



Two-graphs and the Embedded Topology of Smooth Quartics and its Bitangent Lines

Shinzo Bannai and Momoko Ohno

Abstract. In this paper, we study how to distinguish the embedded topology of a smooth quartic and its bitangent lines. In order to do this, we introduce the concept of two-graphs and switching classes from graph theory. This new method improves previous results about a quartic and three bitangent lines considered by E. Artal Bartolo and J. Vallès, four bitangent lines considered by the authors and H. Tokunaga, and enables us to distinguish the embedded topology of a smooth quartic and five or more bitangent lines. As an application, we obtain a new Zariski 5-tuple and a Zariski 9-tuple for arrangements consisting of a smooth quartic and five of its bitangent lines and six of its bitangent lines, respectively.

1 Introduction

Smooth plane quartics and their bitangent lines have attracted the interest of many mathematicians from various points of views. In this paper, we study arrangements consisting of a smooth quartic and its bitangent lines in terms of their embedded topology in the context of Zariski pairs, the concept of which was first introduced in [1].

Let $C_1, C_2 \subset \mathbb{P}^2$ be plane curves. Then C_1 and C_2 are said to be a Zariski pair if they satisfy the following two conditions:

- (1) There exist tubular neighborhoods $T(C_i)$ of C_i ($i = 1, 2$) such that the pairs $(T(C_1), C_1)$ and $(T(C_2), C_2)$ are homeomorphic as pairs.
- (2) The pairs (\mathbb{P}^2, C_1) and (\mathbb{P}^2, C_2) are not homeomorphic as pairs.

The first example of a Zariski pair consisting of irreducible six-cuspidal sextics was found by Zariski himself in [18]. The notion of a Zariski pair can be generalized to any number of curves C_1, \dots, C_r , where we require any two curves to be Zariski pairs. The first condition above is equivalent to saying that the curves C_1 and C_2 have the same combinatorial type that can be calculated algebraically. Thus, a Zariski pair is a pair of plane curves having similar algebraic properties but with subtle differences that contribute to the difference in the embedded topology. Hence, the key to studying Zariski pairs is to find a suitable method to detect the subtle differences. Many invariants have been used to this end, such as the fundamental groups of the complements $\pi(\mathbb{P}^2 \setminus C_i)$, the Alexander polynomials $\Delta_{C_i}(t)$, and the existence/non-existence of certain Galois covers branched along C_i (See [2] for a survey on these topics). Also, other more recent types of invariants called “splitting invariants” have

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been developed and have proved effective in studying reducible curves ([3, 4, 14, 15]). However, as the number of irreducible components increase, these invariants become increasingly more complex, and it becomes difficult to manage the data they provide. One approach in resolving this problem is to consider all sub-arrangements of the curves as done in [5, 6]. In this paper, we take another step further in this direction and apply the concept of two-graphs and switching classes from graph theory (see Section 2 for definitions) in order to clarify the data of the splitting invariants, which enables us to find a new Zariski 5-tuple of degree 9 and a 9-tuple of degree 10.

Let Q be a smooth plane quartic. It is well known that a smooth plane quartic has 28 bitangent lines L_1, \dots, L_{28} . For a subset $I \subset \{1, \dots, 28\}$, let

$$Q + L_I := Q + \sum_{i \in I} L_i.$$

Considering the curves of the above form, the case where $|I| = 3$ has been studied by E. Artal Bartolo and J. Vallès where they found a Zariski pair, about which the authors were informed via private communication. This case has also been studied by B. Guerville-Ballé and J.-B. Meilhan from the perspective of linking invariants ([9]). The case where $|I| = 4$ has been treated by the authors together with H. Tokunaga, where a Zariski triple was found ([7]). In this paper, we consider the cases where $|I| \geq 5$.

For each I , we can associate a two-graph or, equivalently, a switching class \mathcal{L}_I with $Q + L_I$ as will be explained in Section 3. Our first main result is the following theorem.

Theorem 1.1 *Let $I_1, I_2 \subset \{1, \dots, 28\}$ and let $\mathcal{L}_{I_1}, \mathcal{L}_{I_2}$ be their associated two-graphs. If \mathcal{L}_{I_1} and \mathcal{L}_{I_2} are not equivalent as two-graphs, then $(\mathbb{P}^2, Q + L_{I_1})$ and $(\mathbb{P}^2, Q + L_{I_2})$ are not homeomorphic as pairs. If, furthermore, $Q + L_{I_1}$ and $Q + L_{I_2}$ have the same combinatorial type, then they form a Zariski pair.*

Remark 1.2 Two-graphs are well suited for describing the geometric conditions involved, whereas switching classes are more suited in executing the actual computations.

The number t_n of equivalence classes of two-graphs with n vertices, whose OEIS code is A002854, is known ([10]). The values for small n are given in the following table:

| | | | | | | | | | | | |
|-------|---|---|---|---|---|----|----|-----|------|-------|-----|
| n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | ... |
| t_n | 1 | 1 | 2 | 3 | 7 | 16 | 54 | 243 | 2038 | 33120 | ... |

A list of representatives for the above equivalence classes for $n \leq 7$ can be found in [17]. The value $t_3 = 2$ corresponds to the Zariski pair found by E. Artal Bartolo and J. Vallès. The value $t_4 = 3$ corresponds to the Zariski triple found by the authors and H. Tokunaga. However, for $n \geq 5$, there are some two-graphs that do not appear in our situation; hence, we need to check which equivalence classes actually appear. We were able to check the cases for $n = 5, 6$ by hand, which gives our second main result.

Theorem 1.3 Under the notation given above, the numbers of equivalence classes of two-graphs obtained from curves of the form $Q + L_I$ when $|I| = 5, 6$ are 5, 9, respectively. Thus, we have the following.

- (i) There exists a Zariski 5-tuple of curves of the form $Q + L_I$ with $|I| = 5$.
- (ii) There exists a Zariski 9-tuple of curves of the form $Q + L_I$ with $|I| = 6$.

For cases where $n \geq 7$, the discrepancy between the number of equivalence classes of all two-graphs and the number of two-graphs that actually appear in our situation becomes large; hence, a deeper analysis is needed, which we hope to accomplish in the near future.

The organization of this paper is as follows. In Section 2, we state the definition and basic properties of two-graphs and switching classes. In Section 3, we explain how to associate a two-graph with $Q + L_I$. In Section 4, we prove Theorems 1.1 and 1.3.

2 Two-graphs and Switching Classes

In this section, we introduce two-graphs and switching classes that will be used in the proof of Theorem 1.1. We refer the reader to [11] for details. Let $\Omega = \{\omega_1, \dots, \omega_n\}$ be a finite set, and let $\Omega^{(3)}$ be the set of all 3-subsets of Ω .

Definition 2.1 A two-graph (Ω, Δ) is a pair of a vertex set Ω and a triple set $\Delta \subset \Omega^{(3)}$, such that each 4-subset of Ω contains an even number of triples in Δ .

Sometimes, we omit Δ and denote (Ω, Δ) simply by Ω when Δ is clear from the context.

Definition 2.2 Two two-graphs (Ω_1, Δ_1) and (Ω_2, Δ_2) are said to be equivalent if there exists a bijection $\psi: \Omega_1 \rightarrow \Omega_2$ such that $\{\omega_i, \omega_j, \omega_k\} \in \Delta_1$ if and only if $\{\psi(\omega_i), \psi(\omega_j), \psi(\omega_k)\} \in \Delta_2$.

Definition 2.3 An induced sub-two-graph (Ω', Δ') of (Ω, Δ) is a two-graph that satisfies $\Omega' \subset \Omega$ and $\Delta' = \Omega'^{(3)} \cap \Delta$.

In order to handle two-graphs easily, we consider switching equivalence on simple graphs. Let (V, E) be a finite simple graph defined by its labeled vertex set $V = \{v_1, \dots, v_n\}$ and its edge set E , where $|V| = n$ for fixed n . As is well known, the graph (V, E) can be described by its adjacency matrix, which has the elements zero on the diagonal, -1 and $+1$ elsewhere according to whether the corresponding vertices are adjacent or non-adjacent, respectively. Then, as in [17], switching-equivalence is defined as follows.

Definition 2.4 The graphs (V, E) and (V', E') are switching-equivalent if the adjacency matrix A of (V, E) can be changed to the adjacency matrix A' of (V', E') by repeating the following two operations:

- (i) Multiply by -1 any row and the corresponding column.
- (ii) Interchange two rows and, simultaneously, the corresponding columns.

Theorem 2.5 ([11, Theorem 3.2]) *Switching-equivalence is an equivalence relation.*

The equivalence classes with respect to switching equivalence will be called *switching classes*.

Remark 2.6 The name “switching” comes from the fact that the first of the two operations corresponds to “switching” the existence/non-existence of the edges connected to a vertex of the corresponding graph.

When we consider switching-equivalence directly in terms of the graph, the following lemma holds.

Lemma 2.7 ([11, Lemma 3.9]) *Let (V, E) and (V', E') be finite simple graphs. Then the graphs (V, E) and (V', E') are switching-equivalent if and only if there exists a bijection $\psi: V \rightarrow V'$ such that the parity of the number of edges among $\{v_i, v_j, v_k\}$ and $\{\psi(v_i), \psi(v_j), \psi(v_k)\}$ are the same for all $\{v_i, v_j, v_k\} \in V^{(3)}$.*

For any graph (V, E) with four vertices, it can be easily checked that the number of triples $\{v_i, v_j, v_k\} \subset V$ that have an odd number of edges among them is even. Therefore, we can associate a two-graph (Ω_V, Δ_V) with a switching class (V, E) by setting $\Omega_V = V$ and

$$\Delta_V = \{\{v_i, v_j, v_k\} \in V^{(3)} \mid \{v_i, v_j, v_k\} \text{ has an odd number of edges}\}.$$

In fact, this correspondence gives a bijection between switching classes and two-graphs as in the following theorem.

Theorem 2.8 ([11, Theorem 4.2]) *Given n , there is a one-to-one correspondence between the two-graph structures and the switching classes of graphs on the set of n elements.*

3 Two-graphs and Switching Classes Associated with Configurations of Bitangents

In this section, we explain how we associate two-graphs and switching classes with plane curves of the form $Q + L_I$ under our setting.

3.1 Two-graphs Associated with $Q + L_I$

Let Q be a smooth plane quartic. It is well known that a smooth plane quartic has 28 bitangent lines L_1, \dots, L_{28} . For $I \subset \{1, \dots, 28\}$, put $\mathcal{L}_I := \{L_i \mid i \in I\}$. Let Δ_I be the set of triples $\{L_i, L_j, L_k\} \subset \mathcal{L}_I$ such that there exists a conic passing through all six tangent points of $Q \cap (L_i + L_j + L_k)$. As will be shown below, $(\mathcal{L}_I, \Delta_I)$ becomes a two-graph.

Definition 3.1 The two-graph $\mathcal{L}_I = (\mathcal{L}_I, \Delta_I)$ is said to be the two-graph associated with $Q + L_I$.

The existence of a conic is related to the concept of “connected numbers” as seen below. In our case, the definition of connected numbers is as follows.

Definition 3.2 Let $\phi_Q: S \rightarrow \mathbb{P}^2$ be the double cover of \mathbb{P}^2 branched along Q . For a possibly reducible curve $C \subset \mathbb{P}^2$ such that $C \setminus Q$ is connected, the connected number $c_{\phi_Q}(C)$ of C with respect to ϕ_Q is defined to be the number of connected components of $\phi_Q^{-1}(C \setminus Q)$.

Note that since we are considering double coverings, $c_{\phi_Q}(C) = 1$ or 2 . See [15] for a more general treatment of connected numbers.

Proposition 3.3 For all triples $\{L_i, L_j, L_k\} \subset \mathcal{L}_I, \{L_i, L_j, L_k\} \in \Delta_I$ if and only if $c_{\phi_Q}(L_i + L_j + L_k) = 2$.

Proof The same statement for smooth cubic contact curves instead of $C = L_i + L_j + L_k$ is proved in [4, Proposition 3.3]. The proof given there with some considerations also works in our case where C is a reducible, thus singular contact curve. Note that since L_i is a bitangent line, the preimage of L_i under ϕ_Q has two irreducible components. Let $\phi_Q^{-1}(L_i) = L_i^+ + L_i^-$. When $c_{\phi_Q}(C) = 2$, the connected components of $\phi_Q^{-1}(C)$ correspond to two triangles of the form $C^+ = L_i^+ + L_j^+ + L_k^+$ and $C^- = L_i^- + L_j^- + L_k^-$ for a suitable choice of labels. On the other hand, when $c_{\phi_Q}(C) = 1$, the six components $L_i^\pm, L_j^\pm, L_k^\pm$ form a hexagon. The key point in the proof is that the existence of a conic D through the six points of tangency ensures that there exists a fourth line L (which is actually another bitangent line) such that $C + L$ becomes a member of the pencil generated by Q and $2D$. Then we can see that $c_{\phi_Q}(C + L) = 2$ directly from the defining equations, which in turn implies that $c_{\phi_Q}(C) = 2$. On the other hand, if $c_{\phi_Q}(C) = 2$, the set of intersection points of C^+ and C^- is exactly the preimages of the six points of tangency under ϕ_Q where at each point C^+ and C^- intersect transversely. Thus, the intersection number $C^+ \cdot C^- = 6$ and the calculations for the intersection numbers concerning C^\pm given in the proof of [4, Proposition 3.3] become identical to the smooth case. Then we can apply the exact same arguments in [4, Proposition 3.3] to show the existence of a conic D passing through the six points of tangency by considering the properties of C^\pm viewed as tri-sections of a certain rational elliptic surface associated with Q . (See [4] for details.) ■

Theorem 3.4 The pair $(\mathcal{L}_I, \Delta_I)$ is a two-graph for any $I \subset \{1, \dots, 28\}$.

Proof By [7, Lemma 3.5], the number of triples in each 4-subset of \mathcal{L}_I having connected number equal to 2 must be even. Hence, by Proposition 3.3, the statement holds. ■

Note that if $I' \subset I$, then $\mathcal{L}_{I'}$ can be identified with an induced sub-two-graph of \mathcal{L}_I via the obvious inclusion map.

3.2 Switching Classes Associated with $Q + L_I$ via the E_7^* Lattice

With a quartic Q and a smooth point $z \in Q$, a rational elliptic surface $S_{Q,z}$ can be associated as described in [16]. The properties of $S_{Q,z}$ and its group of sections give valuable data in studying configurations involving Q . When Q is a smooth quartic and $z \in Q$ is a general point, by [12], the Mordell–Weil lattice $MW(S_{Q,z})$ is isomorphic to

the dual root lattice E_7^* . The lattice E_7^* can be realized as a sub-lattice of \mathbb{R}^8 such that the 56 minimal vectors of norm $\frac{3}{2}$ in E_7^* are equal to $\pm\frac{1}{4}[-1, \dots, -1, 3, 3] \in \mathbb{R}^8$ up to permutation. We denote these vectors by

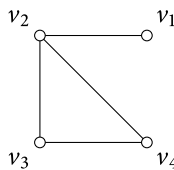
$$\pm u_{jk} = \pm\frac{1}{4}[-1, \dots, \overset{j}{3}, \dots, \overset{k}{3}, \dots, -1],$$

where $1 \leq j < k \leq 8$. We also denote the 56 minimal vectors $\pm u_{jk}$ by $\pm v_1, \dots, \pm v_{28}$ for convenience. On the other hand, there are 56 $\mathbb{C}(t)$ -rational points corresponding to the 56 minimal vectors in E_7^* . Note that by [13], the lines obtained from the x -coordinates of the 56 $\mathbb{C}(t)$ -rational points are bitangents of Q . Hence, each pair $\pm v_i$ corresponds to a bitangent L_i . Conversely, any bitangent can be obtained in this way. Now for $I \subset \{1, \dots, 28\}$ with $|I| = n$ ($1 \leq n \leq 28$), by choosing a representative v_i from each pair $\pm v_i$, we obtain a set of minimal vectors $\{v_i \mid i \in I\}$. Then we construct a simple graph (V_I, E_I) from $\{v_i \mid i \in I\}$ as follows.

- (1) Let $V_I = \{v_i \mid i \in I\}$.
- (2) Define the edge set E_I as follows:
 - (a) if $v_{i_1} \cdot v_{i_2} = -\frac{1}{2}$ for $v_{i_1}, v_{i_2} \in V_I$, add an edge $\overline{v_{i_1}v_{i_2}}$;
 - (b) if $v_{i_1} \cdot v_{i_2} = \frac{1}{2}$ for $v_{i_1}, v_{i_2} \in V_I$, add no edge between v_{i_1} and v_{i_2} .

The construction above depends on the choice of representatives $\{v_i\}$. However, choosing $-v_i$ instead of v_i corresponds to switching the existence/non-existence of the edges of the corresponding graph as in §2. Hence, we have a switching class represented by (V_I, E_I) associated with the set of bitangents \mathcal{L}_I .

Example 3.5 Denote each 56 minimal vectors $\pm u_{jk}$ by $\pm v_i$, and let $v_1 := u_{14}, v_2 := u_{18}, v_3 := u_{28}, v_4 := u_{38}$. Take $I := \{1, 2, 3, 4\} \subset \{1, \dots, 28\}$; then we obtain a simple graph (V_I, E_I) :



The switching class represented by (V_I, E_I) corresponds to the two graph $(\Omega_{V_I}, \Delta_{V_I})$ with

$$\Omega_{V_I} = V_I, \quad \Delta_{V_I} = \{\{v_1, v_3, v_4\}, \{v_2, v_3, v_4\}\}.$$

The parity of the number of edges in each triple of the graph (V_I, E_I) is characterised by the following proposition.

Proposition 3.6 For $\{L_{i_1}, L_{i_2}, L_{i_3}\} \subset \mathcal{L}$, the number of edges among the corresponding triple $\{v_{i_1}, v_{i_2}, v_{i_3}\}$ is odd if and only if $c_{\phi_Q}(L_{i_1} + L_{i_2} + L_{i_3}) = 2$.

Proof By [7, Lemma 3.2], $c_{\phi_Q}(L_{i_1} + L_{i_2} + L_{i_3}) = 2$ if and only if the number of edges of the corresponding triple $\{v_{i_1}, v_{i_2}, v_{i_3}\}$ of the corresponding graph is 1 or 3. ■

So far, we have defined two objects associated with $Q + L_I$, the two-graph \mathcal{L}_I and the switching class (V_I, E_I) . The two objects are actually compatible in the following sense.

Proposition 3.7 *The two graph corresponding to the switching class represented (V_I, E_I) and the two-graph \mathcal{L}_I are equivalent.*

Proof This is a direct consequence of Propositions 3.3 and 3.6. ■

4 Proof of Main Theorems

In this section, we prove Theorems 1.1 and 1.3. First, we consider Theorem 1.1.

Proposition 4.1 *Let $I_1, I_2 \subset \{1, \dots, 28\}$ and let $\mathcal{L}_{I_1}, \mathcal{L}_{I_2}$ be their associated two graphs. If there exists a homeomorphism of pairs $h: (\mathbb{P}^2, Q + L_{I_1}) \rightarrow (\mathbb{P}^2, Q + L_{I_2})$, h induces an equivalence of two-graphs $h_*: \mathcal{L}_{I_1} \rightarrow \mathcal{L}_{I_2}$.*

Proof Since Q is a genus 3 curve and the other irreducible components are lines $h(Q) = Q$ necessarily and h induces a bijection $h_*: \{L_i\}_{i \in I_1} \rightarrow \{L_j\}_{j \in I_2}$. If $\{L_{i_1}, L_{i_2}, L_{i_3}\} \in \Delta_{I_1}$, then $c_{\phi_Q}(L_{i_1} + L_{i_2} + L_{i_3}) = 2$, and by [14, Proposition 1.3], $c_{\phi_Q}(h_*(L_{i_1} + L_{i_2} + L_{i_3})) = c_{\phi}(h_*(L_{i_1}) + h_*(L_{i_2}) + h_*(L_{i_3})) = 2$. Hence, $\{h_*(L_{i_1}), h_*(L_{i_2}), h_*(L_{i_3})\} \in \Delta_{I_2}$. This implies that h_* is an equivalence of two-graphs. ■

The contrapositive of Proposition 4.1 gives Theorem 1.1.

Next, we consider Theorem 1.3. First, we consider the combinatorics of the arrangements of quartics and bitangent lines. For general quartics, we have the following lemma.

Lemma 4.2 *Let Q be a general smooth quartic curve. Then the following hold:*

- (i) *Every bitangent line is a proper bitangent; i.e., it is tangent to Q at two distinct points.*
- (ii) *Any set of three bitangent lines do not intersect at a single point.*

Proof Do to the openness of the conditions, it is enough to find an example satisfying the statements. An example can be constructed using Riemann’s equations for bitangent lines given in [8, Section 6.1.3]. ■

As a direct consequence of Lemma 4.2, we have the following proposition.

Proposition 4.3 *Let Q be a general smooth quartic. Then for a fixed integer r ($1 \leq r \leq 28$), and any $I \subset \{1, \dots, 28\}$ with $|I| = r$, the curves of the form $Q + L_I$ all have the same combinatorics.*

Since Proposition 4.3 ensures that all the curves of the form $Q + L_I$ that we are considering have the same combinatorics, Theorem 1.1 implies that the remaining thing that we need in order to prove Theorem 1.3 is the existence of curves with different two-graph structures on I . For the existence, we provide concrete examples, but not

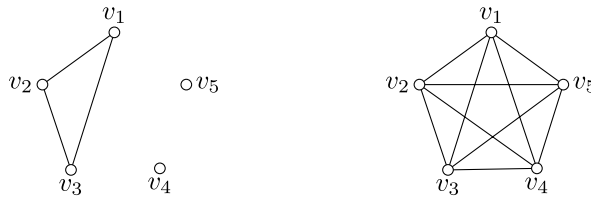


Figure 1: Representatives of switching classes corresponding to two-graphs with $|\Omega| = 5$ and $|\Delta| = 7$ (left), $|\Delta| = 10$ (right).

all possible two-graph structures appear. In order to prove the non-appearance of some of the two-graphs in our case, we prepare some more lemmas.

Lemma 4.4 *Let $I = \{1, \dots, 28\}$. For each triple $\{L_i, L_j, L_k\} \in \Delta_I$, there is a unique bitangent L_l such that every triple of $L_{\{i,j,k,l\}}$ is an element of Δ_I .*

Proof As in the proof of [4, Proposition 3.3], if there exists a conic C through the six points of tangency of Q and L_i, L_j, L_k , the line L_l through the remaining two intersection points of C and Q must be a bitangent line. The existence of the conic implies that the other three triples $\{L_j, L_k, L_l\}, \{L_i, L_k, L_l\}, \{L_i, L_j, L_l\}$ are also elements of Δ_I . The uniqueness of L_l follows from the uniqueness of a conic passing through six points. ■

Remark 4.5 Note that a bitangent line of Q corresponds to an odd theta characteristic of Q . Lemma 4.4 is equivalent to the fact that a syzygetic triad of theta characteristics is contained in a unique syzygetic tetrad.

Lemma 4.6 *For any $I \subset \{1, \dots, 28\}$ with $|I| = 5$, its associated two-graph $\mathcal{L}_I = (\mathcal{L}_I, \Delta_I)$ must satisfy $|\Delta_I| \leq 6$.*

Proof From the classification of two-graphs on 5 elements, the possible values for $|\Delta_I|$ are $|\Delta_I| = 0, 3, 4, 5, 6, 7, 10$. The cases $|\Delta_I| = 7, 10$ can be represented by the switching class in Figure 1. In both cases, $\{v_1, v_2, v_3, v_4\}$ and $\{v_1, v_2, v_3, v_5\}$ are tetrads containing $\{v_1, v_2, v_3\}$ such that every triple in them is contained in Δ_I . This contradicts the uniqueness of Lemma 4.4. Hence, $|\Delta_I| \neq 7, 10$, and we have $|\Delta_I| \leq 6$. ■

Corollary 4.7 *For any $I \subset \{1, \dots, 28\}$, its associated two graph \mathcal{L}_I cannot have an induced sub-two-graph (Ω, Δ) such that $|\Omega| = 5, |\Delta| > 6$.*

Lemma 4.8 *Let $n \geq 4, (\Omega, \Delta)$ be a two graph with $\Omega = \{v_1, \dots, v_n\}$ and let (Ω_i, Δ_i) ($i = 1, \dots, n$) be the induced sub-two-graphs of (Ω, Δ) obtained by deleting v_i (resp. $i = 1, \dots, n$). Let $|\Delta| = d$ and $|\Delta_i| = d_i$. Then there exists at least one i such that $d_i \geq \frac{n-3}{n}d$.*

Proof Since each triple $\omega \in \Delta$ is contained in $n - 3$ of the sub-two-graphs, we have

$$(n - 3)d = \sum_{i=1}^n d_i.$$

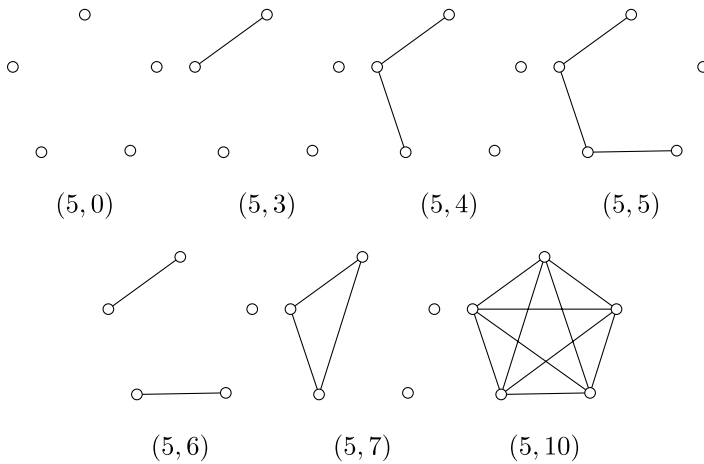


Figure 2: Representatives of switching classes corresponding to two graphs with $|\Omega| = 5$ labeled by $(|\Omega|, |\Delta|)$.

If $d_i < \frac{n-3}{n}d$ for all i , then $\sum_{i=1}^n d_i < (n-3)d$, which is a contradiction. Hence, there must be a d_i satisfying $d_i \geq \frac{n-3}{n}d$. ■

Corollary 4.9 For any subset $I \subset \{1, \dots, 28\}$ with $|I| = 6, |\Delta_I| < 14$.

Proof If $|\Delta_I| = d \geq 14$, then by Lemma 4.8, there must exist a induced sub-two-graph with $|\Delta_i| = d_i \geq 7$, which is impossible by Corollary 4.7. ■

Now we are ready to prove Theorem 1.3. First, we consider the case where $|I| = 5$. Among the switching classes in Figure 2, we have proved in Lemma 4.6 that the cases (5, 7), (5, 10) cannot appear. For the other cases, we can find explicit combinations of minimal vectors $\pm u_{ij} \in E_7^*$ giving the desired switching classes. For example, we have

$$\begin{aligned} (5, 0) &: u_{18}, u_{28}, u_{38}, u_{48}, u_{58} & (5, 3) &: u_{18}, u_{28}, u_{38}, u_{48}, -u_{15} \\ (5, 4) &: u_{18}, u_{28}, u_{38}, u_{48}, -u_{12} & (5, 5) &: u_{18}, u_{28}, u_{38}, u_{23}, -u_{24} \\ (5, 6) &: u_{18}, u_{28}, u_{13}, u_{23}, u_{12} \end{aligned}$$

Next, for the case $|I| = 6$, among the switching classes in Figure 3, (6, 14), (6, 16), (6, 20) cannot appear due to Corollary 4.9. Also, we can easily check that the switching classes $(6, 10)_2, (6, 10)_3, (6, 12)_1, (6, 12)_2$ contain an induced sub switching class equivalent to (5, 7); hence, these cannot appear either due to Corollary 4.7. (We could have done the same for (6, 14), (6, 16), (6, 20).) For the remaining cases (6, 0), (6, 4), (6, 6), (6, 8)₁, (6, 8)₂, (6, 8)₃, (6, 10)₁, (6, 10)₄, and (6, 12)₃ we can find explicit combinations of minimal vectors $\pm u_{ij} \in E_7^*$ giving the desired switching classes.

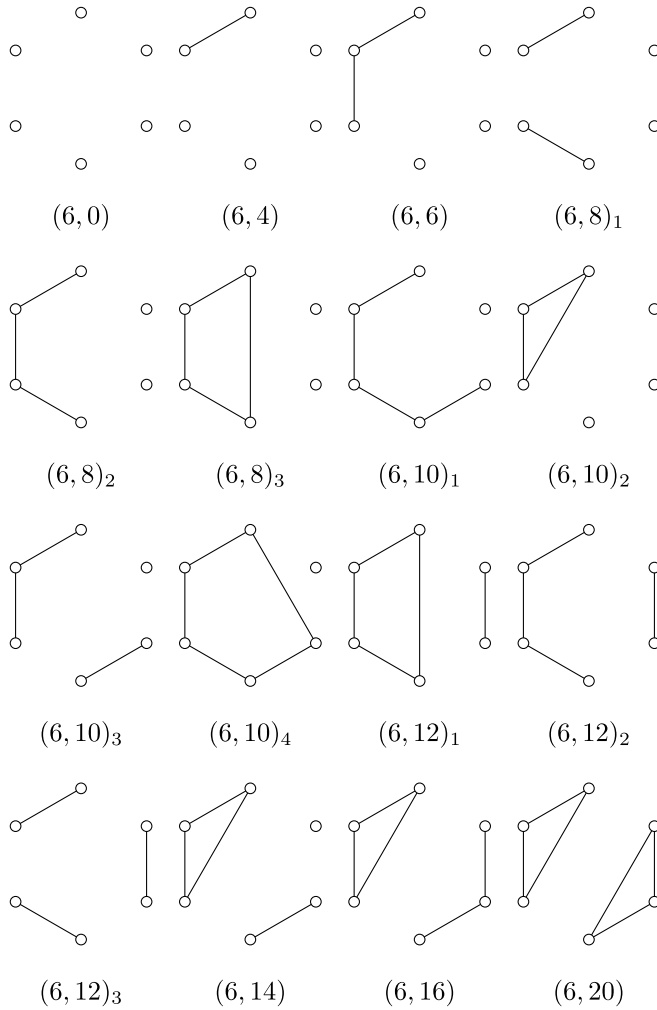


Figure 3: Representatives of switching classes corresponding to two graphs with $|\Omega| = 6$ labeled by $(|\Omega|, |\Delta|)$.

For example, we have

- | | |
|--|---|
| $(6, 0) : u_{18}, u_{28}, u_{38}, u_{48}, u_{58}, u_{68}$ | $(6, 4) : u_{18}, u_{28}, u_{38}, u_{48}, u_{58}, -u_{16}$ |
| $(6, 6) : u_{18}, u_{28}, u_{38}, u_{48}, u_{58}, -u_{12}$ | $(6, 8)_1 : u_{18}, u_{28}, u_{38}, u_{48}, -u_{15}, -u_{26}$ |
| $(6, 8)_2 : u_{18}, u_{28}, u_{38}, u_{48}, -u_{15}, -u_{25}$ | $(6, 8)_3 : u_{18}, u_{28}, u_{38}, -u_{14}, u_{23}, u_{48}$ |
| $(6, 10)_1 : u_{18}, u_{28}, u_{38}, u_{48}, -u_{14}, -u_{34}$ | $(6, 10)_4 : u_{18}, u_{28}, u_{38}, u_{23}, -u_{24}, u_{35}$ |
| $(6, 12)_3 : u_{18}, u_{28}, u_{38}, u_{12}, u_{13}, u_{23}.$ | |

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References

- [1] E. Artal Bartolo, *Sur les couples des Zariski*. J. Algebraic Geom. 3(1994), no. 2, 223–247.
- [2] E. Artal Bartolo, J.-I. Cogolludo, and H. Tokunaga, *A survey on Zariski pairs*. In: *Algebraic geometry in East Asia–Hanoi 2005*. Adv. Stud. Pure Math., 50, Math. Soc. Japan, Tokyo, 2008, pp. 1–100. <https://doi.org/10.2969/aspj/05010001>
- [3] E. Artal Bartolo and H. Tokunaga, *Zariski k -plets of rational curve arrangements and dihedral covers*. Topology Appl. 142(2004), no. 1–3, 227–233. <https://doi.org/10.1016/j.topol.2004.02.003>
- [4] S. Bannai, *A note on splitting curves of plane quartics and multi-sections of rational elliptic surfaces*. Topology Appl. 202(2016), 423–439. <https://doi.org/10.1016/j.topol.2016.02.005>
- [5] S. Bannai, B. Guerville-Ballé, T. Shirane, and H. Tokunaga, *On the topology of arrangements of a cubic and its inflectional tangents*. Proc. Japan Acad. 93(2017), 50–53. <https://doi.org/10.3792/pjaa.93.50>
- [6] S. Bannai and H. Tokunaga, *Geometry of bisections of elliptic surfaces and Zariski N -plets for conic arrangements*. Geom. Dedicata 178(2015), 219–237. <https://doi.org/10.1007/s10711-015-0054-z>
- [7] S. Bannai, H. Tokunaga, and M. Yamamoto, *A note on the topology of arrangements for a smooth plane quartic and its bitangent lines*. Hiroshima Math. J. 49(2019), 289–302. <https://doi.org/10.32917/hmj/1564106549>
- [8] I. V. Dolgachev, *Classical algebraic geometry. A modern view*. Cambridge University Press, Cambridge, 2012. <https://doi.org/10.1017/CBO9781139084437>
- [9] B. Guerville-Ballé and J.-B. Meilhan, *A linking invariant for algebraic curves*. arxiv:1602.04916
- [10] C. L. Mallows and N. J. A. Sloane, *Two-graphs, switching classes and Euler graphs are equal in number*. SIAM J. Appl. Math. 28(1975), 876–880. <https://doi.org/10.1137/0128070>
- [11] J. J. Seidel, *A survey of two-graphs*. Proc. Int. Coll. Theorie Combinatorie, Acc. Naz. Lincei, Rome, 1973.
- [12] T. Shioda, *On the Mordell–Weil lattices*. Comment. Math. Univ. St. Paul. 39(1990), 211–240.
- [13] T. Shioda, *Plane Quartics and Mordell–Weil Lattices of Type E_7^** . Comment. Math. Univ. St. Paul. 42(1993), 61–79.
- [14] T. Shirane, *A note on splitting numbers for Galois covers and π_1 -equivalent Zariski k -plets*. Proc. Amer. Math. Soc. 145(2017), no. 3, 1009–1017. <https://doi.org/10.1090/proc/13298>
- [15] T. Shirane, *Connected numbers and the embedded topology of plane curves*. Canad. Math. Bull. 61(2018), no. 3, 650–658. <https://doi.org/10.4153/CMB-2017-066-5>
- [16] H. Tokunaga, *Sections of elliptic surfaces and Zariski pairs for conic-line arrangements via dihedral covers*. J. Math. Soc. Japan 66(2014), 613–640. <https://doi.org/10.2969/jmsj/06620613>
- [17] J. H. van Lint and J. J. Seidel, *Equilateral point sets in elliptic geometry*. Nederl. Akad. Wetensch. Proc. Ser. A 69=Indag. Math. 28(1966), 335–348.
- [18] O. Zariski, *On the problem of existence of algebraic functions of two variables possessing a given branch curve*. Amer. J. Math. 51(1929), 305–328. <https://doi.org/10.2307/2370712>

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