarantees.

In one sense this paper is not actuarial at all, because of its focus on what are essentially short-term products. For short-term horizons, up to perhaps a year or two, transaction costs are relatively modest and the randomness in the underlying asset price swamps the uncertainty in the quadratic variation, so the Black-Scholes approach is highly illuminating; but over long actuarial time scales, as Mr Green has pointed out, the situation could well be reversed, in which case a new approach is called for. The relative illiquidity of longer-dated structures in the OTC market is circumstantial evidence that banks are rather less confident of their hedging ability over the longer term.

In contrast, actuaries have been in the business of pricing and reserving for long-term guarantees for many years. In his inaugural address, the President referred to "the long-term view that only actuaries can provide". Starting with the Maturity Guarantees Working Party, actuaries have largely rejected arbitrage arguments when assessing such guarantees. Instead, actuaries rely on their understanding of the long term. This often includes an implicit assumption that, whatever the short-term volatility of market prices, the long run growth in asset values is determined by income streams, and is therefore essentially predictable. There is a tension between short-term models, which suggest that performance guarantees should be very expensive, and long-term models, which generally seem to produce rather lower guarantee costs. Perhaps this is why banks, who measure cost on a short-term basis, have difficulty selling options and guarantees to pension funds, who use longer-term methodologies. The latter techniques have the apparent advantage of avoiding any mathematics beyond compound interest. I suggest that this advantage is largely illusory — a very modest change in long-term growth assumptions can have a huge effect on NPV. The accuracy of long-term actuarial forecasts is far from proven, and I strongly favour alternative models which do not rely on them.

If, on the other hand, we really believe that market volatility is essentially a short-term effect, then there are a number of striking consequences. Perhaps the most significant is that, by comparing market prices to a suitable long-term assessed value, profitable trading rules ought to be easily formulated. For the Wilkie model, this adds to the order of 4% p.a. compound; however, if short or geared positions can be achieved by way of derivatives, this margin increases dramatically to around 40% p.a. compound. Similar results arise from other time-series-based asset models. There is a dangerous potential fallacy here — we might be tempted to suggest that an actively managed portfolio is 'worth' more than its market value because of our privately held view that the assets will go up. Users of such models have learnt to apply caution, and Professor Wilkie himself has advised, very sensibly, that "it would be unsafe to rely on the potential profits" from dynamic trading, in the written remarks to his 1995 paper (*B.A.J.* **1**, 777-964). In order to heed this advice, we need a clear method of determining when a particular result does rely on such profits. A corollary of the author's arguments in Section 8.9 is that this may happen, innocently rather than by design, far more often than was previously appreciated.

The author has injected some badly needed clear thinking to this debate. There are two ways of looking at value — either by discounting expected cash flows (DCF), or by arbitrage arguments relative to market values. If markets are not efficient, these two concepts will give different numbers — even more so for derivatives because of the gearing effect. He points out, in ¶8.9.4, that, even if we accept the long-term validity of actuarial forecasts, arbitrage considerations would lead to option prices based effectively on an extrapolation and compounding of short-term volatility. In other words, long and short-term models do not disagree on the arbitrage costs of hedging a guarantee — this says no more than that hedging works — but they do differ dramatically on the likely outcomes of an unhedged position.

There is a huge danger of setting up guarantee reserves with reference to what the author calls "an internally estimated likelihood of a claim being a given size", although that is how actuaries have done things for the past 15 years. The problem is that, for many long-term guarantees, the DCF cost under the Wilkie model is much lower than the arbitrage cost. The use of DCF to value the guarantee is tantamount to capitalising future speculative gains, because the net effect of an unhedged short option position fortuitously stumbles upon the profitable trading strategies, as described in ¶8.9.8. This is precisely the kind of apparently profitable speculative trading for which we should not be taking advance credit. In a world of inherently dynamic liabilities, it is dangerous to suggest that a restriction merely to static asset strategies is an appropriate work-around for the shortcomings of our long-term economic framework as a whole. There are also liability structures whose evaluation requires far greater care than is common practice — ironically precisely those structures for which stochastic modelling was developed in the first place.

It is one thing to give dire apocalyptic warnings about the dangers of using a particular class of model or valuation procedure; it is quite another to demonstrate real examples of damage having been done. However, in this case current practice is sufficiently sloppy to provide several instances. I have recently assisted a client in assessing the cost of guarantees implicit in with-profits business. My first reference was a paper presented to this Institute last year on the subject of 'Asset Shares and their Use in the Financial Management of a With-Profits Fund' (*B.A.J.* **1**, 603-670). Current standard practice is described in Section 5 of that paper, in which the capital required to back guarantees is assessed by reference to adverse scenarios according to the Wilkie model. However, I found that the results were curiously hard to reproduce using more plausible economic models, which typically gave capital requirements about three times higher. The author has provided a comprehensive explanation: that is, there is a danger of being unwittingly duped into capitalising future speculative gains on the part of maturing business, at the expense of new business. At a time when many with-profits funds are contracting, it is not clear to me that such expropriation of wealth is sustainable.

It seems to me that actuarial claims to have tamed the long term are founded largely on wishful thinking. We have been lucky that, largely due to the immense technicalities involved, the long term has not been colonised by other professionals, and, in the U.K. favourable equity performance has spared us some potential scandals. However, I cannot accede to the conventional wisdom that sophisticated economics becomes redundant for sufficiently large t. The author has outlined some of the technical background we need to absorb in making a fresh start. Black-Scholes is a neat introduction, and the mathematics will get worse before it gets better. It is time now to take a firm grasp of this nettle.

Mr T. W. Hewitson, F.F.A.: This paper provides some interesting insights into the similarities between the methodology applied by securities traders and regulators and the application of actuarial judgement to the management and control of insurance portfolios.

The paper may understate the importance of the role of the actuary in describing the so-called Method B in Section 10. For example, the solution to the conundrums posed in ¶¶10.6 and 10.8 lies in part with the application of actuarial reserving methods, including the resilience test that is customarily now applied by life office actuaries. This requires the application of professional actuarial judgement and not simply a mechanical formula, as is largely recognised in our professional guidance notes. Indeed, an actuary seeing a portfolio of assets and liabilities with significant option-like features would, by tradition, take these into account when determining the appropriate reserving basis for an insurer. This is equally true of financial condition reports which utilise scenario or stress tests.

Looking a little more broadly at the paper, it is interesting to consider that the value at risk (VAR) approach, described in ¶10.7, could be applied to many non-life insurance companies to test their financial health. It would then be possible to see whether a company has sufficient capital to cover the adverse experience that could develop over the following year, and still have sufficient assets left to cover the cost of a run-off or portfolio transfer at the end of the year. This depends, of course, on the existence of a suitably liquid market for such a portfolio of insurance liabilities, and it will be interesting to see whether this can be developed.

The VAR concept would, admittedly, be more difficult to apply to a life insurer where there is no ready market at present in portfolios of contracts, and there is the additional complication of defining the interests of with-profits and other policyholders in the ongoing company. The insurance regulators would not be comfortable with the prospect of insurers undertaking dynamic hedging until at least these two issues could be satisfactorily resolved.

Turning to Section 9, I must admit to being puzzled by the apparent suggestion that past or implied volatilities are a good guide to the actual out-turn, even over fairly short future periods. Indeed, the regulatory response from banking and securities regulators, of applying some suitable multiple to any capital requirement derived from such a model, seems much more attuned to our traditional actuarial caution over placing undue reliance on historical data.

I was also interested in the comments, in Section 9.4, about catastrophe risks, which are now often included within insurance derivative contracts. Diversification and limitation of exposure to such risks is crucial, since otherwise, if the catastrophe should occur, then substantial amounts of capital might be needed to maintain solvency. However, full diversification is only feasible if the risks are genuinely uncorrelated, and I have doubts about the implications, in ¶9.4.4, that the U.K. equity market is largely independent of other financial markets.

It is also relevant to remember that there is significant correlation between the assets and liabilities of most U.K. life insurers. Nevertheless, I would hope that there is not an implicit assumption that regulators would have to alter the rules to allow every company to survive in the event of a major fall in the value of assets across world financial markets. The traditional actuarial anecdote of not being at the front of the queue to visit the regulator still seems very apt.

Dr M. W. Baxter (a visitor): I should like to confess that, as a mathematician and academic, I am rather like Daniel in the lion's den here. I am possibly many people's embodiment of all that it standing between them and understanding what derivatives are and how they are priced. It is just I and Itô's formula which are causing all the problems. Nevertheless I shall try to lay to rest a couple of popular misconceptions about derivatives pricing.

Hedging is the key to derivatives. A derivative is a collection of risks all bundled up to give the particular risk profile that the customer desires. The bank's aim is not to be on the other side of those risks, but to break those risks down into risks which are already traded on the market and hedge them away. The bank does not want anything to do with the punter's views on the yen or on the U.K. stock market or on anything else at all. So far as the banks are concerned, hedging is the key thing, despite the limitations which have been expressed by other speakers.

The only way a mathematician, an actuary or an academic can justify a price is by saying that it is a hedging price. Any other price is a guess, and could cause trouble. The theory uses the concept of the martingale measure. This is a set of likelihoods for the possible outcomes which is different from the real world probabilities. Using the martingale measure and taking expectations will give the right number. This works, not because of the expectation, but it works because you can hedge with that number. One misconception is to think of the martingale measure as a risk-neutral measure. Thinking about risk-neutrality can often be more hurtful than it is helpful.

It is hard to state the risk-neutral paradigm accurately, but, roughly, it states that we are trying to calculate prices which, because they are hedging prices, are not investor preference dependent — that is that they do not depend on your ideas about risk utility. Therefore, you could have any risk preference at all, so it might as well be that you are risk indifferent completely.

This is also tied up with the whole concept of market price of risk, which has crept into the literature from the economics side and is a nice interpretation. If, for some derivatives models, you have a riskless bond and a risky asset, the excess return of the risky asset over the riskless asset, normalised out by, say, the volatility of the risky asset, seems to be constant. It is then said that this means that the market has a collective view of the market price of risk.

This is really a rather dangerous idea. The only reason that this number comes out to be the same is because you can hedge things and there is no arbitrage. The market does not have a market price of risk at all. All the market has is an aversion to arbitrage.

One other misconception is how this relates to the risk that actuaries deal with in life offices or general insurance companies. For the securities industry the key concept is the hedge, which, in theory, works with probability one. The theory makes you do the right thing so that your position will be safe.

The actuary's world is slightly different. The contracts at the basic level are annuities and life term assurance contracts. There you cannot hedge. Re-insuring completely takes away the reward — I can sell the risk on, but I cannot hedge it away. Life and pensions tends to work much more by the strong law approach. If you have a whole lot of risks which you hope to have assessed properly, and a large number of them are broadly similar, but independent, then, if you average them out, you should get the average. The mean is close to the experience you actually see. So, if you charge the mean plus a margin for profit and cost, you will be all right.

The derivatives trader is using a completely different idea, hedging the risk away. That is, on risk management the banks and the actuaries are doing different tasks.

Risk management has previously been what has driven most literature in this area. Much of the literature concerns risk management with pricing and hedging, but the banks themselves, the senior managers and the regulators, are becoming increasingly concerned about what could happen, even if you are running your derivatives book as well as you could be. Their attention has broadened from risk management to risk control, where you start thinking about the whole thing as one. How bad can it be? How good can it be? What things can go wrong, even when everyone is meant to be managing their risk as well as they can? There is no short-term profit in this, but there are large losses to be had if you get it wrong.

Risk control was more a subsidiary operation that was left to catch up. Actuaries come to the industry with a reputation for good risk control. The banks are not much better at this than you, and certainly used to be worse. There is an obvious opportunity for the actuarial profession to capitalise on its reputation for prudence and its mathematically skilled membership to play an important part in building and staffing these sophisticated risk-control structures.

Mr D. J. Parsons, F.I.A.: Paragraph 4.4.3 is an important building block for subsequent theories about whether to allow for arbitrage. It refers to a large group of 'irrational investors'; presumably classified as irrational because their approach defies our logic. I am not brave enough to tell anyone that their personal and possibly well thought out investment strategy is irrational. The more I thought about it, the more it seemed to me that everyone must be an irrational investor in the eyes of someone else, otherwise the market system would not work, as no-one would buy what I wanted to sell. Even in ¶¶6.5.3 and 6.5.4 there is a reference to two strongly different views of the utility of Black-Scholes as held by actuaries and bankers. Presumably each thinks the other is irrational.

From my perspective of the public interest, I wondered how we could put derivatives to good use. As a consumer, I would like a fully equity-linked pension policy; I would want to know on day one exactly what pension (in real terms) I am going to get when I retire. A small logical extension of the theories and practices set out in this paper can surely provide this. Given a steady flow of business, it is feasible and very profitable without derivatives — surely it must be feasible with less risk (without the need for a steady flow of business) if derivatives are used. All that is needed is a continuous derivative arrangement, designed to roll over for up to 80 years, which smooths total returns on ordinary shares so that they equate to the long-term average, i.e. an equity-based instrument which, after the cost of the derivatives, guarantees a real rate of return of 4% to 5% over inflation. Just think how this would revolutionise the pensions industry, both occupational and personal, let alone with-profits funds.

Mr J. M. Pemberton, F.I.A.: In Section 13 the author attempts to tie up what he calls derivative concepts with actuarial concepts. Although he has suggested some correspondence of the words, he has not materially managed to tie up their meanings.

The rest of the paper, aside from the factual sections, is mostly concerned with the application of mathematical economics as an approach to the valuation of options. It does not provide any method for the valuation of real options. Several valuation formulae are proposed, but there is no means of determining which formula we should choose, given a real option. Paragraph 4.1.6 of GN25 suggests that any model used must be reasonable in the context of historic experience. This paper makes no mention of how to achieve that and how historic experience can be used to inform our valuations. The mathematical approach which is proposed has, at its heart, a disregard for the facts.

Paragraph 4.1.11 of GN25 suggests that, in assessing the risks, an actuary should have regard to resilience tests of the values. There is a tension between the banking regulation on one hand and the DTI and GN25 on the other. The DTI's provision for adverse changes can, in many senses, be closely related to GN25's proposals for resilience tests. The banking regulations, on the other hand, adopt a probability weighted approach to solvency. This discrepancy is now an urgent matter for this Institute to address, and it is a shame that we have not heard more of it in this paper.

The paper perpetuates a myth that option valuation is difficult and that it requires involved mathematics. From an actuarial standpoint, the mechanics of option valuation are very straightforward. All we need is the machinery of discounted cash flow. The unnecessary mathematical formulations can be replaced with step functions. Step functions are trivial to manage mathematically, and importantly, too, they can ensure the functions chosen have respect for the historic facts of experience. The move to step functions is a lesson with which actuaries will be familiar from a history of mortality tables. The attempts by Gompertz and his contemporaries to develop mathematical laws of mortality were misguided, and have now rightly given way universally to the use of step functions.

Mathematical models, although often good approximations, sometimes lose touch with reality. They may have elegant internal characteristics, such as conforming to no arbitrage constraints in idealised markets, but the mathematical formulation distracts attention from the correct exercise of professional judgement concerning the valuation assumptions. The mathematical formulation gives rise to values which are not immediately compatible with values achieved through the discounted cash flow of other assets and liabilities within the portfolio. It creates a barrier to entry for those who would value options, and particularly those in this Institute who value options as a side product of the valuation of the other assets and liabilities quite naturally.

It is important that the Institute now moves to discourage unnecessary mathematics in this area. As an urgent first step, it is essential that we stop teaching mathematical proofs to our students: the teaching of proofs which purport to show the truth of certain models is incompatible with the tuition of professional judgement concerning the selection of models in the light of experience.

There is a broader issue underlying this point which is to do with the contrast between mathematical economic methods and actuarial methods. Looking at the wider fields, we are at a point in history where the notion that we can prove simple mathematical truths about society is falling into disrepute. An understanding of actuarial methods has become an essential basis for discussions such as this concerning the application of methods in new fields, and it is important that the Institute now takes steps to develop a basis of methodological understanding. I agree wholeheartedly with the author that the field of option valuation is an important example of the areas within which the actuarial profession has an important role to play, but the proposals within the paper are taking us in precisely the wrong direction, away from actuarial science.

Dr S. E. Satchell (a visitor): I am a financial economist from the University of Cambridge, and should like to comment on some of the ill-informed remarks that some speakers have made about empirical evidence and the validity of Black-Scholes.

If we look at option prices and compare them with the theoretical prices, we may get evidence that the theory is wrong, although this is not a trivial matter, because there are various extensions of Black-Scholes that hold under all sorts of circumstances. Considering issues like transactions costs, taxation and arbitrage may also provide evidence that the theory is wrong, or at least needs to be adapted. That is clear. What is also clear, and is not at all understood, is that looking at the behaviour of returns will not provide evidence that Black-Scholes is wrong. This is counter-intuitive, and is something to do with the equivalent Martingale measure. Many different stochastic processes generate returns that are mutually compatible with Black-Scholes. As an example, I believe that it is possible to show that the Wilkie model is compatible with Black-Scholes. Time varying volatility is not incompatible with Black-Scholes.

Remarks have been made about chaos and fuzziness. I am always suspicious when I hear these theories. I have worked in this area and I know that it is very difficult to show that the world is deterministic in a convincing way. One paper that I read said that you require 10^{46} observations before you could be sure. This means that actuaries who go in for long-term forecasts and quarterly or annual data would have to go back much earlier than the sixteenth century, as used by Professor Wilkie, to be confident about using chaotic models. Looking at intra-day data provides, perhaps, 20,000 prices a day. With 10 years' data, say, it is possible to find evidence of rather strange behaviour. That is quite good evidence, but there are several problems. One is that that is not the sort of data that actuaries are customarily concerned with, because they are more usually interested in the long term. Secondly, these things are not, typically, prices of transactions, but quotes. Quotes are often made with no compulsion to trade at any level of volume, so interpreting them as evidence of how the price process works is again problematic.

None of the studies quoted challenge Black-Scholes in any way. Black-Scholes is a remarkably robust model, which should be the centre of some unit of training for any quantitative financial professionals. Actuaries would be very foolish not to make it so in their professional training.

Professor A. D. Wilkie, F.F.A., F.I.A.: Actuaries should be involved very much in derivatives and related work. The Institute and the Faculty have introduced the Advanced Certificate in Derivatives as a post-qualification course, similar to the Fellowship examinations. It is a voluntary course, but I hope that many people will find out about it and sit the examination. The writers of derivatives are taking on risks in the same sort of way as insurance companies; hedging is like reassurance, though hedging is easier. I should like to see the concept of a derivatives actuary developing. Many people in the City and in investment banks who are trading in derivatives need a professional body like ours. It might not stop the crooks, but it would make life a little bit harder for them.

As Dr Satchell has said, the Black-Scholes formula is a first approximation to the option price. It cannot go very far away from it. Mr Pemberton said that Gompertz and Makeham produced formulae of mortality that had been discredited. All small samples of mortality data from age 50 upwards can be fitted with a Gompertz formula, and all small samples that go down to age 20 or so can be fitted with the modification of Makeham's formula that involves reducing mortality rates in the early 20s of age because of accidents. These are the automatic first approximations used in mortality graduations. They have been used for the last 25 or 30 years by the CMI Committee. Mortality tables are fitted, not by step functions, but by continuous functions.

As far as statistical distributions are concerned, while there are certainly occasions when you can use discrete step functions because you cannot find a suitable tractable formula, there are many statistical distributions other than the normal distribution that are smooth and continuous and have nice features. General insurance actuaries have strings of different loss distributions that could be used. An advantage of a mathematically tractable formula is that you can do much analytical work with it. You can also do arithmetic work with a step function, but it is much harder to get nice, analytical results from it.

Considering Mr Green's remarks, the Black-Scholes formula has to be approximately correct. That is evident from plotting the graph of option price against exercise price. It has to go approximately along the sort of curve that is represented by the figures in the paper, and the Black-Scholes formula produces the necessary first approximation for it. Of course there are problems, but you have to start from there. Mr Green produced many arguments as to why it was not exactly accurate. It is rather like saying that Newtonian mechanics is not very accurate because it does not take sufficient account of friction. He was wrong in saying that Walters' papers discuss deterministic chaos. They do not. They discuss α-stable (or Lévy-stable) distributions, which are a different type of statistical distribution, and generate a different type of diffusion-like process that has α-stable increments. Normally distributed increments are the characteristics of a Brownian diffusion process. An α-stable diffusion process is different, and has the possibility of jumps. It is strictly not a diffusion process, but it is very like one. It does have the opportunity of jumps. Mr Smith's paper (*Transactions of the 2nd AFIR International Colloquium*) includes a graph of such a process.

I disagree with the author and Mr Smith when they talk about the Maturity Guarantees Working Party model. The author, in ¶6.5.8, says "although dynamic hedging does not work perfectly when markets 'gap', it does seem to offer some protection, despite the suggestion of the Maturity Guarantees Working Party to the contrary". I agree with that statement. The other members of the

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Maturity Guarantees Working Party were short-sighted in not thinking that dynamic hedging would work a bit. However, they made the sensible point that, if all insurance companies used dynamic hedging, they would all be wanting to sell shares when prices fell and buy them when prices rose, and that would produce October 1987 features rather more often than has actually happened, so, to that extent, it was impracticable.

The fault of the Black-Scholes formulation is that it does not take account of the volume effect. It works if somebody is writing one option or a small number of options, and buys a small number of shares to hedge it exactly. If large volumes of options in one direction are bought, then the action of hedging is going to change the share price as well. The theory works on the assumption that the option activity does not interact with the share price, but it can quite well do so if volumes are large. That is an area where there should be further investigation, but I do not think that the Black-Scholes model should be thrown out just because of that.

In ¶¶8.9.6 and 8.9.7 the author is wrong in suggesting that the work of the Maturity Guarantees Working Party relied on taking advantage of anomalies in the model. It was based entirely on a buy and hold strategy, buying assets when premiums were available and holding them to maturity. The Maturity Guarantees Working Party was looking at reserving for maturity guarantees, not at pricing.

There is a conflict between using the short-term standard deviation, which comes through from the Black-Scholes type of model or the hedging argument where you use the instantaneous standard deviation, and the position where you get, not deterministic trends in income, but (approximately) a random walk for dividends, and fluctuations in dividend yield, so prices fluctuate up and down around the trend of dividends. That produces a narrower spread at maturity than a pure random walk model with the same instantaneous standard deviation.

The author pointed out that the cumulative quadratic variation — that is the amount that the prices jiggle up and down — is just as big with my type of autoregressive model as it is with a random walk model, even though the autoregression means that the spread at maturity is narrower. The narrower spread at maturity produces lower option prices, but I am not sure about the quadratic variation effect. Perhaps there should be lower option prices with a buy and hold type of strategy than with a continuous hedging strategy.

You should not assume that the curious features of the autoregressive model mean that you can sell when shares are dear and buy when they are cheap. That may be possible; it may be a sensible thing to do; but you cannot rely on it. The Maturity Guarantees Working Party did not rely on it. Using option pricing with my model does not rely on it.

Professor R. S. Clarkson, F.F.A.: The abstract of a paper 'An Actuarial Theory of Option Pricing', that I have written and which will be discussed with this one by the Faculty in January 1997, states: "Using an empirical approach to capital market returns analogous to that used for mortality rates by Halley more than three centuries ago, a theory of option pricing is built up in terms of the same three components as for life assurance premiums, namely the expected cost of claims, an allowance for expenses and a contingency margin as a reserve against the risk of insolvency".

The mathematics have to be tractable, but also have to be consistent with the empirical evidence. In 1996 we have seen much evidence that Black-Scholes does not work. Two papers, one by Bouchaud, Iori & Sornette (*Risk Magazine*, March 1996, 61) and the other by Geman & Ané (*Risk Magazine*, September 1996, 145), point to precisely the types of behaviour that I can generate in my new model.

My paper also stresses the disadvantages of the cost of entry through advanced mathematics. Other speakers have picked up that point. I begin the paper with the quotation, "The use of even the most sophisticated forms of mathematics can never be considered as a guarantee of quality... Genuine progress never consists in a purely formal exposition, but always in the discovery of the guiding ideas which underlie any proof". That was written 40 years ago by Professor Maurice Allais, who won the Nobel Prize for economics before Markowitz, Sharpe and Miller. Neither Black nor Scholes has won a Nobel Prize.

Mr P. A. Harlow, F.I.A.: I started working as an actuary in life and pensions, and then went to work in an investment bank, where I have been for the last ten years. I have traded options using the Black-Scholes model, so for me it is not so much theoretical as real and practical. I agree with a lot of the comments made about the difficulties of using Black-Scholes; for the area I am involved in, interest rate derivatives, Black-Scholes is right at the beginning of what is very complex academic research.

Significantly, the industry that involves the use of options and other derivatives has a huge mass of brain power applied to it. Virtually every firm, nowadays, has some Professor of Finance involved, doing some work on it in the U.S.A., in the U.K. or elsewhere. Also, there is an even greater mass of money that is being applied to this industry.

I have been disappointed at the early AFIR colloquiums, and until recently at the Institute, in the lack of serious mathematical work that has been attempted by actuaries in assessing the work that has gone on in the industry that I have been involved in. I am also deeply suspicious about what people mean when they talk about 'tried and tested actuarial techniques'. I am not quite sure that I know any such things except the famous words 'actuarial judgement'.

For me there is a very important point about the derivatives industry which the author has brought out very strongly in his paper; namely the advances that have been made in derivatives and that have enabled firms to do things that they were not able to do before. This is not necessarily connected with option pricing. For instance, the massive increase in swap trading that has taken place has resulted in companies being able to issue complex bonds, and to swap sets of cash flows in ways impossible ten, or even five, years ago. The increases in liquidity and in the possibilities in markets now are truly amazing, and certainly require our attention.

A good analogy here is with computer technology. We now have more availability of tools than ever before on our computers; new products are launched every week, and it is difficult to keep up. One is tempted to dismiss some or all of them as a fad. The same is happening with financial theory — it is a revolution. Not all of the developments are going to be really useful, and not all will be permanent. However, some have lasted, like the swap market, and are worth focusing on.

It is easy to stay firmly on one side of the fence or the other, and to say that option pricing theory is wonderful or that it is rubbish. I should like to encourage people not to do that, but to adopt a more measured approach, to get away from debating the mathematics, and rather to listen to what Dr Baxter was encouraging us to do — to look at the contribution that actuaries can make where banks are not particularly good, which is reserving, measurement and control of risk.

Mr J. A. Gallacher, F.F.A. (closing the discussion): In some of the discussion we have been going round in circles. The profession obviously thinks that derivatives are an important topic, as demonstrated by the papers on financial economics and derivatives pricing written for the *B.A.J.*, for the Staple Inn Actuarial Society and for investment conferences. However, I think that we have to decide what we want from this topic. This paper is extremely useful, because it draws parallels between what actuaries have traditionally done and what people in derivatives pricing are doing now.

Not so long ago the title of the paper would have been a contradiction in terms, but this has changed in the last couple of years. I offer an alternative title: 'Everything you wanted to know about derivatives, but were afraid to bid/ask'.

I have been asked what the difference is between price and value. Maybe traders understand price as value, but we, as actuaries, understand that they are completely different.

In my opinion, the profession should concentrate on the value concept rather than the pricing concept. The actuary's main business is to manage liabilities so that they can be met as and when they arise. An actuary does that by managing the changes in the environment: tax regulation; demographics; politics; economics. That is where actuarial strength lies. I do not think that concentrating on the complex mathematics of derivatives pricing is any more important than becoming demographic experts. I may have forgotten everything that I knew about cubic spline interpolation, but I can still apply a mortality table. Even though we may not know very much about the mathematics of derivatives, it is important that we can apply them for the good of the actuarial profession.

If we concentrate on the value concept — that is: efficient portfolio management; risk management; legislation; reserving techniques; and also adding value for our customers — the policyholders or the pension scheme members — then that will serve our profession much better than concentrating on pricing. The difficulty with option pricing is the very high threshold of knowledge required to get into it: stochastic calculus; partial differential equations; and statistics. This involves much which actuarial training does not prepare us for.

We are encouraged to learn about the pricing of derivatives in order to negotiate with providers of OTC contracts; but we only need to leave the arena of vanilla option pricing and we are back where we started. The incorporation of triggers, for example, and all the different payoffs, will take the mathematics outside our limited knowledge.

It is dangerous for the profession to calculate option prices and develop pricing models. Mr Pemberton asked whether we can be sure that we have the right model and the right parameters. Banks will not be particularly charitable if we get it wrong.

We are not in the industry for pricing derivatives. We are at the user end; the policyholder and pension holder end. When we are selling into the market, even for selling put/call spreads, we are accepting the market price, not pricing.

It is very easy to criticise models, and it is disappointing to hear the profession go round in circles in doing so. We are not saying anything that is new. Banks know that models are merely models — that is something that they have accepted and something which the actuarial profession has not. We are not short of using models ourselves, some with more shortcomings than Black-Scholes. Try to explain the rationale behind the net premium valuation; or consider how the actuarial profession seems inordinately fond of using the gross redemption yield. It is inadequate. Immunisation is a better approach; but it is still a model like the net premium or bonus reserve valuation methods. Physics is littered with models which seem rather simplistic, but give, on the whole, very good results. Some models which are over-specified can be disappointing.

We cannot judge a model by an axiomatic approach — by the fact that it fits 8 out of 10 criteria. We must judge derivative models by how they actually give a price and the hedging methodology they actually demonstrate. Our profession is not adding anything to the debate by lengthy discussions on the limitations of the models, and it is rather naive to think that the banks have not looked at the model limitations themselves. I know of at least one bank which has done a lot of work on volume effects, mentioned by Professor Wilkie.

When any actuarial student goes into an exam, the first thing that he or she must put down on the paper when considering an investigation is: what is the purpose? Though Black-Scholes might be completely useful for equity vanilla trades, it is absolutely unimportant for interest rates or even inflation models.

Any FX trader, who is pricing double trigger box options by straight Black-Scholes, will tell you that he is going to get his fingers singed off at the shoulders. We have all moved on from Black-Scholes. Banks have made vast sums of money from derivatives, and will continue to do so.

If we decide to become involved in option pricing, then we will have to do a lot of work to catch up with the research and effort that the banks have put in. I do not think that we all need to be rocket scientists — if we concentrate on values, then we will miss all the problems of the models. If we want to price — and there probably is a place for pricing within the profession — I believe that we should work through some kind of co-ordinated structure like the Option Pricing Techniques Working Party. This would be analogous to the Continuous Mortality Investigation Bureau. We do not all perform mortality investigations. Why do we not all pool our resources and formally look at the models and their limitations. Let us perform some coherent research on this subject.

I am disappointed with the investment education that I received as a trainee actuary. I realise now that it was completely inadequate. I hope that it is different now. My text books did not give us any concept of what financial economics is all about. The Actuarial Certificate in Derivatives, mentioned by Professor Wilkie, goes some way towards improving that. However, I am not too sure, as an investment banker, that I have anything to gain from taking the exam. My employer will not recognise it; I will not get promoted or sacked if I pass it or fail it.

We should play to the profession's strengths and concentrate on the application of derivatives. That is, after all, what the policyholders and pension holders want. It seems that, over the past couple of years, we have passed more and more of the investment risk back to the policyholder. We can use derivatives to stop this trend.

Recently an investment colleague asked if we are embarrassed, as a profession, to be getting interested in derivatives at this late stage? I do not think that we have anything to get embarrassed about; but we have to get our house in order and concentrate on what is important for us.

Mr M. H. D. Kemp, F.I.A. (replying): I have one comment, on the issue of the short term versus the long term; the long term is, I believe, the sum of lots of short terms. If something is right for all the consecutive short terms simultaneously, it should be right for the long term, and vice-versa. In any case, when does the long term start? Is it one year, five years, ten years?

The President (Mr D. G. R. Ferguson, F.I.A.): The author has given us a paper that not only will be of use to specialists in the field, but also to people who are users of derivatives and to members of the profession who are neither users of derivatives nor traders, but who realise that this is an area that they need to know something about. It is a very well written, easy to read, introduction to the subject, which leads us gently into some of the complexities.

I heartily endorse the conclusion of the paper that actuaries need to be more involved and that there is plenty of work for actuaries in this area.

I agree wholeheartedly with the closer that the actuarial profession should — as I believe that we do — always concentrate on value. We need to distinguish ourselves from Oscar Wilde's definition of accountants — and I wonder whether it applies to derivatives traders as well — who know the price of everything and the value of nothing. All of our work as actuaries concentrates on value.

I express my own thanks to the author and I know that you will all want to join me in thanking him in the usual way.

WRITTEN CONTRIBUTIONS

Mr P. G. Kennedy, F.I.A.: The author is to be congratulated on one of the clearest expositions of the subject that I have yet seen. I can recall, nearly 10 years ago, a distinguished speaker telling the Australian Institute that he understood the importance of option theory, that he was excited by it, and that he had even tried it, *but*, and I remember him pointing his finger as he said it, "I just can't get it into my head!" I hope that this paper will do the trick. Yet the underlying concepts should already be standard for actuaries, and it is surely an indictment of this profession that actuaries are still qualifying, let alone practising, without a sound grasp of the issues raised in this paper.

In the eyes of the Court of Appeal in *Wells* v *Wells* in October 1996, we are already convicted; for not only did they reject the actuarially inspired Ogden Tables as a basis for personal injury awards, but they wished that accountants and, yes, investment managers had been consulted in their construction. We have been warned.

If actuaries wish to see further than other professions, they must be prepared to stand on the shoulders of others. The author concludes with the hope that this paper will encourage younger actuaries to enter the field. We should go further and state that this *is* our field. There should be no more opt-outs. This paper should not have been presented to the Institute today. It should have been a model solution from last month's actuarial examination.

The author subsequently wrote: I agree with Mr Green's comments on the importance of even quite elementary control techniques, such as trading limits, on individual traders. I mention these, but perhaps do not stress them as strongly as I might. Regarding his comments on the Black-Scholes formula:

(1) I agree that originally people probably believed that the Black-Scholes formula required successive price movements to be uncorrelated, but leading practitioners in the field would now agree that this constraint on the price movements is not actually needed for the Black-Scholes formula to be correct (and Dr Satchell confirmed this later in the meeting).

(2) The main reason that I devoted so much space to the Black-Scholes formula is that it is, I believe, a first order approximation to all plausible alternatives. Professor Wilkie clearly agrees with me in this regard. Whilst I agree with Mr Green that markets are not always arbitrage-free, and, with the other weaknesses that he has highlighted, the spreads of returns on diversified asset categories are usually tolerably similar to the normal distribution. Instead of discounting the Black-Scholes

formula, I believe that any 'actuaries aspiring to be academics' should develop formulae which include the Black-Scholes formula as a special case, rather than starting from scratch. An example of the approach I would suggest is the cost of capital model in Appendix B.2.

There are other advantages in having consistency between actuarial approaches and those that are accepted (albeit with caveats) in the banking world. Mr Deighton has commented on the complexities involved in establishing that derivatives meet appropriate DTI defined tests regarding 'efficient portfolio management' or the 'reduction in investment risks'. He makes the useful point that we need to watch out for inconsistencies between the sorts of analyses actuaries might undertake to test these requirements and the approaches used by banks to price the derivatives in question.

Mr Smith concentrated on whether there is some fundamental distinction between the 'short term' and the 'long term'. I agree that the actuarial profession has been lucky that the long term has not been colonised by other professionals and that our claim to have tamed the long term contains an element of wishful thinking, even if it is a useful marketing stratagem. Perhaps the unstated concern some actuaries have regarding the likes of the Black-Scholes formula is that one day experts in the banking world will successfully extend the 'short term' to the 'long term', outflanking this perceived core actuarial skill base. I think that it is only a matter of time before this breakthrough is made (if it has not been made already, since I know of banks willing to provide 30-40 year swap contracts able to match annuity-type liabilities). I hope that it will be actuaries who are in the forefront, rather than other professions. Interestingly, Mr Smith noted that use of supposedly short-term models often results in guarantee costs much *larger* than those produced by supposedly longer-term actuarial approaches. The usual complaint that I have heard from those objecting to the use of Black-Scholes, or the like, is that it understates the true cost. Maybe synthesis would be less painful than some actuaries fear! Our best response, as a profession, is not to stick our heads in the sand, but to accept that we can also learn from others.

Mr Parsons also seems to be looking forward to the day when long-term derivatives become commonplace. I am not convinced that everyone needs to be an irrational investor in the eyes of someone else for the market system to work. My weekly trips to the supermarket, when I want to buy what the retailer wants to sell, are, I hope, far from irrational.

Mr Hewitson made a number of comments from a regulator's perspective. I agree with him about the importance of professional actuarial judgement. I hope that the paper helps actuaries in this respect. He indicated that he was puzzled by the suggestion that past or implied volatilities could be a reasonable guide to actual out-turns. I was concentrating here on the pricing of derivatives, rather than on the setting of prudential capital requirements. It is, after all, commonplace in the pricing of general insurance contracts to assume that the correct price of, say, motor insurance should at least bear some passing resemblance to recent claims experience. However, for solvency purposes, I would agree that it is prudent to assume that the past is not necessarily a good guide to the future.

On the point of diversifying away U.K. equity market risk, Mr Hewitson seems to fall into the same trap as an earlier draft of my paper, in assuming that diversification is only useful if the risks are genuinely *uncorrelated*. Diversification still works, but not so well, if the risks are partly (positively) correlated. I agree that the U.K. equity market is not wholly independent of other financial markets, but price movements in it show only limited links to, say, the movement in the yen versus the U.S. dollar. Even if the degree of independence is limited, the proportion of the world banking industry's total derivative exposures arising from U.K. equity price movements is miniscule, but for U.K. life offices is very substantial, so the general concept of risk pooling ought to encourage greater spreading of this risk.

I am disappointed that Mr Pemberton did not believe that I had been able to tie up derivatives concepts with actuarial concepts. I do not believe that there is, in practice, the discrepancy that he suggested between actuarial theory (as correctly stated) and mathematical economics (as correctly understood). In ¶7.3.6 I note that the 'actuarial' way of valuing a liability is to discount its expected payout at some suitable rate of interest, usually incorporating somewhere a margin for prudence. I show, in ¶7.3.7, that this is the same as is used in the derivatives field, as long as an appropriate probability distribution and discount rate are chosen. However, the main lesson from the derivatives

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industry is that there is a constraint regarding the discount rate that should be used in this calculation. This constraint is that, if we have a whole series of derivatives which, in aggregate, pay out exactly the same as investing in the underlying share or index, then the price of the whole should be the same as the price of the parts. Suppose that I write a life insurance contract and incur two sorts of initial charges, both, say, 1% of the premium invested. Whilst there are circumstances in which the net impact of both is not the same as a single 2% charge (e.g. if one is tax deductible and one is not), it is hard to identify them.

I do not believe that the paper is overly mathematical, except, perhaps, for the appendices, which are usually considered an acceptable repository for things complex. Having recently seen Mr Pemberton's proposed paper on the use of step functions, I do not think it makes the mathematics simpler. Step functions, *per se*, are like trying to find the area under a curve by fitting rectangles underneath it. We were all taught at school that, with sufficient rectangles, the calculation becomes the same as carrying out integration. Is he proposing that integration and differentiation are too difficult for most actuaries? He may be right, but I hope not. They should not be. If they are, we would need to abandon the use of continuously compounded forces of interest!

However, I do agree with Mr Pemberton that my paper provides little explicit guidance on how to choose the parameters to feed into a pricing model. I recommend choosing parameters consistent with market prices to identify the value (or price) of a derivative. I agree that historic experience is an important input in the process of identifying what additional reserves need to be established to protect a derivatives book from default. Listing in detail precisely how to do this for every possible contract would be nigh on impossible. However, I did attempt, in Section 9, to show how it could be done in principle, for just one sort of contract based on the FT-SE Index. Perhaps his main criticism is that I did not distil all of this into a simple formula that could be programmed into a pocket calculator. The real world is more complicated than this.

Dr Satchell seems to share my enthusiasm for not dumping the Black-Scholes baby out with the bath water, as does Professor Wilkie. If the actuarial profession is to have any credibility with academics outside actuarial departments, then we should heed their words.

I can understand Professor Wilkie's reluctance to accept the comments in Section 8.9, since they refer specifically to the Wilkie model. The results surprised me too when I first identified them. Dr Satchell indicated that he believed it possible to show that the Wilkie model was compatible with Black-Scholes. I agree. My recent paper to the Staple Inn Actuarial Society titled 'Asset/liability modelling for pension funds' did just that, and contains details on how to price options under the Wilkie model.

I have not yet studied Professor Clarkson's paper in detail, but I hope that he, too, is willing to accept that Black-Scholes, although subject to flaws, is nevertheless a useful starting point. He refers to a short paper in the March 1996 edition of *Risk Magazine* called 'Real world options' by Bouchaud, Iori & Sornette. If I had spotted this paper earlier I would have included references to it in my own, since it adopts precisely the sort of approach that I tried to propose in my paper, i.e. take the Black-Scholes formula, find the circumstances in which it does not work, and then add appropriate adjustments. The two additions that they incorporate are to allow for jumps (and fat tails), and to allow for transaction costs, i.e. two of the three circumstances that I highlight, in which the generalised Black-Scholes formulation will break down. I would recommend study of their paper, but I fear that it would not meet Mr Pemberton's test of containing less mathematics. It involves more extensive and complicated use of integration. However, it does satisfy my criterion that it should exhibit the Black-Scholes formula as a special case, again justifying my belief that the Black-Scholes formula is right to a first approximation.

I found Mr Harlow's comments on the huge mass of brain power, and even greater mass of money, that has been directed towards the derivatives markets very helpful. The swaps market is many trillions of dollars (measured by notional principal), and the daily volume traded on the forward foreign exchange market is equally mind-boggling. The U.K. life insurance and pension fund industries combined are, on these measures, mere drops in an ocean. That huge brain power has produced an amazing spread of derivatives. I had to change my paper near the end of writing it to avoid claiming that derivatives linked purely to volatility did not trade, when a piece from a leading derivatives house landed on my desk describing just such a trade that they had recently executed.

I thank the closer for his comments. I still have some difficulty in consenting to the distinction that he and others have suggested between 'price' and 'value'. I agree that the most obvious way of checking the price of a derivative is to find out its price in the market place. However, I am not convinced that we can, or should, leave option pricing to others. What about the derivative-like guarantees included in with-profits contracts? Is he suggesting that we do not need to be able to price or to place a value on these, or is it just derivatives that we rarely meet which he thinks that we do not need to be able to price? And what about those in the profession who want this skill to improve their marketability in the job market?

I was interested to hear of his working party on option pricing techniques and wish it well. I would again recommend that they study the Bouchaud, Iori & Sornette paper referred to earlier. I am virtually certain that the three things that they will find most difficult to incorporate are jump risk, uncertain volatility and transaction costs (but not necessarily in that order of importance), because, as I note in ¶8.3.1, if these are not incorporated, then the price of a vanilla option must be expressible in a form akin to the Black-Scholes formula if it is to avoid introducing arbitrage.