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Logarithmic upper bounds for weak solutions to a class of parabolic equations

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It is well known that a weak solution φ to the initial boundary value problem for the uniformly parabolic equation $\partial_t\varphi - \operatorname{div}(A\nabla\varphi) + \omega\varphi = f$ in $\Omega_T \equiv \Omega \times (0,T)$ satisfies the uniform estimate

$$\|\varphi\|_{\infty,\Omega_T} \leqslant \|\varphi\|_{\infty,\partial_p\Omega_T} + c\|f\|_{q,\Omega_T}, \quad c = c(N,\lambda,q,\Omega_T),$$

provided that $q > 1 + N/2$, where Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, $T > 0$, $\partial_p\Omega_T$ is the parabolic boundary of Ω_T, $\omega \in L^1(\Omega_T)$ with $\omega \geqslant 0$, and λ is the smallest eigenvalue of the coefficient matrix A. This estimate is sharp in the sense that it generally fails if $q = 1 + N/2$. In this paper, we show that the linear growth of the upper bound in $\|f\|_{q,\Omega_T}$ can be improved. To be precise, we establish

$$\|\varphi\|_{\infty,\Omega_T} \leqslant \|\varphi_0\|_{\infty,\partial_p\Omega_T} + c\|f\|_{1+N/2,\Omega_T}\left(\ln(\|f\|_{q,\Omega_T} + 1) + 1\right).$$

Keywords: Logarithmic upper bounds; Moser's iteration technique; uniform bounds for weak solutions of parabolic equations

2010 *Mathematics subject classification:* Primary: 35K20; 35B45; 35D30; 35B50

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$. For each $T > 0$ consider the initial boundary value problem

$$\partial_t\varphi - \operatorname{div}(A\nabla\varphi) + \omega\varphi = f \quad \text{in} \quad \Omega_T \equiv \Omega \times (0,T), \tag{1.1}$$

$$\varphi = 0 \quad \text{on} \quad \Sigma_T \equiv \partial\Omega \times (0,T), \tag{1.2}$$

$$\varphi(x,0) = \varphi_0(x) \quad \text{on} \quad \Omega. \tag{1.3}$$

We assume:

(H1) $A = A(x,t)$ is an $N \times N$ matrix whose entries $a_{ij}(x,t)$ satisfy

$$a_{ij}(x,t) \in L^\infty(\Omega_T), \quad \lambda|\xi|^2 \leqslant (A(x,t)\xi\cdot\xi) \quad \text{for } \xi \in \mathbb{R}^N \text{ and a.e. } (x,t) \in \Omega_T, \tag{1.4}$$

where $\lambda > 0$;

(H2) $\omega \in L^1(\Omega_T)$ with $\omega \geqslant 0$ a.e. on Ω_T, $f \in L^q(\Omega_T)$ for some $q > 1 + N/2$, and $\varphi_0 \in L^\infty(\Omega)$.

In the situation considered here, the classical theory for uniformly parabolic equations [5] asserts that there is a unique weak solution φ in the space $L^2(0,T;W_0^{1,2}(\Omega)) \cap L^\infty(\Omega_T)$ and we have the estimate

$$\|\varphi\|_{\infty,\Omega_T} \leqslant \|\varphi_0\|_{\infty,\Omega} + c\|f\|_{q,\Omega_T}, \quad c = c(N,\lambda,q,\Omega_T). \tag{1.5}$$

The result is sharp in the sense that if $q = 1 + N/2$ then the above inequality fails in general. Our objective here is to improve this upper bound for $\|\varphi\|_{\infty,\Omega_T}$. To be precise, we will show:

THEOREM 1.1. *Let* (H1)–(H2) *be satisfied and* φ *be a weak solution to* (1.1)–(1.3) *in the space* $L^2(0,T;W_0^{1,2}(\Omega))$.

Then there is a positive number $c = c(N,\lambda,q,\Omega_T)$ *such that*

$$\|\varphi\|_{\infty,\Omega_T} \leqslant \|\varphi_0\|_{\infty,\Omega} + c\|f\|_{1+N/2,\Omega_T}\left(\ln(\|f\|_{q,\Omega_T}+1)+1\right). \tag{1.6}$$

Similar results for functions in $W^{s,q}(\mathbb{R}^N)$, $sq > N$, have been established in [1, 4]. This theorem can be viewed as the parabolic version of the result in [7]. As in [7], we introduce a change of variable. The equation satisfied by the new unknown function v has the expression

$$\partial_t v - \operatorname{div}(A\nabla v) + \frac{1}{v}A\nabla v\cdot\nabla v + \omega v\ln v = gv \quad \text{in } \Omega_T. \tag{1.7}$$

To prove theorem 1.1, we need to have an estimate like

$$\text{ess sup}_{\Omega_T}|v| \leqslant c\left(\|g\|_{q,\Omega_T}+1\right)^\alpha, \quad c,\alpha > 0, \tag{1.8}$$

where q is given as in (H2).

The approach in ([5], p.185) is to first show the boundedness of v for small t and then extend the result to large t. Unfortunately, this method does not serve our purpose because how ess $\sup_{\Omega_T}|v|$ is bounded by $\|g\|_{q,\Omega_T}$ becomes unclear. By modifying Moser's technique of iteration of L^q norms (see [6]), we are able to obtain (1.8), that is, ess $\sup_{\Omega_T}|v|$ is bounded by a power function of $\|g\|_{q,\Omega_T}+1$. Even though our proof here is inspired by the work of [7] in the elliptic case, we must overcome the complication caused by the time variable. This constitutes the core of our analysis.

Observe that since the coefficient function 'g' in equation (1.7) is of arbitrary sign this equation, in essence, resembles the parabolic version of the Schrödinger equation in [2]. In the elliptic case [2], one can assume that g belongs to the slightly more general Kato class, instead of $L^p(\Omega)$ with $p > N/2$. However, this results in the loss of explicit dependence of ess $\sup|v|$ on g. That is, how ess $\sup|v|$ is bounded by a certain norm of g is hidden. This explains why an elliptic version of theorem 1.1 is established in [7] only under the assumption that $g \in L^p(\Omega)$ with $p > N/2$.

We also would like to point out that the constant c in (1.6) does not depend on the upper bounds of our elliptic coefficients $a_{ij}(x,t)$. This opens the possibility that theorem 1.1 be applicable to certain types of the so-called singular parabolic equations.

Our result can also be extended to more general equations by suitably modifying our proof. The inequality (1.6) can also have different versions. For example, the interested reader might want to try to establish the following

$$\|\varphi\|_{\infty,\Omega_T} \leqslant \|\varphi_0\|_{\infty,\Omega} + c \sup_{0\leqslant t\leqslant T} \|f\|_{N/2,\Omega} \left(\ln(\sup_{0\leqslant t\leqslant T} \|f\|_{q,\Omega} + 1) + 1 \right), \quad q > \frac{N}{2}. \tag{1.9}$$

We refer the reader to [5] for more information on how to balance the integrability of f in the time variable and that of the space variables.

The rest of the paper is dedicated to the proof of Theorem 1.1. In our proof, we will assume that the space dimension N is bigger than 2 due to an application of the Sobolev embedding theorem. Obviously, if $N = 2$, we must take the following version of the Sobolev embedding theorem: for each $s > 2$ there is a positive number c_s such that

$$\int_{\Omega_T} |v|^{2+((2(s-2))/(s))} \mathrm{d}x\mathrm{d}t \leqslant c_s^2 \left(\sup_{0\leqslant t\leqslant T} \int_\Omega |v|^2 \mathrm{d}x \right)^{((s-2)/(2))} \int_{\Omega_T} |\nabla v|^2 \mathrm{d}x\mathrm{d}t \tag{1.10}$$

for each $v \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W_0^{1,2}(\Omega))$. Then all our arguments in the proof of Theorem 1.1 remain valid, that is, theorem 1.1 still holds for $N = 2$.

2. Proof of Theorem 1.1

Proof of Theorem 1.1. We decompose φ into $\varphi_1 + \varphi_2$, where φ_1 and φ_2 are the respective solutions of the following two problems

$$\partial_t\varphi_1 - \operatorname{div}(A\nabla\varphi_1) + \omega\varphi_1 = f \quad \text{in} \quad \Omega_T, \tag{2.1}$$

$$\varphi_1 = 0 \quad \text{on } \Sigma_T, \tag{2.2}$$

$$\varphi_1(x,0) = 0 \quad \text{on } \Omega \text{ and} \tag{2.3}$$

$$\partial_t\varphi_2 - \operatorname{div}(A\nabla\varphi_2) + \omega\varphi_2 = 0 \quad \text{in} \quad \Omega_T, \tag{2.4}$$

$$\varphi_2 = 0 \quad \text{on } \Sigma_T, \tag{2.5}$$

$$\varphi_2(x,0) = \varphi_0(x) \quad \text{on } \Omega. \tag{2.6}$$

Let c_s be the smallest positive number such that

$$\|\phi\|_{((2N)/(N-2)),\Omega} \leqslant c_s \|\nabla\phi\|_{2,\Omega} \tag{2.7}$$

for all $\phi \in W^{1,2}(\Omega)$ with $\phi = 0$ on $\partial\Omega$. It is well known that the Sobolev constant c_s here depends only on N ([3], p. 138). We consider the functions

$$u = \frac{\varphi_1}{\max\{\|f\|_{1+N/2,\Omega_T}, 1\}}, \tag{2.8}$$

$$g = \frac{f}{\max\{\|f\|_{1+N/2,\Omega_T}, 1\}}. \tag{2.9}$$

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Obviously, they satisfy

$$\partial_t u - \operatorname{div}(A\nabla u) + \omega u = g \quad \text{in} \quad \Omega_T, \tag{2.10}$$

$$u = 0 \quad \text{on } \Sigma_T, \tag{2.11}$$

$$u(x,0) = 0 \quad \text{on } \Omega. \tag{2.12}$$

LEMMA 2.1. *Let the assumptions of theorem 1.1 hold. Then to each $\alpha > 0$ sufficiently small there corresponds a positive number $c = c(N, \alpha, c_s, \lambda, \Omega_T, \|f\|_{1+2/N,\Omega_T})$ such that*

$$\int_{\Omega_T} e^{\alpha(1+2/N)u} \mathrm{d}x\mathrm{d}t \leqslant c. \tag{2.13}$$

Proof. First we would like to remark that a weak solution u to (2.10)–(2.12) is unique [5] and can be constructed as the limit of a sequence of smooth approximate solutions, which can be obtained by, for example, regularizing the given functions in the problem. Thus without any loss of generality, we may assume that u is a classical solution in our subsequent calculations. Let $\alpha > 0$ be given. Using $e^{\alpha u} - 1$ a test function in (2.10), which means that we multiply through the equation by the function and integrate the resulting equation over $\Omega_s \equiv \Omega \times (0, s)$, $T \geqslant s > 0$, we obtain

$$\int_\Omega \left(\frac{e^{\alpha u}}{\alpha} - u \right) \mathrm{d}x + \frac{4\lambda}{\alpha} \int_{\Omega_s} \left| \nabla e^{\alpha/2u} \right|^2 \mathrm{d}x\mathrm{d}t + \int_{\Omega_s} \omega u \left(e^{\alpha u} - 1 \right) \mathrm{d}x\mathrm{d}t$$
$$\leqslant \int_{\Omega_s} g(e^{\alpha u} - 1)\mathrm{d}x\mathrm{d}t + \frac{|\Omega|}{\alpha}. \tag{2.14}$$

Note that

$$u \left(e^{\alpha u} - 1 \right) \geqslant 0,$$
$$e^{\alpha u} - 1 = \left(e^{((\alpha u)/(2))} - 1 \right)^2 + 2 \left(e^{((\alpha u)/(2))} - 1 \right), \tag{2.15}$$
$$\|g\|_{1+N/2,\Omega_T} \leqslant 1. \tag{2.16}$$

Keeping these in mind, we estimate from the Sobolev embedding theorem (2.7) that

$$\int_0^T \int_\Omega \left(e^{((\alpha u)/(2))} - 1 \right)^{2+4/N} \mathrm{d}x\mathrm{d}t$$
$$\leqslant \int_0^T \left(\int_\Omega \left(e^{((\alpha u)/(2))} - 1 \right)^{((2N)/(N-2))} \mathrm{d}x \right)^{((N-2)/(N))}$$
$$\times \left(\int_\Omega \left(e^{((\alpha u)/(2))} - 1 \right)^2 \mathrm{d}x \right)^{2/N} \mathrm{d}t$$
$$\leqslant c_s^2 \left(\max_{0 \leqslant t \leqslant T} \int_\Omega \left(e^{((\alpha u)/(2))} - 1 \right)^2 \mathrm{d}x \right)^{2/N} \int_0^T \int_\Omega |\nabla e^{((\alpha u)/(2))}|^2 \mathrm{d}x\mathrm{d}t$$

$$
\begin{aligned}
&\leqslant c_s^2\left(\max_{0\leqslant t\leqslant T}\int_\Omega e^{\alpha u}\mathrm{d}x\right)^{2/N}\int_0^T\int_\Omega |\nabla e^{((\alpha u)/(2))}|^2\mathrm{d}x\mathrm{d}t\\
&+c_s^2|\Omega|^{2/N}\int_0^T\int_\Omega |\nabla e^{((\alpha u)/(2))}|^2\mathrm{d}x\mathrm{d}t\\
&\leqslant \frac{c_s^2\alpha^{1+2/N}}{4\lambda}\left(\int_{\Omega_T} g(e^{\alpha u}-1)\mathrm{d}x\mathrm{d}t+\max_{0\leqslant t\leqslant T}\int_\Omega u\mathrm{d}x+\frac{|\Omega|}{\alpha}\right)^{1+2/N}\\
&+\frac{c_s^2|\Omega|^{2/N}\alpha}{4\lambda}\left(\int_{\Omega_T} g(e^{\alpha u}-1)\mathrm{d}x\mathrm{d}t+\max_{0\leqslant t\leqslant T}\int_\Omega u\mathrm{d}x+\frac{|\Omega|}{\alpha}\right)\\
&\leqslant \left(\frac{c_s^2\alpha^{1+2/N}}{4\lambda}+\delta\right)\left(\int_{\Omega_T} |g(e^{\alpha u}-1)|\mathrm{d}x\mathrm{d}t+\max_{0\leqslant t\leqslant T}\int_\Omega |u|\mathrm{d}x+\frac{|\Omega|}{\alpha}\right)^{1+2/N}\\
&+c(\delta),\quad \delta>0. \qquad (2.17)
\end{aligned}
$$

With the aid of (2.16) and (2.15), we estimate

$$
\begin{aligned}
\int_{\Omega_T}|g(e^{\alpha u}-1)|\mathrm{d}x\mathrm{d}t &\leqslant \int_{\Omega_T}|g|\left(e^{((\alpha u)/(2))}-1\right)^2\mathrm{d}x\mathrm{d}t\\
&\quad+2\int_{\Omega_T}|g||e^{((\alpha u)/(2))}-1|\mathrm{d}x\mathrm{d}t\\
&\leqslant \|g\|_{1+N/2,\Omega_T}\left(\int_{\Omega_T}\left(e^{((\alpha u)/(2))}-1\right)^{((2(N+2))/(N))}\mathrm{d}x\mathrm{d}t\right)^{((N)/(N+2))}\\
&\quad+2\|g\|_{((2(N+2))/(N+4),\Omega_T}\\
&\quad\times\left(\int_{\Omega_T}\left(e^{((\alpha u)/(2))}-1\right)^{((2(N+2))/(N))}\mathrm{d}x\mathrm{d}t\right)^{((N)/(2(N+2)))}\\
&\leqslant \|e^{((\alpha u)/(2))}-1\|^2_{((2(N+2))/(N)),\Omega_T}\\
&\quad+2|\Omega_T|^{((N)/(2(N+2)))}\|e^{((\alpha u)/(2))}-1\|_{((2(N+2))/(N)),\Omega_T}\\
&\leqslant 2\|e^{((\alpha u)/(2))}-1\|^2_{((2(N+2))/(N)),\Omega_T}+c. \qquad (2.18)
\end{aligned}
$$

Plugging this into (2.17), we obtain

$$
\begin{aligned}
&\int_0^T\int_\Omega\left(e^{((\alpha u)/(2))}-1\right)^{2+4/N}\mathrm{d}x\mathrm{d}t\\
&\quad\leqslant 2^{1+4/N}\left(\frac{c_s\alpha^{1+2/N}}{4\lambda}+\delta\right)\int_0^T\int_\Omega\left(e^{((\alpha u)/(2))}-1\right)^{2+4/N}\mathrm{d}x\mathrm{d}t\\
&\qquad+c\left(\max_{0\leqslant t\leqslant T}\int_\Omega|u|\mathrm{d}x\right)^{1+2/N}+c. \qquad (2.19)
\end{aligned}
$$

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Thus choosing α, δ suitably small, we can absorb the first term on the right-hand side to the left-hand side to obtain

$$\int_0^T \int_\Omega \left(e^{((\alpha u)/(2))} - 1\right)^{2+4/N} \mathrm{d}x\mathrm{d}t \leqslant c \left(\max_{0\leqslant t\leqslant T} \int_\Omega |u|\mathrm{d}x\right)^{1+2/N} + c. \tag{2.20}$$

To estimate the right-hand side of the above inequality, we define, for $\varepsilon > 0$,

$$\eta_\varepsilon(s) = \begin{cases} 1 & \text{if } s \geqslant \varepsilon, \\ \dfrac{1}{\varepsilon}s & \text{if } -\varepsilon < s < \varepsilon, \\ -1 & \text{if } s \leqslant -\varepsilon. \end{cases}$$

Use $\eta_\varepsilon(u)$ as a test function in (2.10) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \int_0^u \eta_\varepsilon(s)\mathrm{d}s\mathrm{d}x \leqslant \int_\Omega g\eta_\varepsilon(u)\mathrm{d}x \leqslant \int_\Omega |g|\mathrm{d}x. \tag{2.21}$$

Integrate with respect to t and then let $\varepsilon \to 0^+$ to yield

$$\sup_{0\leqslant t\leqslant T} \int_\Omega |u|\mathrm{d}x \leqslant \int_{\Omega_T} |g|\mathrm{d}x\mathrm{d}t. \tag{2.22}$$

Plugging this into (2.20) gives the desired result. □

To continue the proof of Theorem 1.1, we assume, without any loss of generality, that

$$\text{ess sup}_{\Omega_T} u = \|u\|_{\infty,\Omega_T}. \tag{2.23}$$

Indeed, if (2.23) is not true, we multiply through (2.10) by -1 and consider $-u$.

Let

$$v = e^u. \tag{2.24}$$

Then

$$\begin{aligned} u &= \ln v, \\ A\nabla u &= \frac{1}{v}A\nabla v, \\ \mathrm{div}(A\nabla u) &= -\frac{1}{v^2}A\nabla v \cdot \nabla v + \frac{1}{v}\mathrm{div}(A\nabla v). \end{aligned}$$

Substitute these into (2.10)–(2.12) to obtain (1.7) coupled with

$$v = 1 \quad \text{on } \Sigma_T, \tag{2.25}$$

$$v(x,0) = 1 \quad \text{on } \Omega. \tag{2.26}$$

Set

$$w = \max\{v, 1\}. \tag{2.27}$$

For each $\beta > 0$, we use $w^\beta - 1$ as a test function in (1.7) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\int_0^v (((s-1)^+ + 1)^\beta - 1)\mathrm{d}s\mathrm{d}x \\ + \int_\Omega \beta w^{\beta-1}A(x)\nabla v\cdot\nabla w\mathrm{d}x \leqslant \int_\Omega gv(w^\beta - 1)\mathrm{d}x \leqslant \int_\Omega |g|w^{\beta+1}\mathrm{d}x. \tag{2.28}$$

With (2.27) in mind, we can deduce that

$$\begin{aligned}\int_0^v (((s-1)^+ + 1)^\beta - 1)\mathrm{d}s &= \left(\int_1^v (((s-1)^+ + 1)^\beta - 1)\mathrm{d}s\right)^+ \\ &= \left(\int_1^v (s^\beta - 1)\mathrm{d}s\right)^+ \\ &= \frac{1}{\beta+1}\left(w^{\beta+1} - 1\right) - (w-1)\,.\end{aligned} \tag{2.29}$$

Using this in (2.28) yields

$$\frac{1}{\beta+1}\int_\Omega (w^{\beta+1} - 1)\mathrm{d}x + \frac{4\beta\lambda}{(\beta+1)^2}\int_{\Omega_\tau}\left|\nabla w^{((\beta+1)/(2))}\right|^2\mathrm{d}x\mathrm{d}t \\ \leqslant \int_{\Omega_\tau}|g|w^{\beta+1}\mathrm{d}x\mathrm{d}t + \int_\Omega (w-1)\mathrm{d}x. \tag{2.30}$$

Let η_ε be given as before. We use $\eta_\varepsilon(v-1)$ as a test function in (1.7). With the aid of the proof of (2.22), we arrive at

$$\int_\Omega (w-1)\mathrm{d}x = \int_\Omega (v-1)^+\mathrm{d}x \leqslant \int_\Omega |v-1|\mathrm{d}x \leqslant \int_{\Omega_T}|g|\mathrm{d}x\mathrm{d}t \leqslant \int_{\Omega_T}|g|w^{\beta+1}\mathrm{d}x\mathrm{d}t. \tag{2.31}$$

The last step is due to the fact that $w \geqslant 1$ on Ω_T. Plug this into (2.30) to yield

$$\frac{1}{\beta+1}\sup_{0\leqslant t\leqslant T}\int_\Omega (w^{\beta+1} - 1)\mathrm{d}x + \frac{4\beta\lambda}{(\beta+1)^2}\int_{\Omega_T}\left|\nabla w^{((\beta+1)/(2))}\right|^2\mathrm{d}x\mathrm{d}t \\ \leqslant 2\int_{\Omega_T}|g|w^{\beta+1}\mathrm{d}x\mathrm{d}t. \tag{2.32}$$

Note that

$$\begin{aligned}w^{\beta+1} - 1 &= \left(w^{((\beta+1)/(2))} - 1 + 1\right)^2 - 1 \\ &\leqslant 2\left(w^{((\beta+1)/(2))} - 1\right)^2 + 1.\end{aligned}$$

Keeping this, (2.27), the Sobolev embedding theorem (2.7), and (2.32) in mind, we calculate that

$$
\begin{aligned}
&\int_0^T \int_\Omega w^{(\beta+1)(1+2/N)}\mathrm{d}x\mathrm{d}t \\
&\quad\leqslant 2^{2/N}\int_0^T\int_\Omega \left(w^{\beta+1}-1\right)^{1+2/N}\mathrm{d}x\mathrm{d}t + 2^{2/N}\int_0^T\int_\Omega w^{\beta+1}\mathrm{d}x\mathrm{d}t \\
&\quad\leqslant 2^{2/N}\left(\sup_{0\leqslant t\leqslant T}\int_\Omega (w^{\beta+1}-1)\mathrm{d}x\right)^{2/N} \\
&\qquad\times\int_0^T\left(\int_\Omega \left(w^{\beta+1}-1\right)^{((N)/(N-2))}\mathrm{d}x\right)^{((N-2)/(N))}\mathrm{d}t \\
&\qquad+2^{2/N}\int_0^T\int_\Omega w^{\beta+1}\mathrm{d}x\mathrm{d}t \\
&\quad\leqslant c(\beta+1)^{2/N}\left(\int_{\Omega_T}|g|w^{\beta+1}\mathrm{d}x\mathrm{d}t\right)^{2/N} \\
&\qquad\times\int_0^T\left(\int_\Omega\left(w^{((\beta+1)/(2))}-1\right)^{((2N)/(N-2))}\mathrm{d}x\right)^{((N-2)/(N))}\mathrm{d}t \\
&\qquad+c(\beta+1)^{2/N}\left(\int_{\Omega_T}|g|w^{\beta+1}\mathrm{d}x\mathrm{d}t\right)^{2/N}+2^{2/N}\int_{\Omega_T}w^{\beta+1}\mathrm{d}x\mathrm{d}t \\
&\quad\leqslant c\frac{(\beta+1)^{((2)/(N+2))}}{\beta}\left(\int_{\Omega_T}|g|w^{\beta+1}\mathrm{d}x\mathrm{d}t\right)^{((2)/(N+1))} \\
&\qquad+c(\beta+1)^{2/N}\left(\int_{\Omega_T}|g|w^{\beta+1}\mathrm{d}x\mathrm{d}t\right)^{2/N}+2^{2/N}\int_{\Omega_T}w^{\beta+1}\mathrm{d}x\mathrm{d}t. \qquad (2.33)
\end{aligned}
$$

We estimate from (2.27) that

$$
\begin{aligned}
&\left(\int_{\Omega_T} w^{\beta+1}\mathrm{d}x\mathrm{d}t\right)^{((N)/(N+2))} \\
&\quad\leqslant\left[\left(\int_{\Omega_T} w^{(((\beta+1)q)/(q-1))}\mathrm{d}x\mathrm{d}t\right)^{((q-1)/(q))}|\Omega_T|^{1/q}\right]^{((N)/(N+2))} \\
&\quad=\left[\left(⨍_{\Omega_T} w^{(((\beta+1)q)/(q-1))}\mathrm{d}x\mathrm{d}t\right)^{((q-1)/(q))}|\Omega_T|\right]^{((N)/(N+2))} \\
&\quad\leqslant|\Omega_T|^{((N)/(N+2))-((q-1)/(q))}\left(\int_{\Omega_T} w^{(((\beta+1)q)/(q-1))}\mathrm{d}x\mathrm{d}t\right)^{((q-1)/(q))}. \qquad (2.34)
\end{aligned}
$$

Recall that $q > 1 + N/2$ is given by assumption (H2). Raising both sides of (2.33) to the $((N)/(N+2))$-th power, we derive that

$$\begin{aligned}
\|w^{\beta+1}\|_{((N+2)/(N)),\Omega_T} &\leqslant \frac{c(\beta+1)^{((2(N+1))/(N+2))}}{\beta^{((N)/(N+2))}}\|g\|_{q,\Omega_T} \\
&\times \left(\int_{\Omega_T} w^{(((\beta+1)q)/(q-1))}\mathrm{d}x\mathrm{d}t\right)^{((q-1)/(q))} \\
&+ c(\beta+1)^{((2)/(N+2))}|\Omega_T|^{((2(q-1))/(q(N+2)))-((q)/(q-1))}\|g\|_{q,\Omega_T}^{((2)/(N+2))} \\
&\times \left(\int_{\Omega_T} w^{(((\beta+1)q)/(q-1))}\mathrm{d}x\mathrm{d}t\right)^{((q-1)/(q))} \\
&+ 2^{2/N}|\Omega_T|^{((N)/(N+2))-\frac{q-1}{q}}\left(\int_{\Omega_T} w^{(((\beta+1)q)/(q-1))}\mathrm{d}x\mathrm{d}t\right)^{((q-1)/(q))} \\
&\leqslant \left(c(\beta+1)^{((2(N+1))/(N+2))}\left(\frac{1}{\beta^{((N)/(N+2))}}+1\right)\|g\|_{q,\Omega_T}+c\right) \\
&\quad \times \|w^{\beta+1}\|_{((q)/(q-1),\Omega_T} \\
&\leqslant c(\beta+1)^{((2(N+1))/(N+2))}\left(\frac{1}{\beta^{((N)/(N+2))}}+2\right)(\|g\|_{q,\Omega_T}+1) \\
&\quad \times \|w^{\beta+1}\|_{((q)/(q-1),\Omega_T}.
\end{aligned}$$

Thus we can write the above inequality in the form

$$\begin{aligned}
\|w\|_{(((N+2)(\beta+1))/(N))} &\leqslant c^{((1)/(\beta+1))}(\beta+1)^{((2(N+1))/((N+2)(\beta+1)))} \\
\times &\left(\frac{1}{\beta^{((N)/(N+2))}}+2\right)^{((1)/(\beta+1))}(\|g\|_{q,\Omega_T}+1)^{((1)/(\beta+1))}\|w\|_{((q(\beta+1))/(q-1))}.
\end{aligned} \tag{2.35}$$

Now set

$$\chi = \frac{(N+2)/N}{q/(q-1)} > 1.$$

Fix $\beta_0 > 0$ and let

$$\beta + 1 = (1+\beta_0)\chi^i, \quad i = 0, 1, 2, \ldots.$$

Subsequently,

$$\begin{aligned}
\|w\|_{(1+\beta_0)((q)/(q-1))\chi^{i+1},\Omega_T} &\leqslant [c(\|g\|_{q,\Omega_T}+1)]^{((1)/((1+\beta_0)\chi^i))} \\
&\times [(1+\beta_0)\chi^i]^{((2(N+1))/((1+\beta_0)(N+2)\chi^i))} \\
&\cdot \|w\|_{(1+\beta_0)((q)/(q-1))\chi^i,\Omega_T}
\end{aligned}$$

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$$\leqslant [c(\|g\|_{q,\Omega_T}+1)]^{((1)/((1+\beta_0)))(1/\chi^i+\cdots+1)}$$
$$\times (1+\beta_0)^{((2(N+1))/((1+\beta_0)(N+2)))(1/\chi^i+\cdots+1)}$$
$$\cdot \chi^{((2(N+1))/((1+\beta_0)(N+2)))(i/\chi^i+\cdots+1)}\|w\|_{(1+\beta_0)((q)/(q-1)),\Omega_T}.$$

Taking $i\to\infty$ yields

$$\|w\|_{\infty,\Omega_T}\leqslant c(\|g\|_{q,\Omega_T}+1)^{\alpha_0}\|w\|_{r,\Omega_T}, \tag{2.36}$$

where

$$\alpha_0=\frac{\chi}{(1+\beta_0)(\chi-1)}, \tag{2.37}$$

$$r=(1+\beta_0)\frac{q}{q-1}. \tag{2.38}$$

In view of lemma 2.1, we can find an $\alpha\in(0,r)$ small enough so that

$$\int_{\Omega_T} w^\alpha \mathrm{d}x\mathrm{d}t\leqslant \int_{\Omega_T} e^{\alpha u}\mathrm{d}x\mathrm{d}t+c\leqslant c. \tag{2.39}$$

It follows that

$$\|w\|_{r,\Omega_T}=\left(\int_{\Omega_T} w^{r-\alpha}w^\alpha \mathrm{d}x\right)^{1/r}$$
$$\leqslant \|w\|_{\infty,\Omega_T}^{((r-\alpha)/(r))}\|w\|_{\alpha,\Omega_T}^{((\alpha)/(r))}.$$

This together with (2.36) gives

$$\|w\|_{\infty,\Omega_T}\leqslant c\left(\|g\|_{q,\Omega_T}+1\right)^{\alpha_0}\|w\|_{\infty,\Omega_T}^{((r-\alpha)/(r))}\|w\|_{\alpha,\Omega_T}^{\alpha/r} \tag{2.40}$$

from whence follows

$$\|w\|_{\infty,\Omega_T}\leqslant c\left(\|g\|_{q,\Omega_T}+1\right)^{\alpha_0 r/\alpha}\|w\|_{\alpha,\Omega_T}. \tag{2.41}$$

Recall the definition of w and use (2.39) to obtain

$$\|v\|_{\infty,\Omega_T}\leqslant \|w\|_{\infty,\Omega_T}\leqslant c\left(\|g\|_{q,\Omega_T}+1\right)^{\alpha_0 r/\alpha}\left(\|v\|_{\alpha,\Omega_T}+|\Omega_T|^{1/\alpha}\right)$$
$$\leqslant c\left(\|g\|_{q,\Omega_T}+1\right)^{\alpha_0 r/\alpha}\left(\|g\|_{1,\Omega_T}^{1/\alpha}+c\right)$$
$$\leqslant c\left(\|g\|_{q,\Omega_T}+1\right)^{\alpha_0 r/\alpha+1/\alpha}. \tag{2.42}$$

That is,

$$e^{\|u\|_{\infty,\Omega_T}}\leqslant c\left(\|g\|_q+1\right)^{\alpha_0 r/\alpha+1/\alpha}. \tag{2.43}$$

Take the logarithm and make a note of (2.8) to get

$$\|u\|_{\infty,\Omega_T}\leqslant c(\ln(\|f\|_{q,\Omega_T}+1)+1). \tag{2.44}$$

Finally, we derive

$$\begin{aligned}\|\varphi\|_\infty &\leqslant \|\varphi_1\|_\infty + \|\varphi_2\|_\infty \\ &\leqslant c\|f\|_{1+N/2,\Omega_T}(\ln(\|f\|_{q,\Omega_T}+1)+1) + \|\varphi_0\|_\infty.\end{aligned}$$

This completes the proof. □

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