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ON THE EXISTENCE AND UNIQUENESS OF SOLUTIONS OF THE EQUATION $u_{tt} - \frac{\partial}{\partial x_i}\sigma_i(u_{x_i}) - \Delta_N u_t = f$

BY

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ABSTRACT. The existence and uniqueness of strong global solutions of initial-boundary value problems for the quasilinear equation $u_{tt} - \partial\sigma_i(u_{x_i})/\partial x_i - \Delta_N u_t = f$ is established for functions $\sigma_i(\xi)$, $i=1,\ldots,N$, satisfying: $\sigma_i(\xi) \in C^1(-\infty,\infty)$, $\sigma_i(0)=0$ and $0<\sigma_i'(\xi)\le K_0$ for some constant K_0.

1. **Introduction**. Sufficient conditions on the functions u_0, u_1 and $f(t)$ are established here to ensure the existence and uniqueness of a strong global solution of the initial-boundary value problem

$$(1) \qquad u_{tt} - \frac{\partial}{\partial x_i}\sigma_i(u_{x_i}) - \Delta_N u_t = f, \qquad 0 < t < T$$

$$u\big|_{\partial\Omega} = 0, \qquad u(0) = u_0, \qquad u_t(0) = u_1$$

where $u_t \equiv \partial u/\partial t$, $u_{u_i} \equiv \partial u/\partial x_i$, $\Delta_N \cdot \equiv \partial^2_{\cdot}/\partial x_i^2$ (summation of second term over $i=1,\ldots,N$ is understood), Ω is a bounded domain in N-dimensional Euclidean space E^N with smooth boundary $\partial\Omega$ and σ_i, u_0, u_1 and f are real-valued functions with $\sigma_i(\xi)$, $i=1,\ldots,N$ satisfying

$$(2) \qquad \sigma_i(\xi) \in C^1(-\infty,\infty), \qquad \sigma_i(0) = 0, \qquad 0 < \sigma_i'(\xi) \le K_0$$

for some constant K_0 where $.' \equiv d./d\xi$.

Considerable attention ([4], [5], [6], [8]) has recently been given to quasilinear equations such as that appearing in (1) and related equations which arise in the study of nonlinear elasticity–plasticity theory. For $N=1$ and $f=0$, MacCamy and Mizel [6] have established the existence, uniqueness and stability of a global smooth solution for $\sigma_1(\xi)=\sigma(\xi)$ satisfying

$$\sigma(\xi) \in C^3(-\infty,\infty), \qquad \sigma(0) = 0, \qquad 0 < \sigma'(\xi).$$

Their results follow from the consideration of the differential equation in (1) as two different inhomogeneous equations. For large space dimension N, the investigation of the existence of global classical solutions of quasilinear equations is

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often replaced by the search for weak or perhaps even strong solutions. In what follows, a compactness argument (see e.g. [3], chapter 1) is used to prove the existence of a unique strong solution of (1) for arbitrary N and the $\sigma_i(\xi)$ satisfying conditions (2). In particular, it is shown that the solutions are just as differentiable as the initial data in the Sobolev class $H^{2,2}(\Omega)$.

2. **The existence theorem.** For each p, $1 \leq p \leq \infty$, $L^p(\Omega)$ shall denote the usual real Lebesgue space with norm

$$\|u\|_{0,p}^p \equiv \int_\Omega |u(x)|^p \, dx < \infty \quad \text{if} \quad 1 \leq p < \infty$$

$$\|u\|_{0,\infty} \equiv \operatorname*{essup}_\Omega |u(x)| < \infty \quad \text{if} \quad p = \infty.$$

$L^2(\Omega)$ is a Hilbert space with respect to the scalar product

$$(u, v) = \int_\Omega u(x)v(x) \, dx.$$

For brevity in notation in the $L^2(\Omega)$ norm $\|.\|_{0,2}$ is denoted by $\|.\|$. $H^{m,2}(\Omega) \equiv \{u \in L^2(\Omega) \mid D_\alpha u \equiv (\partial^{|\alpha|}u/\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}) \in L^2(\Omega)$ for every $\alpha_1 + \cdots + \alpha_N = |\alpha| \leq m\}$ with norm $\|u\|_{m,2}^2 \equiv \sum_{|\alpha| \leq m} \|D_x^\alpha u\|^2$ where the derivatives are considered in the weak or distribution sense and by $H_0^{m,2}(\Omega)$ we mean the closure in $H^{m,2}(\Omega)$ of the smooth functions with compact support in Ω.

Let $\|.\|_X$ be the norm and X^* the dual space of a Banach space X. We denote by $L^p(0, T; X)$ $1 \leq p \leq \infty$ the space of (classes of) real functions $f(t): (0, T) \to X$ with

$$\left(\int_0^T \|f(t)\|_X^p \, dt\right)^{1/p} < \infty \quad \text{for} \quad 1 \leq p < \infty$$

and with the usual modification for $p = \infty$.

We shall require the following lemma, the proof of which can be found in ([1], p. 59).

LEMMA 1. *Let Ω be any bounded domain in E^N with smooth boundary and let the functions $w_j(x)$, $j = 1, 2, \ldots,$ form an orthogonal basis in $L^2(\Omega)$. Then for any $\varepsilon > 0$ there exists a number N_ε such that*

$$\|u\| \leq \left(\sum_{j=1}^{N_\varepsilon} (u, w_j)^2\right)^{1/2} + \varepsilon \|u\|_{1,2}$$

for all $u(x)$ in $H^{1,2}(\Omega)$ and the number N_ε does not depend on u.

With the assumption that conditions (2) hold for the $\sigma_i(\xi)$, the following result concerning the existence of a generalized solution of problem (1) is established here.

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THEOREM 1. *For any* $u_0 \in H_0^{1,2}(\Omega) \cap H^{2,2}(\Omega)$, $u_1 \in H_0^{1,2}(\Omega)$ *and* $f \in L^2(0, T; L^2(\Omega))$ *there exists one and only one function u with*

$$u \in L^\infty(0, T; H_0^{1,2}(\Omega) \cap H^{2,2}(\Omega))$$
$$u_t \in L^\infty(0, T; H_0^{1,2}(\Omega)) \cap L^2(0, T; H^{2,2}(\Omega))$$
$$u_{tt} \in L^2(0, T; L^2(\Omega))$$

such that $u(0)=u_0$ *and* $u_t(0)=u_1$ *a.e. on* Ω *and*

$$u_{tt} - \frac{\partial}{\partial x_i} \sigma_i(u_{x_i}) - \Delta_N u_t = f \qquad \text{a.e.}$$

REMARK 1. The precise sense in which the above equation is satisfied is that the L.H.S. and R.H.S. are equivalent a.e. on $(0, T)$ as functions from $(0, T)$ into $L^2(\Omega)$.

REMARK 2. If $u(t): (0, T) \to L^1(\Omega)$ is Lebesgue summable on $(0, T)$, then there exists a function $u(., t)$ defined and measurable on $\Omega \times (0, T)$ which is uniquely determined up to a subset of measure zero on $\Omega \times (0, T)$ and such that $u(t)=u(., t)$ a.e. on $(0, T)$ and $u(x, t) \in L^1(\Omega \times (0, T))$. Furthermore if $u(t)$: $[0, T] \to L^p(\Omega)$, $(1 \le p \le \infty)$, is strongly continuous, then there exists $u(., t)$ measurable on $\Omega \times [0, T]$ such that $u(t)=u(., t)$ for every t in $[0, T]$. It will be clear from the construction of u from the approximate solutions u^n and the corresponding a priori estimates that both u and u_t are strongly continuous from $[0, T]$ into $L^2(\Omega)$.

REMARK 3. A much more difficult but interesting problem is that of proving the existence of unique global classical solutions of (1) when $N=2$ or 3. It is believed that this could be accomplished using techniques similar to those found in [7] by a suitable strengthening of the regularity requirements on the σ_i in (2) and on the data u_0, u_1 and f.

Proof of existence. Let $w_i(x)$, $j=1, 2, \ldots$, be the normalized eigenfunctions associated with the Laplace operator with domain $\mathscr{D}(-\Delta_N)=H_0^{1,2}(\Omega) \cap H^{2,2}(\Omega)$. That is, the functions satisfying

$$-\Delta_N w_j = \mu_j w_j \text{ in } \Omega, \quad w_j = 0 \text{ on } \partial\Omega \qquad (j = 1, 2, \ldots).$$

It is well known that for sufficiently smooth Ω, the functions w_j are in $C^2(\Omega \cup \partial\Omega)$. Let P_n be the projection in $L^2(\Omega)$ onto the subspace $\{w_1, \ldots, w_n\}$ generated by the distinct basis elements $w_1, \ldots, w_n$. It follows from conditions (2) that for each n there exists a solution $u^n(t)=\sum_{k=1}^n c_{nk}(t)w_k$ of the system

$$(u_{tt}^n(t), w_j) - \left(\frac{\partial}{\partial x_i} \sigma_i(u_{x_i}^n(t)), w_j\right) - (\Delta_N u_t^n(t), w_j) = (f(t), w_j) \qquad j = 1, \ldots, n$$

(3) $$u^n(t) \in P_n L^2(\Omega) \quad \text{for all} \quad t \in [0, T]$$
$$u^n(0) = P_n u_0, \qquad u_t^n(0) = P_n u_1$$

which satisfies (3) a.e. on $[0, T_n]$ for some T_n with $0<T_n\leq T$. The a priori estimates which follow allow each $[0, T_n]$ to be taken to be $[0, T]$. One obtains from (3) in the usual way

$$\frac{1}{2}\frac{d}{dt}\left\{\|u_t^n(t)\|^2+2\int_\Omega\left[\int_0^{u^n_{x_i}(t)}\sigma_i(s)\,ds\right]dx\right\}+\|u^n_{x_it}(t)\|^2=(f(t),u_t^n(t))$$

and since $0\leq\int_0^\xi\sigma_i(s)\,ds\leq K_0\xi^2/2$,

$$\|u_t^n(t)\|^2+\|u^n_{x_i}(t)\|^2+\int_0^t\|u^n_{x_is}(s)\|^2\,ds\leq K_1 \tag{4}$$

for every n independent of t in $[0, T]$. Replacing w_j by $-\Delta_N u^n$ in (3) gives

$$\begin{aligned}(u^n_{x_itt}(t), u^n_{x_i}(t))+(\sigma_i'(u^n_{x_i}(t))u^n_{x_ix_i}(t), \Delta_N u^n(t))\\ +\frac{1}{2}\frac{d}{dt}\|\Delta_N u^n(t)\|^2=-(f(t),\Delta_N u^n(t))\end{aligned} \tag{5}$$

and, since $\|u^n_{x_ix_i}(t)\|\leq K_2\|\Delta_N u^n(t)\|$ for all t independent of n ([2]), (5) gives by (4) and conditions (2)

$$\int_0^t\|\Delta_N u^n(s)\|^2\,ds\leq K_3\int_0^t\left(\int_0^s\|\Delta_N u^n(\tau)\|^2\,d\tau\right)ds+K_4t+K_5$$

for all t in $[0, T]$ and K_3, K_4 and K_5 independent of n. Hence,

$$\int_0^t\|\Delta_N u^n(s)\|^2\,ds\leq K_6. \tag{6}$$

Now, by replacing w_j by $-\Delta_N u_t^n(t)$, (3) becomes

$$\frac{1}{2}\frac{d}{dt}\|u^n_{x_it}(t)\|^2+\|\Delta_N u_t^n(t)\|^2=-(\sigma_i'(u^n_{x_i}(t))u^n_{x_ix_i}(t),\Delta_N u_t^n(t))-(f(t),\Delta_N u_t^n(t))$$

and from (6)

$$\|u^n_{x_it}(t)\|^2+\int_0^t\|\Delta_N u_s^n(s)\|^2\,ds\leq K_7 \tag{7}$$

independent of n and t in $[0, T]$. (5) now gives by (7)

$$\|\Delta_N u^n(t)\|^2\leq K_8 \tag{8}$$

independent of n and if t in $[0, T]$. Finally, replacing w_j by $u_{tt}(t)$ gives from (4), (7) and (8)

$$\int_0^t\|u^n_{ss}(s)\|^2\,ds\leq K_9 \tag{9}$$

for some constant K_9 independent of n and of t in $[0, T]$.

Integration of (3) from t_1 to t_2, $t_1, t_2\in[0, T]$ and the subsequent integration of

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that result from t to $t+h$ with respect to t_2 gives by (4), (7), (8), (9) and condition (2)

$$|u^n(t+h)-u^n(t), w_k)| = |c_{nk}(t+h)-c_{nk}(t)| \leq K_{10}(h+h^2)$$

where K_{10} depends on k but not on n for $n \geq k$ or on $t \in [0, T]$. Similarly, integration of (3) from t to $t+h$ gives

$$|(u_t^n(t+h)-u_t^n(t), w_j)| = |c'_{nk}(t+h)-c'_{nk}(t)| \leq K_{11}(h+\sqrt{h})$$

with K_{11} independent of k for $n \geq k$. Thus, the functions $c_{nk}(t)=(u^n(t), w_k)$ and $c'_{nk}(t)=(u_t^n(t), w_k)$, $n=1, 2, \ldots$, are uniformly bounded and equicontinuous for fixed k and arbitrary $n \geq k$. Therefore, by the usual diagonal procedure we can select a subsequence n_m, $m=1, 2, \ldots$, such that for each $k=1, 2, \ldots$, $c_{n_m k}(t)$ and $c'_{n_m k}(t)$ converge uniformly on $[0, T]$ to some continuous functions $c_k(t)$ and $l_k(t)$. These functions determine $u(x, t)=\sum_{k=1}^\infty c_k(t)w_k$ and $\tilde{u}(x, t)=\sum_{k=1}^\infty l_k(t)w_k$ and it follows that

$$\begin{matrix} u^{n_m} \to u \\ u_t^{n_m} \to \tilde{u} \end{matrix} \quad \text{weakly in } L^2(\Omega) \text{ uniformly in } t \in [0, T]. \tag{10}$$

Indeed, for any $v(x) \in L^2(\Omega)$,

$$\begin{aligned} |(u^{n_m}-u, v)| &= \left| \sum_{k=1}^{M} (v, w_k)(u^{n_m}-u, w_k) + \left(u^{n_m}-u, \sum_{k=M+1}^{\infty} (v, w_k) w_k \right) \right| \\ &\leq \left(\sum_{k=1}^{M} |(v, w_k)| \cdot |c_{n_m k}(t)-c_k(t)| \right) + K_{12} \left(\sum_{k=M+1}^{\infty} (v, w_k)^2 \right)^{1/2} \end{aligned}$$

where K_{12} does not depend on n_m and we can choose M and n_m so large that for every $t \in [0, T]$ both terms in the above sum become less than $\varepsilon/2$ for any preassigned $\varepsilon > 0$. Similarly for $u_t^{n_m}-\tilde{u}$. It follows easily that $\tilde{u}=u_t$ in the sense of distributions. For brevity in notation all subsequences u^{n_m} shall again be denoted simply by u^n. Taking further subsequences if necessary, it is clear that the estimates (4), (7), (8) and (9) yield

$$\begin{aligned} u^n &\to u && \text{weak* in } L^\infty(0, T; H_0^{1,2}(\Omega) \cap H^{2,2}(\Omega)) \\ u^n_{x_i t} &\to u_{x_i t} && \text{weak* in } L^\infty(0, T; L^2(\Omega)) \text{ and weakly in } L^2(\Omega \times (0, T)) \\ u^n_{x_i x_i t} &\to u_{x_i x_i t} && \text{weak* in } L^\infty(0, T; L^2(\Omega)) \text{ and weakly in } L^2(\Omega \times (0, T)) \end{aligned}$$

and

$$\begin{aligned} u^n_{tt} &\to u_{tt} && \text{weakly in } L^2(0, T; L^2(\Omega)) \\ \Delta_N u^n_t &\to \Delta_N u && \text{weakly in } L^2(0, T; L^2(\Omega)) \end{aligned}$$

as $n \to \infty$, with all derivatives being considered in the usual weak or distribution sense. Lemma 1 applied to u^n-u and $u_t^n-u_t$ gives

$$\begin{matrix} u^n \to u \\ u_t^n \to u_t \end{matrix} \quad \text{strongly in } L^2(\Omega) \text{ uniformly in } t \text{ on } [0, T]$$

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and $u(x, 0)=u_0(x)$ and $u_t(x, 0)=u_1(x)$ a.e. on Ω. Lemma 1 applied to $u^n_{x_i}-u_{x_i}$ yields $u^n_{x_i}\to u_{x_i}$ strongly in $L^2(\Omega\times(0, T))$ and again extracting further subsequences $u^n_{x_i}\to u_{x_i}$ a.e. on $\Omega\times(0, T)$ for each $i=1,\ldots,N$. Thus, $\sigma_i(u^n_{x_i})\to\sigma_i(u_{x_i})$ a.e. on $\Omega\times(0, T)$ by continuity and by Lebesgue dominated convergence. $\sigma_i'(u^n_{x_i})v\to\sigma_i'(u_{x_i})v$ strongly in $L^2(\Omega\times(0, T))$ as $n\to\infty$ for any fixed function v in $L^2(\Omega\times(0, T))$. Consequently, for any v in $L^2(\Omega\times(0, T))$,

$$\int_0^T (\sigma_i'(u^n_{x_i})u^n_{x_ix_i}, v)\,dt$$

$$=\int_0^T (u^n_{x_ix_i}, \sigma_i'(u^n_{x_i})v)\,dt\to\int_0^T (u_{x_ix_i}, \sigma_i'(u_{x_i})v)\,dt$$

$$=\int_0^T (\sigma_i'(u_{x_i})u_{x_ix_i}, v)\,dt \quad \text{as} \quad n\to\infty.$$

Passage to the limit in (3) as $n\to\infty$ now gives the required result.

Proof of uniqueness. Let $u(x, t)$ and $v(x, t)$ be two strong solutions of problem (1). Then $w=u-v$ is a strong solution of the problem

$$w_{tt}-\Delta_N w-\Delta_N w_t = -\Delta_N(u-c)+\frac{\partial}{\partial x_i}\sigma_i(u_{x_i})-\frac{\partial}{\partial x_i}\sigma_i(c_{x_i}), \qquad 0<t<T$$

$$w|_{\partial\Omega}=0,\ w(x,0=0,\ w_t(x,0)=0.$$

Since $|\sigma_i(\xi)-\sigma_i(\eta)|\le K_0\,|\xi-\eta|$ for all real ξ and η and each $i=1,\ldots,N$, taking the product of this differential equation with w_t and integrating over Ω gives

$$\frac{1}{2}\frac{d}{dt}\{\|w_t(t)\|^2+\|w_{x_i}(t)\|^2\}+\|w_{x_it}(t)\|^2$$

$$=(w_{x_i}(t), w_{x_it}(t))+(\sigma_i(u_{x_i})-\sigma_i(v_{x_i}), w_{x_it})$$

$$\le\|w_{x_it}(t)\|^2+\|w_{x_i}(t)\|^2/2+\frac{1}{2}\int|\sigma_i(u_{x_i})-\sigma_i(v_{x_i})|^2\,dx$$

$$\le\|w_{x_it}(t)\|^2+(K_0+1)\{\|w_t(t)\|^2+\|w_{x_i}(t)\|^2\}.$$

Therefore

$$\|w_t(t)\|^2+\|w(t)\|^2_{1,2}=0$$

and the theorem is proved.

REMARK 4. For $N\le 2$, the uniform bound $\sigma_1'(.)\le K_0$ in conditions (2) is not required since $|u^n_{x_i}(x, t)|\le K_{13}\,\|u^n(t)\|_{2,2}$ a.e. on Ω for every fixed t and some constant K_{13} independent of x in Ω. For $N\ge 3$, the condition $\sigma_i'(0)\le K_0$ amounts to a restriction that the σ_i have at most monomial growth. However, the relaxing of this constraint to permit polynomial growth in the σ_i introduces serious technical problems [9]. It is no longer possible to obtain sufficient a priori estimates to permit the application of a compactness argument.

EXAMPLE. A simple two-dimensional example of (1) is furnished by the model for a clamped vibrating plate if one assumes nonlinear stress relations with a memory term with the equation of motion being given by

$$u_{tt}-\frac{\partial}{\partial x_i}\left(u_{x_i}+\frac{u_{x_i}}{1+u_{x_i}^2}\right)-\Delta_2 u_t = 0$$

and it is clear that each $\sigma_i(\xi)=\xi+(\xi/(1+\xi^2))$, $i=1, 2$, satisfies (2).

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