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On Sequences of Squares with Constant Second Differences

J. Browkin and J. Brzeziński

Abstract. The aim of this paper is to study sequences of integers for which the second differences between their squares are constant. We show that there are infinitely many nontrivial monotone sextuples having this property and discuss some related problems.

1 Introduction

Nontrivial sequences of integers $x_1, x_2, \dots, x_n$ whose squares have constant second differences, that is, $(x_{i+1}^2 - x_i^2) - (x_i^2 - x_{i-1}^2) = \Delta$ for $i = 2, \dots n-1$, create some interest, notably in the case when $\Delta = 2$. The *n squares conjecture*, intimately connected with different versions of Hilbert's tenth problem, claims that every such sequence of positive integers with $\Delta = 2$ and sufficiently large n must consist of consecutive numbers (see [L, V, Y] and the comments in Section 1). In fact, the numerical examples in several papers related to this problem as well as the results of the present paper strongly suggest that already $n = 5$ gives a positive answer, which is the content of Büchi's conjecture (see [L] and the end of Section 4). The main purpose of the paper is to supply more evidence motivating this claim.

It is well known that there are infinitely many increasing sequences of four nonconsecutive positive integers whose squares have second differences 2, for example, the sequence $6, 23, 32, 39$. Constructions of such sequences appear in [A, Ba, Bu, P] in somewhat different contexts (see Section 1 for further comments). In his paper [Bu], Buell asks whether there are (nontrivial) increasing sequences of five integer squares with constant second differences (not necessarily equal 2). Existence of such sequences and possible deviations from the differences 2 could give indications concerning the length of nontrivial sequences in the n squares conjecture. In the present paper, we prove that there are infinitely many such increasing quintuples and sextuples. Of those we were able to find, the two with the smallest $|\Delta|$ are 111, 251, 337, 405, 463 with the second differences of squares -112, and 7, 47, 67, 83, 97 whose second differences of squares are 120. The least example of a sextuple is 54, 229, 316, 381, 434, 479 with the second differences -2110. Recently, Bremner [Br] found two examples of increasing septuples with constant second differences of squares: 572321, 1938531, 2969567, 3938125, 4881537, 5812061, 6735041 with $\Delta = 1630074618608$ and 1633911, 1942706, 3402469, 5106636, 6875821, 8670314, 10477119 with $\Delta = 6698247247010$. In Section 1, we fix the necessary definitions and give more details about the background of the paper. In Section 2, we prove our

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main result that there are infinitely many nontrivial increasing sequences of six integer squares with constant second differences. In Section 3, we discuss an algorithm for construction of such sequences and present the results of computations tabulating known examples of nontrivial integer quintuples and sextuples whose squares have constant and "small" absolute value of the second differences ($|\Delta| \leq 2000$ for quintuples and $|\Delta| \leq 2 \cdot 10^5$ for sextuples and longer sequences). We expect that 112 is the minimum value for quintuples and 2110 for sextuples. In the range of our computations, we did not find any increasing sequence of seven integers whose squares have constant second differences. In the last section, we look at symmetric integer sequences (in the sense defined below) whose squares have constant second differences proving that there are no such nontrivial sequences when $n > 5$ is odd and that their number is at most finite when $n > 8$ is even. These results can be also found in [Br] (see Section 5).

2 Preliminaries

Let $x_1, x_2, \dots, x_n$ be a sequence of positive integers whose squares have constant second differences Δ. Then

$$x_{i-1}^2 - 2x_i^2 + x_{i+1}^2 = \Delta, \tag{1}$$

for $i = 2, \dots, n-1$. If $n \geq 4$, the sequence $x_1, x_2, \dots, x_n$ has constant second differences if and only if

$$x_{i-1}^2 - 3x_i^2 + 3x_{i+1}^2 - x_{i+2}^2 = 0, \tag{2}$$

for $i = 2, \dots, n-2$. We say that such a sequence is *trivial* if there is an arithmetic sequence $y_1, y_2, \dots, y_n$ and $x_i = |y_i|$ for $i = 1, \dots, n$. Observe that the triviality of a sequence satisfying the system (2) is equivalent to the triviality of any three of its consecutive terms. Notice also that if $x_1, x_2, \dots, x_n$ is trivial and the difference of the arithmetic sequence $y_1, y_2, \dots, y_n$ is δ, then $\Delta = 2\delta^2$.

The *n squares conjecture* (see [L, V, Y] for its origin and relations to Hilbert's tenth problem) says that there exists an integer $n > 2$ for which all positive integer solutions of the system

$$x_{i-1}^2 - 2x_i^2 + x_{i+1}^2 = 2,$$

where $i = 2, \dots, n-1$ are sequences of consecutive integers (notice that $\Delta = 2$ implies $\delta = \pm 1$, when the sequence $x_1, x_2, \dots, x_n$ is trivial).

Taking into account the results in the present paper, it is natural to formulate a *generalized n square conjecture*: there is $n > 2$ such that all positive integer solutions of the system (1) are trivial. In particular, if $\Delta \neq 2\delta^2$ and n is sufficiently large, then there are no solutions, *e.g.,* when $\Delta < 0$.

A similar problem when Δ is an arbitrary integer was studied by several authors who constructed quadratics $f(x) = ax^2 + bx + c$ with distinct zeros, assuming integer square values for as many consecutive integers as possible (see [A, Br, P]). It is easy to see that the existence of a quadratic $f(x)$ with the highest coefficient a assuming quadratic values in n consecutive integer points is equivalent to the existence of a sequence x_i, $i = 1, \dots, n$ satisfying the system (1) with $\Delta = 2a$.

If x_i, $i = 1, \dots, n$ satisfy the system (1), then it is easy to see that $\Delta > 0$ and $x_i < x_{i+1}$ imply that the sequence is increasing starting with x_i. Similarly, $\Delta < 0$ and $x_i > x_{i+1}$ imply that the sequence is decreasing from x_i. Since $x_1, x_2, \dots, x_n$ taken in the reverse order also satisfy the system, the observation above shows that the existence of n in the (generalized) n square conjecture for increasing (or decreasing) sequences is equivalent to the existence of suitable n (at most twice as large) for arbitrary sequences.

Therefore our objective is to construct nontrivial increasing sequences of at least five integers whose squares have as small as possible absolute value of the second differences. In Theorem 1, we show that there are infinitely many increasing sequences of six positive integers whose squares have constant second differences.

As we noted before, Buell [Bu], gave a complete description of the infinite set of solutions to equation (1) when $\Delta = 2$ and $n = 4$. In [A], Allison constructs quadratic polynomials (with distinct zeros) assuming square values at 7 and 8 points. For $n = 8$, he shows that there are infinitely many symmetric eight-tuples, the least of them $17, 53, 67, 73, 73, 67, 53, 17$ with $\Delta = -840$. For $n = 7$, this result gives infinitely many "almost-symmetric" (TOYOTA-symmetric in van der Poorten's terminology) septuples. We could not find any nontrivial example of nonsymmetric eight-tuple and any new example of nontrivial septuple besides those which are restrictions of symmetric sequences for $n = 8$ or Allison's two nontrivial examples: $53, 173, 217, 233, 227, 197, 127$ with $\Delta = -9960$, and $526, 337, 160, 113, 274, 461, 652$ with $\Delta = 75138$. Recently, Bremner [Br] has found 13 new examples of nontrivial septuples (two of them are monotone as we mentioned before).

3 Sextuples

In this section, we show that there are infinitely many nontrivial increasing sextuples of positive integers $(x_1, x_2, x_3, x_4, x_5, x_6)$ whose squares have constant second differences. In the construction, we use a particular elliptic curve, whose rational points define the sequences. However, the construction is a special case of a general result giving other elliptic curves, which could be used instead. We comment on this in Remark 1(c).

Let $\varphi\colon \mathbb{Q}^4 \to \mathbb{Q}$, where $\varphi(x_1, x_2, x_3, x_4) = x_1^2 - 3x_2^2 + 3x_3^2 - x_4^2$. Then $(\mathbb{Q}^4, \varphi)$ is a four-dimensional quadratic space over the rational numbers whose signature is 0. If $(a_1, a_2, a_3, a_4) \in \mathbb{Z}^4$ is a nonzero isotropic vector (that is, $\varphi(a_1, a_2, a_3, a_4) = 0$), then there are exactly two totally isotropic subspaces of dimension 2 containing $\alpha = (a_1, a_2, a_3, a_4)$. Let $\beta = (A_1, A_2, A_3, A_4)$ be any integral vector such that α, β is a basis of such a subspace W. Then $W = \{t\alpha + s\beta : t, s \in \mathbb{Q}\}$ and points in W with integer coordinates satisfy (2) for $n = 4$.

Let us observe that if $a_1^2 \neq a_4^2$ or $a_2^2 \neq a_3^2$, then we can take $\beta = (a_4, -a_3, -a_2, a_1)$, *e.g.,* for $\alpha = (1, 2, 3, 4)$, we have $\beta = (4, -3, -2, 1)$, but our choice of the vector $(1, 2, 3, 4)$ has a deeper explanation (see Remark 1(c) below). Put

$$(x_1, x_2, x_3, x_4) = t\alpha + s\beta = (a_1 t + a_4 s, a_2 t - a_3 s, a_3 t - a_2 s, a_4 t + a_1 s).$$

We extend the quadruple (x_1, x_2, x_3, x_4) to a sextuple $(x_0, x_1, x_2, x_3, x_4, x_5)$ such that its sequence of squares has constant second differences. This last condition leads to the system of equations

$$x_0^2 - 3x_1^2 + 3x_2^2 - x_3^2 = 0,$$
$$x_2^2 - 3x_3^2 + 3x_4^2 - x_5^2 = 0,$$

i.e., to the system

$$x_0^2 = 48ts + 25s^2,$$
$$x_5^2 = 25t^2 + 48ts.$$

Without loss of generality we can assume that $s = 1$. Then $x_0^2 = 48t + 25$ and $x_5^2 = 25t^2 + 48t$. Assuming this, we prove:

Theorem 1 *There are infinitely many nontrivial increasing sequences of six integer squares with constant second differences.*

Proof Of course, the second differences of the sequence $48t + 25, (t + 4)^2, (2t - 3)^2, (3t - 2)^2, (4t + 1)^2, 25t^2 + 48t$ are constant (and equal to $\Delta = 2t^2 + 20t + 2$). The sequence is increasing provided $t \geq 41$. It is a trivial sequence (according to the definition in Section 1) if and only if $\pm(t + 4)$, $\pm(2t - 3)$ and $\pm(3t - 2)$ form an arithmetic progression, which does not occur when $t \neq 0, \frac{1}{2}, 2$. We shall prove that there are infinitely many values of $t \in \mathbb{Q}$, $t \geq 41$ such that the first term $48t + 25$ and the last term $25t^2 + 48t$ are squares of rational numbers.

Evidently $48t + 25$ is a rational square if and only if $t = (u^2 - 5^2)/48$ for some $u = x_0 \in \mathbb{Q}$. Assuming that $25t^2 + 48t = w^2$ for some $w = x_5 \in \mathbb{Q}$, we get

$$(5u)^4 + (48^2 - 2 \cdot 25^2) \cdot (5u)^2 + 25^2(25^2 - 48^2) = (5 \cdot 48 \cdot w)^2,$$

i.e.,

$$Y^2 = X^4 + AX^2 + B, \tag{3}$$

where $X = 5u$, $Y = 240w$, $A = 46^2 - 2 \cdot 25^2 = 1054$, $B = 25^2(25^2 - 48^2) = -1049375$.

Applying the well-known birational transformation

$$X = -y/2x,$$
$$Y = -x/2 + y^2/4x^2 + A/2$$

we get a Weierstrass model of the elliptic curve, which we denote by E:

$$y^2 = x^3 - 2Ax^2 + (A^2 - 4B)x,$$

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i.e.,

$$y^2 = x^3 - 2108x^2 + 5308416x.$$

Using the package PARI/GP, we get the torsion points on E: $P = (2304, 115200)$, $2P = (0, 0)$, $3P = (2304, -115200)$ and $4P = 0$. Since $Q = (384, 42240)$ is also an integral point on the curve, its order must be infinite. Thus the rank of the curve is at least one. It is easy to check that E has only one component over the real numbers, so the rational points are dense in it (see [ST, Chap. II, 2], for the relevant facts concerning rational points on real components of elliptic curves). Since

$$t = \frac{X^2 - 625}{1200} = \frac{x^3 - 4608x^2 + 5308416x}{4800x^2},$$

t can be arbitrarily large and, in particular, there are infinitely many rational values of $t \geq 41$ corresponding to rational points on the elliptic curve E ($t \geq 41$ gives $-5.1337 < x < 5.1339$ or $x > 201408$). ■

Remark 1 (a) Using PARI/GP one may check that the rank of E equals 1.

(b) An example of a point on E for which $t \geq 41$ is $5Q$ ($x = \frac{2116939068922464}{808776721}$ giving $t = \frac{47 \cdot 863 \cdot 87966279334847134753571424877205309}{2 \cdot 3^3 \cdot 5^2 \cdot 331^4 \cdot 4729^4 \cdot 28439^2} \approx 544$). The examples of integer sequences of six numbers whose squares have constant second differences constructed from values of t corresponding to the rational points on E are too big to be written here. We give examples constructed in a different way in the next section (see Table 2).

(c) It is interesting to know whether there are other possible choices of the totally isotropic subspace W and its basis in Theorem 1. Assume that $0 < a_1 < a_2 < a_3 < a_4$ and $a_1^2 - 3a_2^2 + 3a_3^2 - a_4^2 = 0$. Then $x_0^2 = 3x_1^2 - 3x_2^2 + x_3^2$ is of degree 1 in t if and only if $a_3^2 = 3a_2^2 - 3a_1^2$. Thus in order to choose suitable a_i, we have to solve the system of two Pell's equations:

$$a_3^2 = 3a_2^2 - 3a_1^2,$$
$$a_4^2 = 6a_2^2 - 8a_1^2.$$

It is not difficult to show that this system has in fact infinitely many solutions, which correspond to the rational points on the elliptic curve $y^2 - 432y = x^3 + 156x^2 + 5616x$, whose group of rational points has rank 1. Some of these rational points lead to an elliptic curve with infinitely many rational points like the one used in the proof of Theorem 1 (all these elliptic curves are on the surface in $\mathbb{P}^5$, which is the intersection of the three quadrics in the system (2) for $n = 6$).

4 An Algorithm and Examples

Let a, b, c be a solution to the equation $a^2 - 2b^2 + c^2 = \Delta$. Denote $b = a + k$, $c = a + l$. Then

$$(a + l)^2 = \Delta + 2(a + k)^2 - a^2,$$

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which gives

$$a = \frac{2k^2 - l^2 + \Delta}{2(l - 2k)},$$
$$b = a + k = \frac{2kl - 2k^2 - l^2 + \Delta}{2(l - 2k)},$$
$$c = a + l = \frac{2k^2 + l^2 - 4kl + \Delta}{2(l - 2k)}.$$

Denote $2k - l = r$. Then:

$$a = \frac{2k^2 - 4kr + r^2 - \Delta}{2r},$$
$$b = \frac{2k^2 - 2kr + r^2 - \Delta}{2r},$$
$$c = \frac{2k^2 - r^2 - \Delta}{2r}.$$

It is easy to see that a, b, c are positive if and only if $a > 0, r > 0$ or $a > 0, r < 0$ and $k^2 + (k - r)^2 < \Delta$. They are increasing if and only if $r < k$, and integer if and only if $2 \mid r$, $r \mid 2k^2 - \Delta$ and Δ is even or $2 \nmid r$, $r \mid 2k^2 - \Delta$ and Δ is odd. If Δ is fixed, there are only finitely many possible pairs (k, r) with $r < 0$ satisfying the required conditions. In order to find all such triples (a, b, c) when Δ is fixed and $r > 0$, one computes for every k the list of the divisors r to $2k^2 - \Delta$ such that $0 < r < k$ and $2(k-r)^2 - r^2 > \Delta$ (the last inequality is equivalent to $a > 0$). For each pair (k, r) satisfying these conditions, we compute the corresponding triples (a, b, c), where $2b \neq a + c$ and then we recursively try to extend the list to the left and right checking whether $\sqrt{\Delta + 2a^2 - b^2}$ and $\sqrt{\Delta + 2d^2 - c^2}$ are integers or not. We also tried to find solutions of length $n \geq 7$ to the system (1), which are not necessarily monotone (increasing). This search was not successful. Observe that for nontrivial quintuples and sextuples the second difference Δ is always even. This is not *a priori* evident, but is easy to prove (there exist solutions to (1), when Δ is odd and $n = 4$).

We have carried out computations for $1 < k < 2000$ and $|\Delta| < 2000$ searching nontrivial sequences (not necessarily monotone) of length $n \geq 5$, which resulted in the following Table 1 of increasing quintuples (and no sextuples) for which the second differences of the squares are constant. Moreover, searching sextuples and arbitrary nontrivial (monotone or not) sequences of length $n \geq 7$, we checked the intervals $1 < k < 2000$, $|\Delta| < 2 \cdot 10^5$. The results of these computations are given in Table 2.

As we noted before, there are two nontrivial septuples already found by Allison, which give four additional nontrivial sextuples for $\Delta = -9960$ and $\Delta = 75138$ (see the end of Section 1).

We end with some final comments on the n squares conjecture based on the numerical results in the paper. Using the same method, which gives a possibility to construct nontrivial solutions to the system (1) in a systematic way, we also looked at

Table 1: Nontrivial increasing quintuples with $|\Delta| \le 2000$

					Δ	k
111	251	337	405	463	-112	68
7	47	67	83	97	120	40
11	50	71	88	103	162	39
17	56	79	98	115	258	39
107	213	281	335	381	-328	54
445	1179	1607	1943	2229	392	734
8	91	130	161	188	402	83
4	41	54	61	64	-430	7
15	47	61	69	73	-472	8
22	53	68	77	82	-510	9
383	1694	2365	2884	3323	642	1311
66	101	124	141	154	-670	17
136	249	326	389	444	770	113
43	521	735	899	1037	-808	164
7	135	193	239	279	848	128
67	133	173	203	227	-960	30
125	337	461	559	643	1008	212
1	79	107	125	137	-1032	18
51	115	151	177	197	-1048	26
21	141	201	249	291	1080	120
47	80	97	106	109	-1182	9
69	575	811	993	1147	1232	506
62	161	216	257	290	-1342	41
2	179	256	317	370	1458	177
33	150	213	264	309	1458	117
116	441	614	749	864	1490	325
46	145	196	233	262	-1518	37
81	116	137	150	157	-1582	13
59	70	89	112	137	1602	11
26	129	176	209	234	-1630	33
452	603	722	823	912	-1630	101
19	125	171	203	227	-1648	32
8	121	166	197	220	-1662	31
64	401	562	685	788	-1662	123
100	103	114	131	152	1778	3
69	281	389	471	539	-1840	82
313	452	559	650	731	1842	139
59	113	155	193	229	1968	54
95	547	769	941	1087	1968	452

all sequences of length $n \ge 5$, which are not monotone with $|\Delta|$ in the same range as in the tables. The least value of $|\Delta|$ when $n > 4$ seems to be 30 for the sequence $2, 7, 8, 7, 2$ ($\Delta = -40$ for $1, 9, 11, 11, 9, 1$ and $\Delta = -48$ for $5, 11, 13, 13, 11, 5$). The least value of $|\Delta|$ for a monotone sequence with $n = 5$ is 112 in Table 1. Thus taking into account our computations, the following conjecture is plausible (see [L] for the case $\Delta = 2$):

Conjecture *Let $x_1, x_2, \dots, x_n$, $n > 2$, be positive integers satisfying*

$$x_{i-1}^2 - 2x_i^2 + x_{i+1}^2 = \Delta$$

for $i = 2, \dots, n-1$.

(a) *If $\Delta = 2$ and $n > 4$, then $x_1, x_2, \dots, x_n$ are consecutive integers.*

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Table 2: Nontrivial nonsymmetric sextuples with $|\Delta| \leq 2 \cdot 10^5$ (increasing in bold face)

						Δ	k
54	**229**	**316**	**381**	**434**	**479**	-2110	87
62	37	32	53	82	113	2130	5
355	**521**	**643**	**743**	**829**	**905**	-3408	86
75	53	71	111	157	205	5048	18
127	79	45	59	103	153	5672	104
103	82	93	128	173	222	5810	11
146	131	144	179	226	279	7730	13
171	122	95	108	151	206	8498	13
155	103	81	109	163	225	9368	22
114	59	76	141	214	289	11810	17
173	83	53	133	227	323	18960	30
186	111	96	159	246	339	19170	15
150	106	142	222	314	410	20192	36
79	**343**	**457**	**529**	**575**	**601**	-20208	46
271	203	173	199	265	349	20952	26
177	98	107	192	293	398	23570	9
166	1	32	175	298	419	28578	31
20	**359**	**478**	**547**	**584**	**595**	-28878	37
239	219	269	361	471	589	33560	20
223	113	101	205	329	457	34392	12
483	**853**	**1087**	**1263**	**1403**	**1517**	-40360	176
252	83	2	207	368	527	49730	81
157	121	245	401	563	727	55392	36
89	472	617	694	725	716	-56958	145
470	331	248	277	394	545	63282	29
557	491	485	541	643	773	63312	6
356	229	206	311	464	631	64290	23
514	**811**	**992**	**1115**	**1198**	**1249**	-67182	83
438	393	432	537	678	837	69570	39
619	**655**	**739**	**857**	**997**	**1151**	71232	36
513	366	285	324	453	618	76482	39
31	**934**	**1291**	**1544**	**1739**	**1894**	-77070	195
153	555	717	801	831	813	-78552	162
445	**886**	**1137**	**1312**	**1439**	**1530**	-79198	127
465	309	243	327	489	675	84312	66
526	469	512	635	802	991	98898	43
434	259	224	371	574	791	104370	35
497	355	333	449	631	837	105848	22
311	146	229	436	659	886	106530	83
527	317	173	257	457	677	106680	84
25	477	589	599	513	245	-107512	10
326	311	452	655	878	1109	117138	141
293	158	287	508	743	982	118290	129
160	523	634	641	548	265	-119502	7
916	831	814	869	984	1141	120530	17
349	61	123	395	653	909	129488	62
371	41	79	389	661	929	140520	32
655	401	183	217	449	705	140912	34
641	**644**	**759**	**946**	**1171**	**1416**	157490	115
794	575	436	457	622	851	159282	21
877	683	577	607	757	973	169080	30
410	769	912	943	874	675	-182878	31
839	659	599	691	889	1139	194160	60
714	407	128	249	550	859	194882	121
835	592	451	506	713	980	199698	141

(b) *If* $\Delta \neq 2$ *and* $n > 8$, *then* $\Delta = 2\delta^2$ *and there is an arithmetic progression* $y_1, y_2, \dots, y_n$ *with difference* δ *such that* $x_i = |y_i|$.

5 Symmetric Sequences

In this last section, we consider nontrivial symmetric sequences $x_1, x_2, \dots, x_n$ satisfying the system (2), that is, the sequences such that $x_i = x_{n-i+1}$ for $i = 1, 2, \dots, n$. The sequences of this type were studied by Allison [A] who proved that there are infinitely many of them when $n = 8$. It is easy to prove that there are infinitely many such sequences when $n = 5$ (they are x_1, x_2, x_3, x_2, x_1, where x_1, x_2, x_3 is a solution of the Diophantine equation $x_1^2 + 3x_3^2 = 4x_2^2$), but for odd $n \geq 7$ there are no symmetric sequences satisfying the system (2). We also prove that the number of nontrivial symmetric sequences of even length $n \geq 10$ is at most finite. Notice that the Conjecture above says that such sequences do not exist. The results of this section can also be found in [Br], where they are formulated in terms of values of quadratic polynomials and proved using very similar arguments. We prove these results again, since the arguments are very short and the curves which we use are somewhat simpler than those in [Br]. We start with a proof of the result about nontrivial sequences of odd length (*cf.* [P, p. 845]).

Theorem 2 *There are no nontrivial symmetric sequences of squares with constant second differences of odd length* $n \geq 7$.

Proof It suffices to prove the Theorem for $n = 7$. Assume that $x_1, x_2, x_3, x_4, x_3, x_2, x_1$ is a symmetric septuple such that the second differences of the sequence of squares x_j^2 are constant. Then we have the relations:

$$x_1^2 - 3x_2^2 + 3x_3^2 - x_4^2 = 0,$$
$$x_2^2 - 3x_3^2 + 3x_4^2 - x_3^2 = 0,$$

which imply

$$x_2^2 = 4x_3^2 - 3x_4^2,$$
$$x_1^2 = 9x_3^2 - 8x_4^2.$$

We consider septuples up to the proportionality, so we can assume that either $x_4 = 0$, $x_3 = 1$ or $x_4 = 1$. The first case leads to $x_2^2 = 4$, $x_1^2 = 9$, so the septuple is trivial. In the second case, we get

$$x_2^2 = 4x_3^2 - 3,$$
$$x_1^2 = 9x_3^2 - 8.$$

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Hence the point $(X, Y) = (x_3, x_1x_2)$ belongs to the elliptic curve

$$Y^2 = (4X^2 - 3)(9X^2 - 8).$$

Multiplying by 36, we see that $(X_0, Y_0) = (6x_3, 6x_1x_2)$ belongs to the elliptic curve

$$Y_0^2 = X_0^4 - 59X_0^2 + 864. \tag{4}$$

The birational mapping

$$X_0 = -Y_1/2X_1,$$
$$Y_0 = -X_1/2 + (Y_1/2X_1)^2 - 59/2$$

leads to the Weierstrass model of the curve (4):

$$E : Y_1^2 = X_1^3 + 118X_1^2 + 25X_1.$$

Using the package GP/PARI, we obtain $L_E(1) = 0.654 \neq 0$, so there is no rational point of infinite order on E. Moreover, the group $E(\mathbb{Q})$ is cyclic of order 6 generated by the point $P = (25, 300)$, and $2P = (1, 12)$, $3P = (0, 0)$. One can verify that to the points $\pm P, \pm 2P$, there correspond the points $(\pm 6, \pm 6)$ on the curve (4), which lead to the trivial septuple of squares $x_1^2 = x_2^2 = x_3^2 = x_4^2 = 1$. ■

Remark 2 To the point $(0, 0) \in E$ there does not correspond any point on the curve (4).

As regards symmetric sequences of squares of even length with constant second differences, Allison [A] showed that their number is infinite for $n = 8$. As a last result, we prove that for even $n \geq 10$ the number of such nontrivial sequences is at most finite.

Theorem 3 *The number of nontrivial symmetric sequences of squares with constant second differences and even length $n \geq 10$ is finite.*

Proof We may assume that $n = 10$ and $x_1, x_2, x_3, x_4, x_5, x_5, x_4, x_3, x_2, x_1$ is a symmetric sequence such that the second differences of the squares x_j^2 are constant. Using similar relations to those in the proof of Theorem 2, we get

$$x_1^2 = 10x_4^2 - 9x_5^2, \tag{5}$$
$$x_2^2 = 6x_4^2 - 5x_5^2,$$
$$x_3^2 = 3x_4^2 - 2x_5^2.$$

We consider sequences up to the proportionality, so we can assume that x_5 is 0 or 1. If $x_5 = 0$, the system (5) gives a contradiction. Let $x_5 = 1$. Then multiplying the equations, we obtain that the point $(x, y) = (x_4, x_1x_2x_3)$ belongs to the hyperelliptic curve

$$y^2 = (10x^2 - 9)(6x^2 - 5)(3x^2 - 2) = 180x^6 - 432x^4 + 343x^2 - 90$$

of genus 2. By Faltings' theorem the number of rational points on this curve is finite. ∎

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Institute of Mathematics
University of Warsaw,
ul. Banacha 2,
PL-02-097 Warsaw
Poland
e-mail: bro@mimuw.edu.pl

Department of Mathematics
Chalmers University of Technology
and Göteborg University
S-41296 Göteborg
Sweden
e-mail: jub@math.chalmers.se

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