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The Froude number for solitary water waves with vorticity

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We consider two-dimensional solitary water waves on a shear flow with an arbitrary distribution of vorticity. Assuming that the horizontal velocity in the fluid never exceeds the wave speed and that the free surface lies everywhere above its asymptotic level, we give a very simple proof that a suitably defined Froude number F must be strictly greater than the critical value $F=1$. We also prove a related upper bound on F, and hence on the amplitude, under more restrictive assumptions on the vorticity.

Key words: solitary waves, surface gravity waves, waves/free-surface flows

1. Introduction

1.1. *Statement of the main results*

We consider the motion of a two-dimensional fluid which is bounded above by a free surface under constant (atmospheric) pressure and below by a horizontal bed. Gravity acts as an external force, and there is no surface tension on the free surface. Inside the fluid, the velocity (u, v) and pressure P satisfy the incompressible Euler equations. We denote the horizontal bed by $y=-d$ and the free surface by $y=\eta(x, t)$. Fixing the constant wave speed $c>0$, we assume that the motion is steady in that η, u, v, and P depend on x and t only through the combination $x-ct$, which we henceforth abbreviate to x. We also assume that the wave is solitary in that

$$\eta \to 0, \quad v \to 0, \quad u \to U(y), \quad \text{as } x \to \pm\infty, \tag{1.1a–c}$$

uniformly in y. We assume that the horizontal velocity u in the fluid is everywhere strictly less than the wave speed c, and similarly for the horizontal velocity U of the shear flow at $x=\pm\infty$. Otherwise U is an arbitrary function of $-d\leqslant y\leqslant 0$. We call a solitary wave *trivial* if $\eta\equiv 0$, $v\equiv 0$, and $u\equiv U$. In the context of this paper, we will also call a solitary wave a *wave of elevation* if $\eta(x)\geqslant 0$ for all x but $\eta\not\equiv 0$. Similarly we call a solitary wave a *wave of depression* if $\eta(x)\leqslant 0$ for all x but $\eta\not\equiv 0$.

The classical Froude number is the dimensionless ratio $c/\sqrt{gd}$, where g is the acceleration due to gravity. When working with solitary waves on shear flows, however, we find it more convenient to define the Froude number F by

$$\frac{1}{F^2} = g\int_{-d}^{0} \frac{\mathrm{d}y}{(c-U(y))^2}, \tag{1.2}$$

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where the denominator in the integrand is uniformly bounded away from zero thanks to our assumption $U < c$. This definition reduces to the classical one when U vanishes identically, and it has the advantage that the critical Froude number is $F = 1$ regardless of the shear flow U. In particular, with this convention the small-amplitude solitary waves with vorticity constructed in Ter-Krikorov (1961), Groves & Wahlén (2008) and Hur (2008*a*) have Froude numbers F slightly bigger than 1.

The reader may assume that u, v, P, η, U are all C^2 or even smooth. On the other hand, our arguments go through unchanged for solutions with the more limited regularity

$$u, v, P \in W^{1,r}_{loc}(\overline{D_\eta}) \subset C^{\alpha}_{loc}(\overline{D_\eta}), \quad \eta \in C^{1+\alpha}(\mathbb{R}), \quad U \in W^{1,r}(-d, 0) \subset C^{\alpha}[-d, 0], \tag{1.3a–c}$$

where here $0 < \alpha < 1$, $r = 2/(1-\alpha)$, and $D_\eta = \{(x, y) : -d < y < \eta(x)\}$ denotes the fluid domain. By $w \in W^{1,r}_{loc}(\overline{D_\eta})$ we mean that $w \in W^{1,r}(D')$ whenever $\overline{D'} \subset \overline{D_\eta}$ is compact, and similarly for $C^{\alpha}_{loc}(\overline{D_\eta})$. The regularity (1.3) is an analogue for solitary waves of the regularity assumed in Theorem 2 of Constantin & Strauss (2011) for periodic waves.

THEOREM 1.1. *Consider a solitary wave with* $\sup u < c$ *and the regularity* (1.3). *Then* $F \neq 1$. *Moreover,* $F > 1$ *if it is a wave of elevation, and* $F < 1$ *if it is a wave of depression.*

In the irrotational case where the vorticity $\omega = v_x - u_y$ vanishes identically and U is constant, the assumption $\sup u < c$ is automatically satisfied (Toland 1996) and the bound $F > 1$ for waves of elevation is well-known (Starr 1947; Amick & Toland 1981; McLeod 1984). While the assumption $\sup u < c$ is still reasonable for waves with vorticity (Constantin & Strauss 2004; Constantin 2011), it rules out the existence of critical layers or stagnation points in flow.

We emphasize that, to our knowledge, there is no evidence of any kind for the existence of solitary waves of depression, or indeed for any solitary waves other than symmetric waves of elevation. In the irrotational case, it is known that there do not exist waves of depression which are in addition symmetric and monotone in that, after a translation, η is an even function of x with $\eta'(x) > 0$ for $x > 0$ (Keady & Pritchard 1974).

In some cases, the argument leading to Theorem 1.1 can be extended to give an upper bound on the Froude number for waves of elevation. Before giving this result, we define a dimensionless quantity $\varLambda \geqslant 1$ by

$$\varLambda = \max_y \frac{c - U(0)}{c - U(y)}. \tag{1.4}$$

We note that $\varLambda$, like F, only depends on the shear flow U at infinity and the wave speed c.

THEOREM 1.2. *For any solitary wave of elevation with the regularity* (1.3), $\sup u < c$, *and* $\varLambda < 2/\sqrt{3}$, *the Froude number* F *satisfies the upper bound*

$$F < (1 - \tfrac{3}{4}\varLambda^2)^{-1/2}. \tag{1.5}$$

For irrotational waves, U is constant, so clearly $\varLambda = 1 < 2/\sqrt{3}$ and hence Theorem 1.2 gives the well-known bound $F < 2$ (Starr 1947; Amick & Toland 1981; McLeod 1984). More generally, if the vorticity $\omega \leqslant 0$, then $U(y) \leqslant U(0)$ for

$-d \leqslant y \leqslant 0$ so that again $\Lambda = 1$, and Theorem 1.2 gives the same bound $F < 2$. In terms of the antiderivative $\Gamma(p)$ of the vorticity function and Bernoulli constant λ defined in § 2, the condition $\Lambda < 2/\sqrt{3}$ can be rephrased as $\lambda > -8 \min_p \Gamma(p)$.

In addition to their own independent interest, Theorems 1.1 and 1.2 have implications for the existence theory for large-amplitude solitary waves; see §§ 1.3 and 6.

1.2. *Historical discussion*

1.2.1. *Irrotational waves*

For irrotational waves, the asymptotic shear flow U is constant, and can be taken to be zero by switching to an appropriate reference frame. Our formula (1.2) for F then reduces to the classical ratio

$$F = \frac{c}{\sqrt{gd}}, \tag{1.6}$$

which is named in honor of William Froude, who in the 1870s argued that in order to compare the resistances felt by scaled models of a ship, the ratio of the speed of the ship to the square root of its length must be kept constant (Froude 1874).

The importance of the critical speed $c = \sqrt{gd}$ corresponding to $F = 1$ was known long before Froude. In 1781, Lagrange showed that long irrotational waves in shallow water travel with nearly this speed (Darrigol 2003), and in 1828 Bélanger showed that a hydraulic jump can occur only if the upstream flow has $F > 1$ (Bélanger 1828; Chanson 2009). More pertinent to this article is John Scott Russell's famous 1844 report (Darrigol 2003), which gives the empirical formula

$$F^2 \approx 1 + \frac{\max \eta}{d} \tag{1.7}$$

for the speed of small-amplitude irrotational solitary waves. Theoretical justifications of (1.7) came decades later with the work of Boussinesq in 1871 and Rayleigh in 1876 (Darrigol 2003).

In 1947, Starr gave a formal proof of the strikingly simple exact formula

$$F^2 = 1 + \frac{3}{2d} \frac{\int \eta^2 \mathrm{d}x}{\int \eta \mathrm{d}x} \tag{1.8}$$

for irrotational solitary waves (Starr 1947); see Longuet-Higgins (1974) for an alternative derivation. Given this identity, the upper and lower bounds $1 < F < 2$ for waves of elevation are straightforward. Indeed, since $\eta \geqslant 0$ does not vanish identically, (1.8) immediately implies $F > 1$. On the other hand, (1.8) also implies the upper bound

$$F^2 < 1 + \frac{3}{2} \frac{\max \eta}{d}. \tag{1.9}$$

Since $\max \eta \leqslant F^2 d/2$ by Bernoulli's law, (1.9) in turn implies $F^2 < 1 + 3F^2/4$ and hence $F < 2$. Substituting $F < 2$ back into Bernoulli's law we also obtain the bound $\max \eta < 2d$ on the amplitude.

In fact, Starr showed the improved upper bound

$$F^2 < 1 + \frac{\max \eta}{d} \tag{1.10}$$

in which the coefficient $3/2$ in (1.9) has been reduced to the 1 appearing in the asymptotic formula (1.7). See Keady & Pritchard (1974) for an alternative derivation. Arguing as in the previous paragraph, (1.10) leads to the bounds $F < \sqrt{2}$ and $\max \eta < d$ on the Froude number and amplitude. We note that the proofs of (1.10) in Starr (1947) and Keady & Pritchard (1974) do not depend on the identity (1.8).

Amick & Toland (1981) gave rigorous proofs of the bounds $1 < F < 2$ in their construction of large-amplitude irrotational solitary waves, which involves reformulating the water wave problem as a Nekrasov-type integral equation on the free surface. They objected to the assumption that the 'mass' $\int \eta \, \mathrm{d}x$ was finite in the earlier proofs, and instead, as McLeod (1984) puts it, 'take sixteen pages and much complicated estimating of integrals to prove $F > 1$ without the assumption of finite mass'. In response, McLeod (1984) showed that the earlier proofs could be easily modified to avoid the assumption of finite mass. Both Amick & Toland (1981) and McLeod (1984) showed as a consequence of $F > 1$ that the mass is necessarily finite.

Craig & Sternberg (1988) proved that all solitary waves with $F \geqslant 1$ are waves of elevation and furthermore that waves with $F > 1$ are symmetric and monotone in that, after a translation, η is an even function of x with $\eta'(x) < 0$ for $x > 0$. Thus a solitary wave has $F > 1$ if and only if it is a wave of elevation, and such waves are also symmetric and monotone. Waves with $F < 1$, if they exist, cannot be of elevation, nor can they be symmetric and monotone waves of depression (Keady & Pritchard 1974). In the infinite depth setting, on the other hand, there do not exist any solitary waves whose free surfaces $\eta(x)$ decay faster than $|x|^{-1-\varepsilon}$ for some $\varepsilon > 0$ (Hur 2012*b*); also see Sun (1999).

For later reference we also mention a third upper bound

$$F^2 < \frac{2(1 + \max \eta/d)^2}{2 + \max \eta/d}, \tag{1.11}$$

which was obtained by Keady & Pritchard (1974) using maximum principle arguments. It is easy to check that (1.11) is strictly weaker than (1.9) and (1.10). In particular, it cannot be combined with Bernoulli's law to obtain a bound on the Froude number which is independent of the amplitude $\max \eta$.

Numerics suggest that there is a one-parameter family of irrotational solitary waves connecting small-amplitude waves with F slightly bigger than 1 and the so-called wave of greatest height, which has a stagnation point at its crest where there is a corner with a 120° interior angle. The maximum value of the Froude number (as well as the maxima of mass, momentum, and energy) for this family is achieved before the wave of greatest height is reached, and there are values of the Froude number for which there exist two distinct solitary waves (Longuet-Higgins & Fenton 1974; Miles 1980). The wave with maximum Froude number has $F = 1.294$ and $\max \eta/d = 0.790$. The maximum amplitude is $\max \eta/d = 0.83322$ (Longuet-Higgins & Tanaka 1997) and hence the corresponding Froude number is $F = 1.29091$. We note that the existence of a wave of extreme form was proved rigorously in Amick & Toland (1981) and Amick, Fraenkel & Toland (1982), while the existence of bifurcation or turning points in the connected set of solutions containing small-amplitude solitary waves was proved in Plotnikov (1991).

1.2.2. *Waves with vorticity*

With vorticity, the importance of the critical value $F = 1$ for long waves in shallow water was recognized by Burns (1953); $F = 1$ is sometimes called the 'Burns condition'. Periodic solutions of the linearized equations exist if and only if $F < 1$, and $F \to 1$ as the period tends to infinity; see for instance Groves & Wahlén (2008), Hur & Lin (2008), and Kozlov, Kuznetsov & Lokharu (2014). On the other hand the small-amplitude solitary waves constructed by Ter-Krikorov (1961) and later by Hur (2008*a*) and then Groves & Wahlén (2008) all have F slightly bigger than 1.

Besides the existence results mentioned above, there are to our knowledge no lower bounds in the literature on the Froude number for solitary waves with vorticity. We note that the lower bound in Ter-Krikorov (1961) applies only to the small-amplitude waves constructed in that paper. The only upper bounds on the Froude number which we know of (Kozlov & Kuznetsov 2012; Wheeler 2013; also see Keady & Norbury 1982) are proved by maximum principle arguments and reduce to (1.11) for irrotational waves. In particular, these bounds seem not to lead to bounds on the Froude number which are independent of the amplitude $\max \eta$.

There are, however, many results on solitary waves with vorticity which require the assumption that $F > 1$. Hur proved exponential asymptotics (Hur 2008*b*) and analyticity of streamlines (Hur 2012*a*) for waves with $F > 1$, and symmetry for waves with $F > 1$ which are also waves of elevation (Hur 2008*b*) (see Matioc & Matioc 2012 for some related results without the assumption $F > 1$). Upper and lower bounds (or the lack thereof) on the Froude number also feature prominently in the construction by the author of large-amplitude solitary waves in Wheeler (2013), where it was also shown that waves with $F \geqslant 1$ are necessarily waves of elevation. See § 5 for more on the consequences of our main results for the amplitude, elevation, symmetry, monotonicity, and decay of solitary waves, and § 6 for more on the existence of large-amplitude waves.

With large positive constant vorticity, numerics seem to suggest the existence of overhanging waves with arbitrarily large Froude number (Vanden-Broeck 1994), a phenomenon which cannot occur for irrotational waves. See § 7 for versions of Theorems 1.1 and 1.2 in the special case of constant vorticity. Note that overhanging waves must have $u = c$ somewhere along their free surfaces, and hence do not satisfy our assumption $\sup u < c$.

It is worth noting that Burns (1953), as well as for instance Thompson (1949) and Freeman & Johnson (1970), are not restricted to waves with $\sup u < c$, nor are the bounds in Kozlov & Kuznetsov (2012). The assumption $\sup u < c$ is made, however, in all of the existence results for solitary waves mentioned earlier (Ter-Krikorov 1961; Groves & Wahlén 2008; Hur 2008*a*; Wheeler 2013). See Wahlén (2009) and Ehrnström, Escher & Wahlén (2011) for the construction of periodic waves where $u - c$ changes sign, and Kozlov & Kuznetsov (2014) for a quite general discussion of the dispersion relation.

Lastly, we emphasize that our definition (1.2) of the Froude number F is made only for convenience and that there are many other conventions in the literature. Fenton (1973), for instance, uses $(c - U(0))/\sqrt{gd}$, while Benjamin (1962) uses the ratio between the mean of $c - U$ and $\sqrt{gd}$. Our definition is essentially that of Ter-Krikorov (1961), who calls our $F\sqrt{gd}$ the 'speed of propagation'.

1.3. *Method of proof and further consequences*

1.3.1. *Integral identities with vorticity*

The main ingredient in the proof of Theorem 1.1 is an integral identity, Lemma 3.1, which seems to be completely new. In particular, we emphasize that this lemma is *not* a straightforward generalization of the identity (1.8) used in the irrotational case. Indeed, following the proof of (1.8) but retaining the extra terms coming from vorticity, one obtains a different identity, Lemma 4.2 (which we use in the proof of Theorem 1.2). Under the relatively strong assumption that vorticity is non-negative, Lemma 4.2 implies a lower bound on the relative speed $c - U(0)$ of asymptotic shear flow at the free surface. Except in the irrotational case where $U(y)$ is constant, however, this bound is not sufficient to show that the Froude number F is greater than the critical value 1. Indeed, the definition (1.2) of F involves the values $U(y)$ for all $-d \leqslant y \leqslant 0$, not just $y = 0$, and the lower bound $F > 1$ is sharp for small-amplitude waves.

After reading a preprint of this paper, Evgeniy Lokharu pointed out to the author that the proof of Lemma 3.1 can be extended to prove an analogue of $F \neq 1$ for periodic waves, or even for waves which are neither periodic nor solitary. Here the relevant quantity is not what we call the Froude number, but instead an appropriately defined Bernoulli constant. This result has since appeared in Kozlov, Kuznetsov & Lokharu (2015).

1.3.2. *Existence of large-amplitude waves*

In Wheeler (2013), the author constructed a connected set $\mathscr{C}$ of solitary waves of elevation whose asymptotic shear flows are given by

$$U(y) = U(y; F) = c - FU^*(y) \tag{1.12}$$

for some arbitrary but fixed positive function U^* satisfying a normalization condition. It is easy to see from (1.12) that the dimensionless parameter Λ is constant along $\mathscr{C}$; when $\Lambda < 2/\sqrt{3}$, we can combine Theorems 1.1 and 1.2 with the results in Wheeler (2013) to show that there exists a sequence of waves in $\mathscr{C}$ which approach stagnation in that $\sup u_n \to c$. This significant improvement is presented in detail in § 6 (Corollary 6.2), along with a related result for $\Lambda \geqslant 2/\sqrt{3}$ (Corollary 6.1).

1.3.3. *Waves with surface tension*

For irrotational waves with surface tension, Amick & Kirchgässner (1989) prove the analogue

$$F^2 = 1 + \frac{3}{2d}\frac{\int \eta^2 \mathrm{d}x}{\int \eta \mathrm{d}x} + \frac{\sigma}{gd}\frac{\int \left(\sqrt{1+\eta_x^2} - 1\right)\mathrm{d}x}{\int \eta \mathrm{d}x} \tag{1.13}$$

of (1.8), where $\sigma > 0$ is the constant coefficient of surface tension. This identity implies that solitary waves of elevation have $F > 1$ while solitary waves of depression have $F < 1$. In § 8, we will show that the same is true for waves with vorticity; indeed, with slightly stronger regularity assumptions, Lemma 3.1 and Theorem 1.1 continue to hold in the presence of surface tension and with nearly identical proofs.

While the existence of solitary waves of depression with Bond number $\sigma/gd^2 > 1/3$ is well-established (Amick & Kirchgässner 1989; Groves & Wahlén 2007), there is both numerical and analytical evidence that solitary waves of elevation do not exist for any values of σ (Sun 1999; Champneys, Vanden-Broeck & Lord 2002). In their place are so-called generalized solitary waves with small trains of periodic waves at infinity (Beale 1991), as well as families of oscillatory waves whose free surfaces vanish at infinity (Iooss & Kirchgässner 1990; Buffoni, Groves & Toland 1996; Groves & Wahlén 2007).

1.4. *Outline*

In § 2, we perform a standard change of variables and collect some formulae which will be used later. The main benefit of this change of variables is that it transforms the fluid domain into a fixed infinite strip, enabling us to multiply the Euler equations by functions defined in terms of the asymptotic shear flow $U(y)$ appearing in the definition (1.2) of the Froude number F.

In § 3, we give a short and elementary proof of Theorem 1.1, based on an integral identity, Lemma 3.1, in the transformed variables. The argument is perhaps even simpler than the argument in McLeod (1984), provided the change of variables in § 2 is taken for granted.

In § 4, we prove the upper bound Theorem 1.2 using two additional integral identities. The first, Lemma 4.2, is essentially (1.8) with an extra term coming from the vorticity, and is proved mostly in the original physical variables. The second, Lemma 4.3 is proved similarly to Lemma 3.1, but with a different test function.

In § 5, we collect some implications of Theorems 1.1 and 1.2 for the amplitude, mass, symmetry, monotonicity, and exponential decay of solitary waves of elevation.

In § 6, we show how Theorems 1.1 and 1.2 can be used to improve the existence theory for large-amplitude solitary waves with vorticity developed by the author in Wheeler (2013).

In § 7, we specialize Theorems 1.1 and 1.2 to the case where the vorticity is constant and more explicit formulae can be given.

Finally, in § 8 we prove that, with slightly modified regularity assumptions, Lemma 3.1 and Theorem 1.1 still hold for waves with surface tension.

2. Preliminaries

With the conventions from § 1.1, we assume that $u, v, P \in W^{1,r}_{loc}(\overline{D_\eta})$ satisfy the stationary incompressible Euler equations

$$(u-c)u_x + vu_y = -P_x, \qquad (2.1a)$$
$$(u-c)v_x + vv_y = -P_y - g, \qquad (2.1b)$$
$$u_x + v_y = 0, \qquad (2.1c)$$

in $L^r_{loc}(\overline{D_\eta})$, together with the boundary conditions

$$v = 0 \quad \text{on } y = -d, \qquad (2.1d)$$
$$v = (u-c)\eta_x \quad \text{on } y = \eta(x), \qquad (2.1e)$$
$$P = 0 \quad \text{on } y = \eta(x), \qquad (2.1f)$$

pointwise, and the asymptotic conditions

$$\eta \to 0, \quad v \to 0, \quad u \to U(y) \quad \text{as } x \to \pm\infty, \qquad (2.1g)$$

uniformly in y. The free surface elevation η has the regularity $\eta \in C^{1+\alpha}(\mathbb{R})$, and $U \in W^{1,r}(-d, 0)$. Here for convenience we have normalized P to vanish on the free surface; P therefore represents the difference between the pressure in the fluid and the atmospheric pressure.

Because of incompressibility (2.1c), there exists a stream function $\psi \in W^{2,r}_{loc}(\overline{D_\eta}) \subset C^{1+\alpha}_{loc}(\overline{D_\eta})$ which satisfies $\psi_y = u - c$ and $\psi_x = v$. From now on we will always assume $\sup u < c$, or equivalently $\sup \psi_y < 0$. By the kinematic boundary conditions

(2.1d)–(2.1e), ψ is constant on $y=-d$ and $y=\eta(x)$. Thus the flux

$$m := \psi(x, -d) - \psi(x, \eta(x)) = \int_{-d}^{\eta(x)} (c - u(x, y))\mathrm{d}y = \int_{-d}^{0} (c - U(y))\mathrm{d}y \tag{2.2}$$

is independent of x. We normalize ψ so that $\psi = 0$ on $y = \eta(x)$ and $\psi = -m$ on $y=-d$. The vorticity ω is given in terms of ψ by

$$\omega = v_x - u_y = -\Delta\psi = \gamma(\psi) \tag{2.3}$$

for some function $\gamma \in L^r[-m, 0]$ called the *vorticity function* (Constantin & Strauss 2011). Because of the asymptotic condition (2.1g),

$$\gamma(\Psi(y)) = -U_y(y), \quad \text{where } \Psi(y) = \int_0^y (U(s) - c)\mathrm{d}s, \tag{2.4}$$

so that γ is completely determined by U and c.

Using

$$q = x, \quad p = -\psi \tag{2.5a,b}$$

as independent variables, we can rewrite (2.1) in terms of the so-called height function $h(q, p)$ defined by

$$h(x, -\psi(x, y)) = y + d. \tag{2.6}$$

The advantage of this formulation is that h is defined on the fixed domain

$$\Omega := \{(q, p) : -m < p < 0\}. \tag{2.7}$$

Defining the asymptotic height function $H(p)$ in a similar way,

$$H(-\Psi(y)) = y + d, \tag{2.8}$$

where $\Psi(y)$ is as in (2.4), we have $H(0) = d$ and $H(-m) = 0$. Since (2.6) implies $h_p^{-1} = -\psi_y$, we necessarily have $\inf h_p > 0$ and $\min H_p > 0$.

The arguments in Constantin & Strauss (2011) show that, under the crucial assumption $\sup u < c$, the solitary water wave problem (2.1) is equivalent to the system

$$\left(-\frac{1+h_q^2}{2h_p^2} + \frac{1}{2H_p^2}\right)_p + \left(\frac{h_q}{h_p}\right)_q = 0 \quad \text{for } -m < p < 0, \tag{2.9a}$$

$$\frac{1+h_q^2}{2h_p^2} - \frac{1}{2H_p^2} + g(h - H) = 0 \quad \text{on } p = 0, \tag{2.9b}$$

$$h = 0 \quad \text{on } p = -m, \tag{2.9c}$$

for $h \in W^{2,r}_{loc}(\overline{\Omega}) \subset C^{1+\alpha}_{loc}(\overline{\Omega})$ and $H \in W^{2,r}[-m, 0]$ together with the asymptotic conditions

$$h_p \to H_p, \quad h_q \to 0 \quad \text{as } q \to \pm\infty, \text{ uniformly in } p. \tag{2.9d}$$

The velocity field (u, v) and free surface η can be recovered from h via

$$c - u = \frac{1}{h_p}, \quad v = -\frac{h_q}{h_p}, \quad \eta(q) = h(q, 0) - H(0). \tag{2.10a–c}$$

We note that the divergence-form equation (2.9a) expresses the balance of the y-component of momentum and that (2.9b) is Bernoulli's law evaluated on the free surface.

Defining the antiderivative $\Gamma \in W^{1,r}[-m, 0] \subset C^{\alpha}[-m, 0]$ of the vorticity function γ and the Bernoulli constant λ by

$$\Gamma(p) = \int_0^p \gamma(-s)\mathrm{d}s, \quad \lambda = (U(0) - c)^2 = \frac{1}{H_p^2(0)}, \tag{2.11a,b}$$

we have the following useful relation between γ, H, and U:

$$(U - c)^2(H(p)) = \frac{1}{H_p^2(p)} = \lambda + 2\Gamma(p). \tag{2.12}$$

In particular, the Froude number F is given in terms of H by

$$\frac{1}{F^2} = g\int_{-d}^0 \frac{\mathrm{d}y}{(U(y) - c)^2} = g\int_{-m}^0 H_p^3(p)\mathrm{d}p. \tag{2.13}$$

In addition, Bernoulli's law can be written as

$$P + \frac{(u - c)^2 + v^2}{2} + gy - \frac{\lambda}{2} - \Gamma(-\psi) \equiv 0. \tag{2.14}$$

Indeed, one can see that the left-hand side is constant by differentiating and using (2.1*a*) and (2.1*b*). The fact that this constant is zero follows by sending $x \to \pm\infty$ in the dynamic boundary condition (2.1*f*).

3. Lower bound

LEMMA 3.1. *Any solitary wave with* $\sup u < c$ *and the regularity (1.3) satisfies*

$$\left(\frac{1}{F^2} - 1\right)\int_{-M}^M \eta\mathrm{d}x + \int_{-M}^M\int_{-m}^0 \frac{H_p^3h_q^2 + (2h_p + H_p)(h_p - H_p)^2}{2h_p^2}\mathrm{d}p\mathrm{d}q \to 0 \quad \textit{as } M \to \infty. \tag{3.1}$$

Proof. Defining the function

$$\Phi(p) = \int_{-m}^p H_p^3(s)\mathrm{d}s \tag{3.2}$$

(which solves the linearization of (2.9*a*) and (2.9*c*) around $h = H$), we note that (2.13) implies $g\Phi(0) = 1/F^2$. Multiplying (2.9*a*) by Φ and integrating by parts, we then have, for any $M > 0$,

$$\begin{aligned} 0 &= \int_{-M}^M\int_{-m}^0 \left[\left(-\frac{1 + h_q^2}{2h_p^2} + \frac{1}{2H_p^2}\right)_p \Phi + \left(\frac{h_q}{h_p}\right)_q \Phi\right]\mathrm{d}p\mathrm{d}q \\ &= \int_{-M}^M\int_{-m}^0 \left(\frac{1 + h_q^2}{2h_p^2} - \frac{1}{2H_p^2}\right)H_p^3\mathrm{d}p\mathrm{d}q \\ &\quad + \frac{1}{gF^2}\int_{-M}^M \left(-\frac{1 + h_q^2}{2h_p^2} + \frac{1}{2H_p^2}\right)(q, 0)\mathrm{d}q + \int_{-m}^0 \frac{h_q}{h_p}\Phi\mathrm{d}p\bigg|_{q=-M}^{q=M}. \end{aligned} \tag{3.3}$$

Since $h_q \to 0$ as $q \to \pm\infty$ by (2.9*d*), the third term in (3.3) vanishes as $M \to \infty$. Using the boundary condition (2.9*b*) to simplify the second term, we obtain

$$\int_{-M}^{M}\int_{-m}^{0}\left(\frac{1+h_q^2}{2h_p^2}-\frac{1}{2H_p^2}\right)H_p^3\,\mathrm{d}p\,\mathrm{d}q+\frac{1}{F^2}\int_{-M}^{M}(h(q,0)-H(0))\,\mathrm{d}q\to 0 \tag{3.4}$$

as $M \to \infty$. Rewriting the first integrand in (3.4) as

$$\left(\frac{1+h_q^2}{2h_p^2}-\frac{1}{2H_p^2}\right)H_p^3=-(h_p-H_p)+\frac{H_p^3h_q^2+(2h_p+H_p)(h_p-H_p)^2}{2h_p^2}, \tag{3.5}$$

we see that

$$\int_{-M}^{M}\int_{-m}^{0}\left(\frac{1+h_q^2}{2h_p^2}-\frac{1}{2H_p^2}\right)H_p^3\,\mathrm{d}p\,\mathrm{d}q=-\int_{-M}^{M}(h(q,0)-H(0))\,\mathrm{d}q \\ +\int_{-M}^{M}\int_{-m}^{0}\frac{H_p^3h_q^2+(2h_p+H_p)(h_p-H_p)^2}{2h_p^2}\,\mathrm{d}p\,\mathrm{d}q. \tag{3.6}$$

Plugging (3.6) into (3.4), rearranging terms, and using the identity $h(q,0)-H(0)=\eta(q)$ from (2.10), we obtain (3.1) as desired. □

Proof of Theorem 1.1. Consider a non-trivial solitary wave. Then H_p and h_p are strictly positive, and $h_q(q,0)=\eta_x(q)$ does not vanish identically. Thus the second integral in (3.1) is a non-decreasing function of M and is strictly positive for M sufficiently large, and therefore the limit in (3.1) implies

$$\limsup_{M\to\infty}\left\{\left(\frac{1}{F^2}-1\right)\int_{-M}^{M}\eta\,\mathrm{d}x\right\}<0. \tag{3.7}$$

Since the left-hand side of (3.7) vanishes if $F=1$, we must have $F\neq 1$. For a wave of elevation, $\eta(x)\geqslant 0$ for all x but $\eta\not\equiv 0$, so (3.7) implies that the coefficient $1/F^2-1$ is strictly negative, i.e. that $F>1$. Similarly, for a wave of depression, $\eta(x)\leqslant 0$ for all x but $\eta\not\equiv 0$, so (3.7) implies that $1/F^2-1$ is strictly positive, i.e. that $F<1$. □

4. Upper bound

In this section we will make use of the vorticity function $\gamma(-p)$, Bernoulli constant $\lambda=(c-U(0))^2$, and antiderivative $\Gamma(p)$ of γ defined in § 2. We begin by giving a formula in our notation for an invariant called the flow force.

LEMMA 4.1. *For any solitary wave with* $\sup u<c$ *and the regularity* (1.3) *and for any* x, *the flow force*

$$S:=\int_{-d}^{\eta(x)}(P+(u-c)^2)(x,y)\,\mathrm{d}y=-2\int_{-m}^{0}\gamma H\,\mathrm{d}p+\lambda d+\frac{gd^2}{2}. \tag{4.1}$$

Proof. That S is independent of x is well-known, and can be proved, for instance, by integrating the identity $(P+(u-c)^2)_x + ((u-c)v)_y = 0$ over a region of the form $\{(x, y) : a < x < b, -d < y < \eta(x)\}$ using the divergence theorem and then applying the boundary conditions. To obtain the formula (4.1), we use Bernoulli's law (2.14) to rewrite

$$P+(u-c)^2 = \frac{(u-c)^2 - v^2}{2} - gy + \frac{\lambda}{2} + \Gamma = \frac{1-h_q^2}{2h_p^2} - g(h-d) + \frac{\lambda}{2} + \Gamma. \tag{4.2}$$

The asymptotic condition (2.9*d*) gives

$$\frac{1-h_q^2}{2h_p^2} \to \frac{1}{2H_p^2} = \frac{\lambda}{2} + \Gamma \quad \text{as } q \to \pm\infty, \tag{4.3}$$

uniformly in p, and hence, since S is independent of x,

$$\begin{aligned} S &= \lim_{q\to\pm\infty} \int_{-m}^{0} \left(\frac{1-h_q^2}{2h_p^2} - g(h-d) + \frac{\lambda}{2} + \Gamma \right) h_p \mathrm{d}p \\ &= \int_{-m}^{0} (\lambda + 2\Gamma - g(H-d)) H_p \mathrm{d}p \\ &= -2\int_{-m}^{0} \gamma H \mathrm{d}p + \lambda d + \frac{gd^2}{2}, \end{aligned} \tag{4.4}$$

where in the last step we integrated by parts using $H(-m) = \Gamma(0) = 0$ and $\Gamma_p = \gamma$. □

The following integral identity is (1.8) but with an extra term coming from the vorticity. Unlike Lemmas 3.1 and 4.3, it is proved mostly in the original physical variables.

LEMMA 4.2. *Any solitary wave with* $\sup u < c$ *and the regularity* (1.3) *satisfies*

$$(\lambda - gd)\int_{-M}^{M} \eta \mathrm{d}x - \frac{3g}{2}\int_{-M}^{M} \eta^2 \mathrm{d}x - 2\int_{-M}^{M}\int_{-m}^{0} \gamma(h-H)\mathrm{d}p\mathrm{d}q \to 0 \quad \textit{as } M\to\infty. \tag{4.5}$$

Proof. Consider the fluid region

$$D = \{(x, y) \in \mathbb{R}^2 : -M < x < M, \ \ -d < y < \eta(x)\}, \tag{4.6}$$

and the two $W^{1,r}(D)$ vector fields

$$\boldsymbol{A} = (\boldsymbol{A}^1, \boldsymbol{A}^2) = (P+(u-c)^2, (u-c)v), \tag{4.7}$$
$$\boldsymbol{B} = (\boldsymbol{B}^1, \boldsymbol{B}^2) = ((u-c)v, P+v^2+gy). \tag{4.8}$$

By the incompressible Euler equations (2.1*a*)–(2.1*c*), $\boldsymbol{A}$ and $\boldsymbol{B}$ are both divergence free. Thus

$$\begin{aligned} \operatorname{div}(x\boldsymbol{A} + (y+d)\boldsymbol{B}) &= \boldsymbol{A}^1 + \boldsymbol{B}^2 \\ &= 2P + (u-c)^2 + v^2 + gy \\ &= \lambda + 2\Gamma(-\psi) - gy \end{aligned} \tag{4.9}$$

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in $L^r(D)$, where in the last step we have used Bernoulli's law (2.14). Integrating (4.9) over D, the divergence theorem gives

$$\int_{\partial D} (x\boldsymbol{A} + (y+d)\boldsymbol{B}) \boldsymbol{\cdot} \boldsymbol{n}\,\mathrm{d}s = \iint_D (\lambda - gy + 2\Gamma(-\psi))\,\mathrm{d}y\,\mathrm{d}x, \tag{4.10}$$

where $\boldsymbol{n}$ is an outward pointing normal. We will obtain (4.5) by simplifying both sides of (4.10) and using Lemma 4.1.

First consider the left-hand side of (4.10). On the free surface, the two boundary conditions $v = \eta_x(u-c)$ and $P = 0$ give

$$(x\boldsymbol{A} + (y+d)\boldsymbol{B}) \boldsymbol{\cdot} \boldsymbol{n} = (x\boldsymbol{A} + (y+d)\boldsymbol{B}) \cdot \frac{(v, c-u)}{\sqrt{(u-c)^2+v^2}} = \frac{gy(y+d)(c-u)}{\sqrt{(u-c)^2+v^2}}, \tag{4.11}$$

while on the bottom $y=-d$ the boundary condition $v=0$ implies $\boldsymbol{A}^2 = (y+d)\boldsymbol{B}^2 = 0$. Thus we see

$$\begin{aligned}\int_{\partial D} (x\boldsymbol{A} + (y+d)\boldsymbol{B}) \boldsymbol{\cdot} \boldsymbol{n}\,\mathrm{d}s &= x\int_{-d}^{\eta(x)} (P + (u-c)^2)\,\mathrm{d}y\bigg|_{x=-M}^{x=M} \\ &\quad + \int_{-d}^{\eta(x)} (y+d)(u-c)v\,\mathrm{d}y\bigg|_{x=-M}^{x=M} + \int_{-M}^{M} g(\eta+d)\eta\,\mathrm{d}x. \end{aligned}\tag{4.12}$$

By Lemma 4.1, we have

$$\int_{-d}^{\eta(x)} (P + (u-c)^2)\,\mathrm{d}y = -2\int_{-m}^{0} \gamma(-p)H\,\mathrm{d}p + \lambda d + \frac{gd^2}{2} \tag{4.13}$$

for all x, and since $v \to 0$ uniformly in y as $x \to \pm\infty$, the second term on the right-hand side of (4.12) vanishes as $M \to \infty$. Thus (4.12) implies

$$\begin{aligned}&\int_{\partial D} (x\boldsymbol{A} + (y+d)\boldsymbol{B}) \boldsymbol{\cdot} \boldsymbol{n}\,\mathrm{d}s - g\int_{-M}^{M} \eta^2\,\mathrm{d}x \\ &\qquad - gd\int_{-M}^{M} \eta\,\mathrm{d}x - 2M\left(\lambda d + \frac{gd^2}{2} - 2\int_{-m}^{0} \gamma H\,\mathrm{d}p\right) \to 0 \end{aligned}\tag{4.14}$$

as $M \to \infty$.

Now we turn to the right-hand side of (4.10). Changing variables and integrating by parts,

$$\int_{-d}^{\eta(x)} \Gamma(-\psi)\,\mathrm{d}y = \int_{-m}^{0} \Gamma(p)h_p\,\mathrm{d}p = -\int_{-m}^{0} \gamma(-p)h\,\mathrm{d}p, \tag{4.15}$$

where we have used that Γ vanishes on $p=0$ while h vanishes on $p=-m$. Thus

$$\begin{aligned}\iint_D (\lambda - gy + 2\Gamma(-\psi))\,\mathrm{d}y\,\mathrm{d}x &= \lambda\int_{-M}^{M} \eta\,\mathrm{d}x - \frac{g}{2}\int_{-M}^{M} \eta^2\,\mathrm{d}x \\ &\quad -2\int_{-M}^{M}\int_{-m}^{0} \gamma h\,\mathrm{d}p\,\mathrm{d}q + 2M\left(\lambda d + \frac{gd^2}{2}\right). \end{aligned}\tag{4.16}$$

Substituting (4.14) and (4.16) into (4.10), most of the terms drop out and we are left with (4.5) as desired. □

LEMMA 4.3. *Any solitary wave with $u < c$ and the regularity (1.3) satisfies*

$$\frac{3g}{2}\int_{-M}^{M}\eta^2\mathrm{d}x - \int_{-M}^{M}\int_{-m}^{0}\frac{H_p^3h_q^2+(2h_p+H_p)(h_p-H_p)^2}{2H_p^2h_p^2}\mathrm{d}p\mathrm{d}q \to 0 \quad \textit{as } M\to\infty. \tag{4.17}$$

Proof. We argue as in the proof of Lemma 3.1, but with the function Φ replaced by H, and then appeal to Lemma 4.2. Multiplying (2.9*a*) by H and integrating by parts, we have, for any $M > 0$,

$$\begin{aligned} 0 &= \int_{-M}^{M}\int_{-m}^{0}\left[\left(-\frac{1+h_q^2}{2h_p^2}+\frac{1}{2H_p^2}\right)_p H + \left(\frac{h_q}{h_p}\right)_q H\right]\mathrm{d}p\mathrm{d}q \\ &= \int_{-M}^{M}\int_{-m}^{0}\left(\frac{1+h_q^2}{2h_p^2}-\frac{1}{2H_p^2}\right)H_p\mathrm{d}p\mathrm{d}q + gd\int_{-M}^{M}\eta\mathrm{d}q \\ &\quad + \int_{-m}^{0}\frac{h_q}{h_p}H\mathrm{d}p\bigg|_{q=-M}^{q=M}, \end{aligned} \tag{4.18}$$

where we have used the boundary condition (2.9*b*) as well as $h(q, 0) - H(0) = \eta(q)$ and $H(0) = d$. As in the proof of Lemma 3.1, the asymptotic conditions (2.9*d*) imply that the last term in (4.18) vanishes as $M \to \infty$. The integrand in the first term of (4.18) can be rewritten as

$$\left(\frac{1+h_q^2}{2h_p^2}-\frac{1}{2H_p^2}\right)H_p = -\frac{h_p-H_p}{H_p^2} + \frac{H_p^3h_q^2+(2h_p+H_p)(h_p-H_p)^2}{2h_p^2H_p^2}, \tag{4.19}$$

as can be seen by dividing (3.5) by H_p^2. Since

$$\left(\frac{1}{H_p^2}\right)_p = (\lambda+2\Gamma)_p = 2\gamma, \qquad \frac{1}{H_p^2(0)} = \lambda, \tag{4.20a,b}$$

we can integrate the first term of (4.19) by parts to get

$$\begin{aligned} -\int_{-m}^{0}\frac{h_p-H_p}{H_p^2}(q,p)\mathrm{d}p &= 2\int_{-m}^{0}\left(\frac{1}{H_p^2}\right)_p (h(q,p)-H(p))\mathrm{d}p - \lambda(h(q,0)-H(0)) \\ &= 2\int_{-m}^{0}\gamma(h(q,p)-H(p))\mathrm{d}p - \lambda\eta(q). \end{aligned} \tag{4.21}$$

Putting everything together, we see that (4.18) implies

$$\begin{aligned} &\int_{-M}^{M}\int_{-m}^{0}\frac{H_p^3h_q^2+(2h_p+H_p)(h_p-H_p)^2}{2h_p^2H_p^2}\mathrm{d}p\mathrm{d}q + 2\int_{-M}^{M}\int_{-m}^{0}\gamma(h-H)\mathrm{d}p\mathrm{d}q \\ &\quad + (gd-\lambda)\int_{-M}^{M}\eta\mathrm{d}q \longrightarrow 0 \quad \text{as } M\to\infty. \end{aligned} \tag{4.22}$$

Adding (4.22) with the conclusion (4.5) of Lemma 4.2, most of the terms cancel and we are left with (4.17) as desired. □

Using Lemmas 3.1 and 4.3, we can now prove Theorem 1.2.

Proof of Theorem 1.2. The key observation is that the integrands of the double integrals in (3.1) and (4.17) differ only by a factor of H_p^2. Plugging the crude estimate

$$\int_{-M}^{M}\int_{-m}^{0}\frac{H_p^3h_q^2+(2h_p+H_p)(h_p-H_p)^2}{2h_p^2}\,\mathrm{d}p\,\mathrm{d}q \leqslant \max H_p^2\int_{-M}^{M}\int_{-m}^{0}\frac{H_p^3h_q^2+(2h_p+H_p)(h_p-H_p)^2}{2h_p^2H_p^2}\,\mathrm{d}p\,\mathrm{d}q \quad (4.23)$$

into the result (3.1) of Lemma 3.1, we obtain

$$\limsup_{M\to\infty}\left\{\left(1-\frac{1}{F^2}\right)\int_{-M}^{M}\eta\,\mathrm{d}x - \max H_p^2\int_{-M}^{M}\int_{-m}^{0}\frac{H_p^3h_q^2+(2h_p+H_p)(h_p-H_p)^2}{2h_p^2H_p^2}\,\mathrm{d}p\,\mathrm{d}q\right\}\leqslant 0. \quad (4.24)$$

Subtracting the conclusion (4.17) of Lemma 4.3 multiplied by $\max H_p^2$, we then get

$$\limsup_{M\to\infty}\left\{\left(1-\frac{1}{F^2}\right)\int_{-M}^{M}\eta\,\mathrm{d}x-\frac{3g\max H_p^2}{2}\int_{-M}^{M}\eta^2\,\mathrm{d}q\right\}\leqslant 0. \quad (4.25)$$

But from Bernoulli's law (2.14) evaluated at the crest we have $g\max\eta<\lambda/2$, so that (4.25) implies

$$\limsup_{M\to\infty}\left(1-\frac{1}{F^2}-\frac{3\lambda}{4}\max H_p^2\right)\int_{-M}^{M}\eta\,\mathrm{d}x<0, \quad (4.26)$$

and hence

$$1-\frac{1}{F^2}-\frac{3\lambda}{4}\max H_p^2<0. \quad (4.27)$$

By (2.11) and (2.12), $\lambda=(U-c)^2(0)$ and $H_p^2=(U-c)^{-2}$, so

$$\lambda\max_p H_p^2=\max_y\frac{(c-U(0))^2}{(c-U(y))^2}=\Lambda^2, \quad (4.28)$$

and (4.27) becomes

$$1-\frac{1}{F^2}-\frac{3\Lambda^2}{4}<0, \quad (4.29)$$

which, assuming $\Lambda<2/\sqrt{3}$, is equivalent to the desired upper bound on F in (1.5). □

5. Amplitude, elevation, symmetry, monotonicity, and decay

In this section we will give several corollaries of Theorems 1.1 and 1.2 and their proofs. Some of these will require stronger regularity and decay assumptions than (1.3) and (2.1*g*), namely

$$h \in C^{2+\beta}(\overline{\Omega}), \quad H \in C^{2+\beta}[-d, 0], \tag{5.1a,b}$$

$$h - H, D(h - H), D^2(h - H) \to 0 \quad \text{as } q \to \pm\infty, \tag{5.2}$$

where $\beta \in (0, 1)$ is arbitrary and the limit in (5.2) is uniform in p.

COROLLARY 5.1 (Bound on the amplitude). *In the setting of Theorem 1.2, the maximum amplitude* $\max_x \eta(x)$ *satisfies the following upper bound:*

$$\frac{\max \eta}{d} < \frac{(c - U(0))^2}{2gd} < \frac{1}{2}\frac{\Lambda^2}{1 - \frac{3}{4}\Lambda^2}. \tag{5.3}$$

Proof. The first inequality in (5.3) is just Bernoulli's law (2.14) evaluated at the crest. Next, we note that the definitions (1.2), (1.4), and (2.11) of F and Λ immediately imply the simple inequality $(c - U(0))^2 \leqslant gd\Lambda^2F^2$. The second inequality in (5.3) then follows from the upper bound on F in Theorem 1.2. □

When $U(y) \leqslant U(0)$ for $-d \leqslant y \leqslant 0$, such as for instance when $\omega \leqslant 0$ so that $U_y \geqslant 0$, we have $\Lambda = 1$ and hence that the second inequality in (5.3) is the simple bound $\max \eta \leqslant 2d$. See § 7 for the case of constant vorticity.

The following is a partial converse of Theorem 1.1.

PROPOSITION 5.2. *A solitary wave with* $\sup u < c$*,* $F \geqslant 1$*, and the regularity* (5.1) *and decay* (5.2) *is a strict wave of elevation in that* $\eta(x) > 0$ *for all* $x \in \mathbb{R}$.

Proof. This is a slightly simplified version of Proposition 2.1 in Wheeler (2013). □

COROLLARY 5.3 (Elevation). *A solitary wave with* $\sup u < c$ *and the regularity* (5.1) *and decay* (5.2) *is a wave of elevation if and only if* $F \geqslant 1$*, and in this case it is a strict wave of elevation in that* $\eta(x) > 0$ *for all* $x \in \mathbb{R}$ *and moreover* $F > 1$.

Proof. This is an immediate consequence of Theorem 1.1 and Proposition 5.2. □

While there are no symmetric and monotone irrotational solitary waves of depression (Keady & Pritchard 1974), it is an open question if the same is true with vorticity. More generally, it is unknown if there are any solitary waves with subcritical Froude number $F < 1$. By Theorem 1.1, no such wave could be a wave of elevation, and in fact (3.1) implies something stronger.

COROLLARY 5.4 (Negative mass for subcritical waves). *Any non-trivial solitary wave with* $\sup u < c$*, the regularity* (1.3)*, and subcritical Froude number* $F < 1$ *must have*

$$\limsup_{M\to\infty} \int_{-M}^{M} \eta \, dx < 0. \tag{5.4}$$

Proof. This follows immediately from (3.7). □

The following corollary is interesting only in the method of proof; the conclusion of Corollary 5.6, which follows from Hur (2008*b*), is much stronger.

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COROLLARY 5.5 (Finite mass). *For any solitary wave of elevation with* $\sup u < c$ *and the regularity* (1.3), *all of the definite integrals appearing in Lemmas* 3.1, 4.2 *and* 4.3, *have a (finite) limit as* $M \to \infty$. *In particular, the limits in* (3.1), (4.5), *and* (4.17) *become equalities when* M *is replaced by* $+\infty$.

Proof. We argue as in Section 3 of McLeod (1984). Assume for contradiction that

$$\int_{-M}^{M} \eta \, \mathrm{d}x \to \infty \quad \text{as } M \to \infty. \tag{5.5}$$

Since $\eta \to 0$ as $x \to \pm\infty$, we have

$$\frac{\displaystyle\int_{-M}^{M} \eta^2 \, \mathrm{d}x}{\displaystyle\int_{-M}^{M} \eta \, \mathrm{d}x} \to 0 \quad \text{as } M \to \infty. \tag{5.6}$$

But then, since $F > 1$ by Theorem 1.1, the left-hand side of (4.25) tends to $+\infty$ as $M \to \infty$, a contradiction. Thus $\int_{-\infty}^{\infty} \eta \, \mathrm{d}x$ and hence $\int_{-\infty}^{\infty} \eta^2 \, \mathrm{d}x$ are both finite. The statement then follows by combining this result with the limits in (3.1), (4.5), and (4.17). □

In order to state the final corollary in this section, we introduce the Sturm–Liouville problem

$$\left(\frac{\varphi_p}{H_p^3}\right)_p + \mu \frac{\varphi}{H_p} = 0 \quad \text{for } -m < p < 0, \tag{5.7a}$$

$$\frac{\varphi_p(0)}{H_p^3} - g\varphi(0) = 0, \quad \varphi(-m) = 0, \tag{5.7b}$$

which appears when studying the linearization of (2.9) around $h = H$. We note that, when $F = 1$, function $\Phi(p)$ used in the proof of Lemma 3.1 solves (5.7) with $\mu = 0$. We define μ_1, μ_2 to be the smallest and second smallest eigenvalues of (5.7), and φ_1 to be the eigenfunction corresponding to μ_1. When $F > 1$, μ_1 is positive, and we can take φ_1 to be positive for $-m < p \leqslant 0$ (Hur 2008*b*). We note that (5.7) is equivalent to the system

$$(U - c)(\tilde{\varphi}_{yy} + \mu\tilde{\varphi}) - U_{yy}\tilde{\varphi} = 0 \quad \text{for } -d < y < 0, \tag{5.8a}$$

$$(U - c)^2\tilde{\varphi}_y(0) - (g + (U - c)U_y)\tilde{\varphi}(0) = 0, \quad \tilde{\varphi}(-d) = 0, \tag{5.8b}$$

for $\tilde{\varphi}(y) := (c - U(y))\varphi(p)$ in the original physical variables; see Lemma 2.3 in Hur & Lin (2008).

COROLLARY 5.6 (Symmetry, monotonicity, and decay). *Any solitary wave of elevation with* $\sup u < c$ *and the regularity* (5.1) *and decay* (5.2) *has the following properties.*

(*a*) *(Symmetry and monotonicity). The wave is symmetric and monotone in that, after shifting the definition of the horizontal variable* q*, the height function* h *is even in* q *and has* $h_q < 0$ *for* $q > 0$ *and* $-m < p \leqslant 0$. *In particular, after this shift* η *is an even function of* x *with* $\eta_x < 0$ *for* $x > 0$.

(*b*) *(Decay and asymptotics). The difference* $w := h - H$ *decays exponentially as* $q \to \pm\infty$, *and satisfies the asymptotic estimate*

$$|D^k(w(q, p) - r\varphi_1(p)\mathrm{e}^{-\sqrt{\mu_1}|q|})| \leqslant C\mathrm{e}^{-s_1|q|} \quad \text{for } k \leqslant 1,\ |q| > 1, \tag{5.9}$$

for some constants $C, r > 0$ *depending on* w, *where* $\mu_1 > 0$ *and* φ_1 *are defined above in terms of the Sturm–Liouville problem* (5.7), *and the exponent* s_1 *appearing on the right-hand side satisfies* $\sqrt{\mu_1} < s_1 < \min(2\sqrt{\mu_1}, \sqrt{\mu_2})$.

Proof. Since the waves considered in this corollary have $F > 1$ by Theorem 1.1, part (*a*) follows immediately from Theorem 3.1 in Hur (2008*b*), while part (*b*) follows from Proposition 4.6 in the same paper. □

6. Existence of large-amplitude waves

In this section we observe the implications of Theorems 1.1 and 1.2 for the existence theory of large-amplitude solitary waves with vorticity developed in Wheeler (2013). This theory involves a one-parameter family of shear flows

$$U(y) = c - FU^*(y), \tag{6.1}$$

where U^* is an arbitrary but fixed strictly positive function satisfying the normalization condition

$$g\int_{-d}^{0} \frac{\mathrm{d}y}{U^*(y)^2} = 1. \tag{6.2}$$

The normalization (6.2) ensures that the parameter F in (6.1) is indeed the Froude number F defined in (1.2). In the corollary below we call a solitary wave *symmetric* if u and η are even in x, and v is odd in x, and *monotone* if in addition $\eta(x)$ is strictly decreasing for $x > 0$.

COROLLARY 6.1 (Existence of large-amplitude waves). *Fix* $g, c, d > 0$, *a Hölder parameter* $0 < \beta \leqslant 1/2$, *and a strictly positive function* $U^* \in C^{2+\beta}[-d, 0]$ *satisfying the normalization condition* (6.2). *Then there exists a connected set* $\mathscr{C}$ *of solitary waves*

$$(u, v, \eta, F) \in C^{1+\beta} \times C^{1+\beta} \times C^{2+\beta}(\mathbb{R}) \times (1, \infty), \tag{6.3}$$

where F *determines the asymptotic shear flow* U *via* (6.1), *with the following properties. Each wave in* $\mathscr{C}$ *is a symmetric and monotone wave of elevation with* $\sup u < c$ *and* $F > 1$. *Moreover, at least one of the following two conditions holds:*

(i) *(Stagnation). There is a sequence of flows* $(u_n, v_n, \eta_n, F_n) \in \mathscr{C}$ *and sequence of points* (x_n, y_n) *such that* $u_n(x_n, y_n) \nearrow c$; *or*

(ii) *(Large Froude number). There exists a sequence of flows* $(u_n, v_n, \eta_n, F_n) \in \mathscr{C}$ *with* $F_n \nearrow \infty$.

Proof. The statement of the corollary is a simplified version of Theorem 1.1 in Wheeler (2013), except that in that theorem there is the additional possibility (iii) that there exists a solitary wave (u, v, η, F) in the closure of $\mathscr{C}$ with critical Froude number $F = 1$. So let (u, v, η, F) be a wave in the closure of $\mathscr{C}$, and assume that condition (i) does not hold. Then $\sup u < c$, so by Theorem 1.1 we have $F \neq 1$. Thus (iii) cannot hold and hence (ii) must hold. □

We note that, even when (i) occurs in Corollary 6.1, the shear flow U is bounded away from c uniformly along the continuum $\mathscr{C}$. Indeed, every wave in $\mathscr{C}$ has $F > 1$ and hence

$$\min(c - U) = F \min U^* > \min U^* > 0. \tag{6.4}$$

In many cases, we can apply Theorem 1.2 to further simplify Corollary 6.1.

COROLLARY 6.2. *In the setting of Corollary 6.1, suppose that the fixed profile U^* satisfies*

$$\Lambda^* := \max_y \frac{U^*(0)}{U^*(y)} < \frac{2}{\sqrt{3}}. \tag{6.5}$$

Then condition (ii) *cannot occur, so that* (i) *must hold.*

Proof. Thanks to (6.1), any wave in $\mathscr{C}$ satisfies

$$\Lambda = \max_y \frac{c - U(0)}{c - U(y)} = \max_y \frac{U^*(0)}{U^*(y)} = \Lambda^*. \tag{6.6}$$

Thus by Theorem 1.2 all waves in $\mathscr{C}$ have $F < (1 - 3(\Lambda^*)^2/4)^{-1/2} < \infty$. □

The conclusion of Corollary 6.2, that there exists a sequence of solutions along the continuum with $\sup u_n$ approaching c, is the same conclusion that was proved for periodic waves in Constantin & Strauss (2004). It remains an open question if the same is true for solitary waves with $\Lambda^* \geqslant 2/\sqrt{3}$, though it seems doubtful that the restriction $\Lambda^* < 2/\sqrt{3}$ is sharp.

7. The case of constant vorticity

In this section we specialize the above results to asymptotic shear flows U which are linear in y, or equivalently to waves whose vorticity

$$\omega(x, y) = \gamma(p) \equiv -U_y \tag{7.1}$$

is constant. In this case more explicit formulae are available; to make these formulae appear simpler, we define the dimensionless constants

$$\lambda^* = \frac{\lambda}{gd} = \frac{(c - U(0))^2}{gd} > 0, \quad \gamma^* = \frac{\gamma d}{\sqrt{\lambda}} = \frac{-U_y d}{(c - U(0))}. \tag{7.2a,b}$$

In terms of λ^* and γ^*, the asymptotic shear flow U is given by

$$c - U(y) = \sqrt{\lambda} + \gamma y = \sqrt{gd\lambda^*}\left(1 + \gamma^* \frac{y}{d}\right). \tag{7.3}$$

Plugging $y = -d$ into (7.3), we see that $\sup U < c$ implies $\gamma^* < 1$. Substituting (7.3) into the definitions (1.2) of F and (1.4) of Λ, we find

$$F^2 = \lambda^*(1 - \gamma^*), \quad \Lambda = \frac{1}{1 - \max(\gamma^*, 0)}. \tag{7.4a,b}$$

Using (7.4) in Theorems 1.1 and 1.2, we can then easily prove the following.

COROLLARY 7.1 (Constant vorticity). *Consider a solitary wave of elevation with constant vorticity* γ, *regularity* (1.3), *and* $\sup u < c$ *so that in particular* $\gamma^* < 1$. *Then we have the following bounds on* λ^* *and* γ^*:

$$\lambda^*(1-\gamma^*) > 1, \tag{7.5}$$

$$\lambda^*(1-\gamma^*) < 4 \quad \textit{for } \gamma^* \leqslant 0, \tag{7.6}$$

$$\lambda^* \frac{1-8\gamma^*+4(\gamma^*)^2}{1-\gamma^*} < 4 \quad \textit{for } 0<\gamma^*<1-\sqrt{3}/2, \tag{7.7}$$

and hence the bounds

$$\frac{\max \eta}{d} < \frac{2}{1-\gamma^*} \qquad \textit{for } \gamma^* \leqslant 0, \tag{7.8}$$

$$\frac{\max \eta}{d} < \frac{2(1-\gamma^*)}{1-8\gamma^*+4(\gamma^*)^2} \quad \textit{for } 0<\gamma^*<1-\sqrt{3}/2 \tag{7.9}$$

on the amplitude.

Proof. The first inequality (7.5) is $F > 1$ from Theorem 1.1, while (7.6) and (7.7) are (1.5) from Theorem 1.2. Since Bernoulli's law (2.14) implies $\max \eta < d\lambda^*/2$, the bounds (7.8) and (7.9) on the amplitude follow immediately from (7.6) and (7.7). □

From (7.6) and (7.7) we see that, for any fixed $\gamma^* < 1-\sqrt{3}/2 \approx 0.134$, waves of elevation with $\sup u < c$ have λ^* bounded above by a constant depending only on γ^*. It is interesting to compare this to the numerical results in Vanden-Broeck (1994), which suggest that for any fixed $\gamma^* > \gamma^*_{cr} \approx 0.33$ there exist overhanging waves with λ^* arbitrarily large. Note however that overhanging waves necessarily violate our assumption $\sup u < c$.

8. Surface tension

In this section we prove that Theorem 1.1 continues to hold in the presence of surface tension. For waves with surface tension, the dynamic boundary condition (2.1*f*) is replaced by

$$P + \sigma\left(\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right)_x = 0 \quad \text{on } y=\eta(x), \tag{8.1}$$

where the constant $\sigma > 0$ is the coefficient of surface tension. The corresponding boundary condition (2.9*b*) in the height equation becomes

$$\frac{1+h_q^2}{2h_p^2} - \frac{1}{2H_p^2} + g(h-H) - \sigma\left(\frac{h_q}{(1+h_q^2)^{1/2}}\right)_q = 0 \quad \text{on } p=0. \tag{8.2}$$

For irrotational solitary waves with surface tension, Amick & Kirchgässner (1989) prove the analogue

$$F^2 = 1 + \frac{3}{2d}\frac{\int \eta^2 \mathrm{d}x}{\int \eta \,\mathrm{d}x} + \frac{\sigma}{gd}\frac{\int\left(\sqrt{1+\eta_x^2}-1\right)\mathrm{d}x}{\int \eta\,\mathrm{d}x} \tag{8.3}$$

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of the integral identity (1.8). For irrotational waves of depression, such as those constructed in Amick & Kirchgässner (1989) with Bond number $\sigma/gd^2 > 1/3$, (8.3) immediately implies that the Froude number $F < 1$. As a consequence of Theorem 8.1 below, the same bound $F < 1$ holds for waves with vorticity.

We will work with solutions which have the regularity

$$\eta = h(\cdot, 0) \in W^{2,r}_{loc}(\mathbb{R}), \quad h \in W^{2,r}_{loc}(\overline{\Omega}) \subset C^{1+\alpha}_{loc}(\overline{\Omega}), \quad H \in W^{2,r}[-m, 0], \qquad (8.4a\text{–}c)$$

where, as in § 1.1, $0 < \alpha < 1$ and $r = 2/(1-\alpha)$; see Martin & Matioc (2014) for the equivalence of various formulations for periodic waves with surface tension.

THEOREM 8.1. *Theorem 1.1 holds for waves with surface tension, provided we replace the regularity (1.3) with (8.4).*

Proof. We argue exactly as in the proof of Theorem 1.1, with Lemma 3.1 replaced by Lemma 8.2 below. □

LEMMA 8.2. *Lemma 3.1 holds for waves with surface tension, provided we replace the regularity (1.3) with (8.4).*

Proof. We will follow the proof of Lemma 3.1 and notice that the term involving surface tension drops out of the calculation entirely. Multiplying (2.9*a*) by $\Phi(p) = \int_{-m}^{p} H_p^3(s)\,\mathrm{d}s$ and integrating by parts, we obtain (3.3) as before,

$$\begin{aligned} 0 &= \int_{-M}^{M}\int_{-m}^{0}\left(\frac{1+h_q^2}{2h_p^2} - \frac{1}{2H_p^2}\right)\Phi_p\,\mathrm{d}p\,\mathrm{d}q \\ &\quad + \frac{1}{gF^2}\int_{-M}^{M}\left(-\frac{1+h_q^2}{2h_p^2} + \frac{1}{2H_p^2}\right)(q,0)\,\mathrm{d}q + \int_{-m}^{0}\frac{h_q}{h_p}\Phi\,\mathrm{d}p\bigg|_{q=-M}^{q=M}. \end{aligned} \qquad (8.5)$$

We claim that (8.5) implies (3.4). Indeed, the first and last terms in (8.5) can be treated as in the proof of Lemma 3.1, so (3.4) follows from the following computation involving the middle term:

$$\begin{aligned} &\frac{1}{gF^2}\int_{-M}^{M}\left(-\frac{1+h_q^2}{2h_p^2} + \frac{1}{2H_p^2}\right)(q,0)\,\mathrm{d}q - \frac{1}{F^2}\int_{-M}^{M}(h(q,0) - H(0))\,\mathrm{d}q \\ &\qquad = -\frac{\sigma}{gF^2}\int_{-M}^{M}\left(\frac{h_q}{(1+h_q^2)^{1/2}}\right)_q(q,0)\,\mathrm{d}q \\ &\qquad = -\frac{\sigma}{gF^2}\frac{h_q}{(1+h_q^2)^{1/2}}\bigg|_{(-M,0)}^{(M,0)} \to 0 \quad \text{as } M \to \infty, \end{aligned} \qquad (8.6)$$

where we have used the boundary condition (8.2) and the asymptotic conditions (2.9*d*). With (3.4) established, we can then complete the argument exactly as in the proof of Lemma 3.1. □

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