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Rings graded by bisimple inverse semigroups

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Sufficient conditions are obtained for a ring R, faithfully graded by a bisimple inverse semigroup S, to be (a) prime and (b) right primitive, these conditions being on the subring R_G consisting of all elements of R with support contained in G, a maximal subgroup of S. Earlier results on semigroup rings arise as special cases.

1. Introduction

Let R be a ring and S a semigroup. Then R is said to be S-*graded* (equivalently, *graded by* S) if and only if (i) its additive group is expressible as a direct sum of subgroups R_x ($x \in S$) and (ii) the multiplication in R is such that, for all x, $y \in S$, $R_x R_y \subseteq R_{xy}$. Such a ring R is said to be *faithful* (equivalently, *faithfully graded by* S) if and only if, for all x, $y \in S$ and each non-zero $a \in R_x$, $aR_y \neq 0$ and $R_y a \neq 0$. For a subgroup G of S, the subring $\oplus_{x \in G} R_x$ of R is denoted by R_G. An important example of an S-graded ring R is the semigroup ring $A[S]$ of S over a given ring A: here $R_x = Ax$ ($x \in S$). Clearly, if A is non-trivial with a unity, then R is faithful; also, if G is a subgroup of S, then R_G is the group ring $A[G]$.

We shall be concerned with the case in which S is an inverse semigroup; that is, a semigroup in which to each element x there corresponds a unique element x^{-1} such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. For a discussion of this important type of semigroup, see, for example [2, ch. V]. In a recent paper [3], Kelarev showed that if R is a ring that is faithfully graded by an inverse semigroup S, and if R_G is semiprimitive for all maximal subgroups G of S, then R is semiprimitive. This generalizes a well-known theorem of Domanov [1, theorem 1] on semigroup rings. It is also shown in [3] that the corresponding result holds if 'semiprimitive' is replaced above by 'semiprime'. In turn, this generalizes [7, theorem 4.2].

An inverse semigroup S is said to be *bisimple* if and only if, for all idempotents e and f in S, there exists $x \in S$ such that $e = xx^{-1}$ and $f = x^{-1}x$. (This condition on the idempotents of S is readily seen to be equivalent to the condition that S consists of a single $\mathcal{D}$-class.) If such a semigroup is not itself a group, then it contains infinitely many idempotents [2, ch. V, § 6] and, moreover, all its maximal subgroups are isomorphic [2, II.3.7]. The purpose of the present paper is to show that if R is a ring faithfully graded by a bisimple inverse semigroup S, and if, for some maximal subgroup G of S, R_G is prime (respectively, right primitive with the additional property that $a \in aR_G$ for all $a \in R_G$), then R is prime (respectively, right primitive). These results generalize [7, theorems 4.4, 4.5], the latter also being a consequence of the proof of theorem 1 in [1].

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2. Preliminaries

Let S be a semigroup and let $R = \oplus_{x \in S} R_x$ be an S-graded ring, where each R_x is a subgroup of $(R, +)$ and, for all $x, y \in S$, $R_x R_y \subseteq R_{xy}$. For any non-empty subset X of S, we write R_X for $\oplus_{x \in X} R_x$. Thus for all non-empty subsets X and Y of S, $R_X R_Y \subseteq R_{XY}$. In particular, if X is a subsemigroup of S, then R_X is a subring of R. For $a \in R$, we denote the R_x-component of a by a_x $(x \in S)$ and we define the *support* of a, $\operatorname{supp}(a)$, by

$$\operatorname{supp}(a) := \{x \in S : a_x \neq 0\}.$$

Observe that $\operatorname{supp}(a)$ is a finite set and is empty if and only if $a = 0$. Now consider a subset X of S. Let $T := X \cap \operatorname{supp}(a)$. Then we write

$$a_X := \begin{cases} \sum_{x \in T} a_x & \text{if } T \neq \emptyset, \\ 0 & \text{if } T = \emptyset. \end{cases}$$

In particular, $a_\emptyset = 0$.

For an idempotent e in a semigroup S, eSe is a subsemigroup of S with identity e. As is customary, the right unit subsemigroup of eSe is denoted by P_e and the maximal subgroup of S with identity e by H_e; thus

$$P_e = \{x \in eSe : xy = e \text{ for some } y \in eSe\}$$

and $H_e (\subseteq P_e)$ is the group of units of eSe. (We avoid here the notation R_e for the $\mathcal{R}$-class of S containing e, since it would conflict with that already introduced for an S-graded ring R.) The set-difference $\{x \in X : x \notin Y\}$ of sets X and Y is denoted by $X \backslash Y$.

We note for later use the following elementary result [7, lemma 1.1].

LEMMA 2.1. *Let S be a semigroup and let $e = e^2 \in S$. Write $G := H_e$, $Q := eS$, $T := eSe \backslash P_e$. Then*

(i) $TS \subseteq Q \backslash G$,

(ii) $G(Q \backslash G) \subseteq Q \backslash G$,

(iii) $GT \subseteq T$.

It is easily checked that, in lemma 2.1, if S is a bisimple inverse semigroup that is not a group, then $T \neq \emptyset$ and $Q \backslash G \neq \emptyset$.

The starting-point of our discussion is [3, lemma 2], stated below as lemma 2.2. This is a direct generalization of a result on semigroup algebras [6, lemma 6]. (See also [7, lemma 4.1].)

LEMMA 2.2 (Kelarev). *Let S be an inverse semigroup, let R be a faithful S-graded ring and let A be a non-zero ideal of R. Then there exist $e = e^2 \in S$ and $a \in A$ such that*

$$e \in \operatorname{supp}(a) \subseteq H_e \cup (eSe \backslash P_e).$$

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As in [7, lemma 3.1], by specializing to the case where S is bisimple, we can improve the result by allowing e to be chosen arbitrarily. This constitutes the next lemma, which, in turn, provides the key to the results in § 3.

LEMMA 2.3. *Let S be a bisimple inverse semigroup, let R be a faithful S-graded ring and let A be a non-zero ideal of R. Then, for all $e = e^2 \in S$, there exists $a \in A$ such that*

$$e \in \text{supp}(a) \subseteq H_e \cup (eSe \backslash P_e).$$

Proof. By lemma 2.2, there exist $f = f^2 \in S$ and $b \in A$ such that

$$f \in \text{supp}(b) \subseteq H_f \cup (fSf \backslash P_f).$$

Let $e = e^2 \in S$. Since S is bisimple, there exists $x \in S$ such that $e = xx^{-1}$ and $f = x^{-1}x$. Since $b_f \neq 0$ and R is faithful, there exists $c \in R_x$ such that $cb_f \neq 0$. Then, since $cb_f \in R_x R_f \subseteq R_{xf} = R_x$ and R is faithful, there exists $d \in R_{x^{-1}}$ such that $cb_f d \neq 0$. Note also that $cb_f d \in R_x R_{x^{-1}} \subseteq R_{xx^{-1}} = R_e$.

Now write $a := cbd$. Then $a \in A$. We show that

$$e \in \text{supp}(a) \subseteq H_e \cup (eSe \backslash P_e).$$

By [5, lemma 1], $\theta : fSf \to eSe$ defined by $y\theta = xyx^{-1} (y \in fSf)$ is an isomorphism. In particular, this implies that

$$[H_f \cup (fSf \backslash P_f)]\theta = H_e \cup (eSe \backslash P_e). \tag{2.1}$$

Let $\text{supp}(b) = \{y_1, y_2, \dots, y_n\}$, where $y_i \neq y_j$ if $i \neq j$. Then, since $\text{supp}(b) \subseteq fSf$ and $a = c(b_{y_1} + b_{y_2} + \cdots + b_{y_n})d$, we see from (2.1) that

$$\text{supp}(a) \subseteq x \,\text{supp}(b) x^{-1} = (\text{supp}(b))\theta \subseteq H_e \cup (eSe \backslash P_e).$$

Without loss of generality, let $f = y_1$. Then, since $cb_f d \in R_e \backslash 0$ and

$$e = xy_1x^{-1} \neq xy_ix^{-1} \quad \text{for } i > 1,$$

we have that $e \in \text{supp}(a)$. □

3. The main results

In this section we give sufficient conditions for a ring R, faithfully graded by a bisimple inverse semigroup S, to be (a) prime and (b) right primitive. In each case, the sufficient condition involves a property of the subring R_G of R, where G is a maximal subgroup of S. It should be noted, in passing, that while any two maximal subgroups G and H of S are necessarily isomorphic, examples can be constructed to show that this need not be true for the corresponding subrings R_G and R_H of R.

THEOREM 3.1. *Let S be a bisimple inverse semigroup, let R be a faithful S-graded ring and let R_G be prime for some maximal subgroup G of S. Then R is prime.*

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Proof. We may assume that $G \neq S$. Let e be the identity of G. Consider non-zero ideals A and B of R. By lemma 2.3, there exist $a \in A$ and $b \in B$ such that

$$e \in \operatorname{supp}(a) \subseteq G \cup T, \qquad e \in \operatorname{supp}(b) \subseteq G \cup T,$$

where G denotes H_e and T denotes $eSe \backslash P_e$. Thus $a = a_G + a_T$ and $b = b_G + b_T$, where $a_G \neq 0$ (since $a_e \neq 0$) and $b_G \neq 0$ (since $b_e \neq 0$). Now the right ideal $a_G R_G$ of R_G is non-zero; for otherwise, if I denotes the principal ideal of R_G generated by a_G, then $I^2 = 0$—contrary to the hypothesis that R_G is prime. Similarly, $b_G R_G$ is non-zero. Hence, since R_G is prime, $(a_G R_G)(b_G R_G) \neq 0$ and so there exist u and v in R_G such that

$$a_G u b_G v \neq 0. \tag{3.1}$$

Clearly,

$$aubv = a_G u b_G v + a_G u b_T v + a_T ubv. \tag{3.2}$$

As before, let Q denote eS. Now

$$a_T ubv,\ b_T v \in R_T R \subseteq R_{TS} \subseteq R_{Q \backslash G},$$

by lemma 2.1 (i). Also, since $a_G u \in R_G$ and $b_T v \in R_{Q \backslash G}$, we have that

$$a_G u b_T v \in R_G R_{Q \backslash G} \subseteq R_{G(Q \backslash G)} \subseteq R_{Q \backslash G},$$

by lemma 2.1 (ii). But $a_G u b_G v \in R_G$. Hence, from (3.2) and (3.1),

$$(aubv)_G = a_G u b_G v \neq 0$$

and so $aubv \neq 0$. Since $au \in A$ and $bv \in B$, this shows that $AB \neq 0$. Thus R is prime. □

We now consider right primitivity. For a right ideal A of a ring R we denote the two-sided ideal $\{r \in R : Rr \subseteq A\}$ by $(A : R)$. In the proof of theorem 3.2, we use the fact that R is right primitive if and only if it contains a modular maximal right ideal A with $(A : R) = 0$ [4, theorem 5.34].

It is convenient to make a further definition. A ring R is *right inclusive* if and only if, for all $a \in R$, $a \in aR$. Clearly, every ring with a (right) unity is right inclusive. We are concerned below with right primitive right inclusive rings. As an example of such a ring with no right unity, we cite the ring of all linear transformations of finite rank (written as right operators) of an infinite-dimensional vector space over a field. (Note also that a right primitive ring need not be right inclusive, as is illustrated by the semigroup ring $F[S]$ of a free semigroup S of rank 2 over a field F.)

THEOREM 3.2. *Let S be a bisimple inverse semigroup, let R be a faithful S-graded ring and let R_G be right primitive and right inclusive for some maximal subgroup G of S. Then R is right primitive.*

Proof. We may again assume that $G \neq S$. Since R_G is right primitive, it contains a modular maximal right ideal B such that $(B : R_G) = 0$. Since B is modular, there

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exists $c \in R_G$ such that $cw - w \in B$ for all $w \in R_G$. Suppose that $c \in B$. Then, for each $w \in R_G$, we have that $w = cw - (cw - w) \in B$. Hence $B = R_G$, which contradicts the definition of B. Thus $c \notin B$.

Let e denote the identity of G and, as in lemma 2.1, put $Q := eS$ and $T := eSe \backslash P_e$. Write $M := \{cw - w : w \in R\}$, where c is as above, and take

$$I := B + BR + R_T + R_T R + M.$$

Since M is a right ideal of R, so also is I. We show that $c \notin I$.

Suppose that, on the contrary, $c \in I$. Then

$$c = u + v + (cw - w)$$

for some $u \in B + BR$, $v \in R_T + R_T R$ and $w \in R$. Thus, since $G \subset Q$,

$$c = u + v + cw_G + cw_{Q\backslash G} + cw_{S\backslash Q} - w_G - w_{Q\backslash G} - w_{S\backslash Q}. \tag{3.3}$$

Since the left-hand side of (3.3) and all terms on the right-hand side preceding the last lie in R_Q, we have that $w_{S\backslash Q} = 0$. Also,

$$u = b + \sum_{i=1}^{n} b_i r_i$$

for some positive integer n, some elements b and b_i of B and some elements r_i of R $(i = 1, 2, \dots, n)$. Let $i \in \{1, 2, \dots, n\}$. Since R_G is right inclusive, there exists $u_i \in R_G$ such that $b_i = b_i u_i$. Then $u_i r_i \in R_G R \subseteq R_{GS} = R_Q$. Hence there exist $p_i \in R_G$ and $q_i \in R_{Q\backslash G}$ such that $u_i r_i = p_i + q_i$. Thus $b_i r_i = b_i u_i r_i = b_i p_i + b_i q_i$. Consequently, from (3.3),

$$c = \left(b + \sum_{i=1}^{n} b_i p_i + cw_G - w_G\right) + \left(\sum_{i=1}^{n} b_i q_i + v + cw_{Q\backslash G} - w_{Q\backslash G}\right). \tag{3.4}$$

Now, since $R_T \subseteq R_{Q\backslash G}$ and $R_T R \subseteq R_{TS} \subseteq R_{Q\backslash G}$, by lemma 2.1 (i), it follows that $v \in R_{Q\backslash G}$. Further, $cw_{Q\backslash G} \in R_G R_{Q\backslash G} \subseteq R_{G(Q\backslash G)} \subseteq R_{Q\backslash G}$, by lemma 2.1 (ii), and, similarly, $b_i q_i \in R_{Q\backslash G}$ for each i. Hence the second bracketed expression on the right-hand side of (3.4) lies in $R_{Q\backslash G}$. But c and the first bracketed expression on the right-hand side lie in R_G. Hence

$$c = b + \sum_{i=1}^{n} b_i p_i + cw_G - w_G.$$

However, $cw_G - w_G \in B$ and so $c \in B$, which is false. Thus $c \notin I$.

By Zorn's lemma, the set of all right ideals of R that contain I and exclude c has a maximal member A, say. Let A' be a right ideal of R strictly containing A. Then $c \in A'$. Thus, for all $w \in R$, $cw \in A'$ and $cw - w \in M \subseteq I \subseteq A'$, from which it follows that $w \in A'$. Hence $A' = R$. This shows that A is a maximal right ideal of R; and, since it contains M, it is modular. Moreover, $B \subseteq I \cap R_G \subseteq A \cap R_G$ and the right ideal $A \cap R_G$ of R_G is proper, since it does not contain c. Hence, by the maximality of B,

$$B = A \cap R_G. \tag{3.5}$$

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To prove that R is right primitive, it suffices to show that $(A : R) = 0$. Suppose that $(A : R) \neq 0$. Then, by lemma 2.3, there exists $a \in (A : R)$ such that

$$e \in \operatorname{supp}(a) \subseteq G \cup T.$$

Hence $a = a_G + a_T$ and $a_G \neq 0$. Let $d \in R_G$. Since $Ra \subseteq A$, we have that $da_G + da_T \in A$. But $da_T \in R_G R_T \subseteq R_{GT} \subseteq R_T$, by lemma 2.1 (iii), and so, since $R_T \subseteq I \subseteq A$, $da_T \in A$. Thus $da_G \in A$. Now $da_G \in R_G$ and, therefore, from (3.5), $da_G \in B$. This shows that $R_G a_G \subseteq B$. Hence $a_G \in (B : R_G)$. But $(B : R_G) = 0$, as noted earlier. Consequently, $a_G = 0$, which is false. It follows that $(A : R) = 0$ and so, since A is a modular maximal right ideal of R, R is right primitive. □

Note that, since every inverse semigroup S has an involution (namely, the mapping $x \mapsto x^{-1} (x \in S)$), the left-right dual of theorem 3.2 also holds. It is not known whether the conclusion of the theorem remains valid if we delete the hypothesis that R_G is right inclusive.

4. Rings graded by 0-bisimple inverse semigroups

To conclude, we observe that the previous results can readily be extended to a class of rings graded by 0-bisimple inverse semigroups.

Two further definitions are required. As usual, for a semigroup S we write '$S = S^0$' to indicate that S has a zero and at least one other element. Given a semigroup $S = S^0$ with zero z and a ring R, we say that R is a *restricted S-graded ring* if and only if it is an S-graded ring with the additional property that $R_z = 0$. A modified definition of faithfulness is appropriate in this context. Such a ring R is termed *faithful* if and only if, for all x, $y \in S\backslash z$, $a \in R_x\backslash 0$,

$$xy \neq z \Rightarrow aR_y \neq 0,$$
$$yx \neq z \Rightarrow R_y a \neq 0.$$

Clearly, for an arbitrary semigroup T, any (faithful) T-graded ring may be regarded as a (faithful) restricted S-graded ring, where S is the semigroup obtained by adjoining a zero to T (whether or not one is already present in T). For a semigroup $S = S^0$ and a ring A, the contracted semigroup ring $R = A_0[S]$ of S over A is a restricted S-graded ring with $R_x = Ax$ for all non-zero x in S; and R is faithful if, for example, A is non-trivial with a unity.

By analogy with lemma 2.2 (see also [7, lemma 4.1]), it can be shown that if $S = S^0$ is an inverse semigroup, R a faithful restricted S-graded ring and A a non-zero ideal of R, then there exist a non-zero idempotent $e \in S$ and an element $a \in A$ such that

$$e \in \operatorname{supp}(a) \subseteq H_e \cup (eSe\backslash P_e).$$

An inverse semigroup S is said to be 0-*bisimple* if $S = S^0$ and, for all non-zero idempotents e and f in S, there exists $x \in S$ such that $e = xx^{-1}$ and $f = x^{-1}x$. Every bisimple inverse semigroup with a zero adjoined is 0-bisimple. Examples not of this type include completely 0-simple inverse semigroups (Brandt semigroups) with at least two non-zero idempotents [2, ch. III]. Restricted S-graded rings, where S is a completely 0-simple inverse semigroup, have been studied in [8].

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The proof of lemma 2.3 can be readily adapted to show that if S is a 0-bisimple inverse semigroup, R a faithful restricted S-graded ring and A a non-zero ideal of R then, for any non-zero idempotent e in S, there exists $a \in A$ such that

$$e \in \operatorname{supp}(a) \subseteq H_e \cup (eSe \backslash P_e).$$

From this result we derive the theorems below. The proofs, which are similar to those of theorems 3.1 and 3.2, are omitted.

THEOREM 4.1. *Let S be a 0-bisimple inverse semigroup, let R be a faithful restricted S-graded ring and let R_G be prime for some non-zero maximal subgroup G of S. Then R is prime.*

THEOREM 4.2. *Let S be a 0-bisimple inverse semigroup, let R be a faithful restricted S-graded ring and let R_G be right primitive and right inclusive for some non-zero maximal subgroup G of S. Then R is right primitive.*

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References

1 O. I. Domanov. Semisimplicity and identities of inverse semigroup algebras. Rings and Modules. *Mat. Issled.* **38** (1976), 123–137.
2 J. M. Howie. *An introduction to semigroup theory* (London: Academic Press, 1976).
3 A. V. Kelarev. Semisimple rings graded by inverse semigroups. *J. Algebra* **205** (1998), 451–459.
4 N. H. McCoy. *The theory of rings* (New York: Macmillan, 1964).
5 W. D. Munn. The idempotent-separating congruences on a regular 0-bisimple semigroup. *Proc. Edinb. Math. Soc.* **15** (1967), 233–240.
6 W. D. Munn. Nil ideals in inverse semigroup algebras. *J. Lond. Math. Soc.* **35** (1987), 433–438.
7 W. D. Munn. On contracted semigroup rings. *Proc. R. Soc. Edinb.* A **115** (1990), 109–117.
8 P. Wauters and E. Jespers. Rings graded by an inverse semigroup with finitely many idempotents. *Houston J. Math.* **15** (1989), 291–304.

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