ON ARITHMETIC PROPERTIES OF THE TAYLOR SERIES OF RATIONAL FUNCTIONS

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Pólya (**3**) has shown that if b_n is a sequence of algebraic integers and $\sum_{n=0}^{\infty} nb_n z^n$ is a rational function, then so is $\sum_{n=0}^{\infty} b_n z^n$. This result was generalized by Uchiyama (**5**) who showed that one may replace the assumption that the b_n are algebraic integers by the assumption that the b_n lie in a finitely generated submodule of the complex numbers, and by the author (**1**) who showed that if p is a non-zero polynomial with complex coefficients and if b_n is a sequence of algebraic integers such that $\sum_{n=0}^{\infty} p(n) b_n z^n$ is a rational function, then so is $\sum_{n=0}^{\infty} b_n z^n$. Our aim in this note is to give a common generalization of all of these theorems.

Let R be the integral closure, in the field of complex numbers $\mathbf{C}$, of a finitely generated subring of $\mathbf{C}$.

THEOREM. *Suppose that b_n is a sequence of elements of R and that p is a non-zero polynomial with complex coefficients. If $\sum_{n=0}^{\infty} p(n) b_n z^n$ is a rational function, then so is $\sum_{n=0}^{\infty} b_n z^n$.*

Note that the theorem proved in (**1**) is the special case of the above theorem for which R is the ring of algebraic integers and that Uchiyama's theorem is obtained from the case $p(n) = n$ by observing that a module generated by $x_1, x_2, \ldots, x_n$ is contained in the integral closure of the ring generated by $x_1, x_2, \ldots, x_n$.

If K is an algebraic number field and S is a finite set of valuations of K containing all Archimedean valuations, we shall say, as usual, that $x \in K$ is an *S-integer* of K if $|x|_v \leqq 1$ for all valuations v of K not in S. The following is a slightly stronger version of (**1**, Lemma 2).

LEMMA 1. *Suppose that $\alpha \in K$ is not a non-negative integer and b_n is a sequence of S-integers of K. Suppose that there exist S-integers $d_0, d_1, d_2, \ldots, d_r$ of K such that $d_0 = 1$, $d_r \neq 0$, and*

$$\sum_{j=0}^{r} d_j (n - j - \alpha) b_{n-j} = 0$$

for all integers $n \geqq r$. Then

$$\sum_{j=0}^{r} d_j b_{n-j} = 0$$

for all integers $n \geqq r$.

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Proof. It is immediate that

$$\sum_{n=0}^{\infty} (n-\alpha)b_n z^n = \frac{C(z)}{D(z)}, \tag{1}$$

where

$$D(z) = \sum_{j=0}^{r} d_j z^j \quad \text{and} \quad C(z) = \sum_{n=0}^{r-1} \left(\sum_{j=0}^{n} d_j (n-j-\alpha) b_{n-j} \right) z^n.$$

The degree of $C(z)$ is less than r, the degree of $D(z)$, and hence we can write the partial fraction expansion

$$\frac{C(z)}{D(z)} = \sum_{i=1}^{s} \frac{C_i(z)}{(1-\theta_i z)^{e_i}}, \tag{2}$$

where $1/\theta_1, 1/\theta_2, \ldots, 1/\theta_s$ are the distinct roots of $D(z)$ with multiplicities $e_1, e_2, \ldots, e_s$, respectively, and the $C_i(z)$ are polynomials of degree less than e_i, respectively. Now

$$\frac{1}{(1-\theta_i z)^{e_i}} = \sum_{n=0}^{\infty} \binom{e_i + n - 1}{e_i - 1} \theta_i^n z^n. \tag{3}$$

Suppose that $C_i(z) = \sum_{j=0}^{e_i-1} c_{ij} z^j$. Then

$$\frac{C_i(z)}{(1-\theta_i z)^{e_i}} = \sum_{n=0}^{\infty} \lambda_i(n) \theta_i^n z^n,$$

where

$$\lambda_i(n) = \sum_{j=0}^{e_i-1} c_{ij} \binom{e_i - n - j - 1}{e_i - 1} \theta_i^{-j}$$

is a polynomial in n with algebraic coefficients of degree less than e_i. Thus, we can expand the right-hand side of (2) and obtain

$$\frac{C(z)}{D(z)} = \sum_{n=0}^{\infty} \sum_{i=1}^{s} \lambda_i(n) \theta_i^n z^n. \tag{4}$$

Let L be an algebraic number field which includes K and contains all of the coefficients of the λ_i and all of the θ_i. Let T be a finite set of valuations of L containing all valuations of L which extend valuations in S, and such that the non-zero coefficients of the λ_i, the θ_i, the differences $\theta_i - \theta_j$, where $i \neq j$, are T-units of L.

By (**1**, Lemma 1), there exist infinitely many pairs (k, P), where k is an integer and P is a prime ideal of L such that $k - \alpha \in P$. Let (k, P) be such a pair with P not corresponding to a valuation in T. Suppose that the norm of P is p^f, where p is a rational prime. Combining (1) and (4) we obtain

$$(n-\alpha)b_n = \sum_{i=1}^{s} \lambda_i(n) \theta_i^n. \tag{5}$$

We substitute $n = k + jp^f$ in (5) and obtain

$$(k + jp^f - \alpha)b_n = \sum_{i=1}^{s} \lambda_i(k + jp^f) \theta_i^k \theta_i^{jp^f}.$$

Reducing modulo P we obtain

$$\sum_{i=1}^{s} \lambda_i(\alpha)\theta_i^{k+j} \equiv 0 \quad (P). \tag{6}$$

The determinant of the homogeneous linear equations in the $\lambda_i(\alpha)$, obtained from (6) by substituting successively $j = 0, 1, 2, \ldots, s-1$, is $\prod_{i=1}^{s}\theta_i^k$ times the Vandermonde determinant $||\theta_i^j||$, and hence is not congruent to 0 (P). It follows that each $\lambda_i(\alpha)$ is congruent to 0 (P). Since this last congruence holds for infinitely many prime ideals P, we see that each $\lambda_i(\alpha) = 0$. Thus, we can write

$$\lambda_i(n) = (n - \alpha)\mu_i(n),$$

where each $\mu_i(n)$ is a polynomial of lower degree than $\lambda_i(n)$. Since $n - \alpha$ is not 0 for $n \geqq 0$, we obtain from (5) that

$$b_n = \sum_{i=1}^{s} \mu_i(n)\theta_i^n. \tag{7}$$

Since for each integer $k \geqq 0$, the function

$$\sum_{n=0}^{\infty} n^k\theta_i^n z^n = \left(z\frac{d}{dz}\right)^k (1 - \theta_i z)^{-1}$$

is rational, it follows from (3) and (7) that $\sum_{n=0}^{\infty} b_n z^n$ can be written in the form $B(z)/D(z)$, where $B(z)$ is a polynomial of degree less than r, and hence $\sum_{i=0}^{r} d_i b_{n-i} = 0$ for all $n \geqq r$.

LEMMA 2. *Suppose that $\alpha \in \mathbf{C}$ and b_n is a sequence of elements of R. If*

$$\sum_{n=0}^{\infty} (n - \alpha)b_n z^n$$

is a rational function, then so is

$$\sum_{n=0}^{\infty} b_n z^n.$$

Proof. We first prove the lemma under the additional assumption that α is not a non-negative integer. We can write $\sum_{n=0}^{\infty}(n - \alpha)b_n z^n$ in the form $A(z) + C(z)/D(z)$, where $A(z)$, $C(z)$, and $D(z)$ are polynomials with complex coefficients, the degree of $C(z)$ is less than d, the degree of $D(z)$, and $D(0) = 1$. By changing a finite number of the b_n, if necessary, we may assume that $A(z) = 0$. By enlarging R, if necessary, we may assume that α, the coefficients of C, the coefficients of D, and 1 are contained in R.

Suppose that ϕ is a homomorphism of R into $\tilde{\mathbf{Q}}$, the field of algebraic numbers. Write

$$C(z) = \sum_{i=0}^{s} c_i z^i \quad \text{and} \quad D(z) = \sum_{i=0}^{r} d_i z^i.$$

Then $d_0 = 1$ and $d_r \neq 0$. Now

$$(8) \qquad \sum_{i=1}^{r} d_i(n - i - \alpha)b_{n-i} = 0$$

if $n \geqq r$. Applying ϕ to (8) shows that

$$\sum_{n=0}^{\infty} \phi(n - \alpha)\phi(b_n)z^n$$

is a rational function and that all of the $\phi(b_n)$ lie in the field K_1 generated by $\phi(\alpha)$, the $\phi(d_i)$, and $\phi(b_0), \phi(b_1), \ldots, \phi(b_{r-1})$. Suppose that R is the integral closure in $\mathbf{C}$ of the ring $R_1 = \mathbf{Z}[x_1, x_2, \ldots, x_h]$ and K is the field generated by the $\phi(x_i)$ over K_1. Let S be a finite set of valuations of the algebraic number field K containing all Archimedean valuations, and such that each $\phi(x_i)$ is an S-integer of K. Each b_n satisfies a monic polynomial with coefficients in R_1, and hence each $\phi(b_n)$ satisfies a monic polynomial whose coefficients are S-integers of K. It is immediate that each $\phi(b_n)$ is an S-integer of K. If $A(z) = \sum_{i=0}^{t} a_i z^i$ is any polynomial with coefficients in R, we write

$$\phi(A)(z) = \sum_{i=0}^{t} \phi(a_i)z^i.$$

Now,

$$\sum_{n=0}^{\infty} (n - \phi(\alpha))\phi(b_n)z^n = \frac{\phi(C)(z)}{\phi(D)(z)}.$$

Now, suppose that ϕ satisfies the additional hypotheses that $\phi(d_r) \neq 0$ and $\phi(\alpha)$ is not a non-negative integer. Then by Lemma 1, $\sum_{n=0}^{\infty} \phi(b_n)z^n$ is a rational function and can be written in the form $A_\phi(z)/\phi(D)(z)$, where A_ϕ is a polynomial with algebraic coefficients of degree less than r. Put $B_n = \det(b_{i+j})$, $0 \leqq i, j \leqq n$; B_n is a Hankel determinant, and by Kronecker's theorem (**4**, p. 5), $\phi(B_n) = 0$ for $n \geqq r$. We now show that $B_n = 0$ for $n \geqq r$. Suppose not, and that $B_{n_0} \neq 0$, where $n_0 \geqq r$. We may suppose that the x_i are non-zero and are chosen so that $x_1, x_2, \ldots, x_g$ form a transcendence basis for R, and, if α is transcendental, that $x_1 = \alpha$. The quantities $x_{g+1}, x_{g+2}, \ldots, x_h, d_r$, and B_{n_0} all satisfy irreducible polynomials with coefficients in $\mathbf{Z}[x_1, x_2, \ldots, x_g]$. Let $q(x_1, x_2, \ldots, x_g)$ be the product of the leading terms and constant terms of these polynomials. If $\beta_1, \beta_2, \ldots, \beta_g$ are algebraic numbers such that $q(\beta_1, \beta_2, \ldots, \beta_g) \neq 0$, then the homomorphism of $\mathbf{Z}[x_1, x_2, \ldots, x_g]$ into the field of algebraic numbers, given by $x_i \rightarrow \beta_i$, $1 \leqq i \leqq g$, can be extended to a homomorphism $\phi_0: R \rightarrow \tilde{\mathbf{Q}}$; see (**2**). Furthermore, $\phi_0(d_r)$ and $\phi_0(B_{n_0})$ satisfy polynomials with non-zero constant terms, hence are not 0. If α is algebraic, then $\phi_0(\alpha) = \alpha$, and hence $\phi_0(\alpha)$ is not a non-negative integer, while if α is transcendental, then $\alpha = x_1$ and $\phi_0(\alpha) = \beta_1$. It is possible to choose β_1 such that β_1 is not a non-negative integer and $q(\beta_1, x_2, \ldots, x_g)$ is not the 0 polynomial. Then $\beta_2, \ldots, \beta_g$ can be chosen so that $q(\beta_1, \beta_2, \ldots, \beta_g) \neq 0$. Thus, we have constructed a homomorphism $\phi_0: R \to \tilde{\mathbf{Q}}$ such that

$$\phi_0(B_{n_0}) \neq 0, \qquad \phi_0(d_r) \neq 0,$$

and such that $\phi_0(\alpha)$ is not a non-negative integer. This contradicts what we proved earlier and shows that $B_{n_0} = 0$. Thus,

$$B_r = B_{r+1} = B_{r+2} = \ldots = 0.$$

By Kronecker's theorem (**4**, p. 5), $\sum_{n=0}^{\infty} b_n z^n$ is a rational function. It remains to prove the lemma when α is a non-negative integer. In this case, put $\alpha' = -1$ and $b_n' = b_{n+\alpha+1}$; it is clear that

$$\sum_{n=0}^{\infty} (n - \alpha')b_n' z^n = \sum_{n=\alpha+1}^{\infty} (n - \alpha) b_n z^{n-\alpha-1}$$

is a rational function. By what we have already proved, $\sum_{n=0}^{\infty} b_n' z^n$ is a rational function, and hence so is $\sum_{n=0}^{\infty} b_n z^n$.

Proof of theorem. Without loss of generality, we may assume that $p(n)$ is monic and factor it as $\prod_{i=1}^{t}(n - \alpha_i)$. Put

$$b_n^{(s)} = b_n \prod_{i=1}^{s} (n - \alpha_i) \quad \text{and} \quad g_s(z) = \sum_{n=0}^{\infty} b_n^{(s)} z^n,$$

where $0 \leqq s \leqq t$. Applying the hypothesis that $g_t(z)$ is rational and using Lemma 2 repeatedly we find successively that

$$g_{t-1}(z), g_{t-2}(z), \ldots, g_0(z) = \sum_{n=0}^{\infty} b_n z^n$$

are rational functions.

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