Probability in the Engineering and Informational Sciences, **26**, 2012, 393–404.
doi:10.1017/S0269964812000071

SOME UNIFIED RESULTS ON COMPARING LINEAR COMBINATIONS OF INDEPENDENT GAMMA RANDOM VARIABLES

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In this paper, a new sufficient condition for comparing linear combinations of independent gamma random variables according to star ordering is given. This unifies some of the newly proved results on this problem. Equivalent characterizations between various stochastic orders are established by utilizing the new condition. The main results in this paper generalize and unify several results in the literature including those of Amiri, Khaledi, and Samaniego [2], Zhao [18], and Kochar and Xu [9].

1. INTRODUCTION

Let X be a gamma random variable $\Gamma(a, \lambda)$ with shape parameter $a \in \mathbb{R}_+$ and scale parameter $\lambda \in \mathbb{R}_+$. The density function of X is

$$f(x) = \frac{\lambda^a}{\Gamma(a)} x^{a-1} \exp\{-\lambda x\}, \quad x \in \mathbb{R}_+.$$

In the literature, linear combinations of independent gamma random variables have attracted considerable attention due to their typical applications in statistics, engineering, insurance, and reliability. Bock et al. [4] and Diaconis and Perlman [5] first

 0269-9648/12 $25.00

studied the tail probabilities of linear combinations of independent gamma random variables. This topic has been followed up by many researchers; see, for example, Korwar [10], Khaledi and Kochar [7], Yu [17], Amiri, Khaledi, and Samaniego [2], Zhao [18], Kochar and Xu [9], and references therein.

Recently, Kochar and Xu [9] studied the problem of comparing the skewness of linear combinations of independent gamma random variables. Let X_1 and X_2 be independent and identically distributed gamma random variables $\Gamma(a, \lambda)$. They proved that for $(\beta_i, \beta_i') \in \mathbb{R}_+^2$, $i = 1, 2$, if either

$$(\beta_1, \beta_2) \overset{\mathrm{m}}{\succeq} (\beta_1', \beta_2') \tag{1.1}$$

or

$$\left(\frac{1}{\beta_1}, \frac{1}{\beta_2}\right) \overset{\mathrm{m}}{\succeq} \left(\frac{1}{\beta_1'}, \frac{1}{\beta_2'}\right), \tag{1.2}$$

then

$$\beta_1 X_1 + \beta_2 X_2 \geq_\star \beta_1' X_1 + \beta_2' X_2, \tag{1.3}$$

where $\geq_\star$ denotes the star order, which is a well-known skewness order in statistics (see Definition 2.1), and $\overset{\mathrm{m}}{\succeq}$ denotes the majorization order (see Definition 2.6). Amiri et al. [2] also independently proved the above results when $a \geq 1$. These results are closely related to the result in Yu [17], who proved that, for $\beta_i \in \mathbb{R}_+$,

$$\sum_{i=1}^{n} \beta_i X_i \geq_\star \sum_{i=1}^{n} X_i, \tag{1.4}$$

where X_i's are gamma random variables $\Gamma(a_i, \lambda)$ for $i = 1, \ldots, n$, respectively. These results reveal that if the coefficients are more dispersed, then the linear combinations are more skewed as compared by star ordering. Conditions like (1.1) and (1.2) are very useful in comparing the the dispersion of linear combinations of gamma random variables. For example, Kochar and Xu [9] showed that for $\beta_i, \beta_i' \in \mathbb{R}_+$ for $i = 1, \ldots, n$, and $a \geq 1$,

$$(\beta_1, \cdots, \beta_n) \succeq_{\mathrm{w}} (\beta_1', \cdots, \beta_n') \Longrightarrow \sum_{i=1}^{n} \beta_i X_i \geq_{\mathrm{RS}} \sum_{i=1}^{n} \beta_i' X_i, \tag{1.5}$$

where $\succeq_{\mathrm{w}}$ denotes the weak submajorization (see Definition 2.6), and $\geq_{\mathrm{RS}}$ denotes the right spread order (see Definition 2.5), which is a dispersion order having wide applications in economics, insurance, reliability and engineering (Kochar and Xu [8]).

Mi, Shi, and Zhou [13] studied linear combinations of independent gamma random variables with different integer shape parameters (i.e., Erlang random variables). They established likelihood ratio ordering between two linear combinations of Erlang random variables under some restrictions on the coefficients and shape parameters. This topic is further pursued by Zhao and Balakrishnan [19] and Zhao [18]. Zhao [18] extended the results of Eqs. (1.1)–(1.3) to the case of independent gamma random variables with different shape parameters. More precisely, let X_1 and X_2 be independent gamma random variables $\Gamma(a_1, \lambda)$ and $\Gamma(a_2, \lambda)$. Zhao [18] proved that, for $\beta_1 \le \beta_2$ and $\beta_1' \le \beta_2'$,

$$(\beta_1, \beta_2) \overset{\mathrm{m}}{\succeq} (\beta_1', \beta_2') \Longrightarrow \beta_1 X_1 + \beta_2 X_2 \ge_{\star} \beta_1' X_1 + \beta_2' X_2, \tag{1.6}$$

and if $\beta_1 \le \beta_2$, $\beta_1' \le \beta_2'$, and $a_1 \le a_2$, then

$$\left(\frac{1}{\beta_1}, \frac{1}{\beta_2}\right) \overset{\mathrm{m}}{\succeq} \left(\frac{1}{\beta_1'}, \frac{1}{\beta_2'}\right) \Longrightarrow \beta_1 X_1 + \beta_2 X_2 \ge_{\star} \beta_1' X_1 + \beta_2' X_2. \tag{1.7}$$

It is interesting to note that Yu [17] proved that

$$\sum_{i=1}^{n} \beta_i X_i \ge_{\mathrm{st}} \sum_{i=1}^{n} \beta X_i \Longleftrightarrow \prod_{i=1}^{n} \beta_i^{a_i} \ge \prod_{i=1}^{n} \beta^{a_i}, \tag{1.8}$$

where $\beta, \beta_i \in \mathbb{R}_+$, and X_i's are gamma random variables $\Gamma(a_i, \lambda)$ for $i = 1, \ldots, n$, respectively.

In this paper, we will give a simple sufficient condition for comparing the skewness of linear combinations of two independent gamma random variables which unifies the existing results on this problem. It is shown that

$$\frac{\beta_{(2)}}{\beta_{(1)}} \ge \frac{\beta_{(2)}'}{\beta_{(1)}'} \Longrightarrow \beta_{(1)} X_1 + \beta_{(2)} X_2 \ge_{\star} \beta_{(1)}' X_1 + \beta_{(2)}' X_2,$$

where $\{\beta_{(1)}, \beta_{(2)}\}$ denotes the increasing arrangement of the components of the vector $(\beta_1, \beta_2) \in \mathbb{R}_+^2$. As consequences, results (1.6) and (1.7) in Zhao [18] follow immediately from this. It is worth pointing out that the restriction ($a_1 \le a_2$) in (1.7) is unnecessary and hence can be dropped. As further consequences, equivalent characterizations for comparing linear combinations of independent gamma random variables in the sense of dispersive order and right spread order are developed.

The rest of the paper is organized as follows. In Section 2, we first recall some stochastic orders and majorization orders that are pertinent to the present discussion. In Section 3, sufficient conditions for comparing linear combinations of independent gamma random variables according to usual stochastic order and dispersive order are established. In the last section, we give some sufficient conditions for comparison according to right spread order.

2. PRELIMINARIES

In this section, we will review some notions of stochastic orders and majorization orders, which will be used in the sequel.

Assume random variables X and Y have distribution functions F and G, and survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, respectively.

Definition 2.1: *X is said to be less skewed than Y according to star ordering, denoted by $X \leq_\star Y$ if $G^{-1}F(x)/x$ is increasing in x on the support of X, where G^{-1} denotes the right continuous inverse.*

The star order is also called more IFRA (increasing failure rate in average) order in reliability theory (see Kochar and Xu [9] for details). It is known in the literature (Marshall and Olkin [[12], p. 69]) that

$$X \leq_\star Y \Longrightarrow \mathrm{cv}(X) \leq \mathrm{cv}(Y),$$

where $\mathrm{cv}(X) = \sqrt{\mathrm{Var}(X)}/\mathrm{E}(X)$ denote the coefficient of variation of X.

Dispersive ordering as defined below is a basic notion for comparing the variabilities of two random variables.

Definition 2.2: *X is said to be less dispersed than Y (denoted by $X \leq_{\mathrm{disp}} Y$) if*

$$F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha)$$

for all $0 < \alpha \leq \beta < 1$, where F^{-1} and G^{-1} denote their corresponding right continuous inverses.

The dispersive order is closely related to the hazard rate order.

Definition 2.3: *X is said to be smaller than Y according to the hazard rate order (denoted by $X \leq_{\mathrm{hr}} Y$) if $\bar{G}(x)/\bar{F}(x)$ is increasing.*

Bagai and Kochar [3] showed that if $X \leq_{\mathrm{disp}} Y$ and F or G is IFR (increasing failure rate), then $X \leq_{\mathrm{hr}} Y$.

Definition 2.4: *X is said to be smaller than Y in the usual stochastic order (denoted by $X \leq_{\mathrm{st}} Y$), if $\bar{F}(x) \leq \bar{G}(x)$ for all x.*

A weaker variability order called the right spread order in Fernández-Ponce, Kochar, and Muñoz-Perez [6], or *excess wealth* order in Shaked and Shanthikumar [15] has also been proposed to compare the variabilities of two distributions.

Definition 2.5: *X is said to be less right spread than Y (denoted by $X \leq_{\mathrm{RS}} Y$) if*

$$\int_{F^{-1}(p)}^{\infty} \bar{F}(x)\,dx \leq \int_{G^{-1}(p)}^{\infty} \bar{G}(x)\,dx, \quad \text{for all } 0 \leq p \leq 1.$$

It is known that

$$X \leq_{\mathrm{disp}} Y \Longrightarrow X \leq_{\mathrm{RS}} Y \Longrightarrow \mathrm{Var}(X) \leq \mathrm{Var}(Y).$$

https://doi.org/10.1017/S0269964812000071 Published online by Cambridge University Press

For more discussion on the these stochastic orders, please refer to Shaked and Shanthikumar (2007) and references therein.

We shall also be using the concept of majorization in our discussion. Let $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ be the increasing arrangement of the components of the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

DEFINITION 2.6: *For vectors* $\mathbf{x}, \mathbf{y}$, $\mathbf{x}$ *is said to be*

- *majorized by* $\mathbf{y}$ *(denoted by* $\mathbf{x} \overset{\mathrm{m}}{\preceq} \mathbf{y}$*) if*

$$\sum_{i=1}^{j} x_{(i)} \geq \sum_{i=1}^{j} y_{(i)}$$

for $j = 1, \ldots, n-1$ *and* $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$.

- *weakly submajorized by* $\mathbf{y}$ *(denoted by* $\mathbf{x} \preceq_{\mathrm{w}} \mathbf{y}$*) if*

$$\sum_{i=j}^{n} x_{(i)} \leq \sum_{i=j}^{n} y_{(i)}$$

for $j = 1, \ldots, n$

For extensive and comprehensive details on the theory of majorization orders and their applications, please refer to the excellent book of Marshall and Olkin [11].

3. STAR ORDER AND DISPERSIVE ORDER

The following result gives a new sufficient condition for comparing the skewness of linear combinations of two independent gamma random variables with arbitrary shape parameters.

PROPOSITION 3.1: *Let* X_1 *and* X_2 *be independent gamma random variables* $\Gamma(a_1, \lambda)$ *and* $\Gamma(a_2, \lambda)$*, respectively. Then*

$$\frac{\beta_{(2)}}{\beta_{(1)}} \geq \frac{\beta'_{(2)}}{\beta'_{(1)}} \Longrightarrow \beta_{(1)} X_1 + \beta_{(2)} X_2 \geq_{\star} \beta'_{(1)} X_1 + \beta'_{(2)} X_2,$$

where $\{\beta_{(1)}, \beta_{(2)}\}$ *denotes the increasing arrangement of the components of the vector* $(\beta_1, \beta_2) \in \mathbb{R}^2_+$.

PROOF: *Case 1*: $\beta_1 + \beta_2 = \beta'_1 + \beta'_2$. Without loss of generality, let $\lambda = 1$ and

$$\beta_1 + \beta_2 = \beta'_1 + \beta'_2 = 1.$$

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Note that the assumption in this case implies

$$(\beta_1, \beta_2) \overset{\text{m}}{\succeq} (\beta_1', \beta_2').$$

Now, let $\beta_{(2)} = \beta$ and $\beta'_{(2)} = \beta'$. Then,

$$(\beta_1, \beta_2) \overset{\text{m}}{\succeq} (\beta_1', \beta_2') \Longleftrightarrow \beta \geq \beta'.$$

Note that, the distribution function of $(1-\beta)X_1 + \beta X_2$ is

$$F_\beta(t) = \iint_A \frac{x_1^{a_1-1} e^{-x_1}}{\Gamma(a_1)} \frac{x_2^{a_2-1} e^{-x_2}}{\Gamma(a_2)} dx_2\, dx_1,$$

where $A = \{(x_1, x_2) \in \mathbb{R}_+^2 : (1-\beta)x_1 + \beta x_2 \leq t\}$. The rest of the proof can be completed by mimicking the proof of Theorem 3.6 of Kochar and Xu [9]; see also Theorem 4.3 of Zhao [18]. ■

Case 2: $\beta_1 + \beta_2 \neq \beta_1' + \beta_2'$. Without loss of generality, assume

$$\beta_1 + \beta_2 = c(\beta_1' + \beta_2'),$$

where $c \in \mathbb{R}_+$ is a scalar. Now

$$(\beta_1, \beta_2) \overset{\text{m}}{\succeq} (c\beta_1', c\beta_2').$$

Thus,

$$\beta_{(1)}X_1 + \beta_{(2)}X_2 \geq_\star c\beta'_{(1)}X_1 + c\beta'_{(2)}X_2,$$

Since the star order is scale invariant, it follows that,

$$\beta_{(1)}X_1 + \beta_{(2)}X_2 \geq_\star \beta'_{(1)}X_1 + \beta'_{(2)}X_2.$$

Hence, the required result follows. ■

Remark 1: The condition given in the Proposition 3.1 is very general. It is weaker than any of the following conditions, which are commonly used in the literature:

(a) $(\beta_1, \beta_2) \overset{\text{m}}{\succeq} (\beta_1', \beta_2')$;

(b) $(\log(\beta_1), \log(\beta_2)) \overset{\text{m}}{\succeq} (\log(\beta_1'), \log(\beta_2'))$;

(c) $(1/\beta_1, 1/\beta_2) \overset{\text{m}}{\succeq} (1/\beta_1', 1/\beta_2')$.

Remark 2: Conditions (a) and (c) have been used to prove Theorems 4.2 and 4.3 in Zhao [18] (see also Eqs. (1.6) and (1.7)). The proof of Theorem 4.2 of Zhao [18] is quite involved. However, it follows immediately from Remark 1.

The following lemma gives conditions under which star ordering implies stochastic ordering and dispersive ordering.

LEMMA 3.2: *If $X \leq_\star Y$, then the following statements are equivalent:*

(a) $X \leq_{\text{disp}} Y$;
(b) $X \leq_{\text{st}} Y$;
(c) $\lim_{x\to 0^+} F(x)/G(x) \geq 1$.

PROOF: Under star ordering, the equivalence of (a) and (b) follows from Theorem 3 in Ahmed et al. [1]. Hence, it is enough to prove that (b) and (c) are equivalent. Note that $X \leq_\star Y$ implies that $F(x) - G(x)$ crosses zero at most once, and if the crossing occurs, the sign change would be from negative to positive. Hence, $\lim_{x\to 0^+} F(x)/G(x) \geq 1$ implies there is no crossing for $F(x) - G(x)$, and $F(x) \geq G(x)$ for $x \geq 0$. ■

Remark: Yu [17] established the following equivalent characterization of stochastic order: if $l(x) = \log(f(x)/g(x))$ is continuous and concave. Then,

$$X \leq_{st} Y \Longleftrightarrow \lim_{x\to 0^+} l(x) \geq 0.$$

Lemma 3.2 provides a new characterization for stochastic order under a weak condition.

The following result gives an equivalent characterization of stochastic ordering between two linear combinations of independent gamma random variables.

LEMMA 3.3: *Let X_1 and X_2 be independent gamma random variables $\Gamma(a_1, \lambda)$ and $\Gamma(a_2, \lambda)$, respectively. If $\beta_{(2)}/\beta_{(1)} \geq \beta'_{(2)}/\beta'_{(1)}$, then the following statements are equivalent:*

(a) $\beta_{(1)}^{a_1}\beta_{(2)}^{a_2} \geq {\beta'}_{(1)}^{a_1}{\beta'}_{(2)}^{a_2}$;
(b) $\beta_{(1)}X_1 + \beta_{(2)}X_2 \geq_{st} \beta'_{(1)}X_1 + \beta'_{(2)}X_2$.

PROOF: From Sim [16], the distribution function of $\beta_{(1)}X_1 + \beta_{(2)}X_2$ can be expressed as

$$F(x) = \frac{1}{\Gamma(a_1+a_2)}\left(\frac{1}{\beta_{(1)}}\right)^{a_1}\left(\frac{1}{\beta_{(2)}}\right)^{a_2} \times \sum_{r=0}^{\infty}\frac{(a_1)_r}{(a_2)_r r!}\left(\frac{1}{\beta_{(2)}} - \frac{1}{\beta_{(1)}}\right)^r \beta_{(2)}^{r+a_1+a_2}\gamma(r+a_1+a_2, x/\beta_{(2)}),$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$, and

$$\gamma(s,z) = \int_0^z t^{s-1}e^{-t}\,dt$$

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is the lower gamma function. Note that the distribution function can be further represented as

$$F(x) = \frac{1}{\Gamma(a_1+a_2)}\left(\frac{1}{\beta_{(1)}}\right)^{a_1}\left(\frac{1}{\beta_{(2)}}\right)^{a_2} \times \sum_{r=0}^{\infty}\frac{(a_1)_r}{(a_2)_r r!}\left(\frac{1}{\beta_{(2)}}-\frac{1}{\beta_{(1)}}\right)^r \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\beta_{(2)}^k}\frac{x^{r+a_1+a_2+k}}{r+k+a_1+a_2}.$$

For $x \to 0^+$, one may rewrite it as

$$F(x) = \frac{x^{a_1+a_2}}{\Gamma(a_1+a_2)}\left(\frac{1}{\beta_{(1)}}\right)^{a_1}\left(\frac{1}{\beta_{(2)}}\right)^{a_2}\frac{1}{a_1+a_2} + o(x^{a_1+a_2}).$$

Hence, the required result follows from Lemma 3.2 and Proposition 3.1. ■

Combing Proposition 3.1 with Lemmas 3.2 and 3.3, one may immediately have the following result.

PROPOSITION 3.4: *Let X_1 and X_2 be independent gamma random variables $\Gamma(a_1, \lambda)$ and $\Gamma(a_2, \lambda)$, respectively. If $\beta_{(2)}/\beta_{(1)} \geq \beta'_{(2)}/\beta'_{(1)}$, then*

$$\beta_{(1)}^{a_1}\beta_{(2)}^{a_2} \geq \beta'^{a_1}_{(1)}\beta'^{a_2}_{(2)} \Longleftrightarrow \beta_{(1)}X_1 + \beta_{(2)}X_2 \geq_{\text{disp}} \beta'_{(1)}X_1 + \beta'_{(2)}X_2.$$

Remark: Theorem 3.2 in Zhao [18] proves that under the condition $1 \leq a_1 \leq a_2$,

$$(\log(\beta_1), \log(\beta_2)) \succeq_{\text{w}} (\log(\beta'_1), \log(\beta'_2)) \Longrightarrow \beta_{(1)}X_1 + \beta_{(2)}X_2 \geq_{\text{disp}} \beta'_{(1)}X_1 + \beta'_{(2)}X_2.$$

Our result here removes the restriction on the shape parameters.

As a direct consequence, we have the following result.

COROLLARY 3.5: *Let X_1 and X_2 be independent gamma random variables $\Gamma(a_1, \lambda)$ and $\Gamma(a_2, \lambda)$, respectively. If $a_1 \leq a_2$, then*

$$(\log(\beta_1), \log(\beta_2)) \overset{\text{m}}{\succeq} (\log(\beta'_1), \log(\beta'_2)) \Longrightarrow \beta_{(1)}X_1 + \beta_{(2)}X_2 \geq_{\text{disp}} \beta'_{(1)}X_1 + \beta'_{(2)}X_2.$$

The following result is a natural extension of Corollary 3.5, which recovers Theorem 3.3 in Zhao [18].

PROPOSITION 3.6: *Let $X_1, \ldots, X_n$ be independent gamma random variables $\Gamma(a_1, \lambda), \ldots, \Gamma(a_n, \lambda)$, respectively. If $1 \leq a_1 \leq a_2 \leq \cdots \leq a_n$, then*

$$(\log(\beta_1), \ldots, \log(\beta_n)) \succeq_{\text{w}} (\log(\beta'_1), \ldots, \log(\beta'_n)) \Longrightarrow \sum_{i=1}^{n}\beta_{(i)}X_i \geq_{\text{disp}} \sum_{i=1}^{n}\beta'_{(i)}X_i.$$

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PROOF: According to Proposition 5.A.9 in Marshall and Olkin [11], if

$$(\log(\beta_1), \dots, \log(\beta_n)) \succeq_{\mathrm{w}} (\log(\beta_1'), \dots, \log(\beta_n')),$$

then there exists a real vector $(u_1, \dots, u_n)$ such that

$$(\log(\beta_1), \dots, \log(\beta_n)) \overset{\mathrm{m}}{\succeq} (\log(u_{(1)}), \dots, \log(u_{(n)})) \geq (\log(\beta_{(1)}'), \dots, \log(\beta_{(n)}')).$$

Since, for $c \geq c' > 0$,

$$cX_i \geq_{\mathrm{disp}} c'X_i$$

and a gamma random variable with shape parameter greater or equal to 1 has a log-concave density, using Theorem 3.B.9 of Shaked and Shanthikumar (2007), it holds that

$$\sum_{i=1}^{n} u_{(i)}X_i \geq_{\mathrm{disp}} \sum_{i=1}^{n} \beta_{(i)}'X_i.$$

Hence, it is enough to prove that

$$(\log(\beta_1), \dots, \log(\beta_n)) \overset{\mathrm{m}}{\succeq} (\log(u_1), \dots, \log(u_n)) \Longrightarrow \sum_{i=1}^{n} \beta_{(i)}X_i \geq_{\mathrm{disp}} \sum_{i=1}^{n} u_iX_i.$$

By the nature of majorization order, it is enough to prove the case for $n = 2$, which follows from Corollary 3.5. ■

4. RIGHT SPREAD ORDER

In this section, we will discuss stochastic comparisons of linear combinations of gamma random variables. The first result presents an equivalent characterization of right spread order for linear combinations of two gamma random variables.

LEMMA 4.1: *Let X_1 and X_2 be independent gamma random variables $\Gamma(a_1, \lambda)$ and $\Gamma(a_2, \lambda)$, respectively. If $\beta_{(2)}/\beta_{(1)} \geq \beta_{(2)}'/\beta_{(1)}'$, then the following statements are equivalent:*

(a) $\beta_{(1)}a_1 + \beta_{(2)}a_2 \geq \beta_{(1)}'a_1 + \beta_{(2)}'a_2$;
(b) $\beta_{(1)}X_1 + \beta_{(2)}X_2 \geq_{\mathrm{RS}} \beta_{(1)}'X_1 + \beta_{(2)}'X_2$.

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Proof: It follows from Theorem 4.3 in Fernández-Ponce et al. [6] that for two nonnegative random variables X and Y, if $X \leq_\star Y$, then

$$EX \leq EY \iff X \leq_{\mathrm{RS}} Y.$$

From Proposition 3.1, it holds that

$$\beta_{(1)}X_1 + \beta_{(2)}X_2 \geq_\star \beta'_{(1)}X_1 + \beta'_{(2)}X_2.$$

Hence,

$$\beta_{(1)}X_1 + \beta_{(2)}X_2 \geq_{\mathrm{RS}} \beta'_{(1)}X_1 + \beta'_{(2)}X_2$$

is equivalent to

$$E(\beta_{(1)}X_1 + \beta_{(2)}X_2) \geq E(\beta'_{(1)}X_1 + \beta'_{(2)}X_2).$$

So, the required result follows. ■

Remark: Theorem 4.5 in Zhao [18] proved that if $1 \leq a_1 \leq a_2$, then

$$(\beta_1, \beta_2) \succeq_{\mathrm{w}} (\beta'_1, \beta'_2) \Longrightarrow \beta_{(1)}X_1 + \beta_{(2)}X_2 \geq_{\mathrm{RS}} \beta'_{(1)}X_1 + \beta'_{(2)}X_2.$$

Lemma 4.1 removes the restriction on the shape parameters.

As a direct consequence, we have the following result.

Corollary 4.2: *Let X_1 and X_2 be independent gamma random variables $\Gamma(a_1, \lambda)$ and $\Gamma(a_2, \lambda)$, respectively. Then*

$$(\beta_1, \beta_2) \overset{\mathrm{m}}{\succeq} (\beta'_1, \beta'_2) \Longrightarrow \beta_{(1)}X_1 + \beta_{(2)}X_2 \geq_{\mathrm{RS}} \beta'_{(1)}X_1 + \beta'_{(2)}X_2.$$

The following result of Zhao [18] immediately follows from Corollary 4.2, Theorem 3.C.7 of Shaked and Shanthikumar (2007), and the similar argument to Proposition 3.6.

Corollary 4.3: *Let $X_1, \ldots, X_n$ be independent gamma random variables $\Gamma(a_1, \lambda), \ldots, \Gamma(a_n, \lambda)$, respectively. If $1 \leq a_1 \leq a_2 \leq \ldots \leq a_n$, then*

$$(\beta_1, \ldots, \beta_n) \succeq_{\mathrm{w}} (\beta'_1, \ldots, \beta'_n) \Longrightarrow \sum_{i=1}^{n} \beta_{(i)}X_i \geq_{\mathrm{RS}} \sum_{i=1}^{n} \beta'_{(i)}X_i.$$

Yu [17] gave necessary and sufficient conditions for stochastically comparing linear combinations of heterogeneous and homogeneous gamma random variables. The following result gives necessary and sufficient conditions for comparing linear combinations of gamma random variables according to right spread order.

https://doi.org/10.1017/S0269964812000071 Published online by Cambridge University Press

PROPOSITION 4.4: *Let $X_1, \ldots, X_n$ be independent gamma random variables $\Gamma(a_1, \lambda), \ldots, \Gamma(a_n, \lambda)$, respectively. Then*

$$\sum_{i=1}^{n} \beta_i X_i \geq_{\mathrm{RS}} \beta \sum_{i=1}^{n} X_i \Longleftrightarrow \beta \leq \frac{\sum_{i=1}^{n} \beta_i a_i}{\sum_{i=1}^{n} a_i}.$$

PROOF: It follows from Yu [17] (see also (1.4)) that

$$\sum_{i=1}^{n} \beta_i X_i \geq_{\star} \beta \sum_{i=1}^{n} X_i.$$

Using Theorem 4.3 in Fernández-Ponce et al. [6] again, we have

$$\sum_{i=1}^{n} \beta_i X_i \geq_{\mathrm{RS}} \beta \sum_{i=1}^{n} X_i \Longleftrightarrow E\left(\sum_{i=1}^{n} \beta_i X_i\right) \leq E\left(\sum_{i=1}^{n} \beta X_i\right).$$

Hence, the required result follows. ■

Remark: Compared to Corollary 4.3, there is no restriction on the shape parameters.

Acknowledgment

The authors are grateful to the anonymous referee for some constructive comments and suggestions which led to this improved version of the paper. The second author was supported by NFIG at Illinois State University.

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