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# A REMARK ON THE NILPOTENCY INDEX OF THE RADICAL OF A GROUP ALGEBRA OF A $p$-SOLVABLE GROUP

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Let $K$ be a field of characteristic $p>0$, $G$ a finite $p$-solvable group, $P$ a $p$-Sylow subgroup of $G$ of order $p^a$, $KG$ the group algebra of $G$ over $K$, and $J(KG)$ the Jacobson radical of $KG$. In the present paper we study the nilpotency index $t(G)$ of $J(KG)$, which is the least positive integer $t$ with $J(KG)^t=0$. Since $J(EG)=E\otimes_K J(KG)$ for any extension field $E$ of $K$ (cf. [**7**, Proposition 12.11]), we may assume that $K$ is algebraically closed.

D. A. R. Wallace [**12**] proved that

$$t(G)\geqq a(p-1)+1.$$

There is a problem to determine the structure of $G$ with the property $t(G)=a(p-1)+1$. When $G$ is of $p$-length 1, by the results of S. A. Jennings [**6**] and K. Morita [**8**], $t(G)=a(p-1)+1$ if and only if $P$ is elementary abelian (cf. [**10**, Corollary 1]). But for $p$-solvable groups $G$ of $p$-length $\geqq 2$ the assertion does not hold in general. Indeed, K. Motose and Y. Ninomiya [**10**] showed that when $p=2$ and $G=S_4$ (which denotes the symmetric group of degree 4), $t(G)=4$ though $P$ is dihedral of order 8. Recently, K. Motose [**9**] proved that if $p=2$, $P$ is metacyclic and $G/O_{2'}(G)\neq S_4$, then $t(G)=a+1$ if and only if $P$ is elementary abelian. The purpose of this paper is to consider the proposition for the case where $p$ is odd. If $p$ is odd and $P$ is metacyclic, then $P$ is a regular $p$-group (cf. [**5**, III 10.2 Satz (c)]). Y. Tsushima [**11**] claimed that when $P$ is regular, $t(G)=a(p-1)+1$ if and only if $P$ is elementary abelian. At line 11 of page 37 in [**11**], he says that since $P$ has exponent $p$, $G$ is of $p$-length 1 from [**4**, Theorem A (ii)]. However, Tsushima's assertion is not correct. There exists an example (to be given later) of a $p$-solvable group $G$ of $p$-length $\geqq 2$ such that $P$ has exponent $p$ and so that $P$ is regular. Our main result can be stated as follows: If $p$ is odd and $P$ is metacyclic, then $t(G)=a(p-1)+1$ if and only if $P$ is elementary abelian.

Throughout this paper we use the following notation. We write $O_{p'}(G)$ and $O_p(G)$ for the maximal normal subgroup of $G$ of order prime to $p$ and the maximal normal $p$-subgroup of $G$, respectively. We define $O_{p',p}(G)$ by $O_p(G/O_{p'}(G))=O_{p',p}(G)/O_{p'}(G)$. We write $H\lhd G$ if $H$ is a normal subgroup of $G$. For a finite group $Y$, $|Y|$ and Aut $(Y)$ denote the order of $Y$ and the group of all automorphisms of $Y$, respectively. When $X$ is a subgroup of $G$, we write $N_G(X)$, $C_G(X)$ and $|G:X|$ for the normaliser of $X$ in $G$, the centraliser of $X$ in $G$ and the index of $X$ in $G$, respectively. If $x_1,\dots,x_n$ are in $G$, we write $\langle x_1,\dots,x_n\rangle$ for the subgroup of $G$ generated by $\{x_1,\dots,x_n\}$. When $H$ is a subgroup of $G$ and $g\in G$, let $[H, g]=\langle h^{-1}g^{-1}hg \mid h\in H\rangle$ and $[H, g, g]=[[H, g], g]$. We write $GL(2, p)$ and $SL(2, p)$ for the general linear group and the special linear group, respectively (cf. [**3**, p. 40]).

For an odd prime $p$, we say that $G$ is $p$-stable in the sense of [**1**, p. 1104 Definition 2.3].

We write $Qd(p)$ for the group defined in [**1**, p. 1104] and [**2**, p. 32]. Then $Qd(p)$ is the semi-direct product of $R$ by $SL(2, p)$ with respect to the identity map $SL(2, p)\to SL(2, p)\subseteq GL(2, p)\simeq \mathrm{Aut}(R)$, where $R$ is an elementary abelian group of order $p^2$. It is noted that if $p$ is odd then the $p$-Sylow subgroup of $Qd(p)$ is nonabelian of order $p^3$ of exponent $p$ (cf. [**2**, p. 32 and p. 33 Example 11.4]).

To begin with, we state the next two lemmas which are useful for our aim.

**Lemma 1.** *Let $G$ be a finite group and $p$ an odd prime. If the $p$-Sylow subgroup of $G$ is of order $p^3$ with exponent $p^2$, then $G$ is $p$-stable.*

**Proof.** By [**1**, Lemma 6.3], it suffices to show that $X/Y\not\simeq Qd(p)$ for any subgroup $X$ of $G$ and any $Y\lhd X$ (see [**1**, p. 1103] for the term "involved"). Assume that $X/Y\simeq Qd(p)$ for some subgroup $X$ of $G$ and some $Y\lhd X$. Since the order of the $p$-Sylow subgroup of $Qd(p)$ is $p^3$ by [**2**, p. 32], $p\nmid |G:X|$ and $p\nmid |Y|$. Let $P$ be a $p$-Sylow subgroup of $X$. Then $P$ is a $p$-Sylow subgroup of $G$, so that $P$ has exponent $p^2$. On the other hand, $(PY)/Y$ is a $p$-Sylow subgroup of $X/Y$. Hence $(PY)/Y$ has exponent $p$ from [**2**, p. 33 Example 11.4]. This is a contradiction since $(PY)/Y\simeq P/(P\cap Y)\simeq P$. This completes the proof.

**Lemma 2** [**3**, Theorem 8.1.3]. *Let $p$ be an odd prime, and let $G$ be a finite group with a $p$-Sylow subgroup $P$ such that $O_p(G)\neq 1$ and $G$ is $p$-stable and $p$-solvable. If $A$ is an abelian normal subgroup of $P$, then $A\subseteq O_{p',p}(G)$.*

**Proof.** Let $H=O_{p',p}(G)$, $Q=P\cap H$, $N=N_G(Q)$ and $C=C_G(Q)$. Then $O_{p'}(G)\cdot Q=H\lhd G$. Take any $x\in A$. Clearly $x\in N$. Since $A\lhd P\supseteq Q$, $[Q, x]\subseteq A$. Since $A$ is abelian, $[Q, x, x]\subseteq[A, x]=1$, so that $[Q, x, x]=1$. Since $G$ is $p$-stable, $xC\in O_p(N/C)$. This shows $(AC)/C\subseteq O_p(N/C)$. Since $G$ is $p$-solvable, $C\subseteq H$ by [**3**, Theorem 6.3.3], so that $C\subseteq H\cap N$. By the Frattini argument [**3**, Theorem 1.3.7], $G=HN$. Then we have the following epimorphism

$$\begin{array}{ccl} N/C & \xrightarrow{f} & N/(H\cap N)\simeq(HN)/H=G/H. \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ yC & \longrightarrow & y(H\cap N) \end{array}$$

Since $H=O_{p',p}(G)$, $O_p(G/H)=1$, so that $O_p(N/(H\cap N))=1$. Since $f$ is an epimorphism, $f(O_p(N/C))\subseteq O_p(N/(H\cap N))$. This implies $f((AC)/C)=1$, so that $A\subseteq H\cap N$.

Using these lemmas we can prove the next main result of this paper.

**Theorem.** *Let $p$ be an odd prime, and let $G$ be a finite $p$-solvable group with a metacyclic $p$-Sylow subgroup $P$ of order $p^a$. Then $t(G)=a(p-1)+1$ if and only if $P$ is elementary abelian.*

**Proof.** Assume that $P$ is elementary abelian. By [**3,** Theorem 6.3.3], $P \subseteq O_{p',p}(G)$. This implies that $G$ is of $p$-length 1. So that $t(G)=a(p-1)+1$ by [**10,** Corollary 1].

Suppose $t(G)=a(p-1)+1$. We use induction on $|G|$. Assume $G \neq 1$. Let $H = O_{p'}(G)$. By [**12,** Theorems 2.2 and 3.3], $a(p-1)+1 \leqq t(G/H) \leqq t(G)=a(p-1)+1$. Hence we may assume $H=1$ by induction. Let $R=O_p(G)$ and $|R|=p^b$, so that $1 \leqq b \leqq a$. Then $a(p-1)+1=t(G) \geqq t(R)+t(G/R)-1$ by [**12,** Theorem 2.4]. Since $t(R) \geqq b(p-1)+1$ and $t(G/R) \geqq (a-b)(p-1)+1$ by [**12,** Theorem 3.3], we have $t(R)= b(p-1)+1$. So that $R$ is elementary abelian from [**10,** Theorem 1]. Since $P$ is metacyclic, $R$ is cyclic of order $p$ or is elementary abelian of order $p^2$. Then $C_G(R)=R$ by [**3**, Theorem 6.3.3], so that

$$G/R = N_G(R)/C_G(R) \hookrightarrow \mathrm{Aut}\,(R). \qquad (*)$$

If $R$ is cyclic of order $p$, then $p \nmid |G/R|$ by (*), so that $P$ is cyclic of order $p$. Hence we may assume that $R$ is elementary abelian of order $p^2$. By [**10**, Corollary 1], it suffices to show that $G$ is of $p$-length 1. Suppose that $G$ is of $p$-length $\geqq 2$. Since $|\mathrm{Aut}\,(R)| = |GL(2,p)| = p(p-1)^2(p+1)$ by [**3,** Theorem 2.8.1], $|P/R|=1$ or $p$ from (*). This shows that $|P|=p^2$ or $p^3$. Since $G$ is of $p$-length $\geqq 2$, $P$ is nonabelian from [**3,** Theorem 6.3.3]. Hence $|P|=p^3$. Since $P$ is metacyclic, we can write

$$P = M_3(p) = \langle x, y \mid x^p = y^{p^2} = 1, x^{-1}yx = y^{p+1} \rangle$$

by [**3,** Theorem 5.5.1]. Then $G$ is $p$-stable by Lemma 1. Since $\langle x, y^p \rangle \lhd P$ and $\langle y \rangle \lhd P$ and since $R \neq 1$, we have that $\langle x, y^p \rangle \subseteq R$ and $\langle y \rangle \subseteq R$ by Lemma 2. Then $x, y \in R$, so that $P=R$. Hence $G$ is of $p$-length 1, a contradiction. This completes the proof.

Finally we give an example as mentioned in the introduction.

**Example.** Let $p=3$, and let $R$ be an elementary abelian group of order 9. Let $G$ be the semi-direct product of $R$ by $SL(2,3)$ with respect to the identity map $SL(2,3) \to SL(2,3) \subseteq GL(2,3) \simeq \mathrm{Aut}\,(R)$. Then $G \simeq Qd(3)$ (cf. [**1,** p. 1104] and [**2,** p. 32]). Let $R=\langle b, c \rangle$ and $S=SL(2,3)$. For each $x=\left(\begin{smallmatrix} s & t \\ u & v \end{smallmatrix}\right) \in S$, we can write that $x^{-1}bx=b^s c^t$ and $x^{-1}cx=b^u c^v$. Let $a=\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in S$, then $a$ is of order 3, so that we can write $P=\langle a, b, c \mid a^3=b^3=c^3=1, a^{-1}ba=bc, a^{-1}ca=c, b^{-1}cb=c \rangle$, where $P$ is a 3-Sylow subgroup of $G$. Then $P$ has exponent 3 (cf. [**2,** pp. 32–33] and [**3,** p. 203]). Let $Q$ be a 2-Sylow subgroup of $S$. Since $Q \lhd S$ and since $Q$ is quaternion of order 8, $S$ has the unique involution $z=\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ in $Q$. Let $H=O_{3'}(G)$. Since $|G|=2^3 \cdot 3^3=216$, $H=O_2(G)$. Since $Q$ is a 2-Sylow subgroup of $G$, $H \subseteq Q$. Evidently, $HR=H \times R$. If $H \neq 1$, then $z \in H$, so that $z \in C_G(R)$, a contradiction. Hence $H=1$. On the other hand, $P$ is not normal in $G$. So that $G$ is of 3-length 2.

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