Glasgow Math. J. **64** (2022) 434–453. © The Author(s), 2021. Published by Cambridge University Press on behalf of Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089521000203.

EVALUATION OF CONVOLUTION SUMS $\sum_{l+km=n} \sigma(l)\sigma(m)$ AND $\sum_{al+bm=n} \sigma(l)\sigma(m)$ FOR $k = a \cdot b = 21, 33,$ AND 35

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(Received 28 September 2020; revised 5 June 2021; accepted 24 June 2021;
first published online 16 July 2021)

Abstract. The article focuses on the evaluation of convolution sums $W_k(n) := \sum_{m<\frac{n}{k}} \sigma(m)\sigma(n-km)$ involving the sum of divisor function $\sigma(n)$ for $k = 21, 33,$ and 35. In this article, our aim is to obtain certain Eisenstein series of level 21 and use them to evaluate the convolution sums for level 21. We also make use of the existing Eisenstein series identities for level 33 and 35 in evaluating the convolution sums for level 33 and 35. Most of the convolution sums were evaluated using the theory of modular forms, whereas we have devised a technique which is free from the theory of modular forms. As an application, we determine a formula for the number of representations of a positive integer n by the octonary quadratic form

$$\left(x_1^2 + x_1x_2 + ax_2^2 + x_3^2 + x_3x_4 + ax_4^2\right) + b\left(x_5^2 + x_5x_6 + ax_6^2 + x_7^2 + x_7x_8 + ax_8^2\right),$$

for $(a, b) = (1, 7), (1, 11), (2, 3),$ and $(2, 5)$.

2020 *Mathematics Subject Classification*. Primary 11A25, 11M36; Secondary 11E20, 11E25, 11F20

1. Introduction. Let $\mathbb{N}$ denote the set of all natural numbers. As usual let

$$\sigma_k(n) := \sum_{d|n} d^k, \qquad n, d, k \in \mathbb{N} \quad \text{and} \quad \sigma_k(n) = 0 \quad \forall \quad n \notin \mathbb{N}.$$

For convenience, we set $\sigma_1(n) = \sigma(n)$. We define the convolution sum $W_k(n)$ by

$$W_k(n) := \sum_{m<\frac{n}{k}} \sigma(m)\sigma(n-km) \quad \text{and} \quad W_{(a,b)}(n) := \sum_{\substack{l,m \\ al+bm=n}} \sigma(l)\sigma(m),$$

where $a, b, k, n \in \mathbb{N}$. Note that $W_{(1,k)}(n) = W_{(k,1)}(n) = W_k(n)$ and $W_{(a,b)}(n) = W_{(b,a)}(n)$. M. Besge in his paper [13] obtained the formula

$$W_1 = \frac{5}{12}\sigma_3(n) + \frac{1-6n}{12}\sigma(n). \tag{1.1}$$

The above found to be the first work in the evaluation of convolution sum for divisor function. Glaisher [19, 20] and Ramanujan [32] have also deduced (1.1). Below is the listed table on the works of convolution sum $W_k(n)$and $W_{(a,b)}(n)$ motivated by the above works.

k and (a, b)	Authors	References
1	Besge, Glaisher, Ramanujan	[13, 19, 32]
2, 3, 4	Huard et.al	[21]
5, 7	Lemire and Williams, Cooper and Toh	[22, 17]
6, (2,3)	Alaca and Williams	[6]
8, 9	Williams	[36, 35]
10, 11 , 13, 14	Royer	[33]
(2,5), (4,5), 20	Cooper and Ye	[18]
12, (3,4), 16, 18, (2,9), 24, (3,8)	Alaca et al.	[1, 4, 2, 3]
15, (3,5)	Ramakrishnan and Sahu	[29]
23	Chan and Cooper	[15]
25	Xia et al.	[37]
27, 32	Alaca and Kesicioğlu	[5]
36, (4,9)	Ye	[38]
14, (2,7), 26, (2,13), 28, (4,7), 30, (2,15), (3,10), (5,6)	Ntienjem	[25]
22, (2,11), 44, (4,11), 52, (4,13)	Ntienjem	[27, 26]
33, (3,11), 40, (5,8), 45, (5,9), 50, (2,25), 54, (2,27), 56, (7,8)	Ntienjem	[28]

Most of the convolution sums have been evaluated by the method using the theory of modular forms. The main objective of this paper is to evaluate the convolution sums for level 21, 33, and 35 i.e., W_{21}, $W_{(3,7)}$, W_{33}, $W_{(3,11)}$, W_{35}, and $W_{(5,7)}$ by devising a method free from the theory of modular forms.

Ramanujan recorded $P_1 - nP_n$ in terms of his theta functions for many positive integer n but not for $n = 21$. In [24, Theorem 5.8, p. 88], one can see that $P_1 - nP_n$ is a modular form in $M_2[\Gamma_0(n)]$. In the process of evaluating the convolution sums, we deduce certain Eisenstein series identities for level 21 in terms of Ramanujan's theta functions using the identities known to the Ramanujan. As an application, we determine a formula for the number of representations of a positive integer n by the octonary quadratic form

$$\left(x_1^2 + x_1x_2 + ax_2^2 + x_3^2 + x_3x_4 + ax_4^2\right) + b\left(x_5^2 + x_5x_6 + ax_6^2 + x_7^2 + x_7x_8 + ax_8^2\right),$$

for $(a, b) = (1, 7)$, $(1, 11)$, $(2, 3)$, and $(2, 5)$.

2. Preliminary results. In this section, we recall definitions and known results which are required to prove our main identities. For any complex number q with $|q| < 1$, we define

$$(q; q)_\infty = \prod_{n=1}^{\infty}(1 - q^n) = f(-q),$$

https://doi.org/10.1017/S0017089521000203 Published online by Cambridge University Press

where $f(-q)$ is one of the Ramanujan's theta functions. For any positive integer n, we set

$$f_n = f(-q^n),\ A_n = \frac{f_n}{q^{\frac{n}{12}} f_{3n}},\ B_n = \frac{f_n}{q^{\frac{n}{6}} f_{5n}},\ C_n = \frac{f_n}{q^{\frac{n}{4}} f_{7n}}.$$

Further for convenience throughout this paper, we set

$$u = \frac{1}{C_1C_3},\ v = \frac{1}{A_1A_7},\ w = \frac{C_1}{C_3},\ s = \frac{1}{w^2} + w^2,$$
$$v_1 = \frac{1}{A_1A_{11}},\ u_2 = \frac{B_1}{B_7},\ v_2 = \frac{1}{B_1B_7}\ \textit{and}\ w_2 = \frac{1}{C_1C_5}.$$

We make use of the following interesting four theta function identities of Ramanujan found in the unorganized portion of his second notebook [30]:

THEOREM 2.1. *[11, Entry 68, p. 236], [30, p. 323] We have*

$$(C_1C_3) + \frac{7}{(C_1C_3)} = \left(\frac{C_3}{C_1}\right)^2 - 3 + \left(\frac{C_1}{C_3}\right)^2,$$

equivalently

$$\frac{1}{u} + 7u = \frac{1}{w^2} - 3 + w^2. \tag{2.1}$$

From the above theorem, one can find that

$$C_1^4 + \frac{49}{C_1^4} + 13 = \frac{1}{2}\left[s^3 - 6s^2 - 5s + 26 - (s-3)\sqrt{(s^2-4)\left(s^2-6s-19\right)}\right] \tag{2.2}$$

and

$$C_3^4 + \frac{49}{C_3^4} + 13 = \frac{1}{2}\left[s^3 - 6s^2 - 5s + 26 + (s-3)\sqrt{(s^2-4)\left(s^2-6s-19\right)}\right]. \tag{2.3}$$

THEOREM 2.2. *[11, Entry 69, p. 236], [30, p. 323] We have*

$$(A_1A_7)^3 + \frac{27}{(A_1A_7)^3} = \left(\frac{A_7}{A_1}\right)^4 - 7\left(\frac{A_7}{A_1}\right)^2 + 7\left(\frac{A_1}{A_7}\right)^2 - \left(\frac{A_1}{A_7}\right)^4,$$

equivalently

$$\frac{1}{v^3} + 27v^3 = \frac{1}{w^4} - \frac{7}{w^2} + 7w^2 - w^4.$$

From the above theorem, we have

$$\frac{1}{v^3} + 27v^3 = (s-7)\sqrt{s^2-4}, \tag{2.4}$$

$$A_1^6 + \frac{27}{A_1^6} = \frac{\sqrt{s-2}}{2}\left[(s-7)(s-1)(s+2) - (s-4)(s+1)\sqrt{\left(s^2-6s-19\right)}\right],$$

and

$$A_7^6 + \frac{27}{A_7^6} = \frac{\sqrt{s-2}}{2}\left[(s-7)(s-1)(s+2) + (s-4)(s+1)\sqrt{\left(s^2-6s-19\right)}\right].$$

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Clearly, $s = \frac{1}{u} + 7u + 3$. Upon substituting this value of s in the above two equations, we obtain

$$A_1^6 + \frac{27}{A_1^6} = (1+7u)^3 \left(\frac{1}{u} + 7u + 1\right)^{\frac{1}{2}} \tag{2.5}$$

and

$$A_7^6 + \frac{27}{A_7^6} = \left(1 + \frac{1}{u}\right)^3 \left(\frac{1}{u} + 7u + 1\right)^{\frac{1}{2}}. \tag{2.6}$$

THEOREM 2.3. *[**11**, Entry 70, p. 236], [**30**, p. 323] We have*

$$\frac{1}{v^3} - 27v^3 = \frac{1}{u^2} - \frac{1}{u} + 7u - 49u^2.$$

Employing Theorem 2.1 in the right side of the above, we have

$$\frac{1}{v^3} - 27v^3 = (s-4)\sqrt{s^2 - 6s - 19}.$$

From (2.4) and the above, we have

$$\frac{1}{v^3} = \frac{1}{2}\left[(s-7)\sqrt{s^2-4} + (s-4)\sqrt{s^2-6s-19}\right] \tag{2.7}$$

and

$$27v^3 = \frac{1}{2}\left[(s-7)\sqrt{s^2-4} - (s-4)\sqrt{s^2-6s-19}\right]. \tag{2.8}$$

From (2.2) and (2.7), we have

$$\frac{1}{v^3}\left(C_1^4 + \frac{49}{C_1^4} + 13\right)^2 = (s-2)\left(2\sqrt{s^2-4} - \sqrt{s^2-6s-19}\right)^3. \tag{2.9}$$

From (2.3) and (2.8), we have

$$27v^3\left(C_3^4 + \frac{49}{C_3^4} + 13\right)^2 = (s-2)\left(2\sqrt{s^2-4} + \sqrt{s^2-6s-19}\right)^3. \tag{2.10}$$

Upon substituting the value of $s = \frac{1}{u} + 7u + 3$ in the above identity, we obtain

$$27v^3\left(C_3^4 + \frac{49}{C_3^4} + 13\right)^2 = \left(\frac{1}{u} + 7u + 1\right)\left[2\sqrt{\left(\frac{1}{u} + 7u + 3\right)^2 - 4} + \frac{1}{u} - 7u\right]^3. \tag{2.11}$$

Proof of Theorems 2.1–2.3 can found in [**14**], which is free from the theory of modular forms.

THEOREM 2.4. *[**11**, Entry 71, p. 236], [**30**, p. 303] We have*

$$(C_1C_5)^2 - 5 + \frac{49}{(C_1C_5)^2} = \left(\frac{C_5}{C_1}\right)^3 - 5\left(\frac{C_5}{C_1}\right)^2 - 5\left(\frac{C_1}{C_5}\right)^2 - \left(\frac{C_1}{C_5}\right)^3,$$

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equivalently

$$\frac{1}{w_2^2} - 5 + 49w_2^2 = \frac{1}{u_2^3} - \frac{5}{u_2^2} - 5u_2^2 - u_2^3. \tag{2.12}$$

From the above theorem, we have

$$\frac{1}{w_2^2} - 49w_2^2 = \left(\frac{1}{u_2} - 3 - u_2\right)\sqrt{\left(\frac{1}{u_2} + 1 - u_2\right)\left(\frac{1}{u_2^3} - \frac{5}{u_2^2} - 9 - 5u_2^2 - u_2^3\right)}. \tag{2.13}$$

We also require the following theorem:

THEOREM 2.5. *[31, p. 55], [7, p. 378] We have*

$$(B_1B_7)^3 + \frac{125}{(B_1B_7)^3} = \left(\frac{B_7}{B_1}\right)^4 - 7\left(\frac{B_7}{B_1}\right)^3 + 7\left(\frac{B_7}{B_1}\right)^2 + 14\left(\frac{B_7}{B_1}\right) + 14\left(\frac{B_1}{B_7}\right) - 7\left(\frac{B_1}{B_7}\right)^2 - 7\left(\frac{B_1}{B_7}\right)^3 - \left(\frac{B_1}{B_7}\right)^4,$$

equivalently

$$\frac{1}{v_2^3} + 125v_2^3 = \frac{1}{u_2^4} - \frac{7}{u_2^3} + \frac{7}{u_2^2} + \frac{14}{u_2} + 14u_2 - 7u_2^2 - 7u_2^3 - u_2^4.$$

From the above theorem, one can deduce that

$$B_1^6 - \frac{125}{B_1^6} = \frac{1}{2}\Bigg[\left(\frac{1}{u_2^2} - \frac{5}{u_2} + 2 + 5u_2 + u_2^2\right)\left(\frac{1}{u_2^3} + u_2^3\right)\sqrt{\beta} - \left(\frac{1}{u_2^3} - u_2^3\right)\left(\frac{1}{u_2^4} - \frac{7}{u_2^3} + \frac{7}{u_2^2} + \frac{14}{u_2} + 14u_2 - 7u_2^2 - 7u_2^3 - u_2^4\right)\Bigg] \tag{2.14}$$

and

$$B_7^6 - \frac{125}{B_7^6} = \frac{1}{2}\Bigg[\left(\frac{1}{u_2^2} - \frac{5}{u_2} + 2 + 5u_2 + u_2^2\right)\left(\frac{1}{u_2^3} + u_2^3\right)\sqrt{\beta} + \left(\frac{1}{u_2^3} - u_2^3\right)\left(\frac{1}{u_2^4} - \frac{7}{u_2^3} + \frac{7}{u_2^2} + \frac{14}{u_2} + 14u_2 - 7u_2^2 - 7u_2^3 - u_2^4\right)\Bigg], \tag{2.15}$$

where

$$\beta = \left(\frac{1}{u_2} + 1 - u_2\right)\left(\frac{1}{u_2^3} - \frac{5}{u_2^2} - 9 - 5u_2^2 - u_2^3\right). \tag{2.16}$$

Let $P(q)$ and $Q(q)$ denote the Eisenstein series of weight 2 and 4, respectively, defined by

$$P(q) := 1 - 24\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k} \quad \text{and} \quad Q(q) := 1 + 240\sum_{k=1}^{\infty}\frac{k^3q^k}{1-q^k}.$$

For any positive integer n, we set $P_n := P(q^n)$ and $Q_n := Q(q^n)$. We also require the following Eisenstein series identities:

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THEOREM 2.6. *We have*

$$P_n = 1 - 24 \sum_{k=1}^{\infty} \sigma\,(k)\, q^{kn}, \tag{2.17}$$

$$Q_n = 1 + 240 \sum_{k=1}^{\infty} \sigma_3\,(k)\, q^{kn}, \tag{2.18}$$

and

$$(P(q))^2 = 1 + \sum_{n=1}^{\infty} [240\sigma_3\,(n) - 288n\sigma\,(n)]\, q^n. \tag{2.19}$$

For a proof of (2.17) and (2.18), see [**9**, p. 318] and for a proof of (2.19), see Glaisher [**19**].

THEOREM 2.7. *[**30**], [**10**, Chapter 21] We have*

$$-P_1 + 3P_3 = 2q^{\frac{1}{3}} f_1^2 f_3^2 \left(A_1^6 + \frac{27}{A_1^6}\right)^{\frac{2}{3}} \tag{2.20}$$

and

$$-P_1 + 7P_7 = 6q^{\frac{2}{3}} f_1^2 f_7^2 \left(C_1^4 + \frac{49}{C_1^4} + 13\right)^{\frac{2}{3}}. \tag{2.21}$$

Employing [**34**, Theorem 4.1] in (2.21), one can find that

$$-P_1 + 7P_7 = 6\left(\frac{f_1^2 f_7^2}{f_2 f_{14}} + 4q\frac{f_2^2 f_{14}^2}{f_1 f_7}\right)^2.$$

THEOREM 2.8. *[**16**, p. 228] We have*

$$(-P_1 + 3P_3)^2 = \frac{2}{5}\,(Q_1 + 9Q_3)\,. \tag{2.22}$$

In his lost notebook [**31**], Ramanujan recorded the following interesting Eisenstein series identities:

THEOREM 2.9. *We have*

$$Q_1 = q f_1^4 f_5^4 \left(B_1^6 + \frac{3125}{B_1^6} + 250\right), \tag{2.23}$$

$$Q_5 = q f_1^4 f_5^4 \left(B_1^6 + \frac{5}{B_1^6} + 10\right), \tag{2.24}$$

$$Q_1 = q^{\frac{4}{3}} f_1^4 f_7^4 \left(C_1^4 + \frac{49}{C_1^4} + 13\right)^{\frac{1}{3}} \left(C_1^4 + \frac{7^4}{C_1^4} + 245\right), \tag{2.25}$$

and

$$Q_7 = q^{\frac{4}{3}} f_1^4 f_7^4 \left(C_1^4 + \frac{49}{C_1^4} + 13\right)^{\frac{1}{3}} \left(C_1^4 + \frac{1}{C_1^4} + 5\right). \tag{2.26}$$

For a proof of the above theorem, see Berndt et al [**12**].

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3. Main theorems. In this section, we state and prove our main results.

3.1. Level 21. The following theorem seems to be new and the $c_7(n)$ defined below is same as in [22, Theorem 2].

THEOREM 3.1. *Let*

$$\sum_{n=1}^{\infty} c_{21}(n)q^n = 16q^{\frac{8}{3}} f_1^2 f_3^2 f_7^2 f_{21}^2 \left(\frac{1}{w^2} + w^2 - 2\right)^{\frac{2}{3}} \left(24w^4 - 28w^2 - \frac{40}{w^2} - 116\right),$$

$$384 + \sum_{n=1}^{\infty} c_{(3,7)}(n)q^n = 16q^{\frac{8}{3}} f_1^2 f_3^2 f_7^2 f_{21}^2 \left(\frac{1}{w^2} + w^2 - 2\right)^{\frac{2}{3}} \left(\frac{24}{w^4} - \frac{28}{w^2} - 40w^2 - 116\right),$$

and

$$\sum_{n=0}^{\infty} c_7(n)q^n = q^{\frac{4}{3}} f_1^4 f_7^4 \left(C_1^4 + \frac{49}{C_1^4} + 13\right)^{\frac{1}{3}} = q f_1^3 f_7^3 \left(\frac{f_1^2 f_7^2}{f_2 f_{14}} + 4q \frac{f_2^2 f_{14}^2}{f_1 f_7}\right).$$

Then

$$W_{21}(n) = \frac{1}{504}\sigma_3(n) - \frac{1}{14}\sigma_3\left(\frac{n}{3}\right) - \frac{7}{18}\sigma_3\left(\frac{n}{7}\right) + \frac{7}{8}\sigma_3\left(\frac{n}{21}\right) + \left(\frac{1}{24} - \frac{n}{84}\right)\sigma(n)$$
$$+ \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{21}\right) - \frac{1}{189}c_7(n) - \frac{1}{21}c_7\left(\frac{n}{3}\right) + \frac{c_{21}(n)}{24192} \tag{3.1}$$

and

$$W_{(3,7)}(n) = \frac{1}{56}\sigma_3\left(\frac{n}{3}\right) - \frac{1}{126}\sigma_3(n) + \frac{7}{72}\sigma_3\left(\frac{n}{7}\right) - \frac{7}{2}\sigma_3\left(\frac{n}{21}\right) + \left(\frac{1}{24} - \frac{n}{28}\right)\sigma\left(\frac{n}{3}\right)$$
$$+ \left(\frac{1}{24} - \frac{n}{12}\right)\sigma\left(\frac{n}{7}\right) - \frac{1}{189}c_7(n) - \frac{1}{21}c_7\left(\frac{n}{3}\right) + \frac{c_{(3,7)}(n)}{24192}. \tag{3.2}$$

Proof. Set $G = q^{\frac{4}{3}} f_1 f_3 f_7 f_{21}$. We have

$$-5P_1 + 3P_3 - 7P_7 + 105P_{21} = 5(-P_1 + 3P_3) + 7(-P_7 + 3P_{21}) + 12(-P_3 + 7P_{21}).$$

Replacing q by q^7 in (2.20) and q by q^3 in (2.21) and substituting the resulting identities in the above and using (2.20), we obtain

$$-5P_1 + 3P_3 - 7P_7 + 105P_{21} = 2G\Bigg[\frac{5}{u}\left(A_1^6 + \frac{27}{A_1^6}\right)^{\frac{2}{3}} + 7u\left(A_7^6 + \frac{27}{A_7^6}\right)^{\frac{2}{3}}$$
$$+ 36v\left(C_3^4 + \frac{49}{C_3^4} + 13\right)^{\frac{2}{3}}\Bigg].$$

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Using (2.5), (2.6), and (2.11) in the above, we deduce that

$$-5P_1+3P_3-7P_7+105P_{21}=24G\left(\frac{1}{w^2}+w^2-2\right)^{\frac{1}{3}}\left[2\left(\frac{1}{u}+7u\right)+7 + 2\sqrt{\left(\frac{1}{u}+7u+3\right)^2-4}\right].$$

Employing (2.1) in the above, we obtain

$$-5P_1+3P_3-7P_7+105P_{21}=24G\left(\frac{1}{w^2}+w^2-2\right)^{\frac{1}{3}}\left(\frac{4}{w^2}+1\right). \tag{3.3}$$

We have

$$-P_1+15P_3-35P_7+21P_{21}=(-P_1+3P_3)+35(-P_7+3P_{21})-12(-P_3+7P_{21}).$$

Replacing q by q^7 in (2.20) and q by q^3 in (2.21) and substituting the resulting identities in the above and using (2.20), we obtain

$$-5P_1+3P_3-7P_7+105P_{21}=2G\left[\frac{1}{u}\left(A_1^6+\frac{27}{A_1^6}\right)^{\frac{2}{3}}+35u\left(A_7^6+\frac{27}{A_7^6}\right)^{\frac{2}{3}} - 36v\left(C_3^4+\frac{49}{C_3^4}+13\right)^{\frac{2}{3}}\right].$$

Using (2.5), (2.6), and (2.11) in the above, we deduce that

$$-5P_1+3P_3-7P_7+105P_{21}=24G\left(\frac{1}{w^2}+w^2-2\right)^{\frac{1}{3}}\left[2\left(\frac{1}{u}+7u\right)+7 - 2\sqrt{\left(\frac{1}{u}+7u+3\right)^2-4}\right].$$

Employing (2.1) in the above, we obtain

$$-P_1+15P_3-35P_7+21P_{21}=24G\left(\frac{1}{w^2}+w^2-2\right)^{\frac{1}{3}}\left(4w^2+1\right). \tag{3.4}$$

Subtracting (3.4) from five times (3.3), we have

$$-P_1+21P_{21}=4G\left(\frac{1}{w^2}+w^2-2\right)^{\frac{1}{3}}\left(\frac{5}{w^2}-w^2+1\right). \tag{3.5}$$

Squaring the above identity on both sides, we obtain

$$P_1^2+441P_{21}^2-42P_1P_{21}=16G^2T^2M^2, \tag{3.6}$$

where

$$T=\left(\frac{1}{w^2}+w^2-2\right)^{\frac{1}{3}} \text{ and } M=\left(\frac{5}{w^2}-w^2+1\right).$$

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Replacing q by q^{21} in (2.17) and in (2.19) and substituting the resulting identities in the left-hand side of (3.6) and using (2.17) and (2.19), we obtain

$$24,192\sum_{n=1}^{\infty}\left(\sum_{m<\frac{n}{21}}\sigma(m)\sigma(n-21m)\right)q^n$$
$$=400+441\sum_{n=1}^{\infty}\left(240\sigma_3\left(\frac{n}{21}\right)-\frac{96}{7}n\sigma\left(\frac{n}{21}\right)\right)q^n$$
$$+1008\sum_{n=1}^{\infty}\sigma\left(\frac{n}{21}\right)q^n+\sum_{n=1}^{\infty}(240\sigma_3(n)-288n\sigma(n))q^n$$
$$+1008\sum_{n=1}^{\infty}\sigma(n)q^n-16G^2T^2M^2. \tag{3.7}$$

Consider

$$Q_1+9Q_3+49Q_7+441Q_{21}=(Q_1+49Q_7)+9(Q_3+49Q_{21}).$$

Replacing q by q^3 in (2.25) and in (2.26) and using the resulting identities in the right side of the above along with (2.25) and (2.26), we find that

$$Q_1+9Q_3+49Q_7+441Q_{21}=\left[\frac{50G^2}{v^2}\left(C_1^4+\frac{49}{C_1^4}+13\right)^{\frac{4}{3}}-160\sum_{n=0}^{\infty}c_7(n)q^n\right.$$
$$\left.-1440\sum_{n=0}^{\infty}c_7\left(\frac{n}{3}\right)q^n+50G^2v^2\left(C_3^4+\frac{49}{C_3^4}+13\right)^{\frac{4}{3}}\right].$$

Using (2.9) and (2.10) in the above and employing $s=\frac{1}{w^2}+w^2$, we obtain

$$Q_1+9Q_3+49Q_7+441Q_{21}=100G^2T^2\left(\frac{5}{w^4}+5w^4-\frac{6}{w^2}-6w^2-25\right)$$
$$-160\sum_{n=0}^{\infty}c_7(n)q^n-1440\sum_{n=0}^{\infty}c_7\left(\frac{n}{3}\right)q^n. \tag{3.8}$$

By rewriting the above, it is easy to see that

$$16G^2T^2M^2=\frac{16}{20}\left[Q_1+9Q_3+49Q_7+441Q_{21}\right]+128\sum_{n=0}^{\infty}c_7(n)q^n$$
$$+1152\sum_{n=0}^{\infty}c_7\left(\frac{n}{3}\right)q^n-\sum_{n=1}^{\infty}c_{21}(n)q^n.$$

Using the above in (3.7) to eliminate $16G^2T^2M^2$ and then using (2.18) and then equating the coefficients of q^n on both sides of the resulting identity, we obtain (3.1).

Subtracting five times (3.4) from (3.3), we have

$$-3P_3+7P_7=4GT\left(\frac{1}{w^2}-5w^2-1\right). \tag{3.9}$$

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Squaring the above identity on both sides, we obtain

$$9P_3^2 + 49P_7^2 - 42P_3P_7 = 16G^2T^2S^2, \tag{3.10}$$

where

$$S = \left(\frac{1}{w^2} - 5w^2 - 1\right).$$

Replacing q by q^3 in (2.17) and in (2.19) and q by q^7 in (2.17) and in (2.19) and substituting the resulting identities in the left-hand side of (3.10), we find that

$$\begin{aligned}24,192\sum_{n=1}^{\infty}\left(\sum_{3l+7m=n}\sigma(l)\sigma(m)\right)q^n &= 16 + 9\sum_{n=1}^{\infty}\left(240\sigma_3\left(\frac{n}{3}\right) - 96n\sigma\left(\frac{n}{3}\right)\right)q^n \\ &+1008\sum_{n=1}^{\infty}\sigma\left(\frac{n}{3}\right)q^n + 49\sum_{n=1}^{\infty}\left(240\sigma_3\left(\frac{n}{7}\right) - \frac{288}{7}n\sigma\left(\frac{n}{7}\right)\right)q^n \\ &+1008\sum_{n=1}^{\infty}\sigma\left(\frac{n}{7}\right)q^n - 16G^2T^2S^2. \end{aligned}\tag{3.11}$$

Using (3.8), it is also easy to see that

$$\begin{aligned}16G^2T^2S^2 &= \frac{16}{20}\left[Q_1 + 9Q_3 + 49Q_7 + 441Q_{21}\right] + 128\sum_{n=0}^{\infty}c_7(n)q^n \\ &+ 1152\sum_{n=0}^{\infty}c_7\left(\frac{n}{3}\right)q^n - 384 - \sum_{n=1}^{\infty}c_{(3,7)}(n)q^n.\end{aligned}$$

Using the above in (3.11) to eliminate $16G^2T^2S^2$ and then using (2.18) and then equating the coefficients of q^n on both sides of the resulting identity, we obtain (3.2). □

REMARK 1. From (3.5) and (3.9), it is clear that the coefficients of q^n of the resulting series of the right-hand side are integers, which in turn implies that $c_{21}(n)$ and $c_{(3,7)}(n)$ are integers.

3.2. Level 33. The convolution sum for level 33 is quite challenging to evaluate as the constants involved are huge.

THEOREM 3.2. *Let*

$$\begin{aligned}610 + \sum_{n=1}^{\infty}c_{33}(n)q^n &= q^4f_1^2f_3^2f_{11}^2f_{33}^2\Bigg[\frac{420}{v_1^3} + \frac{3135}{v_1^2} + \frac{11340}{v_1} + 24693 + 34020v_1 \\ &+ 28215v_1^2 + 11340v_1^3 + \left(\frac{610}{v_1^2} + \frac{1647}{v_1} + 1647v_1 + 2745\right) \\ &\sqrt{\left(\frac{1}{v_1} + 1 + 3v_1\right)\left(\frac{1}{v_1^3} + \frac{7}{v_1^2} + \frac{28}{v_1} + 59 + 84v_1 + 63v_1^2 + 27v_1^3\right)}\Bigg]\end{aligned}$$

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and

$$-610+\sum_{n=1}^{\infty}c_{(3,11)}(n)q^n$$
$$=q^4f_1^2f_3^2f_{11}^2f_{33}^2\left[\frac{420}{v_1^3}+\frac{3135}{v_1^2}+\frac{11340}{v_1}+24693+34020v_1\right.$$
$$+28215v_1^2+11340v_1^3-\left(\frac{610}{v_1^2}+\frac{1647}{v_1}+1647v_1+2745\right)$$
$$\left.\sqrt{\left(\frac{1}{v_1}+1+3v_1\right)\left(\frac{1}{v_1^3}+\frac{7}{v_1^2}+\frac{28}{v_1}+59+84v_1+63v_1^2+27v_1^3\right)}\right].$$

Then

$$W_{33}(n)=\frac{47}{48312}\sigma_3(n)-\frac{129}{2684}\sigma_3\left(\frac{n}{3}\right)-\frac{77}{2196}\sigma_3\left(\frac{n}{11}\right)+\frac{3201}{488}\sigma_3\left(\frac{n}{33}\right)$$
$$+\left(\frac{1}{24}-\frac{n}{132}\right)\sigma(n)+\left(\frac{1}{24}-\frac{n}{4}\right)\sigma\left(\frac{n}{33}\right)-\frac{c_{33}(n)}{24156} \tag{3.12}$$

and

$$W_{(3,11)}(n)=\frac{291}{5368}\sigma_3\left(\frac{n}{3}\right)-\frac{7}{24156}\sigma_3(n)+\frac{517}{4392}\sigma_3\left(\frac{n}{11}\right)-\frac{1419}{244}\sigma_3\left(\frac{n}{33}\right)$$
$$+\left(\frac{1}{24}-\frac{n}{44}\right)\sigma\left(\frac{n}{3}\right)+\left(\frac{1}{24}-\frac{n}{12}\right)\sigma\left(\frac{n}{11}\right)-\frac{c_{(3,11)}(n)}{24156}. \tag{3.13}$$

Proof. Let $G_1=q^2f_1f_3f_{11}f_{33}$ and α be defined by

$$\alpha=\left(\frac{1}{v_1}+1+3v_1\right)\left(\frac{1}{v_1^3}+\frac{7}{v_1^2}+\frac{28}{v_1}+59+84v_1+63v_1^2+27v_1^3\right).$$

From [8], we have the following identities for level 33:

$$33P_{33}-11P_{11}+3P_3-P_1=24G_1\sqrt{\alpha}, \tag{3.14}$$

$$33P_{33}+11P_{11}-3P_3-P_1=40G_1\left(\frac{1}{v_1^2}+\frac{18}{5v_1}+9+\frac{54}{5}v_1+9v_1^2\right), \tag{3.15}$$

and

$$33P_{33}-11P_{11}-3P_3+P_1=20G_1\left(\frac{1}{v_1^2}+\frac{9}{5v_1}-\frac{27}{5}v_1-9v_1^2\right). \tag{3.16}$$

Adding (3.14) and (3.15) and then squaring the resulting identity, we have

$$P_1^2+1089P_{33}^2-66P_1P_{33}=16G_1^2T_1^2, \tag{3.17}$$

where

$$T_1=\frac{5}{v_1^2}+\frac{18}{v_1}+45+54v_1+45v_1^2+3\sqrt{\alpha}$$

https://doi.org/10.1017/S0017089521000203 Published online by Cambridge University Press

Replacing q by q^{33} in (2.17) and in (2.19) and substituting the resulting identities in the left-hand side of (3.17) and using (2.17) and (2.19), we obtain

$$\begin{aligned}38,016\sum_{n=1}^{\infty}\left(\sum_{m<\frac{n}{33}}\sigma(m)\sigma(n-33m)\right)q^n &= 1024+1089\sum_{n=1}^{\infty}\left(240\sigma_3\left(\frac{n}{33}\right)-\frac{96}{11}n\sigma\left(\frac{n}{33}\right)\right)q^n\\ &+1584\sum_{n=1}^{\infty}\sigma\left(\frac{n}{33}\right)q^n+\sum_{n=1}^{\infty}(240\sigma_3(n)-288n\sigma(n))\,q^n\\ &+1584\sum_{n=1}^{\infty}\sigma(n)\,q^n-16G_1^2T_1^2. \qquad (3.18)\end{aligned}$$

Squaring the identity obtained by subtracting (3.16) from (3.14) and then using (2.22) in the resulting identity, we find that

$$\frac{2}{5}(Q_1+9Q_3)=G_1^2\left(12\sqrt{\alpha}-10\left(\frac{1}{v_1^2}+\frac{9}{5v_1}-\frac{27}{5}v_1-9v_1^2\right)\right)^2. \qquad (3.19)$$

Squaring the identity obtained by adding (3.14) and (3.16) and then using the resulting identity along with the identity obtained by replacing q by q^{11} in (2.22), we see that

$$\frac{242}{5}(Q_{11}+9Q_{33})=G_1^2\left(12\sqrt{\alpha}+10\left(\frac{1}{v_1^2}+\frac{9}{5v_1}-\frac{27}{5}v_1-9v_1^2\right)\right)^2. \qquad (3.20)$$

From the above two identities, we find that

$$16G_1^2T_1^2=\frac{258}{305}(Q_1+9Q_3)+\frac{1694}{305}(Q_{11}+9Q_{33})+960+\frac{96}{61}\sum_{n=1}^{\infty}c_{33}(n)q^n.$$

Using the above in (3.18) to eliminate $16G_1^2T_1^2$ and then using (2.18) and then equating the coefficients of q^n on both sides of the resulting identity, we obtain (3.12).

Subtracting (3.14) from (3.15) and then squaring the resulting identity, we have

$$9P_3^2+121P_{11}^2-66P_3P_{11}=16G_1^2M_1^2, \qquad (3.21)$$

where

$$M_1=\frac{5}{v_1^2}+\frac{18}{v_1}+45+54v_1+45v_1^2-3\sqrt{\alpha}.$$

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Replacing q by q^3 in (2.17) and in (2.19) and q by q^{11} in (2.17) and in (2.19) and substituting the resulting identities in the left-hand side of (3.21), we find that

$$38,016\sum_{n=1}^{\infty}\left(\sum_{3l+11m=n}\sigma(l)\sigma(m)\right)q^n = 64+9\sum_{n=1}^{\infty}\left(240\sigma_3\left(\frac{n}{3}\right)-96n\sigma\left(\frac{n}{3}\right)\right)q^n$$
$$+1584\sum_{n=1}^{\infty}\sigma\left(\frac{n}{3}\right)q^n+121\sum_{n=1}^{\infty}\left(240\sigma_3\left(\frac{n}{11}\right)-\frac{288}{11}n\sigma\left(\frac{n}{11}\right)\right)q^n$$
$$+1584\sum_{n=1}^{\infty}\sigma\left(\frac{n}{11}\right)q^n-16G_1^2M_1^2. \quad (3.22)$$

From (3.19) and (3.20), we find that

$$16G_1^2M_1^2=\frac{14}{305}(Q_1+9Q_3)+\frac{31,218}{305}(Q_{11}+9Q_{33})-960+\frac{96}{61}\sum_{n=1}^{\infty}c_{(3,11)}(n)q^n.$$

Using the above in (3.22) to eliminate $16G_1^2M_1^2$ and then using (2.18) and then equating the coefficients of q^n on both sides of the resulting identity, we obtain (3.13). □

3.3. Level 35. The constants involved in the convolution sums for level 35 are free from any fractional powers and hence the convolution sums for this level is a bit in an elegant form.

THEOREM 3.3. *Let*

$$6960+\sum_{n=1}^{\infty}c_{35}(n)q^n=q^4f_1^2f_5^2f_7^2f_{35}^2\left(\frac{6960}{u_2^4}-\frac{4123}{u_2^3}+\frac{2200}{u_2^2}-\frac{1310}{u_2}\right.$$
$$\left.-480+130u_2-40u_2^2+5u_2^3\right)$$

and

$$6960+\sum_{n=1}^{\infty}c_{(5,7)}(n)q^n=q^4f_1^2f_5^2f_7^2f_{35}^2\left(\frac{6960}{u_2^4}-\frac{12475}{u_2^3}-\frac{9320}{u_2^2}-\frac{4190}{u_2}\right.$$
$$\left.-480-2750u_2+11,480u_2^2-8347u_2^3\right).$$

Then

$$W_{35}(n)=\frac{289}{48,384}\sigma_3(n)-\frac{25}{48,384}\sigma_3\left(\frac{n}{5}\right)-\frac{7}{6912}\sigma_3\left(\frac{n}{7}\right)+\frac{50,575}{6912}\sigma_3\left(\frac{n}{35}\right)$$
$$+\left(\frac{1}{24}-\frac{n}{140}\right)\sigma(n)+\left(\frac{1}{24}-\frac{n}{4}\right)\sigma\left(\frac{n}{35}\right)-\frac{c_{35}(n)}{241,920} \quad (3.23)$$

and

$$W_{(5,7)}(n)=\frac{289}{48,384}\sigma_3(n)-\frac{25}{48,384}\sigma_3\left(\frac{n}{5}\right)-\frac{7}{6912}\sigma_3\left(\frac{n}{7}\right)+\frac{50,575}{6912}\sigma_3\left(\frac{n}{35}\right)$$
$$+\left(\frac{1}{24}-\frac{n}{28}\right)\sigma\left(\frac{n}{5}\right)+\left(\frac{1}{24}-\frac{n}{20}\right)\sigma\left(\frac{n}{7}\right)-\frac{c_{(5,7)}(n)}{241,920}. \quad (3.24)$$

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Proof. Let $G_2 = q^2 f_1 f_5 f_7 f_{35}$ and β be as defined in (2.16). From [8], we have the following identities for level 35:

$$35P_{35} - 7P_7 + 5P_5 - P_1 = 8G_2 \left(\frac{4}{u_2^2} - \frac{1}{u_2} + 1 + u_2 + 4u_2^2 \right) \tag{3.25}$$

and

$$35P_{35} + 7P_7 - 5P_5 - P_1 = 12G_2 \left(\frac{3}{u_2^2} - \frac{1}{u_2} - u_2 - 3u_2^2 \right). \tag{3.26}$$

Adding (3.25) and (3.26) and then squaring the resulting identity, we have

$$P_1^2 + 1225P_{35}^2 - 70P_1P_{35} = 4G_2^2 T_2^2, \tag{3.27}$$

where

$$T_2 = \frac{17}{u_2^2} - \frac{5}{u_2} + 2 - u_2 - u_2^2.$$

Replacing q by q^{35} in (2.17) and in (2.19) and substituting the resulting identities in the left-hand side of (3.27) and using (2.17) and (2.19), we obtain

$$\begin{aligned} &40,320 \sum_{n=1}^{\infty} \left(\sum_{m<\frac{n}{35}} \sigma(m)\sigma(n-35m) \right) q^n \\ &= 1156 + 1225 \sum_{n=1}^{\infty} \left(240\sigma_3\left(\frac{n}{35}\right) - \frac{288}{35} n\sigma\left(\frac{n}{35}\right) \right) q^n \\ &+ 1680 \sum_{n=1}^{\infty} \sigma\left(\frac{n}{35}\right) q^n + \sum_{n=1}^{\infty} \left(240\sigma_3(n) - 288n\sigma(n)\right) q^n \\ &+ 1680 \sum_{n=1}^{\infty} \sigma(n) q^n - 4G_2^2 T_2^2. \end{aligned} \tag{3.28}$$

Subtracting (2.23) from 25 times of (2.24) and then using (2.12), (2.13) and (2.14), we deduce that

$$\begin{aligned} 25Q_5 - Q_1 = \frac{24G_2^2}{w_2^2} \left(B_1^6 - \frac{125}{B_1^6} \right) = 24G_2^2 \Big(&24u_2^4 + 43u_2^3 - 32u_2^2 + 14u_2 + \frac{14}{u_2} + \frac{32}{u_2^2} + \frac{43}{u_2^3} \\ &- \frac{24}{u_2^4} - \left(25u_2^2 + 5u_2 + \frac{5}{u_2} - \frac{25}{u_2^2} \right) \sqrt{\beta} \Big). \end{aligned} \tag{3.29}$$

Replacing q by q^7 in (2.23) and (2.24) and using the resulting identity along with (2.12), (2.13), and (2.15), we find that

$$\begin{aligned} 25Q_{35} - Q_7 = \frac{24G_2^2 w_2^2}{49} \left(B_7^6 - \frac{125}{B_7^6} \right) = 24G_2^2 \Big(&-24u_2^4 - 43u_2^3 + 32u_2^2 - 14u_2 - \frac{14}{u_2} \\ &- \frac{32}{u_2^2} - \frac{43}{u_2^3} + \frac{24}{u_2^4} - \left(25u_2^2 + 5u_2 - \frac{25}{u_2^2} + \frac{5}{u_2} \right) \sqrt{\beta} \Big). \end{aligned} \tag{3.30}$$

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From the above two identities, we find that

$$4G_2^2T_2^2 = \frac{1}{288}\left(-Q_1 + 25Q_5\right) + \frac{49}{288}\left(Q_7 - 25Q_{35}\right) + 1160 + \frac{1}{6}\sum_{n=1}^{\infty} c_{35}(n)q^n.$$

Using the above in (3.28) to eliminate $4G_2^2T_2^2$ and then using (2.18) and then equating the coefficients of q^n on both sides of the resulting identity, we obtain (3.23).

Subtracting (3.25) from (3.26) and then squaring the resulting identity, we have

$$25P_5^2 + 49P_7^2 - 70P_5P_7 = 4G_2^2M_2^2, \tag{3.31}$$

where

$$M_2 = \frac{1}{u_2^2} - \frac{1}{u_2} - 2 - 5u_2 - 17u_2^2.$$

Replacing q by q^5 in (2.17) and in (2.19) and q by q^7 in (2.17) and in (2.19) and substituting the resulting identities in the left-hand side of (3.31), we find that

$$\begin{aligned} 40,320\sum_{n=1}^{\infty}\left(\sum_{5l+7m=n}\sigma(l)\sigma(m)\right)q^n &= 4 + 25\sum_{n=1}^{\infty}\left(240\sigma_3\left(\frac{n}{5}\right) - \frac{288}{5}n\sigma\left(\frac{n}{5}\right)\right)q^n \\ &\quad + 1680\sum_{n=1}^{\infty}\sigma\left(\frac{n}{5}\right)q^n + 49\sum_{n=1}^{\infty}\left(240\sigma_3\left(\frac{n}{7}\right) - \frac{288}{7}n\sigma\left(\frac{n}{7}\right)\right)q^n \\ &\quad + 1680\sum_{n=1}^{\infty}\sigma\left(\frac{n}{7}\right)q^n - 4G_2^2M_2^2. \end{aligned} \tag{3.32}$$

From (3.29) and (3.30), we find that

$$4G_2^2M_2^2 = \frac{289}{288}\left(-Q_1 + 25Q_5\right) + \frac{14,161}{288}\left(Q_7 - 25Q_{35}\right) + 1160 + \frac{1}{6}\sum_{n=1}^{\infty} c_{(5,7)}(n)q^n.$$

Using the above in (3.32) to eliminate $4G_2^2M_2^2$ and then using (2.18) and then equating the coefficients of q^n on both sides of the resulting identity, we obtain (3.24). □

4. Applications.

Let $\mathbb{Z}$ denote the set of integers and $x_i \in \mathbb{Z}$ for $1 \leqslant i \leqslant 8$. For $a, b, n \in \mathbb{N}$, let $T_{(a,b)}(n)$ be the number of representations of n by the form

$$\left(x_1^2 + x_1x_2 + ax_2^2 + x_3^2 + x_3x_4 + ax_4^2\right) + b\left(x_5^2 + x_5x_6 + ax_6^2 + x_7^2 + x_7x_8 + ax_8^2\right).$$

THEOREM 4.1. *If $n \in \mathbb{N}$, then*

$$\begin{aligned} T_{(1,7)}(n) &= \frac{132}{35}\sigma_3(n) + \frac{1188}{35}\sigma_3\left(\frac{n}{3}\right) + \frac{924}{5}\sigma_3\left(\frac{n}{7}\right) + \frac{8316}{5}\sigma_3\left(\frac{n}{21}\right) \\ &\quad + \frac{88}{35}c_7(n) + \frac{792}{35}c_7\left(\frac{n}{3}\right) - \frac{1}{56}c_{21}(n) - \frac{1}{56}c_{(3,7)}(n). \end{aligned}$$

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Proof. For $k\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, let

$$s_{(a,4)}(k)=card\left\{(x_1,x_2,x_3,x_4)\in\mathbb{Z}^4|k=x_1^2+x_1x_2+ax_2^2+x_3^2+x_3x_4+ax_4^2\right\}.$$

Clearly $s_{(a,4)}(0)=1$. It is known that [21, Theorem 13, p. 266], [23].

$$s_{(1,4)}(k)=12\sigma(k)-36\sigma\left(\frac{k}{3}\right),\qquad k\in\mathbb{N}. \tag{4.1}$$

We have

$$\begin{aligned}T_{(1,7)}(n)&=\sum_{\substack{(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)\in\mathbb{Z}^8\\ n=(x_1^2+x_1x_2+x_2^2+x_3^2+x_3x_4+x_4^2)+7(x_5^2+x_5x_6+x_6^2+x_7^2+x_7x_8+x_8^2)}}1\\
&=\sum_{\substack{l,m\in\mathbb{Z}\\ l,m\geqslant 0\\ l+7m=n}}s_{(1,4)}(l)s_{(1,4)}(m)=s_{(1,4)}(0)s_{(1,4)}\left(\frac{n}{7}\right)+s_{(1,4)}(n)s_{(1,4)}(0)\\
&\quad+\sum_{\substack{l,m\in\mathbb{N}\\ l+7m=n}}s_{(1,4)}(l)s_{(1,4)}(m).\end{aligned}$$

Using (4.1) in the above, we obtain

$$\begin{aligned}T_{(1,7)}(n)&=\Bigg[12\sigma\left(\frac{n}{7}\right)-36\sigma\left(\frac{n}{21}\right)+12\sigma(n)-36\sigma\left(\frac{n}{3}\right)\\
&\quad+\sum_{\substack{l,m\in\mathbb{N}\\ l+7m=n}}\left(12\sigma(l)-36\sigma\left(\frac{l}{3}\right)\right)\left(12\sigma(m)-36\sigma\left(\frac{m}{3}\right)\right)\Bigg]\\
&=12\sigma\left(\frac{n}{7}\right)-36\sigma\left(\frac{n}{21}\right)+12\sigma(n)-36\sigma\left(\frac{n}{3}\right)+144\sum_{l+7m=n}\sigma(l)\sigma(m)\\
&\quad-432\sum_{l+7m=n}\sigma\left(\frac{l}{3}\right)\sigma(m)-432\sum_{l+7m=n}\sigma(l)\sigma\left(\frac{m}{3}\right)\\
&\quad+1296\sum_{l+7m=n}\sigma\left(\frac{l}{3}\right)\sigma\left(\frac{m}{3}\right)\\
&=12\sigma\left(\frac{n}{7}\right)-36\sigma\left(\frac{n}{21}\right)+12\sigma(n)-36\sigma\left(\frac{n}{3}\right)+144W_7(n)\\
&\quad-432W_{(3,7)}(n)-432W_{21}(n)+1296W_7\left(\frac{n}{3}\right).\end{aligned}$$

Using the convolution sum $W_7(n)$ from [22, Theorem 2] and employing (3.1) and (3.2) in the above, we complete the proof. □

THEOREM 4.2. *Let*

$$\begin{aligned}7625+\sum_{n=1}^{\infty}b_{11}(n)q^n&=q^4f_1^2f_3^2f_{11}^2f_{33}^2\Bigg(\frac{7625}{v_1^4}+\frac{39,330}{v_1^3}+\frac{118,557}{v_1^2}+\frac{238,410}{v_1}+428,508\\
&\quad+715,230v_1+1,067,013v_1^2+1,061,910v_1^3+617,625v_1^4\Bigg).\end{aligned}$$

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Then,

$$T_{(1,11)}(n) = \frac{1632}{671}\sigma_3(n) + \frac{14,688}{671}\sigma_3\left(\frac{n}{3}\right) + \frac{17,952}{61}\sigma_3\left(\frac{n}{11}\right) + \frac{161,568}{61}\sigma_3\left(\frac{n}{33}\right) - \frac{b_{11}(n)}{671}.$$

Proof. From (3.15) and (3.16), one may obtain

$$W_{11}(n) = \frac{5}{264}\sigma_3(n) + \frac{55}{24}\sigma_3\left(\frac{n}{11}\right) + \left(\frac{1}{24} - \frac{n}{44}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{11}\right) - \frac{1}{3168}c_{11}(n), \tag{4.2}$$

where

$$25 + \sum_{n=1}^{\infty} c_{11}(n)q^n = q^4 f_1^2 f_3^2 f_{11}^2 f_{33}^2 \left(\frac{5}{v_1^2} + \frac{27}{v_1} + 90 + 135v_1 + 135v_1^2\right)^2$$

and

$$25 + \sum_{n=1}^{\infty} c_{11}\left(\frac{n}{3}\right) q^n = q^4 f_1^2 f_3^2 f_{11}^2 f_{33}^2 \left(\frac{5}{v_1^2} + \frac{15}{v_1} + 30 + 27v_1 + 15v_1^2\right)^2.$$

Proceeding as in the proof of the previous theorem, we have

$$T_{(1,11)}(n) = 12\sigma\left(\frac{n}{11}\right) - 36\sigma\left(\frac{n}{33}\right) + 12\sigma(n) - 36\sigma\left(\frac{n}{3}\right) + 144W_{11}(n) - 432W_{(3,11)}(n) - 432W_{33}(n) + 1296W_{11}\left(\frac{n}{3}\right).$$

Using (4.2), (3.12), and (3.13) in the above and then using the definition of the constants, we obtain the desired result. □

We have used (4.2) instead of $W_{11}(n)$ from [**33**] to reduce the complexity of the formula of $T_{(1,11)}(n)$.

THEOREM 4.3. *If* $n \in \mathbb{N}$, *then*

$$T_{(2,3)}(n) = \frac{4}{3}\sigma_3(n) + 12\sigma_3\left(\frac{n}{3}\right) + \frac{196}{3}\sigma_3\left(\frac{n}{7}\right) + 588\sigma_3\left(\frac{n}{21}\right) + \frac{32}{27}c_7(n) + \frac{32}{3}c_7\left(\frac{n}{3}\right) - \frac{1}{216}c_{21}(n) - \frac{1}{216}c_{(3,7)}(n).$$

Proof. From [**15**, Example 3, p. 6]

$$1 + \sum_{j=1}^{\infty} s_{(2,4)}(j)q^j = \left(\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+mn+2n^2}\right)^2 = -\frac{1}{6}P_1 + \frac{7}{6}P_7.$$

(See [**7**, Entry 18.2.15, p. 405], [**10**, Entry 5, p. 467] for further details.) From the above identity, we have

$$s_{(2,4)}(k) = 4\sigma(k) - 28\sigma\left(\frac{k}{7}\right), k \in \mathbb{N}. \tag{4.3}$$

https://doi.org/10.1017/S0017089521000203 Published online by Cambridge University Press

We have

$$T_{(2,3)}(n) = \sum_{\substack{l,m\in\mathbb{Z}\\ l,m\geqslant 0\\ l+3m=n}} s_{(2,4)}(l)s_{(2,4)}(m) = s_{(2,4)}(0)s_{(2,4)}\left(\frac{n}{3}\right) + s_{(2,4)}(n)s_{(2,4)}(0) + \sum_{\substack{l,m\in\mathbb{N}\\ l+3m=n}} s_{(2,4)}(l)s_{(2,4)}(m).$$

Using (4.3) in the above, we obtain

$$T_{(2,3)}(n) = 4\sigma\left(\frac{n}{3}\right) - 28\sigma\left(\frac{n}{21}\right) + 4\sigma(n) - 28\sigma\left(\frac{n}{7}\right) + 16W_3(n) - 112W_{(3,7)}(n) - 112W_{21}(n) + 784W_3\left(\frac{n}{7}\right).$$

Using the convolution sum $W_3(n)$ from [21, Theorem 3] and employing (3.1) and (3.2) in the above, we complete the proof. □

The $c_5(n)$ defined below is same as in [22, Theorem 1].

THEOREM 4.4. *Let*

$$\sum_{n=1}^{\infty} c_5(n)q^n = qf_1^4 f_5^4.$$

Then

$$T_{(2,5)}(n) = -\frac{3037}{2808}\sigma_3(n) + \frac{18,325}{2808}\sigma_3\left(\frac{n}{5}\right) + \frac{35,917}{2808}\sigma_3\left(\frac{n}{7}\right) - \frac{3,720,325}{2808}\sigma_3\left(\frac{n}{35}\right) - \frac{8}{65}c_5(n) - \frac{392}{65}c_5\left(\frac{n}{7}\right) + \frac{1}{2160}c_{35}(n) + \frac{1}{2160}c_{(5,7)}(n).$$

Proof. Proceeding as in the proof of the previous theorem, we have

$$T_{(2,5)}(n) = 4\sigma\left(\frac{n}{5}\right) - 28\sigma\left(\frac{n}{35}\right) + 4\sigma(n) - 28\sigma\left(\frac{n}{7}\right) + 16W_5(n) - 112W_{(5,7)}(n) - 112W_{35}(n) + 784W_5\left(\frac{n}{7}\right).$$

Using the convolution sum $W_5(n)$ from [22, Theorem 1] and employing (3.23) and (3.24) in the above, we obtain the required result. □

ACKNOWLEDGMENTS. We are thankful for the valuable suggestions of the anonymous referee for the betterment of this article in particular for bring our attention to [24, 15] and for his/her suggestion for the extension of the work for level 33 and 35. The research of the first author is supported by grant F.No.16-6(DEC, 2018)/2019(NET/CSIR) by the funding agency UGC, New Delhi, India and second author is supported by SERB, Govt. of India (EMR/2016/001064).

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