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Dumbbell micro-robot driven by flow oscillations

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In this paper we study the self-propulsion of a dumbbell micro-robot submerged in a viscous fluid. The micro-robot consists of two rigid spherical beads connected by a rod or a spring; the rod/spring length changes periodically. The constant density of each sphere differs from the density of the fluid, while the whole micro-robot has neutral buoyancy. An effective oscillating gravity field is created via rigid-body oscillations of the fluid. Our calculations show that the micro-robot undertakes both translational and rotational motion. Using an asymptotic procedure containing a two-time method and a distinguished limit, we obtain analytic expressions for the averaged self-propulsion velocity and averaged angular velocity. The important special case of zero angular velocity represents rectilinear self-propulsion with constant velocity. In particular, we have shown that: (*a*) no unidirectional oscillations of a fluid result in self-propulsion; and (*b*) for the oscillations of a fluid in two directions rectilinear motion of a micro-robot can be achieved.

Key words: micro-/nano-fluid dynamics, propulsion, stokesian dynamics

1. Introduction

The study of self-propelling micro-robots is a flourishing current research topic, striving to create a fundamental base for modern applications in medicine and technology, see e.g. Purcell (1977), Becker, Koelher & Stone (2003), Najafi & Golestanian (2004), Dreyfus *et al.* (2005), Felderhof (2006), Chang *et al.* (2007), Earl *et al.* (2007), Alouges, DeSimone & Lefebvre (2008), Golestanian & Ajdari (2008), Alexander, Pooley & Yeomans (2009), Belovs & Cēbers (2009), Leoni *et al.* (2009), Gilbert *et al.* (2010), Lauga (2011) and Romanczuk *et al.* (2012). We define *self-propulsion* as the motion of a micro-robot which is subjected to zero external total force. The simplicity of the micro-robot geometry represents a major advantage in contrast to the extreme complexity of self-swimming micro-organisms. This advantage allows us to describe the motion of micro-robots in greater depth. The major problem in the designing of a micro-robot is the need for an external source of energy to

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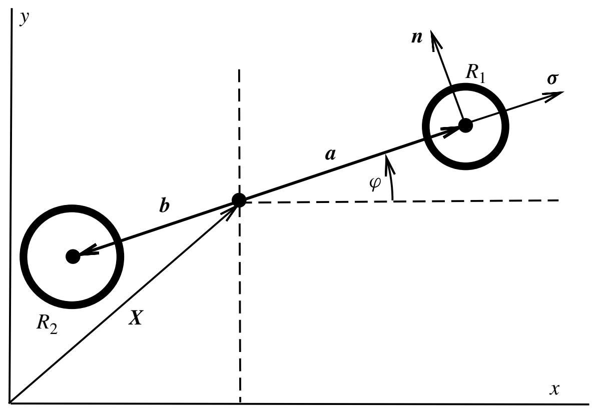

FIGURE 1. Two spheres, linked by a rod of periodically changing length.

provide its oscillatory behaviour. Proposed sources include an oscillating (or rotating) magnetic field (see Dreyfus *et al.* 2005; Belovs & Cēbers 2009; Gilbert *et al.* 2010), an electric field (see Chang *et al.* 2007), and even molecular Brownian forces (see Romanczuk *et al.* 2012). At the same time, the major oscillatory forces available in fluid have not been exploited; these are the forces caused by fluid oscillations which are imposed by periodically varying boundary conditions, waves, or turbulence. The ratio of characteristic spatial scales (several microns for a micro-robot versus millimetres, centimetres, or greater scales for flow oscillations) makes it clear that the first problem to study is the behaviour of a micro-robot in a fluid that oscillates as a rigid body.

In this paper, we consider the self-propulsion of a two-sphere *buoyancy-driven dumbbell* micro-robot (which we call a BD-robot), see figure 1. The whole micro-robot is neutrally buoyant (in order to avoid sedimentation); one of its spherical beads is positively buoyant and the other is negatively buoyant. We study two versions of BD-robots. In the first one the beads are connected by a rod of prescribed oscillating length, in the second one the beads are linked by an elastic spring. First, we study the case of a rod and, next, we consider the changes that appear after replacing the rod with a spring. A mathematical formulation of the problem leads us to the study of creeping motion with time-periodic forces. The problem is solved by employing a version of the two-time method and distinguished limit arguments, developed in Vladimirov (2005, 2008, 2012*a*). The approach allows any motion of the BD-robot to be described analytically. Our calculations show that, generally, the BD-robot undergoes both translational and rotational motion. Rectilinear translational self-propulsion with constant velocity represents a special case of this solution. We have calculated the velocity of rectilinear self-propulsion and the ranges of governing parameters that correspond to translational motion.

2. Problem formulation

The BD-robot represents a dumbbell configuration, which consists of two homogeneous rigid spherical beads of different radii R_ν, $\nu = 1, 2$ connected by a rod of length l, see figure 1. We study two-dimensional motion of a three-dimensional dumbbell in Cartesian coordinates (x, y). The centres of the spheres $\boldsymbol{x}^{(\nu)}$ are described by

$$\boldsymbol{x}^{(1)} = \boldsymbol{X} + \boldsymbol{a}, \quad \boldsymbol{x}^{(2)} = \boldsymbol{X} + \boldsymbol{b}, \quad R_1\boldsymbol{a} + R_2\boldsymbol{b} = 0, \tag{2.1a}$$

$$\boldsymbol{a} = a\boldsymbol{\sigma} = r_1 l\boldsymbol{\sigma}, \quad \boldsymbol{b} = b\boldsymbol{\sigma} = -r_2 l\boldsymbol{\sigma}; \quad r_1 \equiv R_2/(R_1+R_2), \quad r_2 \equiv R_1/(R_1+R_2), \tag{2.1b}$$

where $\boldsymbol{X} = (X, Y)$ is the radius-vector of a centre of reaction. The axis of symmetry of a dumbbell is given by the vector $\boldsymbol{l} \equiv \boldsymbol{x}^{(1)} - \boldsymbol{x}^{(2)}$, $l \equiv |\boldsymbol{l}|$. The unit vectors $\boldsymbol{\sigma}, \boldsymbol{n}$ and the angle φ are given by

$$\boldsymbol{\sigma} \equiv \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}, \quad \boldsymbol{n} \equiv \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}, \quad \boldsymbol{n} = \boldsymbol{\sigma}_\varphi, \quad \boldsymbol{\sigma} = -\boldsymbol{n}_\varphi, \quad \boldsymbol{\sigma}\cdot\boldsymbol{n} = 0, \tag{2.2}$$

where the subscript φ stands for $\mathrm{d}/\mathrm{d}\varphi$. The length l is changing periodically

$$l = L + \varepsilon\widetilde{l}(\tau); \quad \tau \equiv \omega t; \quad \omega = \text{const}, \quad \varepsilon = \text{const} \tag{2.3}$$

where L is a constant averaged value and $\widetilde{l}$ is a 2π-periodic function of τ with zero average value (throughout the paper a 'tilde' above a function of time denotes that this function is oscillating and has zero mean value). The spheres experience external friction forces $\boldsymbol{F}^{(\nu)} = (F_1^{(\nu)}, F_2^{(\nu)})$ while the rod is so thin (in comparison with either R_ν) that its interaction with the fluid can be considered negligible.

We consider the motion of a BD-robot in a viscous incompressible fluid which, in the absence of the BD-robot, oscillates as a rigid body. These rigid-body oscillations are prescribed as a two-dimensional translational spatial displacement $\widetilde{\boldsymbol{\xi}}(\tau) = (\xi_1(\tau), \xi_2(\tau))$ of fluid particles (at infinity in space); the related acceleration is $\widetilde{\boldsymbol{\xi}}_{tt} = \omega^2\widetilde{\boldsymbol{\xi}}_{\tau\tau}$, where the subscripts stand for related derivatives. The problem can be studied in an oscillating (non-inertial) system of reference, in which a fluid at infinity is in a state of rest. In this frame, according to Einstein's principle of equivalence, or according to a related transformation of a Lagrangian function, the equations of fluid motion are standard; however, they contain an additional oscillating gravity force

$$\widetilde{\boldsymbol{g}} = -\omega^2\widetilde{\boldsymbol{\xi}}_{\tau\tau} \tag{2.4}$$

which causes buoyancy forces $-M^{(\nu)}\widetilde{\boldsymbol{g}}$, where the coefficient $M^{(\nu)}$ is equal to the difference in the mass of a sphere and the mass of displaced fluid; $M^{(\nu)}$ can be either positive or negative. The potential energy of a sphere is $\Pi^{(\nu)} = M^{(\nu)}\widetilde{\boldsymbol{g}}\cdot\boldsymbol{x}^{(\nu)}$. We consider a BD-robot of neutral total buoyancy, with total potential energy

$$\Pi = \Pi^{(1)} + \Pi^{(2)} = M\widetilde{\boldsymbol{g}}\cdot\boldsymbol{l}, \quad M \equiv M^{(1)} = -M^{(2)} > 0. \tag{2.5}$$

The problem formulation contains three characteristic lengths: the length of the rod L, the radius of the spheres R, and the amplitude of the rod oscillation a. In addition we have the characteristic time scale T, excess mass M, gravity g, and viscous force F. We have chosen these scales as

$$R \equiv (R_1+R_2)/2, \quad T \equiv 1/\omega, \quad a \equiv \varepsilon L, \quad F \equiv 6\pi\eta RL/T, \quad g \equiv \max|\widetilde{\boldsymbol{g}}(\tau)|, \tag{2.6}$$

where η is the fluid viscosity. The dimensionless variables (marked with asterisks) are $\boldsymbol{x} = L\boldsymbol{x}^*$, $t = T\,t^*$, $F_i = F\,F_i^*$. Three independent small parameters of the problem are

$$\varepsilon \equiv a/L, \quad \delta \equiv 3R/(4L), \quad m \equiv Mg/F. \tag{2.7}$$

Below we use only dimensionless variables, but omit the asterisks. Note that in the chosen dimensionless units, $R_1 + R_2 = 2$.

We choose the generalized coordinates of the BD-robot to be $\boldsymbol{q} = (q_1, q_2, q_3, q_4) \equiv (X, Y, l, \varphi)$. The motion of the BD-robot, with a given $l(\tau)$ (2.3), is described by the Lagrangian function $\mathscr{L} = \mathscr{L}(\boldsymbol{q}, \boldsymbol{q}_t)$, which includes the constraint (2.3) with Lagrangian multiplier (reaction of constraint) N

$$\mathscr{L}(\boldsymbol{q}, \boldsymbol{q}_t) = \mathscr{K} - \varPi + N(l - L - \varepsilon\widetilde{l}), \tag{2.8}$$

where $\mathscr{K}$ and $\varPi$ are the kinetic energy and potential energy (2.5) of the BD-robot; N represents an additional unknown function of time. The Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial q_{nt}} - \frac{\partial \mathscr{L}}{\partial q_n} = Q_n, \quad Q_n = \sum_{\nu=1}^{2}\sum_{k=1}^{2} F_k^{(\nu)} \frac{\partial x_k^{(\nu)}}{\partial q_n}, \tag{2.9}$$

where $\boldsymbol{Q} = (Q_1, Q_2, Q_3, Q_4)$ is the generalized external viscous force exerted by the fluid on the BD-robot. As one can see, we use subscripts $i, k = 1, 2$ for Cartesian components of vectors and tensors, subscript $n = 1, 2, 3, 4$ for generalized coordinates, and subscripts (or superscripts) $\mu, \nu = 1, 2$ to identify the spheres. We assume that the fluid flow past a BD-robot is described by the Stokes equations, where all inertial terms are neglected. Correspondingly, the masses of the rod and the spheres are negligible, hence $\mathscr{K} \equiv 0$. Therefore (2.8), (2.9), (2.1) give rise to the following system of equations:

$$\boldsymbol{F}^{(1)} + \boldsymbol{F}^{(2)} = 0, \tag{2.10}$$

$$\varepsilon\alpha\boldsymbol{g}\boldsymbol{\cdot}\boldsymbol{n} = \boldsymbol{F}^{-}\boldsymbol{\cdot}\boldsymbol{n}, \quad 2\boldsymbol{F}^{-} \equiv R_2\boldsymbol{F}^{(1)} - R_1\boldsymbol{F}^{(2)}, \tag{2.11}$$

$$\varepsilon\alpha\boldsymbol{g}\boldsymbol{\cdot}\boldsymbol{\sigma} - N = \boldsymbol{F}^{-}\boldsymbol{\cdot}\boldsymbol{\sigma}, \tag{2.12}$$

which is supplemented by constraint (2.3). The great advantage of a Lagrangian formalism is its self-sufficiency. In particular, the conditions of zero force (2.10) and the balance of torques (2.11) appear automatically, while (2.12) allows us to find the reaction of constraint N. In (2.11), (2.12) we have assumed that

$$m = \varepsilon\alpha, \quad \alpha = \mathrm{const} = O(1). \tag{2.13}$$

This is our physical assumption, which states that two small parameters ε and m (2.7) are of the same order. Physically, it means that the difference between the densities (of each sphere and the fluid) or the amplitude of oscillations of the fluid is small (or both these parameters are small). The explicit expressions for $\boldsymbol{F}^{(\nu)}$ are

$$\boldsymbol{F}^{(1)} \simeq -R_1\boldsymbol{x}_t^{(1)} + \delta R_{12}\mathbb{S}\boldsymbol{x}_t^{(2)}, \quad \boldsymbol{F}^{(2)} \simeq -R_2\boldsymbol{x}_t^{(2)} + \delta R_{12}\mathbb{S}\boldsymbol{x}_t^{(1)}, \tag{2.14a}$$

$$l^3\mathbb{S} = l^3 S_{ik} \equiv l^2\delta_{ik} + l_i l_k, \quad R_{12} \equiv R_1 R_2. \tag{2.14b}$$

Each force $\boldsymbol{F}^{(\nu)}$ represents the first approximation for the Stokes friction force exerted on a sphere moving in a flow field generated by another sphere. To construct (2.14) we use a classical explicit formula for the fluid velocity past a moving sphere, see Lamb (1932), Landau & Lifshitz (1959) and Moffatt (1996). Equations (2.10)–(2.12) and (2.14) represent a system of four equations for four unknown functions of time: X, Y, φ, and N. For the prescribed l (2.3), equation (2.12) need not to be considered if we are interested only in the motion of the micro-robot and are not calculating reaction force N. For future use, we rewrite (2.10), (2.11) as

$$\boldsymbol{X}_t - \delta R_{12}\mathbb{S}[\boldsymbol{X}_t - \widehat{R}\boldsymbol{l}_t/4] = 0, \tag{2.15}$$

$$\boldsymbol{n}\boldsymbol{\cdot}[\boldsymbol{l}_t + \delta\mathbb{S}(\widehat{R}\boldsymbol{X}_t + R_{12}\boldsymbol{l}_t)] = -2\varepsilon\alpha\boldsymbol{n}\boldsymbol{\cdot}\widetilde{\boldsymbol{g}}/R_{12}, \tag{2.16}$$

where $\widehat{R} \equiv R_1 - R_2$.

3. Two-time method and asymptotic procedure

3.1. *Functions and notation*

The following dimensionless notation and definitions are used:

(i) s and τ denote slow and fast times; subscripts τ and s stand for related partial derivatives.

(ii) A dimensionless function, say $h = h(s, \tau)$, belongs to the class $\mathscr{I}$ if $h = O(1)$ and all partial s- and τ-derivatives of h (required for our consideration) are also $O(1)$. In this paper all functions belong to class $\mathscr{I}$, while all small parameters appear as explicit multipliers.

(iii) We consider only *functions periodic in* τ $\{h \in \mathscr{P} : \quad h(s, \tau) = h(s, \tau + 2\pi)\}$, where s-dependence is not specified. Hence, all functions considered below belong to $\mathscr{P} \bigcap \mathscr{I}$.

(iv) For arbitrary $h \in \mathscr{P}$ the *averaging operation* is

$$\langle h \rangle \equiv \frac{1}{2\pi} \int_{\tau_0}^{\tau_0 + 2\pi} h(s, \tau)\, \mathrm{d}\tau \equiv \overline{h}(s), \quad \forall\ \tau_0. \tag{3.1}$$

(v) *The oscillating part of an integral* is:

$$\widetilde{h}^{\tau} \equiv \int_0^{\tau} \widetilde{h}(\boldsymbol{x}, s, \nu)\, \mathrm{d}\nu - \frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^{\mu} \widetilde{h}(\boldsymbol{x}, s, \nu)\, \mathrm{d}\nu \right) \mathrm{d}\mu. \tag{3.2}$$

(vi) *The tilde-function* (or purely oscillating function) represents a special case of the $\mathscr{P}$-function with zero average $\langle \widetilde{h} \rangle = 0$. The *bar-function* (or mean-function) $\overline{h} = \overline{h}(s)$ does not depend on τ. For any periodic function h a unique decomposition $h = \overline{h} + \widetilde{h}$ is valid.

3.2. *Asymptotic procedure and successive approximations*

The introduction of a fast time variable τ and a slow time variable s represents a crucial step in our asymptotic procedure. We choose $\tau = t$ and $s = \varepsilon^2 t$. This choice can be justified by the same distinguished limit arguments as in Vladimirov (2012*a*). Here we present this choice without proof; however, its most important aspect (that this choice leads to a valid asymptotic procedure) is exposed and exploited below. We use the chain rule

$$\mathrm{d}/\mathrm{d}t = \partial/\partial\tau + \varepsilon^2 \partial/\partial s \tag{3.3}$$

and then assume (temporarily) that τ and s represent two independent variables. Furthermore we consider series expansions in the small parameter ε and restrict our attention to terms which are at most $O(\varepsilon^2)$. Simultaneously, we keep at most linear-in-δ terms. It does not mean that in our setting $\delta \sim \varepsilon^2$, since in all expressions δ appears not separately but as a product with various degrees of ε. Hence, we do not specify the dependence of unknown functions on δ; such dependence reveals itself naturally during the calculations. The unknown functions are taken as regular series in ε

$$\boldsymbol{X}(\tau, s) = \boldsymbol{X}_0(\tau, s) + \varepsilon \boldsymbol{X}_1(\tau, s) + \varepsilon^2 \boldsymbol{X}_2(\tau, s) + \cdots, \tag{3.4}$$

with a similar expression for $\varphi(\tau, s)$. We are looking for solutions with

$$\widetilde{\boldsymbol{X}}_0(s, \tau) \equiv 0 \quad \text{and} \quad \widetilde{\varphi}_0(s, \tau) \equiv 0 \quad \text{while} \quad \overline{\boldsymbol{X}}_0(s) \neq 0 \quad \text{and} \quad \overline{\varphi}_0(s) \neq 0, \tag{3.5}$$

which express the target property: long distances of self-swimming and large angles of rotation are caused by small oscillations. The application of (3.3) to (3.4) gives

$$\boldsymbol{X}_t = \varepsilon \widetilde{\boldsymbol{X}}_{1\tau} + \varepsilon^2 (\widetilde{\boldsymbol{X}}_{2\tau} + \overline{\boldsymbol{X}}_{0s}) + O(\varepsilon^2) \tag{3.6}$$

and a similar expression for φ_t. In the calculations below all bar-functions belong to the zero approximation, while all tilde-functions belong to the first approximation; therefore we omit the related subscripts, e.g. $\widetilde{\boldsymbol{X}}_1$ and $\overline{\boldsymbol{n}}_0$ are replaced with $\widetilde{\boldsymbol{X}}$, $\overline{\boldsymbol{n}}$. The successive approximations of (2.15), (2.16) yield:
terms $O(\varepsilon^0)$ give the identities $0 = 0$; *terms* $O(\varepsilon^1)$ lead to

$$\widetilde{\boldsymbol{X}}_\tau = \delta R_{12} \overline{\mathbb{S}} (\widetilde{\boldsymbol{X}}_\tau - \widehat{R} \widetilde{\boldsymbol{l}}_\tau /4), \quad \widetilde{\boldsymbol{l}}_\tau \boldsymbol{\cdot} \overline{\boldsymbol{n}} = -2\alpha \widetilde{\boldsymbol{g}} \boldsymbol{\cdot} \overline{\boldsymbol{n}} / R_{12}. \tag{3.7}$$

The use of (2.2) transforms the second equation to the form $\widetilde{\varphi}_\tau = -2\alpha \widetilde{\boldsymbol{g}} \boldsymbol{\cdot} \overline{\boldsymbol{n}} / R_{12}$. Then integration of (3.7) in the class of periodic functions yields

$$\widetilde{\boldsymbol{X}} = \delta R_{12} \overline{\mathbb{S}} (\widetilde{\boldsymbol{X}} - \widehat{R} \widetilde{\boldsymbol{l}} /4), \quad \widetilde{\varphi} = -2\alpha \widetilde{\boldsymbol{g}}^\tau \boldsymbol{\cdot} \overline{\boldsymbol{n}} / R_{12} \tag{3.8}$$

where the notation (3.2) is used. *Terms* $O(\varepsilon^2)$ of (2.15) with the use of (3.8) (and with linear in $\delta \to 0$ precision) give

$$\overline{\boldsymbol{X}}_s = \tfrac{1}{4} \delta R_{12} \widehat{R} \langle \widetilde{\mathbb{S}}_\tau \widetilde{\boldsymbol{l}} \rangle, \tag{3.9}$$

where have used integration by parts in the average operation (3.1). The use of the definition of matrix $\mathbb{S}$ (2.14) and (2.2) yield $\langle \widetilde{\mathbb{S}}_\tau \widetilde{\boldsymbol{l}} \rangle = 2 \langle \widetilde{l} \widetilde{\varphi}_\tau \rangle \overline{\boldsymbol{n}}$. Then (3.9), (3.8) takes the form

$$\overline{\boldsymbol{X}}_s = -\delta \alpha \widehat{R} \langle \widetilde{l} \widetilde{\boldsymbol{g}} \boldsymbol{\cdot} \overline{\boldsymbol{n}} \rangle \overline{\boldsymbol{n}}. \tag{3.10}$$

Similarly, from (2.16), we can derive the equation for $\overline{\varphi}$ and obtain the system of equations

$$\overline{\boldsymbol{X}}_{s'} = -\delta \mu U \overline{\boldsymbol{n}} / \gamma, \quad \overline{\varphi}_{s'} = U - \gamma G; \tag{3.11}$$

$$s' \equiv \gamma s, \quad \mu \equiv \alpha \widehat{R}, \quad \gamma \equiv 2\alpha / R_{12}, \quad G \equiv \langle \widetilde{g}_1^\tau \widetilde{g}_2 \rangle, \tag{3.12}$$

$$U \equiv \langle \widetilde{l} (\widetilde{\boldsymbol{g}} \boldsymbol{\cdot} \overline{\boldsymbol{n}}) \rangle = -G_1 \sin \overline{\varphi} + G_2 \cos \overline{\varphi}, \quad G_1 \equiv \langle \widetilde{l} \widetilde{g}_1 \rangle, \quad G_2 \equiv \langle \widetilde{l} \widetilde{g}_2 \rangle, \tag{3.13}$$

where we have used the equality $\langle (\widetilde{\boldsymbol{g}}^\tau \boldsymbol{\cdot} \overline{\boldsymbol{\sigma}})(\widetilde{\boldsymbol{g}} \boldsymbol{\cdot} \overline{\boldsymbol{n}}) \rangle = G$, which is valid by virtue of (2.1), (3.1), and $\widetilde{\boldsymbol{g}} = (\widetilde{g}_1, \widetilde{g}_2)$. One can see that the dynamics of a dumbbell is determined by the values of three parameters δ, μ, γ and by three correlations G, G_1, and G_2.

The slow time scale $s \equiv \varepsilon^2 t$ implies that in order to obtain physical dimensionless velocities we have to multiply $\overline{\boldsymbol{X}}_{s'}$ and $\overline{\varphi}_{s'}$ (3.11) by $\gamma \varepsilon^2$. Accordingly, the mean translational velocity $\overline{\boldsymbol{V}}$ and the mean angular velocity $\overline{\varOmega}$ are

$$\overline{\boldsymbol{V}} = O(\varepsilon^2 \delta), \quad \overline{\varOmega} = O(\varepsilon^2), \tag{3.14}$$

hence the self-rotation is 'much faster' than the translational motion. We also can draw a general conclusion, based on (3.11), that $\overline{\boldsymbol{V}}$ is always directed along the normal vector $\overline{\boldsymbol{n}}$.

4. Prescribed oscillations of the BD-robot

4.1. *Unidirectional oscillations of a fluid do not cause rectilinear propulsion*

The simplest motion takes place when $G\equiv 0$. Physically, it means that fluid oscillations are unidirectional, see (2.4). In this case (3.11) produce the integral

$$\overline{\boldsymbol{X}}-\boldsymbol{C}=-\frac{\delta\mu}{\gamma}\overline{\boldsymbol{\sigma}}\quad\text{or}\quad(\overline{\boldsymbol{X}}-\boldsymbol{C})^2=\delta^2\mu^2/\gamma^2 \tag{4.1}$$

with a vectorial constant of integration $\boldsymbol{C}$. This equality shows that $\overline{\boldsymbol{X}}(s')$ changes along a circular path (or along an arc of a circle) of small radius $\delta\mu/\gamma=\delta R_{12}\widehat{R}/2$. The equation for $\overline{\varphi}$ (3.11) can be integrated exactly. For unidirectional oscillations along the y-axis, when $\widetilde{\boldsymbol{g}}=(0,\widetilde{g}_2)$ (2.4), the second equation (3.11) takes the form $\overline{\varphi}_{s'}=G_2\cos\overline{\varphi}$. It can be integrated, having an initial value $\varphi(0)=\Phi$, as

$$\sin\overline{\varphi}=\frac{(1+\sin\Phi)\mathrm{e}^{2G_2s'}-(1-\sin\Phi)}{(1+\sin\Phi)\mathrm{e}^{2G_2s'}+(1-\sin\Phi)} \tag{4.2}$$

which shows that for $s\to\infty$ we have $\overline{\varphi}\to\pm\pi/2$; it means that the axis of symmetry of a dumbbell is turning monotonically towards the direction of oscillations. Equation (4.1) describes the simultaneous change of $\overline{\boldsymbol{X}}$ along the arc of a small circle. It is clear that in the general case of unidirectional oscillations along any direction (different from y), the result is the same: the axis of the dumbbell asymptotically approaches the direction parallel to the oscillations, and $\overline{\boldsymbol{X}}$ changes along the arc of a small circle. Therefore, we can conclude that no unidirectional oscillations of the fluid result in the self-propulsion of the BD-robot.

4.2. *Rectilinear self-propulsion without rotation*

For $G\neq 0$ we first consider motion without rotation $\overline{\varphi}_{s'}=0$. In this case the angular part of (3.11) gives $-G_1\sin\overline{\varphi}+G_2\cos\overline{\varphi}=\gamma G$, which immediately leads to

$$\overline{\varphi}=\Phi=-\arctan(G_1/G_2)+\arccos\left(\gamma G\bigg/\sqrt{{G_1}^2+{G_2}^2}\right)=\text{const} \tag{4.3}$$

when

$$|\gamma G|\leqslant\sqrt{{G_1}^2+{G_2}^2}. \tag{4.4}$$

Physically, the restriction (4.4) means that the BD-robot can move without rotation if the oscillations $\widetilde{l}$ are 'strong enough'. In this case the first equation in (3.11) gives $\overline{\boldsymbol{X}}_{s'}=-\delta\mu G\overline{\boldsymbol{n}}=\text{const}$, which shows that the BD-robot moves with constant speed $|\delta\mu G|$ in the fixed direction $\overline{\boldsymbol{n}}$, which is given by the angle $\Phi\pm\pi/2$ (4.3), where the sign is determined by the correlation G. Seeking more general results, one can show that the system (3.11) can be integrated analytically in the general case $\overline{\varphi}_{s'}\neq 0$, with the conclusion that if the parameters satisfy (4.4) then a trajectory with any initial data $\overline{\varphi}(0)$ asymptotically (when $s\to\infty$) approaches the same straight paths (4.3) as described above. Exact integration outside of the range of parameters (4.4) is also accessible analytically; it produces motion with rotation $|\overline{\varphi}_s|>\text{const}$, which we do not consider in this paper.

Let us consider a particular example

$$\widetilde{g}_1=-\widehat{g}_1\sin\tau,\quad \widetilde{g}_2=\widehat{g}_2\cos\tau,\quad \widetilde{l}=\widehat{l}\sin\tau; \tag{4.5}$$

$$G_1 = -\widehat{g}_1\widehat{l}/2, \quad G_2 = 0, \quad G = \widehat{g}_1\widehat{g}_2/2, \quad \Phi = \frac{\pi}{2} + \arccos(\gamma\widehat{g}_2/\widehat{l}), \tag{4.6}$$

with constants $\widehat{g}_1 > 0$, $\widehat{g}_2 > 0$, and $\widehat{l} > 0$. For $|\gamma\widehat{g}_2/\widehat{l}| \leqslant 1$ the BD-robot propels itself with constant speed $|\overline{\boldsymbol{X}}_{s'}| = \delta\mu\widehat{g}_1\widehat{g}_2/2$ along a straight path $\varphi = \Phi \pm \pi/2$. It is remarkable that this self-propulsion speed does not depend on the amplitude $\widehat{l}$. However, one should keep in mind that such solutions are available only for 'strong' oscillations, when $|\widehat{l}| \geqslant |\gamma\widehat{g}_2|$; for 'very strong' oscillations, when $|\widehat{l}/\widehat{g}_2| \to \infty$, we have $\Phi \to \pi$.

5. Elastic BD-robot

The above results correspond to the prescribed periodic function $\widetilde{l}(\tau)$ (2.3), which can be chosen arbitrarily and represents a given time-dependent constraint. However, in practice, the oscillations $\widetilde{l}$ produced by the forces exerted by an oscillating fluid on the beads are more interesting. A simple way to consider such oscillations is to replace the rod with a spring of stiffness $k = \text{const}$. In this case the dimensionless Lagrangian function (2.8) and potential energy (2.5) become

$$\mathscr{L}(\boldsymbol{q}, \boldsymbol{q}_t) = \mathscr{K} - \Pi, \quad \Pi = \varepsilon\alpha\widetilde{\boldsymbol{g}}\cdot\boldsymbol{l} + k\,(l-1)^2/2. \tag{5.1}$$

One can check that for this Lagrangian function the equations for total force and torque (2.10), (2.11) remain the same, while the equation for the reaction of constraint (2.12) must be replaced with

$$\varepsilon\alpha\boldsymbol{g}\cdot\boldsymbol{\sigma} + kl = -R_{12}\boldsymbol{l}_t\cdot\boldsymbol{\sigma} + O(\varepsilon\delta). \tag{5.2}$$

The relation between the problem for an elastic *BD*-robot and the previous problem for a *BD*-robot with arbitrary oscillation of a rod is evident: in the latter all possible solutions are considered, while the former corresponds to a special subclass of $\widetilde{l}(\tau)$ only, that appears as the result of spring oscillations. Hence the ability for self-propulsion can only worsen after the introduction of a spring. The $O(\varepsilon)$ equation (5.2) produces a linear equation for $\widetilde{l}$

$$\widetilde{l}_\tau + K\widetilde{l} = -\gamma\widetilde{\boldsymbol{g}}\cdot\overline{\boldsymbol{\sigma}}, \quad K \equiv 2k/R_{12}. \tag{5.3}$$

It gives us $\widetilde{l}(\tau)$ which must be substituted into G_1 and G_2 in (3.11) instead of an arbitrarily chosen function $\widetilde{l}$. The rest of the problem remains unchanged. The general solution of (5.3) can be obtained analytically in an integral form, or in the form of a Fourier series. Both forms are rather cumbersome and are not considered in this paper. Instead, we present an example for a gravity field $\widetilde{g}_1 = -\widehat{g}_1\sin\tau$, $\widetilde{g}_2 = \widehat{g}_2\cos\tau$ that coincides with (4.5). The related solution of (5.3) is

$$\left.\begin{aligned} \widetilde{l} &= \frac{1}{1+K^2}[(P_2 + KP_1)\sin\tau + (KP_2 - P_1)\cos\tau]; \\ P_1 &\equiv \gamma\widehat{g}_1\cos\overline{\varphi}, \quad P_2 \equiv -\gamma\widehat{g}_2\sin\overline{\varphi}, \end{aligned}\right\} \tag{5.4}$$

where an exponentially decreasing complementary solution has been dropped. It leads to an explicit formula for U (3.13)

$$U = \frac{\gamma}{2(1+K^2)}\left[\frac{1}{2}K(\widehat{g}_1^2 - \widehat{g}_2^2)\sin(2\overline{\varphi}) - \widehat{g}_1\widehat{g}_2\right] \tag{5.5}$$

which determines the system of equations (3.11). In this case we obtain the following equations for the motion without rotation: ($\overline{\varphi}_s \equiv 0$)

$$\overline{\boldsymbol{X}}_{s'} = -\frac{1}{2}\mu \widehat{g}_1 \widehat{g}_2 \overline{\boldsymbol{n}}, \quad \overline{\varphi} = \varPhi = \frac{1}{2}\arcsin \kappa = \text{const}, \quad \kappa \equiv \frac{2\widehat{g}_1\widehat{g}_2}{\widehat{g}_1^2 - \widehat{g}_2^2}\left(\frac{2}{K} + K\right). \tag{5.6}$$

One can see that any direction of rectilinear self-propulsion can be arranged by an appropriate choice of $\widehat{g}_1$ and $\widehat{g}_2$. It is also interesting that the speed of self-propulsion, $\mu\widehat{g}_1\widehat{g}_2/2$, does not depend on the spring stiffness k; however, the required condition $|\kappa| \leqslant 1$ shows that neither a small nor large stiffness leads to rectilinear motion. Another interesting conclusion is that for rectilinear motion to exist, the values of vibrational amplitudes a and b cannot be chosen close to each other. It means that imposed vibrations (2.4) must be anisotropic (the circular vibrations with $\widehat{g}_1 = \widehat{g}_2$ and close to them are excluded). Again, for $\overline{\varphi}_{s'} \neq 0$, $|\kappa| \leqslant 1$ the system of equations (3.11), (5.5) can be integrated analytically. The integration shows that any trajectory asymptotically (for $s \to \infty$) approaches (5.6). In the case $\kappa > 1$ the full system (3.11), (5.5) also allows explicit analytical integration; it leads to motion with rotation $|\overline{\varphi}_s| >$ const, which we do not consider in this paper.

6. Discussion

(i) Our choice of slow time $s = \varepsilon^2 t$ (3.3) agrees with classical studies of self-propulsion for low Reynolds numbers, see Taylor (1951), Blake (1971) and Childress (1981), as well as the geometric studies of Shapere & Wilczek (1989).

(ii) It is remarkable that the magnitude $O(\varepsilon^2)$ of the averaged angular velocity is $1/\delta$ times higher than the magnitude of translational velocity (3.14). It means that a BD-robot rotates much faster than propagates. This result shows that self-rotation exists without taking into account hydrodynamic interactions between the spheres; it is caused only by the standard Stokes drag force (in an infinite fluid) and the reactions of constraints. Such a high angular velocity was first obtained in Dreyfus, Baudry & Stone (2005) for a different micro-robot; the authors have suggested that this rotation is similar to that in the 'falling cat problem'.

(iii) We have constructed an asymptotic procedure with two small parameters: $\varepsilon \to 0$ and $\delta \to 0$. Such a setting usually requires the consideration of different asymptotic paths on the plane (ε, δ) when, say, $\delta = \delta(\varepsilon)$. In our case we can avoid this additional analysis, since δ does not appear separately, but only in combinations like $\varepsilon^2\delta$.

(iv) In this paper we consider only plane motion of a three-dimensional dumbbell. This class of motion corresponds to two-dimensional oscillations/gravity (2.4). At the same time, for experimental realization, it could be necessary to solve a full three-dimensional problem.

(v) It is well known that an oscillating dumbbell is able to self-swim when an oscillating external torque, exerted on the dumbbell, is present; the related discussion can be found in Felderhof (2006, 2007) and Friedman (2007). The self-swimming of a magnetically driven oscillating dumbbell has been studied by Gilbert *et al.* (2010).

(vi) The form of the two-time method (TTM) used is more elaborate than its traditional form, see e.g. Kevorkian & Cole (1991). In particular, we systematically use the explicit separation between the oscillatory and mean parts of motion, which has been introduced in § 3.1. This upgraded form of TTM works well in the complex situation considered, with different leading orders of angular and translational velocities. It also allows justification of the chosen slow-time scale, which can be obtained from distinguish limit arguments, similar to Vladimirov (2012*a*).

Studies of different micro-robots by the same method can be found in Vladimirov (2012*b*,*c*, 2013). In Vladimirov (2012*a*,*d*) the same form of TTM resulted in a new asymptotic model and a new equation for the averaged flows generated by acoustic waves and for MHD-flows. A form of TTM has been used for the description of micro-rotors by Leoni & Liverpoole (2010); however these authors have not exposed their asymptotic procedure explicitly.

(vii) A mathematical justification of the results presented can be performed by the estimation of an error in the original equation, as in Vladimirov (2010, 2011). It is also possible to derive higher approximations of $\overline{V}$ and Ω, as has been done by Vladimirov (2010, 2011) for different cases. These approximations can be useful to study the cases when self-propulsion in the main order vanishes.

(viii) For the first experimental studies of self-propulsion of the BD-robot one could consider: rigid-body oscillations of a fluid enclosed within a vibrating container; viscous flows, caused by oscillatory boundary conditions; or oscillations of a fluid due to an external acoustic wave. The experimental velocities of self-propulsion can be smaller than flow oscillations and can be even smaller than some secondary flows (like acoustic streaming). To identify the relative motion of a micro-robot one could seed the ambient fluid with passive tracers.

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