

THE UNITARY SPHERICAL SPECTRUM FOR SPLIT CLASSICAL GROUPS

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Abstract This paper gives a complete description of the spherical unitary dual of split classical real and p -adic groups. The proof makes heavy use of the affine graded Hecke algebra.

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1. Introduction

This paper gives a complete classification of the spherical unitary dual of the split groups $\mathrm{Sp}(n)$ and $\mathrm{So}(n)$ over the real and p -adic field. Most of the results were known earlier from [2], [3], [4] and [12]. As is explained in these references, in the p -adic case the classification of the spherical unitary dual is equivalent to the classification of the unitary generic (in the sense of admitting Whittaker models) Iwahori-spherical modules. The new result is the proof of necessary conditions for unitarity in the real case. Following a suggestion of Vogan, I find a set of K -types which I call *relevant* which detect the nonunitarity. They have the property that they are in 1–1 correspondence with certain irreducible Weyl group representations (called relevant) so that the intertwining operators are *the same* in the real and p -adic case. The fact that these relevant W -types detect unitarity in the p -adic case is also new. Thus the same proof applies in both cases. Since the answer is independent of the field, this establishes a form of the Lefschetz principle.

Let G be a split symplectic or orthogonal group over a local field \mathbb{F} which is either \mathbb{R} or a p -adic field. Fix a maximal compact subgroup K . In the real case, there is only one conjugacy class. In the p -adic case, let $K = G(\mathcal{R})$ where $\mathbb{F} \supset \mathcal{R} \supset \mathcal{P}$, with \mathcal{R} the ring of integers and \mathcal{P} the maximal prime ideal. Fix also a rational Borel subgroup $B = AN$. Then $G = KB$. A representation (π, V) (admissible) is called spherical if $V^K \neq (0)$.

The classification of irreducible admissible spherical modules is well known. For every irreducible spherical π , there is a character $\chi \in \hat{A}$ such that $\chi|_{A \cap K} = \mathrm{triv}$, and π is the unique spherical subquotient of $\mathrm{Ind}_B^G[\chi \otimes \mathbf{1}]$. We will call a character χ whose restriction

to $A \cap K$ is trivial, *unramified*. Write $X(\chi)$ for the induced module (principal series) and $L(\chi)$ for the irreducible spherical subquotient. Two such modules $L(\chi)$ and $L(\chi')$ are equivalent if and only if there is an element in the Weyl group W such that $w\chi = \chi'$. An $L(\chi)$ admits a non-degenerate hermitian form if and only if there is $w \in W$ such that $w\chi = -\bar{\chi}$.

The character χ is called *real* if it takes only positive real values. For real groups, χ is real if and only if $L(\chi)$ has real infinitesimal character [22, Chapter 16]. As is proved there, any unitary representation of a real reductive group with non-real infinitesimal character is unitarily induced from a unitary representation with real infinitesimal character on a proper Levi component. So for real groups it makes sense to consider only real infinitesimal character. In the p -adic case, χ is called real if the infinitesimal character is real in the sense of [11]. The results in [10] show that the problem of determining the unitary irreducible representations with Iwahori fixed vectors is equivalent to the same problem for the Iwahori–Hecke algebra. In [11], it is shown that the problem of classifying the unitary dual for the Hecke algebra reduces to determining the unitary dual with real infinitesimal character of some smaller Hecke algebra (not necessarily one for a proper Levi subgroup). So for p -adic groups as well it is sufficient to consider only real χ .

So we start by parametrizing real unramified characters of A . Since G is split, $A \cong (\mathbb{F}^\times)^n$ where n is the rank. Define

$$\mathfrak{a}^* = X^*(A) \otimes_{\mathbb{Z}} \mathbb{R}, \tag{1.1}$$

where $X^*(A)$ is the lattice of characters of the algebraic torus A . Each element $\nu \in \mathfrak{a}^*$ defines an unramified character χ_ν of A , characterized by the formula

$$\chi_\nu(\tau(f)) = |f|^{\langle \tau, \nu \rangle}, \quad f \in \mathbb{F}^\times, \tag{1.2}$$

where τ is an element of the lattice of one parameter subgroups $X_*(A)$. Since the torus is split, each element of $X_*(A)$ can be regarded as a homomorphism of \mathbb{F}^\times into A . The pairing in the exponent in (1.2) corresponds to the natural identification of \mathfrak{a}^* with $\text{Hom}[X_*(A), \mathbb{R}]$. The map $\nu \rightarrow \chi_\nu$ from \mathfrak{a}^* to real unramified characters of A is an isomorphism. We will often identify the two sets writing simply $\chi \in \mathfrak{a}^*$.

Let \check{G} be the (complex) dual group, and let \check{A} be the torus dual to A . Then $\mathfrak{a}^* \otimes_{\mathbb{R}} \mathbb{C}$ is canonically isomorphic to $\check{\mathfrak{a}}$, the Lie algebra of \check{A} . So we can regard χ as an element of $\check{\mathfrak{a}}$. We attach to each χ a nilpotent orbit $\check{\mathcal{O}}(\chi)$ as follows. By the Jacobson–Morozov theorem, there is a 1–1 correspondence between nilpotent orbits $\check{\mathcal{O}}$ and \check{G} -conjugacy classes of Lie triples $\{\check{e}, \check{h}, \check{f}\}$; the correspondence satisfies $\check{e} \in \check{\mathcal{O}}$. Choose the Lie triple such that $\check{h} \in \check{\mathfrak{a}}$. Then there are many $\check{\mathcal{O}}$ such that χ can be written as $w\chi = \frac{1}{2}\check{h} + \nu$ with $\nu \in \mathfrak{z}(\check{e}, \check{h}, \check{f})$, the centralizer in $\check{\mathfrak{g}}$ of the triple. For example this is always possible with $\check{\mathcal{O}} = (0)$. The results in [10] guarantee that for any χ there is a unique $\check{\mathcal{O}}(\chi)$ satisfying

- (1) there exists $w \in W$ such that $w\chi = \frac{1}{2}\check{h} + \nu$ with $\nu \in \mathfrak{z}(\check{e}, \check{h}, \check{f})$,
- (2) if χ satisfies property (1) for any other $\check{\mathcal{O}}'$, then $\check{\mathcal{O}}' \subset \overline{\check{\mathcal{O}}(\chi)}$.

Here is another characterization of the orbit $\check{\mathcal{O}}(\chi)$. Let

$$\check{\mathfrak{g}}_1 := \{x \in \check{\mathfrak{g}} : [\chi, x] = x\}, \quad \check{\mathfrak{g}}_0 := \{x \in \check{\mathfrak{g}} : [\chi, x] = 0\}.$$

Then \check{G}_0 , the Lie group corresponding to the Lie algebra $\check{\mathfrak{g}}_0$ has an open dense orbit in $\check{\mathfrak{g}}_1$. Its \check{G} saturation in $\check{\mathfrak{g}}$ is $\check{\mathcal{O}}(\chi)$.

The pair $(\check{\mathcal{O}}(\chi), \nu)$ has further nice properties. For example assume that $\nu = 0$ in (1) above. Then the representation $L(\chi)$ is one of the parameters that the Arthur conjectures predict to play a role in the residual spectrum. In particular, $L(\chi)$ should be unitary. In the p -adic case one can verify the unitarity directly as follows. In [10] it is shown how to calculate the Iwahori–Matsumoto dual of $L(\chi)$ in the Kazhdan–Lusztig classification of representations with Iwahori-fixed vector. It turns out that in the case $\nu = 0$, it is a tempered module, and therefore unitary. Since the results in [10] show that the Iwahori–Matsumoto involution preserves unitarity, $L(\chi)$ is unitary as well. In the real case, a direct proof of the unitarity of $L(\chi)$ (still with $\nu = 0$ as in (1) above) is given in [4], and in §9 of this paper.

In the classical Lie algebras, the centralizer $\mathfrak{z}(\check{e}, \check{h}, \check{f})$ is a product of symplectic and orthogonal Lie algebras. We will often abbreviate it as $\mathfrak{z}(\check{\mathcal{O}})$. The orbit $\check{\mathcal{O}}$ is called *distinguished* if $\mathfrak{z}(\check{\mathcal{O}})$ does not contain a non-trivial torus; equivalently, the orbit does not meet any proper Levi component. Let $\check{\mathfrak{m}}_{BC}$ be the centralizer of a Cartan subalgebra in $\mathfrak{z}(\check{\mathcal{O}})$. This is the Levi component of a parabolic subalgebra. The subalgebra $\check{\mathfrak{m}}_{BC}$ is the Levi subalgebra attached to $\check{\mathcal{O}}$ by the Bala–Carter classification of nilpotent orbits. The intersection of $\check{\mathcal{O}}$ with $\check{\mathfrak{m}}_{BC}$ is the other datum attached to $\check{\mathcal{O}}$, a distinguished orbit in $\check{\mathfrak{m}}_{BC}$. We will usually denote it $\check{\mathfrak{m}}_{BC}(\check{\mathcal{O}})$ if we need to emphasize the dependence on the nilpotent orbit. Let $M_{BC} \subset G$ be the Levi subgroup whose Lie algebra \mathfrak{m}_{BC} has $\check{\mathfrak{m}}_{BC}$ as its dual.

The parameter χ gives rise to a spherical irreducible representation $L_{M_{BC}}(\chi)$ on M_{BC} as well as a $L(\chi)$. Then $L(\chi)$ is the unique spherical irreducible subquotient of

$$I_{M_{BC}}(\chi) := \text{Ind}_{M_{BC}}^G [L_{M_{BC}}(\chi)]. \tag{1.3}$$

To motivate why we consider $M_{BC}(\check{\mathcal{O}})$, we need to recall some facts about the Kazhdan–Lusztig classification of representations with Iwahori fixed vectors in the p -adic case. Denote by τ the Iwahori–Matsumoto involution. Then the space of Iwahori fixed vectors of $\tau(L(\chi))$ is a W -representation (see §5.2 and [10]), and contains the W -representation ‘sgn’. Irreducible representations with Iwahori-fixed vectors are parametrized by Kazhdan–Lusztig data; these are \check{G} conjugacy classes of (\check{e}, χ, ψ) where $\check{e} \in \check{\mathfrak{g}}$ is such that $[\chi, \check{e}] = \check{e}$, and ψ is an irreducible representation of the component group $A(\chi, \check{e})$. To each such parameter there is associated a standard module $X(\check{e}, \chi, \psi)$ which contains a unique irreducible submodule $L(\check{e}, \chi, \psi)$. All other factors have parameters $(\check{e}', \chi', \psi')$ such that

$$\check{\mathcal{O}}(\check{e}) \subset \overline{\check{\mathcal{O}}(\check{e}')}, \quad \check{\mathcal{O}}(\check{e}) \neq \check{\mathcal{O}}(\check{e}').$$

As explained in §§4 and 8 in [10], $X(\check{e}', \chi', \psi')$ contains sgn if and only if $\psi' = \text{triv}$. Thus if we assume $\check{\mathcal{O}}$ satisfies (1) and (2) with respect to χ , it follows that $X(\check{e}, \chi, \text{triv}) =$

$L(\check{e}, \chi, \text{triv})$. We would like it to equal $I_{M_{\text{BC}}}$ but this is not true. In general (for an M which contains M_{BC}), $L(\check{e}, \chi, \text{triv}) = \text{Ind}_M^G[X_M(\check{e}, \chi, \text{triv})]$ if and only if the component $A_M(\check{e}, \chi)$ equals the component group $A(\check{e}, \chi)$. We will enlarge $M_{\text{BC}}(\check{O})$ to an M_{KL} so that $A_{M_{\text{KL}}}(\check{e}, \chi) = A(\check{e}, \chi)$. Note that if $\check{\mathfrak{m}} \subset \check{\mathfrak{m}}'$, then $A_M(\check{e}, \chi) \subset A_{M'}(\check{e}, \chi)$. Then

$$\text{Ind}_{M_{\text{KL}}}^G[X_{M_{\text{KL}}}(\check{e}, \chi, \text{triv})] = X(\check{e}, \chi, \text{triv}) = L(\check{e}, \chi, \text{triv}), \tag{1.4}$$

and defining

$$I_{M_{\text{KL}}}(\chi) := \text{Ind}_{M_{\text{KL}}}^G[L_{M_{\text{KL}}}(\chi)], \tag{1.5}$$

the equality

$$L(\chi) = I_{M_{\text{KL}}}(\chi) \tag{1.6}$$

follows by applying τ . We remark that M_{KL} depends on χ as well as \check{e} . It will be described explicitly in § 2. A more general discussion about how canonical $\check{\mathfrak{m}}_{\text{KL}}$ is, appears in [9].

In the real case, we use the same Levi components as in the p -adic case. Then equality (1.6) does not hold for any proper Levi component. A result essential for the paper is that equality *does* hold at the level of multiplicities of the *relevant* K -types (defined in §§ 4.2–4.5).

We will use the data (\check{O}, ν) to parametrize the unitary dual. Fix an \check{O} . A representation $L(\chi)$ will be called a *complementary series attached to \check{O}* , if it is unitary, and $\check{O}(\chi) = \check{O}$. To describe it, we need to give the set of ν such that $L(\chi)$ with $\chi = \frac{1}{2}\check{h} + \nu$ is unitary. Viewed as an element of $\mathfrak{z}(\check{O})$, the element ν gives rise to a spherical parameter $((0), \nu)$ where (0) denotes the trivial nilpotent orbit. The main result in § 3.2 says that the ν giving rise to the complementary series for \check{O} coincide with the ones giving rise to the complementary series for (0) on $\mathfrak{z}(\check{O})$. This is suggestive of Langlands functoriality.

It is natural to conjecture that such a result will hold for all split groups. Recent work of Ciubotaru for F_4 , and by Ciubotaru and myself for the other exceptional cases, show that this is generally true, but there are exceptions.

I give a more detailed outline of the paper. Section 2 reviews notation from earlier papers. Section 3 gives a statement of the main results. A representation is called *spherical unipotent* if its parameter is of the form $\frac{1}{2}\check{h}$ for the neutral element of a Lie triple associated to a nilpotent orbit \check{O} . The unitarity of the spherical unipotent representations is dealt with in § 8. For the p -adic case I simply cite [12]. The real case (sketched in [3]) is redone in § 9.4. The proofs are simpler than the original ones because I take advantage of the fact that wavefront sets, asymptotic supports and associated varieties ‘coincide’ due to [28]. Section 10.1 proves an irreducibility result in the real case which is clear in the p -adic case from the work of Kazhdan–Lusztig. This is needed for determining the complementary series (Definition 3.1 in § 3.1).

Sections 4 and 5 deal with the nonunitarity. The decomposition $\chi = \frac{1}{2}\check{h} + \nu$ is introduced in § 3. It is more common to parametrize the χ by representatives in $\check{\mathfrak{a}}$ which are dominant with respect to some positive root system. We use Bourbaki’s standard realization of the positive system. It is quite messy to determine the data (\check{O}, ν) from a dominant parameter, because of the nature of the nilpotent orbits and the Weyl group. Sections 2.3 and 2.8 give a combinatorial description of (\check{O}, ν) starting from a dominant χ .

In the classical cases, the orbit \check{O} is given in terms of partitions. To such a partition we associate the Levi component

$$\mathfrak{m}_{\check{B}C} := \mathfrak{gl}(a_1) \times \cdots \times \mathfrak{gl}(a_k) \times \check{\mathfrak{g}}_0(n_0)$$

given by the Bala–Carter classification. (The $\check{\mathfrak{g}}_0$ in this formula is *not related* to the one just after conditions (1) and (2).) The intersection of \check{O} with $\mathfrak{m}_{\check{B}C}$ is an orbit of the form

$$(a_1) \times \cdots \times (a_r) \times \check{O}_0$$

where \check{O}_0 is a distinguished nilpotent orbit, and (a_i) is the principal nilpotent orbit of $\mathfrak{gl}(a_i)$. This is the distinguished orbit associated to \check{O} by Bala–Carter. Then χ gives rise to irreducible spherical modules $L_M(\chi)$, $L(\chi)$ and $I_M(\chi)$ as in (1.3) and (1.6). The module $L(\chi)$ is the irreducible spherical subquotient of $I_M(\chi)$. As already mentioned, $I_{M_{\text{KL}}}(\chi) = L(\chi)$ in the p -adic case, but not the real case. In all cases, the multiplicities of the *relevant K -types* in $L(\chi)$, $I_M(\chi)$ coincide. These are representations of the Weyl group in the p -adic case, representations of the maximal compact subgroup in the real case. Their definition is in §§ 4.2–4.7, and in § 4.8; they are a small finite set of representations which provide necessary conditions for unitarity which are also sufficient. Their role is outlined at the beginning of § 4. The relationship between the real and p -adic case is investigated in § 4. In particular the issue is addressed of how the relevant K -types allow us to deal with the p -adic case only. A more general class of K -types for split real groups (named *petite K -types*), on which the intertwining operator is equal to the p -adic operator, is defined in [6], and the proofs are more conceptual. In [27] a more general notion of *single petaled* is studied independently, and for different purposes. Sections 4.4, and 4.5 are included for completeness. The interested reader can consult [6] and [8] for results where these kinds of K -types and W -types are useful. The determination of the non-unitary parameters proceeds by induction on the rank of $\check{\mathfrak{g}}$ and by the inclusion relations of the closure of the orbit \check{O} . Section 5 completes the induction step; it shows that conditions (B) in § 3.1 is necessary. The last part of the induction step is actually done in § 3.1.

2. Description of the spherical parameters

2.1. Explicit Langlands parameters

We consider spherical irreducible representations of the split connected classical groups of rank n of type B, C, D, precisely, $G = \text{So}(2n + 1)$, $G = \text{Sp}(2n)$ and $G = \text{So}(2n)$. These groups will be denoted by $G(n)$ when there is no danger of confusion (n is the rank). Levi components will be written as

$$M = \text{GL}(k_1) \times \cdots \times \text{GL}(k_r) \times G_0(n_0), \tag{2.1}$$

where $G_0(n_0)$ is the factor of the same type as G . The corresponding complex Lie algebras are denoted $\mathfrak{g}(n)$ and $\mathfrak{m} = \mathfrak{gl}(k_1) \times \cdots \times \mathfrak{gl}(k_r) \times \mathfrak{g}_0(n_0)$.

As already explained in the introduction, we deal with real unramified characters only, and this is sufficient for determining the full spherical unitary dual. In the case of classical groups, such a character can be represented by a vector of size the rank of the group. Two such vectors parametrize the same irreducible spherical module if they are conjugate via the Weyl group which acts by permutations and sign changes for types B and C and by permutations and an even number of sign changes in type D.

For a given *unramified* χ , let $L(\chi)$ be the corresponding irreducible spherical module. We will occasionally refer to χ as the infinitesimal character.

For any nilpotent orbit $\check{O} \subset \check{\mathfrak{g}}$ we attach an unramified parameter $\chi_{\check{O}} \in \mathfrak{a}^*$ as follows. Recall from the introduction that $\mathfrak{a}^* \otimes_{\mathbb{R}} \mathbb{C}$ is canonically isomorphic to $\check{\mathfrak{a}}$. Let $\{\check{e}, \check{h}, \check{f}\}$ be representatives for the Lie triple associated to a nilpotent orbit \check{O} . Then $\chi_{\check{O}} := \frac{1}{2}\check{h}$.

Conversely, to each χ we will attach a nilpotent orbit $\check{O} \subset \check{\mathfrak{g}}$ and Levi components M_{BC} , $M_{KL} := GL(k_1) \times \dots \times GL(k_r) \times G_0(n_0)$. In addition we will specify an even nilpotent orbit $\check{O}_0 \subset \check{\mathfrak{g}}_0(n_0)$ with unramified character $\chi_0 := \chi_{\check{O}_0}$ on $\mathfrak{g}_0(n_0)$, and unramified characters χ_i on the $GL(k_i)$. These data have the property that $L(\chi)$ is the spherical subquotient of

$$\text{Ind}_{M_{KL}}^G \left[\bigotimes_i L(\chi_i) \otimes L(\chi_0) \right]. \tag{2.2}$$

2.2. We introduce the following notation (a variant of the one used by Zelevinski [37]).

Definition 2.1. A *string* is a sequence

$$(a, a + 1, \dots, b - 1, b)$$

of numbers increasing by 1 from a to b . A set of strings is called *nested* if for any two strings either the coordinates *do not* differ by integers, or if they do, then their coordinates, say (a_1, \dots, b_1) and (a_2, \dots, b_2) , satisfy

$$a_1 \leq a_2 \leq b_2 \leq b_1 \quad \text{or} \quad a_2 \leq a_1 \leq b_1 \leq b_2 \tag{2.3}$$

or

$$b_1 + 1 < a_2 \quad \text{or} \quad b_2 + 1 < a_1. \tag{2.4}$$

A set of strings is called *strongly nested* if the coordinates of any two strings either *do not* differ by integers or else satisfy (2.3). □

Each string represents a one-dimensional spherical representation of a $GL(n_i)$ with $n_i = b_i - a_i + 1$. The match up is

$$(a, \dots, b) \longleftrightarrow |\det|^{(a+b)/2}, \quad \text{of } GL(b - a + 1), \tag{2.5}$$

where \det is the determinant character of $GL(n)$. In the case of $G = GL(n)$, we record the following result. For the p -adic case, it originates in the work of Zelevinski and Bernstein–Zelevinski (see [37] and references therein). To each set of strings $(a_1, \dots, b_1; \dots; a_k, \dots, b_k)$ we can attach a Levi component

$$M_{BC} := \prod_{1 \leq i \leq k} GL(n_i),$$

and an induced module

$$I(\chi) := \text{Ind}_{M_{\text{BC}}}^{\text{GL}(n)} \left[\bigotimes L(\chi_i) \right], \tag{2.6}$$

where χ_i are obtained from the strings as in (2.5).

In general, if the set of strings is not nested, then the corresponding induced module is not irreducible. The coordinates of χ in $\mathfrak{a}^* \simeq \mathbb{R}^n$ determine a set of nested strings as follows. Extract the longest string starting with the smallest coordinate in χ . Continue to extract sequences from the remainder until there are no elements left. This set of strings is, up to the order of the strings, the unique set of nested strings one can form out of the entries of χ .

Theorem 2.2. *Suppose \mathbb{F} is p -adic. Let $(a_1, \dots, b_1; \dots; a_r, \dots, b_r)$ be a set of nested strings, and $M := \text{GL}(b_1 - a_1 + 1) \times \dots \times \text{GL}(b_r - a_r + 1)$. Then*

$$L(\chi) = \text{Ind}_M^{\text{GL}(n)} [|\det|^{(a_1+b_1)/2} \cdot \dots \cdot |\det|^{(a_r+b_r)/2}].$$

In the language of § 2.1, $M_{\text{BC}} = M_{\text{KL}} = M$, where M is the one defined in the theorem. The nilpotent orbit $\check{\mathcal{O}}$ corresponds to the partition of n with entries $(b_i - a_i + 1)$; it is the orbit $\check{\mathcal{O}}(\chi)$ satisfying (1) and (2) in the introduction, with respect to $\chi = (a_1, \dots, b_1; \dots; a_r, \dots, b_r)$.

For the real case (still $\text{GL}(n)$), the induced module in Theorem 2.2 fails to be irreducible. However

$$[\mu : L(\chi)] = [\mu : \text{Ind}_M^{\text{GL}(n)} [|\det|^{(a_1+b_1)/2} \cdot \dots \cdot |\det|^{(a_r+b_r)/2}]] \tag{2.7}$$

whenever μ is a *relevant* K -type.

This is essential for many of the arguments. Relevant K -types are defined in §§ 4.2–4.7.

We will generalize this procedure to the other classical groups. As before, the induced modules that we construct fail to be irreducible in the real case. Instead we will establish that

$$\dim \text{Hom}_K[\mu : L(\chi)] = \dim \text{Hom}_K[\mu : I_{M_{\text{KL}}}(\chi)] \tag{2.8}$$

for *relevant* K -types, where $I_{M_{\text{KL}}}(\chi)$ is defined by (1.5), and make essential use of this fact.

2.3. Nilpotent orbits

In this section we attach a set of parameters to each nilpotent orbit $\check{\mathcal{O}} \subset \check{\mathfrak{g}}$. Let $\{\check{e}, \check{h}, \check{f}\}$ be a Lie triple so that $\check{e} \in \check{\mathcal{O}}$, and let $\mathfrak{z}(\check{\mathcal{O}})$ be its centralizer. In order for χ to be a parameter attached to $\check{\mathcal{O}}$ we require that

$$\chi = \frac{1}{2}\check{h} + \nu, \quad \nu \in \mathfrak{z}(\check{\mathcal{O}}), \text{ semisimple}, \tag{2.9}$$

but also that if

$$\chi = \frac{1}{2}\check{h}' + \nu', \quad \nu' \in \mathfrak{z}(\check{\mathcal{O}}'), \text{ semisimple} \tag{2.10}$$

for another nilpotent orbit $\check{O}' \subset \check{\mathfrak{g}}$, then $\check{O}' \subset \overline{\check{O}}$. In [10], it is shown that the orbit of χ , uniquely determines \check{O} and the conjugacy class of $\nu \in \mathfrak{z}(\check{O})$. We describe the pairs (\check{O}, ν) explicitly in the classical cases.

Nilpotent orbits are parametrized by partitions

$$(\underbrace{1, \dots, 1}_{r_1}, \underbrace{2, \dots, 2}_{r_2}, \dots, \underbrace{j, \dots, j}_{r_j}, \dots), \tag{2.11}$$

satisfying the following constraints.

$\check{\mathfrak{g}} = A_{n-1}$: $\mathfrak{gl}(n)$, partitions of n .

$\check{\mathfrak{g}} = B_n$: $\mathfrak{so}(2n+1)$, partitions of $2n+1$ such that every even part occurs an even number of times.

$\check{\mathfrak{g}} = C_n$: $\mathfrak{sp}(2n)$, partitions of $2n$ such that every odd part occurs an even number of times.

$\check{\mathfrak{g}} = D_n$: $\mathfrak{so}(2n)$, partitions of $2n$ such that every even part occurs an even number of times. In the case when every part of the partition is even, there are two conjugacy classes of nilpotent orbits with the same Jordan blocks, labelled (I) and (II). The two orbits are conjugate under the action of $O(2n)$.

The Bala–Carter classification is particularly well suited for describing the parameter spaces attached to the $\check{O} \subset \check{\mathfrak{g}}$. An orbit is called *distinguished* if it does not meet any proper Levi component. In type A, the only distinguished orbit is the principal nilpotent orbit, where the partition has only one part. In the other cases, the distinguished orbits are the ones where each part of the partition occurs at most once. In particular, these are *even nilpotent orbits*, i.e. $\text{ad } \check{h}$ has even eigenvalues only on $\check{\mathfrak{g}}$. Let $\check{O} \subset \check{\mathfrak{g}}$ be an arbitrary nilpotent orbit. We need to put it into as small as possible Levi component $\check{\mathfrak{m}}$. In type A, if the partition is (a_1, \dots, a_k) , the Levi component is $\check{\mathfrak{m}}_{\text{BC}} = \mathfrak{gl}(a_1) \times \dots \times \mathfrak{gl}(a_k)$. In the other classical types, the orbit \check{O} meets a proper Levi component if and only if one of the $r_j > 1$. So separate as many pairs (a, a) from the partition as possible, and rewrite it as

$$((a_1, a_1), \dots, (a_k, a_k); d_1, \dots, d_l), \tag{2.12}$$

with $d_i < d_{i+1}$. The Levi component $\check{\mathfrak{m}}_{\text{BC}}$ attached to this nilpotent by Bala–Carter is

$$\check{\mathfrak{m}}_{\text{BC}} = \mathfrak{gl}(a_1) \times \dots \times \mathfrak{gl}(a_k) \times \check{\mathfrak{g}}_0(n_0), \quad n_0 := n - \sum a_i. \tag{2.13}$$

The distinguished nilpotent orbit is the one with partition (d_i) on $\check{\mathfrak{g}}(n_0)$, principal nilpotent on each $\mathfrak{gl}(a_j)$. The χ of the form $\frac{1}{2}\check{h} + \nu$ are the ones with ν an element of the centre of $\check{\mathfrak{m}}_{\text{BC}}$. The explicit form is

$$(\dots; -\frac{1}{2}(a_i - 1) + \nu_i, \dots, \frac{1}{2}(a_i - 1) + \nu_i, \dots; \frac{1}{2}\check{h}_0), \tag{2.14}$$

where \check{h}_0 is the middle element of a triple corresponding to (d_i) . We will write out (d_i) and $\frac{1}{2}\check{h}_0$ in §§ 2.4–2.7.

We will consider more general cases where we write the partition of \check{O} in the form (2.12) so that the d_i are not necessarily distinct, but (d_i) forms an even nilpotent orbit in $\check{\mathfrak{g}}_0(n_0)$. This will be the situation for $\check{\mathfrak{m}}_{\text{KL}}$.

The parameter χ determines an irreducible spherical module $L(\chi)$ for G as well as an $L_M(\chi)$ for $M = M_{\text{BC}}$ or M_{KL} of the form

$$L_1(\chi_1) \otimes \cdots \otimes L_k(\chi_k) \otimes L_0(\chi_0), \tag{2.15}$$

where the $L_i(\chi_i), i = 1, \dots, k$, are one dimensional. We will consider the relation between the induced module

$$I_M(\chi) := \text{Ind}_M^G[L_M(\chi)], \tag{2.16}$$

and $L(\chi)$.

2.4. G of type A

We write the $\frac{1}{2}\check{h}$ for a nilpotent \check{O} corresponding to (a_1, \dots, a_k) with $a_i \leq a_{i+1}$ as

$$(\dots; -\frac{1}{2}(a_i - 1), \dots, \frac{1}{2}(a_i - 1); \dots).$$

The parameters of the form $\chi = \frac{1}{2}\check{h} + \nu$ are then

$$(\dots; -\frac{1}{2}(a_i - 1) + \nu_i, \dots, \frac{1}{2}(a_i - 1) + \nu_i; \dots). \tag{2.17}$$

Conversely, given a parameter as a concatenation of strings

$$\chi = (\dots; A_i, \dots, B_i; \dots), \tag{2.18}$$

it is of the form $\frac{1}{2}\check{h} + \nu$, where \check{h} is the neutral element for the nilpotent orbit with partition $(A_i + B_i + 1)$ (the parts need not be in any particular order) and $\nu_i = \frac{1}{2}(A_i - B_i)$. We recall the following well-known result about closures of nilpotent orbits.

Lemma 2.3. Assume \check{O} and \check{O}' correspond to the (increasing) partitions (a_1, \dots, a_k) and (b_1, \dots, b_k) respectively, where some of the a_i or b_j may be zero in order to have the same number k . The following are equivalent:

- (1) $\check{O}' \subset \overline{\check{O}}$;
- (2) $\sum_{i \geq s} a_i \geq \sum_{i \geq s} b_i$ for all $k \geq s \geq 1$.

Proposition 2.4. A parameter χ as in (2.17) is attached to \check{O} in the sense of satisfying (2.9) and (2.10) if and only if it is nested.

Proof. Assume the strings are not nested. There must be two strings

$$(A, \dots, B), \quad (C, \dots, D) \tag{2.19}$$

such that $A - C \in \mathbb{Z}$, and $A < C \leq B < D$, or $C = B + 1$. Then by conjugating χ by the Weyl group to a χ' , we can rearrange the coordinates of the two strings in (2.19) so that the strings

$$(A, \dots, D), \quad (C, \dots, B) \quad \text{or} \quad (A, \dots, B, B + 1 = C, \dots, D) \tag{2.20}$$

appear. Then by the lemma, $\chi' = \frac{1}{2}\check{h}' + \nu'$ for a strictly larger nilpotent \check{O}' .

Conversely, assume $\chi = \frac{1}{2}\check{h} + \nu$, so it is written as strings, and they are nested. The nilpotent orbit for which the neutral element is $\frac{1}{2}\check{h}$ has partition given by the lengths of the strings, say (a_1, \dots, a_k) in increasing order. If χ is nested, then a_k is the length of the longest string of entries we can extract from the coordinates of χ , a_{k-1} the longest string we can extract from the remaining coordinates and so on. Then (2) of Lemma 2.3 precludes the possibility that some conjugate χ' equals $\frac{1}{2}\check{h}' + \nu'$ for a strictly larger nilpotent orbit. \square

In type A, $\check{m}_{KL} = \check{m}_{BC}$.

2.5. G of type B

Rearrange the parts of the partition of $\check{O} \subset \text{sp}(2n, \mathbb{C})$, in the form (2.12),

$$((a_1, a_1), \dots, (a_k, a_k); 2x_0, \dots, 2x_{2m}). \tag{2.21}$$

The d_i have been relabelled as $2x_i$ and a $2x_0 = 0$ is added if necessary, to insure that there is an odd number. The x_i are integers, because all the odd parts of the partition of \check{O} occur an even number of times, and were therefore extracted as (a_i, a_i) . The χ of the form $\frac{1}{2}\check{h} + \nu$ are

$$\left(\dots; -\frac{1}{2}(a_i - 1) + \nu_i, \dots, \frac{1}{2}(a_i - 1) + \nu_i; \dots; \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{n_{1/2}}, \dots, \underbrace{x_{2m} - \frac{1}{2}, \dots, x_{2m} - \frac{1}{2}}_{n_{x_{2m}-1/2}}\right), \tag{2.22}$$

where

$$n_{l-1/2} = \#\{x_i \geq l\}. \tag{2.23}$$

Lemma 2.3 holds for this type verbatim. So the following proposition holds.

Proposition 2.5. *A parameter $\chi = \frac{1}{2}\check{h} + \nu$ cannot be conjugated to one of the form $\frac{1}{2}\check{h}' + \nu'$ for any larger nilpotent \check{O}' if and only if*

- (1) *the set of strings satisfying $\frac{1}{2}(a_i - 1) + \nu_i - \frac{1}{2}(a_j - 1) - \nu_j \in \mathbb{Z}$ are nested;*
- (2) *the strings satisfying $\frac{1}{2}(a_i - 1) + \nu_i \in \frac{1}{2} + \mathbb{Z}$ satisfy the additional condition that either $x_{2m} + \frac{1}{2} < -\frac{1}{2}(a_i - 1) + \nu_i$ or there is j such that*

$$x_j + \frac{1}{2} < -\frac{1}{2}(a_i - 1) + \nu_i \leq \frac{1}{2}(a_i - 1) + \nu_i \leq x_{j+1} - \frac{1}{2}. \tag{2.24}$$

The Levi component \check{m}_{KL} is obtained from \check{m}_{BC} as follows. Consider the strings for which a_i is even, and $\nu_i = 0$. If a_i is not equal to any $2x_j$, then remove one pair (a_i, a_i) , and add two $2x_j = a_i$ to the last part of (2.21). For example, if the nilpotent orbit \check{O} is

$$(2, 2, 2, 3, 3, 4, 4), \tag{2.25}$$

then the parameters of the form $\frac{1}{2}\check{h} + \nu$ are

$$\left(-\frac{1}{2} + \nu_1, \frac{1}{2} + \nu_1; -1 + \nu_2, \nu_2, 1 + \nu_2; -\frac{3}{2} + \nu_3, -\frac{1}{2} + \nu_3, \frac{1}{2} + \nu_3, \frac{3}{2} + \nu_3; \frac{1}{2}\right). \tag{2.26}$$

The Levi component is $\check{\mathfrak{m}}_{\text{BC}} = \mathfrak{gl}(2) \times \mathfrak{gl}(3) \times \mathfrak{gl}(4) \times \check{\mathfrak{g}}(1)$. If $\nu_3 \neq 0$, then $\check{\mathfrak{m}}_{\text{BC}} = \check{\mathfrak{m}}_{\text{KL}}$. But if $\nu_3 = 0$, then $\check{\mathfrak{m}}_{\text{KL}} = \mathfrak{gl}(2) \times \mathfrak{gl}(3) \times \check{\mathfrak{g}}(5)$. The parameter is rewritten

$$\left. \begin{aligned} \check{O} &\longleftrightarrow ((2, 2)(3, 3); 2, 4, 4), \\ \chi &\longleftrightarrow \left(-\frac{1}{2} + \nu_1, \frac{1}{2} + \nu_1; -1 + \nu_2, \nu_2, 1 + \nu_2; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}\right). \end{aligned} \right\} \tag{2.27}$$

The explanation is as follows. For a partition (2.11),

$$\mathfrak{z}(\check{O}) = \mathfrak{sp}(r_1) \times \mathfrak{so}(r_2) \times \mathfrak{sp}(r_3) \times \cdots \tag{2.28}$$

and the centralizer in \check{G} is a product of $\text{Sp}(r_{2j+1})$ and $\text{O}(r_{2j})$, i.e. Sp for the odd parts, O for the even parts. Thus the component group $A(\check{h}, \check{e})$, which by [16] also equals $A(\check{e})$, is a product of \mathbb{Z}_2 , one for each $r_{2j} \neq 0$. Then $A(\chi, \check{e}) = A(\nu, \check{h}, \check{e})$. In general, $A_{M_{\text{BC}}}(\chi, \check{e}) = A_{M_{\text{BC}}}(\check{e})$ embeds canonically into $A(\chi, \check{e})$, but the two are not necessarily equal. In this case they are unless one of the $\nu_i = 0$ for an even a_i with the additional property that there is no $2x_j = a_i$.

We can rewrite each of the remaining strings

$$\left(-\frac{1}{2}(a_i - 1) + \nu_i, \dots, \frac{1}{2}(a_i - 1) + \nu_i\right) \tag{2.29}$$

as

$$\chi_i := (f_i + \tau_i, f_i + 1 + \tau_i, \dots, F_i + \tau_i), \tag{2.30}$$

satisfying

$$f_i \in \mathbb{Z} + \frac{1}{2}, \quad 0 \leq \tau_i \leq \frac{1}{2}, \quad F_i = f_i + a_i - 1, \quad |f_i + \tau_i| \geq |F_i + \tau_i| \quad \text{if } \tau_i = \frac{1}{2}. \tag{2.31}$$

This is done as follows. We can immediately get an expression like (2.30) with $0 \leq \tau_i < 1$, by defining f_i to be the largest element in $\mathbb{Z} + \frac{1}{2}$ less than or equal to $-\frac{1}{2}(a_i - 1) + \nu_i$. If $\tau_i \leq \frac{1}{2}$ we are done. Otherwise, use the Weyl group to change the signs of all entries of the string, and put them in increasing order. This replaces f_i by $-F_i - 1$, and τ_i by $1 - \tau_i$. The presentation of the strings subject to (2.31) is unique except when $\tau_i = \frac{1}{2}$. In this case the argument just given provides the presentation $(f_i + \frac{1}{2}, \dots, F_i + \frac{1}{2})$, but also provides the presentation

$$\left(-F_i - 1 + \frac{1}{2}, \dots, -f_i - 1 + \frac{1}{2}\right). \tag{2.32}$$

We choose between (2.30) and (2.32) the one whose leftmost term is larger in absolute value. That is, we require $|f_i + \tau_i| \geq |F_i + \tau_i|$ whenever $\tau_i = \frac{1}{2}$.

2.6. G of type C

Rearrange the parts of the partition of $\check{O} \subset \mathfrak{so}(2n + 1, \mathbb{C})$, in the form (2.12),

$$\left((a_1, a_1), \dots, (a_k, a_k); 2x_0 + 1, \dots, 2x_{2m} + 1\right). \tag{2.33}$$

The d_i have been relabelled as $2x_i + 1$. In this case it is automatic that there is an odd number of non-zero x_i . The x_i are integers, because all the even parts of the partition of

\check{O} occur an even number of times, and were thretherefore extracted as (a_i, a_i) . The χ of the form $\frac{1}{2}\check{h} + \nu$ are

$$(\dots; -\frac{1}{2}(a_i - 1) + \nu_i, \dots, \frac{1}{2}(a_i - 1) + \nu_i; \dots; \underbrace{0, \dots, 0}_{n_0}, \dots, \underbrace{x_{2m}, \dots, x_{2m}}_{n_{x_{2m}}}), \tag{2.34}$$

where

$$n_l = \begin{cases} m & \text{if } l = 0, \\ \#\{x_i \geq l\} & \text{if } l \neq 0. \end{cases} \tag{2.35}$$

Lemma 2.3 holds for this type verbatim. So the following proposition holds.

Proposition 2.6. *A parameter $\chi = \frac{1}{2}\check{h} + \nu$ cannot be conjugated to one of the form $\frac{1}{2}\check{h}' + \nu'$ for any larger nilpotent \check{O}' if and only if*

- (1) *the set of strings satisfying $\frac{1}{2}(a_i - 1) + \nu_i - \frac{1}{2}(a_j - 1) - \nu_j \in \mathbb{Z}$ are nested;*
- (2) *the strings satisfying $\frac{1}{2}(a_i - 1) + \nu_i \in \mathbb{Z}$ satisfy the additional condition that either $x_{2m} + 1 < -\frac{1}{2}(a_i - 1) + \nu_i$ or there is j such that*

$$x_j + 1 < -\frac{1}{2}(a_i - 1) + \nu_i \leq \frac{1}{2}(a_i - 1) + \nu_i \leq x_{j+1}. \tag{2.36}$$

The Levi component \check{m}_{KL} is obtained from \check{m}_{BC} as follows. Consider the strings for which a_i is odd and $\nu_i = 0$. If a_i is not equal to any $2x_j + 1$, then remove one pair (a_i, a_i) , and add two $2x_j + 1 = a_i$ to the last part of (2.33). For example, if the nilpotent orbit is

$$(1, 1, 1, 3, 3, 4, 4) = ((1, 1), (3, 3), (4, 4); 1), \tag{2.37}$$

then the parameters of the form $\frac{1}{2}\check{h} + \nu$ are

$$(\nu_1; -1 + \nu_2, \nu_2, 1 + \nu_2; -\frac{3}{2} + \nu_3, -\frac{1}{2} + \nu_3, \frac{1}{2} + \nu_3, \frac{3}{2} + \nu_3). \tag{2.38}$$

The Levi component is $\check{m}_{BC} = \mathfrak{gl}(1) \times \mathfrak{gl}(3) \times \mathfrak{gl}(4)$. If $\nu_2 \neq 0$, then $\check{m}_{BC} = \check{m}_{KL}$. But if $\nu_2 = 0$, then $\check{m}_{KL} = \mathfrak{gl}(1) \times \mathfrak{gl}(4) \times \check{\mathfrak{g}}(3)$. The parameter is rewritten

$$\left. \begin{aligned} \check{O} &\longleftrightarrow ((1, 1), (4, 4); 1, 3, 3), \\ \chi &\longleftrightarrow (\nu_1; -\frac{3}{2} + \nu_3, -\frac{1}{2} + \nu_3, \frac{1}{2} + \nu_3, \frac{3}{2} + \nu_3; 0, 1, 1). \end{aligned} \right\} \tag{2.39}$$

The Levi component \check{m}_{KL} is unchanged if $\nu_1 = 0$.

The explanation is as follows. For a partition (2.11),

$$\mathfrak{z}(\check{O}) = \mathfrak{so}(r_1) \times \mathfrak{sp}(r_2) \times \mathfrak{so}(r_3) \times \dots \tag{2.40}$$

and the centralizer in \check{G} is a product of $O(r_{2j+1})$ and $Sp(r_{2j})$, i.e. O for the odd parts, Sp for the even parts. Thus the component group is a product of \mathbb{Z}_2 , one for each $r_{2j+1} \neq 0$. Then $A(\chi, \check{e}) = A(\nu, \check{h}, \check{e})$, and so $A_{M_{BC}}(\chi, \check{e}) = A(\chi, \check{e})$ unless one of the $\nu_i = 0$ for an odd a_i with the additional property that there is no $2x_j + 1 = a_i$.

We can rewrite each of the remaining strings

$$(-\frac{1}{2}(a_i - 1) + \nu_i, \dots, \frac{1}{2}(a_i - 1) + \nu_i) \tag{2.41}$$

as

$$\chi_i := (f_i + \tau_i, f_i + 1 + \tau_i, \dots, F_i + \tau_i), \tag{2.42}$$

satisfying

$$f_i \in \mathbb{Z}, \quad 0 \leq \tau_i \leq \frac{1}{2}, \quad F_i = f_i + a_i - 1, \quad |f_i + \tau_i| \geq |F_i + \tau_i| \quad \text{if } \tau_i = \frac{1}{2}. \tag{2.43}$$

This is done as follows. We can immediately get an expression like (2.42) with $0 \leq \tau_i < 1$, by defining f_i to be the largest element in \mathbb{Z} less than or equal to $-\frac{1}{2}(a_i - 1) + \nu_i$. If $\tau_i \leq \frac{1}{2}$ we are done. Otherwise, use the Weyl group to change the signs of all entries of the string, and put them in increasing order. This replaces f_i by $-F_i - 1$, and τ_i by $1 - \tau_i$. The presentation of the strings subject to (2.43) is unique except when $\tau_i = \frac{1}{2}$. In this case the argument just given also provides the presentation

$$(-F_i - 1 + \frac{1}{2}, \dots, -f_i - 1 + \frac{1}{2}). \tag{2.44}$$

We choose between (2.42) and (2.44) the one whose leftmost term is larger in absolute value. That is, we require $|f_i + \tau_i| \geq |F_i + \tau_i|$ whenever $\tau_i = \frac{1}{2}$.

2.7. G of type D

Rearrange the parts of the partition of $\check{O} \subset \mathfrak{so}(2n, \mathbb{C})$, in the form (2.12),

$$((a_1, a_1), \dots, (a_k, a_k); 2x_0 + 1, \dots, 2x_{2m-1} + 1). \tag{2.45}$$

The d_i have been relabelled as $2x_i + 1$. In this case it is automatic that there is an even number of non-zero $2x_i + 1$. The x_i are integers, because all the even parts of the partition of \check{O} occur an even number of times, and were therefore extracted as (a_i, a_i) . The χ of the form $\frac{1}{2}\check{h} + \nu$ are

$$(\dots; -\frac{1}{2}(a_i - 1) + \nu_i, \dots, \frac{1}{2}(a_i - 1) + \nu_i; \dots; \underbrace{0, \dots, 0}_{n_0}, \dots, \underbrace{x_{2m-1}, \dots, x_{2m-1}}_{n_{x_{2m-1}}}), \tag{2.46}$$

where

$$n_l = \begin{cases} m & \text{if } l = 0, \\ \#\{x_i \geq l\} & \text{if } l \neq 0. \end{cases} \tag{2.47}$$

Lemma 2.3 holds for this type verbatim. So the following proposition holds.

Proposition 2.7. *A parameter $\chi = \frac{1}{2}\check{h} + \nu$ cannot be conjugated to one of the form $\frac{1}{2}\check{h}' + \nu'$ for any larger nilpotent \check{O}' if and only if*

- (1) the set of strings satisfying $\frac{1}{2}(a_i - 1) + \nu_i - \frac{1}{2}(a_j - 1) - \nu_j \in \mathbb{Z}$ are nested;
- (2) the strings satisfying $\frac{1}{2}(a_i - 1) + \nu_i \in \mathbb{Z}$ satisfy the additional condition that either $x_{2m-1} + 1 < -\frac{1}{2}(a_i - 1) + \nu_i$ or there is j such that

$$x_j + 1 < -\frac{1}{2}(a_i - 1) + \nu_i \leq \frac{1}{2}(a_i - 1) + \nu_i \leq x_{j+1}. \tag{2.48}$$

The Levi component \check{m}_{KL} is obtained from \check{m}_{BC} as follows. Consider the strings for which a_i is odd and $\nu_i = 0$. If a_i is not equal to any $2x_j + 1$, then remove one pair (a_i, a_i) , and add two $2x_j + 1 = a_i$ to the last part of (2.45). For example, if the nilpotent orbit is

$$(1, 1, 3, 3, 4, 4), \tag{2.49}$$

then the parameters of the form $\frac{1}{2}\check{h} + \nu$ are

$$(\nu_1; -1 + \nu_2, \nu_2, 1 + \nu_2; -\frac{3}{2} + \nu_3, -\frac{1}{2} + \nu_3, \frac{1}{2} + \nu_3, \frac{3}{2} + \nu_3). \tag{2.50}$$

The Levi component is $\check{m}_{BC} = \mathfrak{gl}(1) \times \mathfrak{gl}(3) \times \mathfrak{gl}(4)$. If $\nu_2 \neq 0$ and $\nu_1 \neq 0$, then $\check{m}_{BC} = \check{m}_{KL}$. If $\nu_2 = 0$ and $\nu_1 \neq 0$, then $\check{m}_{KL} = \check{\mathfrak{g}}(3) \times \mathfrak{gl}(1) \times \mathfrak{gl}(4)$. If $\nu_2 \neq 0$ and $\nu_1 = 0$, then $\check{m}_{KL} = \mathfrak{gl}(3) \times \mathfrak{gl}(4) \times \check{\mathfrak{g}}(1)$. If $\nu_1 = \nu_2 = 0$, then $\check{m}_{KL} = \mathfrak{gl}(4) \times \check{\mathfrak{g}}(4)$. The parameter is rewritten

$$\left. \begin{aligned} \check{O} &\longleftrightarrow ((1, 1), (4, 4); 3, 3), \\ \chi &\longleftrightarrow (\nu_1; -\frac{3}{2} + \nu_3, -\frac{1}{2} + \nu_3; \frac{1}{2} + \nu_3, \frac{3}{2} + \nu_3; 0, 1, 1). \end{aligned} \right\} \tag{2.51}$$

The explanation is as follows. For a partition (2.11),

$$\mathfrak{z}(\check{O}) = \mathfrak{so}(r_1) \times \mathfrak{sp}(r_2) \times \mathfrak{so}(r_3) \times \dots \tag{2.52}$$

and the centralizer in \check{G} is a product of $O(r_{2j+1})$ and $Sp(r_{2j})$, i.e. O for the odd parts, Sp for the even parts. Thus the component group is a product of \mathbb{Z}_2 , one for each $r_{2j+1} \neq 0$. Then $A(\chi, \check{e}) = A(\nu, \check{h}, \check{e})$, and so $A_{M_{BC}}(\chi, \check{e}) = A(\chi, \check{e})$ unless one of the $\nu_i = 0$ for an odd a_i with the additional property that there is no $2x_j + 1 = a_i$.

We can rewrite each of the remaining strings

$$(-\frac{1}{2}(a_i - 1) + \nu_i, \dots, \frac{1}{2}(a_i - 1) + \nu_i) \tag{2.53}$$

as

$$\chi_i := (f_i + \tau_i, f_i + 1 + \tau_i, \dots, F_i + \tau_i), \tag{2.54}$$

satisfying

$$f_i \in \mathbb{Z}, \quad 0 \leq \tau_i \leq \frac{1}{2}, \quad F_i = f_i + a_i - 1, \quad |f_i + \tau_i| \geq |F_i + \tau_i| \quad \text{if } \tau_i = \frac{1}{2}. \tag{2.55}$$

This is done as in types B and C, but see the remarks which have to do with the fact that $-\text{Id}$ is not in the Weyl group. We can immediately get an expression like (2.54) with $0 \leq \tau_i < 1$, by defining f_i to be the largest element in \mathbb{Z} less than or equal to $-\frac{1}{2}(a_i - 1) + \nu_i$. If $\tau_i \leq \frac{1}{2}$ we are done. Otherwise, use the Weyl group to change the signs of all entries of the string, and put them in increasing order. This replaces f_i by $-F_i - 1$,

and τ_i by $1 - \tau_i$. The presentation of the strings subject to (2.55) is unique except when $\tau_i = \frac{1}{2}$. In this case the argument just given also provides the presentation

$$(-F_i - 1 + \frac{1}{2}, \dots, -f_i - 1 + \frac{1}{2}). \tag{2.56}$$

We choose between (2.54) and (2.56) the one whose leftmost term is larger in absolute value. That is, we require $|f_i + \tau_i| \geq |F_i + \tau_i|$ whenever $\tau_i = \frac{1}{2}$.

Remark 2.8. (1) A (real) spherical parameter χ is hermitian if and only if there is $w \in W(D_n)$ such that $w\chi = -\chi$. This is the case if the parameter has a coordinate equal to zero, or if none of the coordinates are 0, but then n must be even.

(2) Assume the nilpotent orbit $\tilde{\mathcal{O}}$ is very even, i.e. all the parts of the partition are even (and therefore occur an even number of times). The nilpotent orbits labelled (I) and (II) are characterized by the fact that \check{m}_{BC} is of the form

$$\begin{aligned} \text{(I)} &\longleftrightarrow \mathfrak{gl}(a_1) \times \dots \times \mathfrak{gl}(a_{k-1}) \times \mathfrak{gl}(a_k), \\ \text{(II)} &\longleftrightarrow \mathfrak{gl}(a_1) \times \dots \times \mathfrak{gl}(a_{k-1}) \times \mathfrak{gl}(a_k)'. \end{aligned}$$

The last \mathfrak{gl} factors differ by which extremal root of the fork at the end of the diagram for D_n is in the Levi component. The string for k is

$$\begin{aligned} \text{(I)} &\longleftrightarrow (-\frac{1}{2}(a_k - 1) + \nu_k, \dots, \frac{1}{2}(a_k - 1) + \nu_k), \\ \text{(II)} &\longleftrightarrow (-\frac{1}{2}(a_k - 1) + \nu_k, \dots, \frac{1}{2}(a_k - 3) + \nu_k, -\frac{1}{2}(a_k - 1) - \nu_k). \end{aligned}$$

We can put the parameter in the form (2.54) and (2.55), because all strings are even length. In any case (I) and (II) are conjugate by the outer automorphism, and for unitarity it is enough to consider the case of (I).

The assignment of a nilpotent orbit (I) or (II) to a parameter is unambiguous. If a χ has a coordinate equal to 0, it might be written as $\frac{1}{2}h_I + \nu_I$ or $\frac{1}{2}h_{II} + \nu_{II}$. But then it can also be written as $\frac{1}{2}h' + \nu'$ for a larger nilpotent orbit. For example, in type D_2 , the two cases are $(2, 2)_I$ and $(2, 2)_{II}$, and we can write

$$\begin{aligned} \text{(I)} &\longleftrightarrow (\frac{1}{2}, -\frac{1}{2}) + (\nu, \nu), \\ \text{(II)} &\longleftrightarrow (\frac{1}{2}, \frac{1}{2}) + (\nu, -\nu). \end{aligned}$$

The two forms are not conjugate unless the parameter contains a 0. But then it has to be $(1, 0)$ and this corresponds to $(1, 3)$, the larger principal nilpotent orbit.

(3) Because we can only change an even number of signs using the Weyl group, we might not be able to change all the signs of a string. We can always do this if the parameter contains a coordinate equal to 0, or if the length of the string is even. If there is an odd length string, and none of the coordinates of χ are 0, changing all of the signs of the string cannot be achieved unless some other coordinate changes sign as well. However, if $\chi = \frac{1}{2}\check{h} + \nu$ cannot be made to satisfy (2.54) and (2.55), then χ' , the parameter obtained from χ by applying the outer automorphism, can. Since $L(\chi)$ and $L(\chi')$ are either both

unitary of both non-unitary, it is enough to consider just the cases that can be made to satisfy (2.54) and (2.55). For example, the parameters

$$\begin{aligned} & \left(-\frac{1}{3}, \frac{2}{3}, \frac{5}{3}, -\frac{7}{4}, -\frac{3}{4}, \frac{1}{4}\right), \\ & \left(-\frac{5}{3}, -\frac{2}{3}, \frac{1}{3}, -\frac{7}{4}, -\frac{3}{4}, \frac{1}{4}\right) \end{aligned}$$

in type D_6 are of this kind. Both parameters are in a form satisfying (2.54) but only the second one satisfies (2.55). The first one cannot be conjugated by $W(D_6)$ to one satisfying (2.55).

2.8. Relation between infinitesimal characters and strings

In the previous sections we described for each nilpotent orbit \check{O} the parameters of the form $\frac{1}{2}\check{h} + \nu$ with $\nu \in \mathfrak{z}(\check{O})$ semisimple, along with condition (2.10). In this section we give algorithms to find the data (\check{O}, ν) satisfying (2.9) and (2.10), and the various Levi components from a $\chi \in \check{\mathfrak{a}}$. The formulation was suggested by Sahi. Given a $\chi \in \check{\mathfrak{a}}$, we need to specify

- (a) strings, same as sequences of coordinates with increment 1,
- (b) a partition, same as a nilpotent orbit $\check{O} \subset \check{\mathfrak{g}}$,
- (c) the centralizer of a Lie triple corresponding to $\mathfrak{z}(\check{O})$,
- (d) coordinates of the parameter ν , coming from the decomposition $\chi = \frac{1}{2}\check{h} + \nu$.

Furthermore, we give algorithms for

- (e) two Levi components $\check{\mathfrak{m}}_{BC}$ and $\check{\mathfrak{m}}_{KL}$,
- (f) another two Levi components $\check{\mathfrak{m}}_e$ and $\check{\mathfrak{m}}_o$,
- (g) one-dimensional characters χ_e and χ_o of the Levi components $\check{\mathfrak{m}}_e$ and $\check{\mathfrak{m}}_o$.

Parts (f) and (g) are described in detail in §5.3. These Levi components are used to compute multiplicities of relevant K -types in $L(\chi)$.

Algorithms for (a) and (b)

Step 0

G of type C. Double the number of 0s and add one more.

G of type D. Double the number of 0s. If there are no coordinates equal to 0 and the rank is odd, the parameter is not hermitian. If the rank is even, only an even number of sign changes are allowed in the subsequent steps.

Step 1

G of types C and D. Extract maximal strings of the form $(0, 1, \dots)$. Each contributes a part in the partition of size ‘2(length of string) – 1’ to \check{O} .

G of type B. Extract maximal strings of type $(\frac{1}{2}, \frac{3}{2}, \dots)$. Each contributes a part of size ‘2(length)’ to \check{O} .

Step 2

For all types, extract maximal strings from the remaining entries after Step 1, changing signs if necessary. Each string contributes two parts of size ‘(length of string)’ to $\check{\mathcal{O}}$. In type D, if the rank is odd and no coordinate of the original χ is 0, the parameter is not hermitian. If there are no 0s and the rank of type D is even, only an even number of sign changes is allowed. In this case, the last string might be $(\dots, b, -b - 1)$. If so, and all strings are of even size, $\check{\mathcal{O}}$ is very even, and is labelled II. If all strings are of the form $(\dots, b, b + 1)$, then the very even orbit is labelled I.

Algorithms for (c)

G of types C and D. $\mathfrak{z}(\check{\mathcal{O}}) = \mathfrak{so}(m_1) \times \mathfrak{sp}(m_2) \times \mathfrak{so}(m_3) \times \dots$, where m_i are the number of parts of $\check{\mathcal{O}}$ equal to i .

G of type B. $\mathfrak{z}(\check{\mathcal{O}}) = \mathfrak{sp}(m_1) \times \mathfrak{so}(m_2) \times \dots \times \mathfrak{sp}(m_3) \times \dots$, where again m_i is the number of parts of $\check{\mathcal{O}}$ equal to i .

Algorithms for (d)

The parameter ν is a vector of size equal to $\text{rk } G$. For each \mathfrak{z}_i , of $\mathfrak{z}(\mathcal{O})$, add $\text{rk } \mathfrak{z}_i$ coordinates each equal to the average of the string corresponding to the size i part of $\check{\mathcal{O}}$. For each factor \mathfrak{z}_i , of $\mathfrak{z}(\check{\mathcal{O}})$, the coordinates are the averages of the corresponding strings. The remaining coordinates of ν are all zero.

Algorithm for (e)

The Levi subgroups are determined by specifying the GL factors. There is at most one other factor $G_0(n_0)$ of the same type as the group.

For $\check{\mathfrak{m}}_{\text{BC}}$, each pair of parts (k, k) yields a $\text{GL}(k)$. If the corresponding string comes from Step 2, then the character on $\text{GL}(k)$ is given by $|\det(*)|^{\text{average of string}}$. Otherwise it is the trivial character. The parts of the remaining partition have multiplicity 1 corresponding to a distinguished orbit in $\check{\mathfrak{g}}_0(n_0)$.

For $\check{\mathfrak{m}}_{\text{KL}}$, apply the same procedure as for $\check{\mathfrak{m}}_{\text{BC}}$, except for pairs coming from Step 1. If originally there was an odd number of parts, then there is no change. If there was an even number, leave behind one pair. The parts in the remaining partition have multiplicity 1 or 2 corresponding to an even orbit in $\check{\mathfrak{g}}_0(n_0)$.

Algorithms for (f) and (g)

Both $\check{\mathfrak{m}}_e$ and $\check{\mathfrak{m}}_o$ acquire a $\text{GL}(k)$ factor for each pair of parts (k, k) in Step 2, with character given by the average of the corresponding string as before.

For the parts coming from Step 1, write them in decreasing order $a_r \geq \dots \geq a_1 > 0$.

For $\check{\mathfrak{m}}_e$, there are the following additional GL factors.

G of type B: $(a_1 + a_2)/2, (a_3 + a_4)/2, \dots, (a_{r-2} + a_{r-1})/2$ if r is odd, $(a_1)/2, (a_2 + a_3)/2, \dots, (a_{r-2} + a_{r-1})/2$ when r is even. The characters are given by the averages of the strings $(-(a_{r-1} - 1)/2, \dots, (a_{r-2} - 1)/2)$ and so on, and $(-(a_1 - 1)/2, \dots, -\frac{1}{2}), \dots$ when r is even. Recall that the a_i are all even because $\check{\mathcal{O}}$ is a nilpotent orbit in type C.

G of type C: $(a_1 + a_2)/2, (a_3 + a_4)/2, \dots$, with characters given by the averages of the strings $(-(a_1 - 1)/2, \dots, (a_2 - 1)/2)$, and so on. In this case \check{O} is type B, so there are an odd number of odd parts.

G of type D: $(a_1 + a_2)/2, (a_3 + a_4)/2, \dots$, with characters obtained by the same procedure as in type C. In this case \check{O} has an even number of odd parts.

For \check{m}_o , there are the following additional GL factors.

G of type B: $(a_2 + a_3)/2, (a_4 + a_5)/2, \dots, (a_{r-1} + a_r)/2$ leaving a_1 out if r is odd, $(a_1 + a_2)/2, (a_3 + a_4)/2, \dots, (a_{r-1} + a_r)/2$ if r is even. The characters are given by the averages of the strings $(-(a_r - 1)/2, \dots, (a_{r-1} - 1)/2) \dots$ and so on.

G of type C: $(a_2 + a_3)/2, (a_4 + a_5)/2, \dots$, and characters given by the averages of the strings $(-(a_3 - 1)/2, \dots, (a_2 - 1)/2), \dots$. In this case \check{O} has an odd number of odd sized parts.

G of type D: $(a_1 + a_2)/2, \dots, (a_{r-3} + a_{r-2})/2, ((a_{r-1} - 1)/2)$ with characters obtained by the averages of the strings $(-(a_2 - 1)/2, \dots, (a_1 - 1)/2), \dots, ((-a_{r-1} - 1)/2, \dots, -1)$. In this case \check{O} has an even number of odd sized parts.

2.9. Let $\chi = \frac{1}{2}\check{h} + \nu$ be associated to the orbit \check{O} . Recall from 2.3

$$I_M(\chi) := \text{Ind}_M^G[L_M(\chi)], \tag{2.57}$$

where $L_M(\chi)$ is the irreducible spherical module of M with parameter χ . Write the nilpotent orbit in (2.12) with the (d_1, \dots, d_l) as in §§ 2.5–2.7 depending on the Lie algebra type. Then $\check{m}_{BC} = \mathfrak{gl}(a_1) \times \dots \times \mathfrak{gl}(a_k) \times \mathfrak{g}_0(n_0)$ is as in (2.13). Thus χ determines a spherical irreducible module

$$L_{M_{BC}}(\chi) = L_1(\chi_1) \otimes \dots \otimes L_k(\chi_k) \otimes L_0(\chi_0), \tag{2.58}$$

with $\chi_i = (-\frac{1}{2}(a_i - 1) + \nu_i, \dots, \frac{1}{2}(a_i - 1) + \nu_i)$, while $\chi_0 = \frac{1}{2}\check{h}_0$ for the nilpotent (d_i) .

Let \check{m}_{KL} be the Levi component attached to $\chi = \frac{1}{2}\check{h} + \nu$ in §§ 2.5–2.7. As for \check{m}_{BC} we have a parameter $L_{M_{KL}}(\chi)$. In this case $\check{O} = ((a'_1, a'_1), \dots, (a'_r, a'_r); d'_1, \dots, d'_l)$ as described in §§ 2.5–2.7. Then (a W -conjugate of) χ can be written as in (2.22)–(2.46), and

$$\left. \begin{aligned} \check{m}_{KL} &= \mathfrak{gl}(a'_1) \times \dots \times \mathfrak{gl}(a'_r) \times \mathfrak{g}_0(n'_0), \\ L_{M_{KL}}(\chi) &= L_1(\chi'_1) \otimes \dots \otimes L_r(\chi'_r) \otimes L_0(\chi'_0). \end{aligned} \right\} \tag{2.59}$$

Theorem 2.9. *In the p -adic case*

$$I_{M_{KL}}(\chi) = L(\chi).$$

Proof. This is in [10], \check{m}_{KL} was defined in such a way that this result holds. □

Corollary 2.10. *The module $I_{M_{BC}}(\chi)$ equals $L(\chi)$ in the p -adic case if all the $\nu_i \neq 0$.*

3. The main result

3.1. Recall that \check{G} is the (complex) dual group, and $\check{A} \subset \check{G}$ the maximal torus dual to A . Assuming as we may that the parameter is real, a spherical irreducible representation corresponds to an orbit of a hyperbolic element $\chi \in \check{\mathfrak{a}}$, the Lie algebra of \check{A} . In §2 we attached a nilpotent orbit \check{O} in $\check{\mathfrak{g}}$ with partition

$$\underbrace{(a_1, \dots, a_1)}_{r_1}, \dots, \underbrace{(a_k, \dots, a_k)}_{r_k}$$

to such a parameter. Let $\{\check{e}, \check{h}, \check{f}\}$ be a Lie triple attached to \check{O} . Let $\chi := \frac{1}{2}\check{h} + \nu$ satisfy (2.9), (2.10).

Definition 3.1. A representation $L(\chi)$ is said to be in the complementary series for \check{O} , if the parameter χ is attached to \check{O} in the sense of satisfying (2.9) and (2.10), and is unitary.

We will describe the complementary series explicitly in coordinates.

The centralizer $Z_{\check{G}}(\check{e}, \check{h}, \check{f})$ has Lie algebra $\mathfrak{z}(\check{O})$ which is a product of $\mathfrak{sp}(r_l, \mathbb{C})$ or $\mathfrak{so}(r_l, \mathbb{C})$, $1 \leq l \leq k$, according to the rule

\check{G} of types B and D: $\mathfrak{sp}(r_l)$ for a_l even, $\mathfrak{so}(r_l)$ for a_l odd,

\check{G} of type C: $\mathfrak{sp}(r_l)$ for a_l odd, $\mathfrak{so}(r_l)$ for a_l even.

On $\mathfrak{z}(\check{O})$, ν determines a spherical irreducible module $L_{\check{O}}(\nu)$ for the split group whose dual is $Z_{\check{G}}(\check{e}, \check{h}, \check{f})^0$. Applying (2.9), (2.10), we find that ν is attached to the trivial orbit in $\mathfrak{z}(\check{O})$. If it were not, then $\nu = \frac{1}{2}\check{h}' + \nu'$, for a triple $\{\check{e}', \check{h}', \check{f}'\}$, with $\check{e}' \neq 0$, so $\chi = (\frac{1}{2}\check{h} + \frac{1}{2}\check{h}') + \nu'$, and $\{\check{e} + \check{e}', \check{h} + \check{h}', \check{f} + \check{f}'\}$ is a Lie triple such that the orbit of $\check{e} + \check{e}'$ is strictly larger than the orbit of \check{e} , and contains it in its closure.

Theorem 3.2. *The complementary series attached to \check{O} coincides with the one attached to the trivial orbit in $\mathfrak{z}(\check{O})$. For the trivial orbit (0) in each of the classical cases, the complementary series are as follows.*

G of type B:

$$0 \leq \nu_1 \leq \dots \leq \nu_k < \frac{1}{2}.$$

G of types C and D:

$$0 \leq \nu_1 \leq \dots \leq \nu_k \leq \frac{1}{2} < \nu_{k+1} < \dots < \nu_{k+l} < 1$$

so that $\nu_i + \nu_j \neq 1$. There are

- (1) an even number of ν_i such that $1 - \nu_{k+1} < \nu_i \leq \frac{1}{2}$,
- (2) for every $1 \leq j \leq l$, there is an odd number of ν_i such that $1 - \nu_{k+j+1} < \nu_i < 1 - \nu_{k+j}$,
- (3) in type D of odd rank, $\nu_1 = 0$ or else the parameter is not hermitian.

Remark 3.3. (1) The complementary series for $\check{O} = (0)$ consists of representations which are both spherical and generic in the sense that they have Whittaker models.

(2) The condition that $\nu_i + \nu_j \neq 1$ implies that in types C and D there is at most one $\nu_k = \frac{1}{2}$.

(3) In the case of $\check{\mathcal{O}} \neq (0)$, $\chi = \frac{1}{2}\check{h} + \nu$, and each of the coordinates ν_i for the parameter on $\mathfrak{z}(\check{\mathcal{O}})$ comes from a string, i.e. each ν_i comes from $(-\frac{1}{2}(a_i - 1) + \nu_i, \dots, \frac{1}{2}(a_i - 1) + \nu_i)$. The parameter does not satisfy (2.55). For (2.55) to hold, it suffices to change ν_{k+j} for types C and D to $1 - \nu_{k+j}$. More precisely, for $\frac{1}{2} < \nu_{k+j} < 1$ the connection with the strings in the form (2.54) and (2.55) is as follows. Write

$$\left(-\frac{1}{2}(a_{k+j} - 1) + \nu_{k+j}, \dots, \frac{1}{2}(a_{k+j} - 1) + \nu_{k+j}\right)$$

as

$$\left(-\frac{1}{2}(a_{k+j} - 3) + (\nu_{k+j} - 1), \dots, \frac{1}{2}(a_{k+j} + 1) + (\nu_{k+j} - 1)\right)$$

and then conjugate each entry to its negative to form

$$\left(-\frac{1}{2}(a_{k+j} - 3) + \nu'_{k+j}, \dots, \frac{1}{2}(a_{k+j} + 1) + \nu'_{k+j}\right),$$

with $0 < \nu'_{k+j} = 1 - \nu_{k+j} < \frac{1}{2}$.

An algorithm

For types C and D we give an algorithm, due to Sahi, to decide whether a parameter in types C and D is unitary. This algorithm is for the complementary series for $\check{\mathcal{O}} = (0)$. For arbitrary $\check{\mathcal{O}}$ it applies to the parameter for $\mathfrak{z}(\check{\mathcal{O}})$ obtained as in Remark 3.3 (3) above.

Order the parameter in dominant form,

$$\left. \begin{aligned} 0 \leq \nu_1 \leq \dots \leq \nu_n, & \quad \text{for type C,} \\ 0 \leq |\nu_1| \leq \dots \leq \nu_n, & \quad \text{for type D.} \end{aligned} \right\} \tag{3.1}$$

The first condition is that $\nu_n < 1$, and in addition that if the type is D_n with n odd, then $\nu_1 = 0$. Next replace each coordinate $\frac{1}{2} < \nu_i$ by $1 - \nu_i$. Reorder the new coordinates in increasing order as in (3.1). Let $F(\nu)$ be the set of new positions of the $1 - \nu_i$. If any position is ambiguous, the parameter is not unitary, or is attached to a different nilpotent orbit. This corresponds to either a $\nu_i + \nu_j = 1$, or a $\frac{1}{2} < \nu_i = \nu_j$. Finally, $L(\nu)$ is unitary if and only if $F(\nu)$ consists of odd numbers only.

3.2. We prove the unitarity of the parameters in the theorem for $\check{\mathcal{O}} = (0)$ for types B, C and D. First we record some facts.

Let $G := GL(2a)$ and

$$\chi := \left(-\frac{1}{2}(a - 1) - \nu, \dots, \frac{1}{2}(a - 1) - \nu; -\frac{1}{2}(a - 1) + \nu, \dots, \frac{1}{2}(a - 1) + \nu\right). \tag{3.2}$$

Let $M := GL(a) \times GL(a) \subset GL(2a)$. Then the two strings of χ determine an irreducible spherical (one-dimensional) representation $L_M(\chi)$ on M . Recall $I_M(\chi) := \text{Ind}_M^G[L_M(\chi)]$.

Lemma 3.4. *The representation $I_M(\chi)$ is unitary irreducible for $0 \leq \nu < \frac{1}{2}$. The irreducible spherical module $L(\chi)$ is not unitary for $\nu > \frac{1}{2}$, $2\nu \notin \mathbb{Z}$.*

Proof. This is well known and goes back to [29] (see also [30] and [34]). □

We also recall the following well-known result due to Kostant in the real case, Casselman in the p -adic case.

Lemma 3.5. *If none of the $\langle \chi, \alpha \rangle$ for $\alpha \in \Delta(\check{\mathfrak{g}}, \check{\mathfrak{a}})$ is a non-zero integer, then $X(\chi)$ is irreducible. In particular, if $\chi = 0$, then*

$$L(\chi) = X(\chi) = \text{Ind}_A^G[\chi],$$

and it is unitary.

Let $\check{\mathfrak{m}} \subset \check{\mathfrak{g}}$ be a Levi component, and $\xi_t \in \mathfrak{z}(\check{\mathfrak{m}})$, where $\mathfrak{z}(\check{\mathfrak{m}})$ is the centre of $\check{\mathfrak{m}}$, depending continuously on $t \in [a, b]$.

Lemma 3.6. *Let ξ_t be a character of M , depending continuously on $t \in \mathbb{R}$. Assume that*

$$I_M(\chi_t) := \text{Ind}_M^G[L_M(\eta_0) \otimes \xi_t]$$

is irreducible and hermitian for $a \leq t \leq b$. Then if $I_M(\chi_t)$ (equal to $L(\chi_t)$) is unitary for some $a \leq t_0 \leq b$, it is unitary for all $a \leq t \leq b$.

This is well known, and amounts to the fact that if a hermitian matrix is non-degenerate and depends continuously on a parameter $a \leq t \leq b$, if it is positive definite for some $a \leq t_0 \leq b$, it is positive definite throughout the interval; to change from positive to negative, it would have to go through a zero, i.e. become degenerate. This happens for example when $L_M(\eta_0) \otimes \xi_{t_0}$ is unitary for some t_0 , and $I_M(\chi_t)$ is irreducible. I do not know the original reference.

When the conditions of Lemma 3.6 are satisfied, we say that $I_M(\chi_t)$ is a *continuous deformation* of $L(\chi_0)$.

We now start the proof of the unitarity.

Type B

In this case there are no roots $\alpha \in \Delta(\check{\mathfrak{g}}, \check{\mathfrak{a}})$ such that $\langle \chi, \alpha \rangle$ is a non-zero integer. Thus

$$L(\chi) = \text{Ind}_A^G[\chi]$$

as well. When deforming χ to 0 continuously, the induced module stays irreducible. Since $\text{Ind}_A^G[0]$ is unitary, so is $L(\chi)$.

Types C and D

There is no root such that $\langle \chi, \alpha \rangle$ is a non-zero integer, so $L(\chi) = \text{Ind}_A^G[\chi]$. If there are no $\nu_{k+i} > \frac{1}{2}$ the argument for type B carries over word for word. When there are $\nu_{k+i} > \frac{1}{2}$ we have to be more careful with the deformation. We will do an induction on the rank. Suppose that $\nu_{j-1} = \nu_j$ for some j . Necessarily, $\nu_j < \frac{1}{2}$. Conjugate χ by the Weyl group so that

$$\chi = (\nu_1, \dots, \nu_i, \dots, \widehat{\nu_{j-1}}, \widehat{\nu_j}, \dots; \nu_{j-1}; \nu_j) := (\chi_0; \nu_{j-1}; \nu_j). \tag{3.3}$$

Let $\check{m} := \check{g}(n - 2) \times \mathfrak{gl}(2)$, and denote by M the corresponding Levi component. Then by induction in stages,

$$L(\chi) = \text{Ind}_M^G[L_M(\chi)], \tag{3.4}$$

where $L_M(\chi) = L_0(\chi_0) \otimes L_1(\nu_{j-1}, \nu_j)$. By Lemma 3.4, $L_1(\nu_{j-1}, \nu_j)$ is unitary. Thus $L(\chi)$ is unitary if and only if $L_0(\chi_0)$ is unitary. If χ satisfies the assumptions of the theorem, then so does χ_0 . By the induction hypothesis, $L_0(\chi_0)$ is unitary, and therefore so is $L(\chi)$. Thus we may assume that

$$0 \leq \nu_1 < \dots < \nu_k \leq \frac{1}{2} < \nu_{k+1} < \dots < \nu_{k+l}. \tag{3.5}$$

If $\nu_k < 1 - \nu_{k+1}$, then the assumptions imply $1 - \nu_{k+2} < \nu_k$. Consider the parameter

$$\chi_t := (\dots, \nu_k, \nu_{k+1} - t, \dots). \tag{3.6}$$

Then

$$L(\chi_t) = \text{Ind}_A^G[\chi_t], \quad \text{for } 0 \leq t \leq \nu_{k+1} - \nu_k, \tag{3.7}$$

because no $\langle \chi_t, \alpha \rangle$ is a non-zero integer. At $t = \nu_{k+1} - \nu_k$, the parameter is in the case just considered earlier. By induction we are done.

If on the other hand $1 - \nu_{k+1} < \nu_k$, the assumptions on the parameter are such that necessarily $1 - \nu_{k+1} < \nu_{k-1} < \nu_k$. Then repeat the argument with

$$\chi_t := (\dots, \nu_{k-1}, \nu_k - t, \dots), \quad 0 \leq t \leq \nu_k - \nu_{k-1}. \tag{3.8}$$

This completes the proof of the unitarity of the parameters in Theorem 3.2 when $\check{O} = (0)$.

3.3. We prove the unitarity of the parameters in Theorem 3.2 in the general case when $\check{O} \neq (0)$.

The proof is essentially the same as for $\check{O} = (0)$, but special care is needed to justify the irreducibility of the modules. Recall the notation of the partition of \check{O} (2.11).

The factors of $\mathfrak{z}(\check{O})$ isomorphic to $\mathfrak{sp}(r_j)$, contribute $r_j/2$ factors of the form $\mathfrak{gl}(a_i)$ to \check{m}_{KL} . The factors of type $\mathfrak{so}(r_j)$ with r_j odd, contribute a d_i (notation (2.12)) to the expression (2.12) of the partition of \check{O} , and $\frac{1}{2}(r_j - 1)$ factors of the form $\mathfrak{gl}(a_i)$. The factors $\mathfrak{so}(r_j)$ of type D (r_j even) are more complicated. Write the strings coming from this factor as in (2.14),

$$\left(-\frac{1}{2}(a_i - 1) + \nu_i, \dots, \frac{1}{2}(a_i - 1) + \nu_i\right)$$

with the ν_i satisfying the assumptions of Theorem 3.2. If r_j is not divisible by 4, then there must be a $\nu_1 = 0$ (otherwise the corresponding spherical parameter is not hermitian), and $\check{m}_{\text{BC}} \neq \check{m}_{\text{KL}}$. Similarly when r_j is divisible by 4 and $\nu_1 = 0$, $\check{m}_{\text{BC}} \neq \check{m}_{\text{KL}}$. In all situations, we consider

$$I_{M_{\text{KL}}}(\chi) \tag{3.9}$$

as for $\check{O} = (0)$. We aim to show that this module stays irreducible under the deformations used for $\check{O} = (0)$, separately for the ν_i for the same partition size or equivalently simple

factor of $\mathfrak{z}(\check{O})$. It is enough to prove that under these deformations the strings stay strongly nested at all the values of the parameters. Then Proposition 3.8 applies.

Using the conventions of § 2, the strings are of the form

$$(-A - 1 + \nu_1, \dots, A - 1 + \nu_1), \tag{3.10}$$

$$(-B + \nu_2, \dots, B - 1 + \nu_2), \tag{3.11}$$

$$(-C + \nu_3, \dots, C + \nu_3). \tag{3.12}$$

The ν_i satisfy $0 \leq \nu_i \leq \frac{1}{2}$. When $\nu_1 = \frac{1}{2}$, the string is $(-A - \frac{1}{2}, \dots, A - \frac{1}{2})$ so it conforms to (2.31), (2.43), (2.55). Similarly when $\nu_2 = \frac{1}{2}$, the string is $(-B + \frac{1}{2}, \dots, B - \frac{1}{2})$. But when $\nu_3 = \frac{1}{2}$, the string is $(-C + \frac{1}{2}, \dots, C + \frac{1}{2})$ and it must be replaced by $(-C - \frac{1}{2}, \dots, C - \frac{1}{2})$ to conform to (2.31), (2.43), (2.55). The string in (3.10) gives a $\nu_{1,3}(\check{O}) = 1 - \nu_1$ (as explained in Remark 3.3 (3)). The other ones ν_2 and ν_3 give $\nu_{j,3}(\check{O}) = \nu_2$ or ν_3 respectively. Suppose first that there is only one size of string. This means that the corresponding entries $\nu_{j,3}(\check{O})$ belong to the same simple factor of $\mathfrak{z}(\check{O})$. Then the strings are either all of the form (3.10) and (3.12) with $A = C$ or all of the form (3.11). Consider the first case. If there is a string (3.10) then $\nu_{1,3}(\check{O}) = 1 - \nu_1 \geq \frac{1}{2}$, and $\nu_{1,3}(\check{O})$ is deformed downward. By the assumptions $\nu_{1,3}(\check{O})$ does not equal any $\nu_{j,3}(\check{O})$ nor does $\nu_{1,3}(\check{O}) + \nu_{j,3}(\check{O}) = 1$ for any j . The module stays irreducible. When $\nu_{1,3}(\check{O})$ crosses $\frac{1}{2}$, the string becomes $(-A + \nu'_1, \dots, A + \nu'_1)$, and ν'_1 is deformed downward from $\frac{1}{2}$ to either some ν_3 or to 0. Again $\nu'_1 + \nu_{j,3}(\check{O}) \neq 1$ for any j , so no reducibility occurs. The irreducibility in the case when ν'_1 reaches 0 is dealt with by § 10.

It remains to check that in these deformations no reducibility occurs because the string interacts with one of a different length, in other words, when a $\nu_{i,3}(\check{O})$ for one size string or equivalently factor of $\mathfrak{z}(\check{O})$ becomes equal to a $\nu_{j,3}(\check{O})$ from a distinct factors of $\mathfrak{z}(\check{O})$. We explain the case when the deformation involves a string of type (3.10), the others are similar and easier. Consider the deformation of ν_1 from 0 to $\frac{1}{2}$. If the module is to become reducible ν_1 must reach a ν_3 so that there is a string of the form (3.12) satisfying

$$-A - 1 < -C, \quad A - 1 < C. \tag{3.13}$$

This is the condition that the strings are not nested for some value of the parameter. This implies that $A - 1 < C < A + 1$ must hold. Thus $A = C$, and the two strings correspond to the same size partition or simple factor in $\mathfrak{z}(\check{O})$.

In the p -adic case, the irreducibility of (3.9) in the case of strongly nested strings follows from the results of Kazhdan–Lusztig. In the case of real groups, the same irreducibility results hold, but are harder to prove. Given χ , consider the root system

$$\check{\Delta}_\chi := \{\check{\alpha} \in \check{\Delta} : \langle \chi, \check{\alpha} \rangle \in \mathbb{Z}\}. \tag{3.14}$$

Let G_χ be the connected split real group whose dual root system is $\check{\Delta}_\chi$.

The Kazhdan–Lusztig–Vogan conjectures for non-integral infinitesimal character relate statements about the character theory of admissible (\mathfrak{g}_c, K) -modules of G with infinitesimal character χ to similar statements for characters of admissible modules for G_χ but at integral infinitesimal character, for which the Kazhdan–Lusztig–Vogan conjectures for

regular integral infinitesimal character can be used. This is beyond the scope of this paper (or my competence) so I refer to [1, Chapters 16 and 17] for an explanation.

In short, χ determines an irreducible spherical representation $L(\chi)$, and a spherical $L_{G_\chi}(\chi)$. To prove the irreducibility results we need, we will prove them for $L_{G_\chi}(\chi)$, where we may assume that the infinitesimal character is integral. Since G_χ is not simple, it is sufficient to prove the needed irreducibility result for each simple factor. This root system is a product of classical systems as follows.

Definition 3.7. For each $0 \leq \tau \leq \frac{1}{2}$ let A_τ be the set of coordinates of χ congruent to τ modulo \mathbb{Z} .

Each A_τ contributes to $\check{\Delta}_\chi$ as follows.

G of type B: every $0 < \tau < \frac{1}{2}$ contributes a type A of size equal to the number of coordinates in A_τ . Every $\tau = 0, \frac{1}{2}$ contributes a type C of rank equal to the number of coordinates in A_τ .

G of type C: every $0 < \tau < \frac{1}{2}$ contributes a type A as for type B. Every $\tau = 0$ contributes a type B, while $\tau = \frac{1}{2}$ contributes type D.

G of type D: every $0 < \tau < \frac{1}{2}$ contributes a type A. Every $\tau = 0, \frac{1}{2}$ contributes a type D.

The irreducibility results for $I_{M_{KL}}(\chi_t)$ needed to carry out the proof are contained in the following proposition.

Proposition 3.8. *Let*

$$\chi := (\dots; -\frac{1}{2}(a_i - 1) + \nu_i, \dots, \frac{1}{2}(a_i - 1) + \nu_i; \dots)$$

be given in terms of strings, and let $\check{\mathfrak{m}} = \mathfrak{gl}(a_1) \times \dots \times \mathfrak{gl}(a_k)$ be the corresponding Levi component. Assume that χ is integral (i.e. $\langle \chi, \check{\alpha} \rangle \in \mathbb{Z}$). In addition assume that the coordinates of χ are

- *in \mathbb{Z} , in type B,*
- *in $\frac{1}{2} + \mathbb{Z}$ for type D.*

If the strings are strongly nested, then

$$I_M(\chi) = \text{Ind}_M^G[L_M(\chi)].$$

The proof of the proposition will be given in § 10.

Remark. For the ν_j attached to factors of type D in $\mathfrak{z}(\check{\mathcal{O}})$, it is important in the argument that we do not deform to (0). The next example illustrates why.

Assume $\check{\mathcal{O}} = (2, 2, 2, 2) \subset \mathfrak{sp}(4)$. The parameters of the form $\frac{1}{2}\check{h} + \nu$ are

$$(-\frac{1}{2} + \nu_1, \frac{1}{2} + \nu_1; -\frac{1}{2} + \nu_2, \frac{1}{2} + \nu_2), \tag{3.15}$$

and, because parameters are up to W -conjugacy, we may restrict attention to the region $0 \leq \nu_1 \leq \nu_2$. In this case $\mathfrak{z}(\check{O}) = \mathfrak{so}(4)$, and the unitarity region is $0 \leq \pm\nu_1 + \nu_2 < 1$. Furthermore, $\check{\mathfrak{m}}_{BC} = \mathfrak{gl}(2) \times \mathfrak{gl}(2)$, but $\check{\mathfrak{m}}_{KL} = \check{\mathfrak{m}}_{BC}$ only if $0 < \nu_1$. When $\nu_1 = 0$, $\check{\mathfrak{m}}_{KL} = \mathfrak{sp}(2) \times \mathfrak{gl}(2)$, the nilpotent orbit is rewritten $(2, 2; (2, 2))$, and $\frac{1}{2}\check{h}_0 = (\frac{1}{2}, \frac{1}{2})$. For $\nu_1 = 0$, the induced representations

$$I_{M_{KL}}(\chi_{\nu_2}) := \text{Ind}_{\text{Sp}(2) \times \text{GL}(2)}^{\text{Sp}(4)} [L_0(\frac{1}{2}, \frac{1}{2}) \otimes L_1(-\frac{1}{2} + \nu_2, \frac{1}{2} + \nu_2)] \tag{3.16}$$

are induced irreducible in the range $0 \leq \nu_2 < 1$. For $0 < \nu_1$ the representation

$$\text{Ind}_{\text{GL}(4)}^{\text{Sp}(4)} [L((-\frac{1}{2} + \nu_1 + t, \frac{1}{2} + \nu_1 + t); (-\frac{1}{2} - \nu_1 - t, \frac{1}{2} - \nu_1 - t))] \tag{3.17}$$

is induced irreducible for $0 \leq t \leq \frac{1}{2} - \nu_1$.

The main point of the example is that $\text{Ind}_{\text{GL}(2)}^{\text{Sp}(2)} [L(-\frac{1}{2} + t, \frac{1}{2} + t)]$ is *reducible* at $t = 0$. So we cannot conclude that $L(\chi)$ is unitary for a (ν_1, ν_2) with $0 < \nu_1$ from the unitarity of $L(\chi)$ for a parameter with $\nu_1 = 0$. Instead we conclude that the representation is unitary in the region $0 \leq \pm\nu_1 + \nu_2 < 1$ because it is a deformation of the irreducible module for $\nu_1 = \nu_2$ which is unitarily induced irreducible from a Stein complementary series on $\text{GL}(4)$.

4. Relevant K -types

In this section we define a special set of K -types (occurring in the spherical principal series) which we call relevant. Their first important property is that the intertwining operators used to compute the hermitian form are particularly simple. This is the property of *petite*. A more general notion was introduced and studied independently by Oda [27]. In the notation of §5 these operators only depend on the W -structure of $V^{K \cap B}$. In §4 we determine these Weyl group representations for the relevant K -types. In §6 we carry out the intertwining operator calculations in terms of the Weyl group representations. These calculations are used to obtain necessary conditions for unitarity in §7. But in §§9 and 10 they also prove to be crucial for determining unitarity and irreducibility of certain modules needed to complete the determination of the unitary dual.

4.1. In the real case we will call a K -type (μ, V) *quasi-spherical* if it occurs in the spherical principal series. In §4, we will use the notation $M = K \cap B$. By Frobenius reciprocity (μ, V) is quasi-spherical if and only if $V^{K \cap B} \neq 0$. Because the Weyl group $W(G, A)$ may be realized as $N_K(A)/Z_K(A)$, this Weyl group acts naturally on this space.

The representations of $W(A_{n-1}) = S_n$ are parametrized by partitions $(a) := (a_1, \dots, a_k)$, $a_i \leq a_{i+1}$, of n , and we write $\sigma((a))$ for the corresponding representation. The representations of $W(B_n) \cong W(C_n)$ are parametrized as in [24] by pairs of partitions, and we write as

$$\sigma((a_1, \dots, a_r), (b_1, \dots, b_s)), \quad a_i \leq a_{i+1}, \quad b_j \leq b_{j+1}, \quad \sum a_i + \sum b_j = n. \tag{4.1}$$

Precisely the representation parametrized by (4.1) is as follows. Let $k = \sum a_i$, $l = \sum b_j$. Recall that $W \cong S_n \times \mathbb{Z}_2^n$. Let χ be the character of \mathbb{Z}_2^n which is trivial on the

first k \mathbb{Z}_2 s, sign on the last l . Its centralizer in S_n is $S_k \times S_l$. Let $\sigma((a))$ and $\sigma((b))$ be the representations of S_k, S_l corresponding to the partitions (a) and (b) . Then let $\sigma((a), (b), \chi)$ be the unique representation of $(S_k \times S_l) \ltimes \mathbb{Z}_2^n$ which is a multiple of χ when restricted to \mathbb{Z}_2^n , and $\sigma(a) \otimes \sigma(b)$ when restricted to $S_k \times S_l$. The representation in (4.1) is

$$\sigma((a), (b)) = \text{Ind}_{(S_k \times S_l) \ltimes \mathbb{Z}_2^n}^W [\sigma(a, b, \chi)]. \tag{4.2}$$

If $(a) \neq (b)$, the representations $\sigma((a), (b))$ and $\sigma((b), (a))$ restrict to the same irreducible representation of $W(D_n)$, which we denote again by the same symbol. When $a = b$, the restriction is a sum of two inequivalent representations which we denote $\sigma((a), (a))_{\text{I,II}}$. Let $W_{(a),\text{I}} := S_{a_1} \times \dots \times S_{a_r}$ and $W_{(a),\text{II}} := S_{a_1} \times \dots \times S'_{a_r}$, be the Weyl groups corresponding to the Levi components considered in Remark 2.8 (2) in §2.7. Then $\sigma((a), (a))_{\text{I}}$ is characterized by the fact that its restriction to $W_{(a),\text{I}}$ contains the trivial representation. Similarly $\sigma((a), (a))_{\text{II}}$ is the one that contains the trivial representation of $W_{(a),\text{II}}$.

4.2. Symplectic groups

The group is $\text{Sp}(n)$ and the maximal compact subgroup is $\text{U}(n)$. The highest weight of a K -type will be written as $\mu(a_1, \dots, a_n)$ with $a_i \geq a_{i+1}$ and $a_i \in \mathbb{Z}$, or

$$\mu(a_1^{r_1}, \dots, a_k^{r_k}) := \underbrace{(a_1, \dots, a_1)}_{r_1}, \dots, \underbrace{(a_k, \dots, a_k)}_{r_k} \tag{4.3}$$

when we want to emphasize the repetitions. We will repeatedly use the following restriction formula.

Lemma 4.1. *The restriction of $\mu(a_1, \dots, a_n)$ to $\text{U}(n - 1) \times \text{U}(1)$ is*

$$\sum \mu(b_1, \dots, b_{n-1}) \otimes \mu(b_n),$$

where the sum ranges over all possible $a_1 \geq b_1 \geq a_2 \geq \dots \geq b_{n-1} \geq a_n$, and

$$b_n = \sum_{1 \leq i \leq n} a_i - \sum_{1 \leq j \leq n-1} b_j.$$

Definition 4.2. The representations $\mu_e(n - r, r) := \mu(2^r, 0^{n-r})$ and $\mu_o(k, n - k) := \mu(1^k, 0^{n-2k}, -1^k)$ are called *relevant*.

Proposition 4.3. *The relevant K -types are quasi-spherical. The representation of $W(C_n)$ on V^M is*

$$\begin{aligned} \mu_e(n - r, r) &\longleftrightarrow \sigma[(n - r), (r)], \\ \mu_o(k, n - k) &\longleftrightarrow \sigma[(k, n - k), (0)]. \end{aligned}$$

The K -types $\mu(0^{n-r}, (-2)^r)$, dual to $\mu_e(n - r, r)$, are also quasi-spherical, and could be used in the same way as $\mu_e(n - r, r)$.

Proof. We do an induction on n . When $n = 1$, the only relevant representations of $U(1)$ are $\mu_e(1, 0) = \mu(0)$ and $\mu_e(0, 1) = (2)$, which correspond to the trivial and the sign representations of $W(C_1) = \mathbb{Z}/2\mathbb{Z}$, respectively. Consider the case $n = 2$. There are four relevant representations of $U(2)$ with highest weights $(2, 0)$, $(1, -1)$, $(2, 2)$ and $(0, 0)$. The first representation is the symmetric square of the standard representation, the second one is the adjoint representation and the fourth one is the trivial representation. The normalizer of A in K can be identified with the diagonal subgroup $(\pm 1, \pm 1)$ inside $U(1) \times U(1) \subset U(2)$. The Weyl group is generated by the elements

$$\begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{4.4}$$

The restriction to $U(1) \times U(1)$ of the four representations of $U(2)$ is

$$\left. \begin{aligned} (2, 0) &\rightarrow (2) \otimes (0) + (1) \otimes (1) + (0) \otimes (2), \\ (1, -1) &\rightarrow (1) \otimes (-1) + (0) \otimes (0) + (-1) \otimes (1), \\ (2, 2) &\rightarrow (2) \otimes (2), \\ (0, 0) &\rightarrow (0) \otimes (0). \end{aligned} \right\} \tag{4.5}$$

The space V^M is the sum of all the weight spaces $(p) \otimes (q)$ with both p and q even. For the last one, the representation of W on V^M is $\sigma[(2), (0)]$. The third one is one dimensional so V^M is one dimensional; the Weyl group representation is $\sigma((0), (2))$. The second one has V^M one dimensional and the Weyl group representation is $\sigma((11), (0))$. For the first one, V^M is two dimensional and the Weyl group representation is $\sigma((1), (1))$. These facts can be read off from explicit realizations of the representations.

Assume that the claim is proved for $n - 1$. Choose a parabolic subgroup so that its Levi component is $M' = \text{Sp}(n - 1) \times \text{GL}(1)$ and M is contained in it. Let $H = U(n - 1) \times U(1)$ be such that $M \subset M' \cap K \subset H$.

Suppose that μ is relevant. The cases when $k = 0$ or $r = 0$ are one dimensional and are straightforward. So we only consider $k, r > 0$. The K -type $\mu(2^r, 0^{n-r})$ restricts to the sum of

$$\mu(2^r, 0^{n-r-1}) \otimes \mu(0), \tag{4.6}$$

$$\mu(2^{r-1}, 1, 0^{n-r-1}) \otimes \mu(1), \tag{4.7}$$

$$\mu(2^{r-1}, 0^{n-r}) \otimes \mu(2). \tag{4.8}$$

Of the representations appearing, only $\mu(2^r, 0^{n-r-1}) \otimes \mu(0)$ and $\mu(2^{r-1}, 1, 0^{n-r-1}) \otimes \mu(2)$ are quasi-spherical. So the restriction of V^M to $W(C_{n-1}) \times W(C_1)$ is the sum of

$$\sigma[(n - r - 1), (r)] \otimes \sigma[(1), (0)], \tag{4.9}$$

$$\sigma[(n - r), (r - 1)] \otimes \sigma[(0), (1)]. \tag{4.10}$$

The only irreducible representations of $W(C_n)$ containing (4.9) in their restrictions to $W(C_{n-1})$ are

$$\sigma[(1, n - r - 1), (r)], \tag{4.11}$$

$$\sigma[(n - r), (r)]. \tag{4.12}$$

But the restriction of $\sigma[(1, n - r - 1), (r)]$ to $W(C_{n-1}) \times W(C_1)$ contains $\sigma[(1, n - r - 1), (r - 1)] \otimes \sigma[(0), (1)]$, and this does not appear in (4.9), (4.10). Thus the representation of $W(C_n)$ on V^M for (4.11) must be (4.7), and the claim is proved in this case.

Consider the case $\mu(1^k, 0^l, -1^k)$ for $k > 0, 2k + l = n$. The restriction of this K -type to $U(n - 1) \times U(1)$ is the sum of

$$\mu(1^k, 0^l, -1^{k-1}) \otimes \mu(-1), \tag{4.13}$$

$$\mu(1^{k-1}, 0^l, -1^k) \otimes \mu(1), \tag{4.14}$$

$$\mu(1^{k-1}, 0^{l+1}, -1^{k-1}) \otimes \mu(0), \tag{4.15}$$

$$\mu(1^k, 0^{l-1}, -1^k) \otimes \mu(0). \tag{4.16}$$

Of the representations appearing, only (4.15) and (4.16) are quasi-spherical. So the restriction of V^M to $W(C_{n-1}) \times W(C_1)$ is the sum of

$$\sigma[(k - 1, k + l), (0)] \otimes \sigma[(1), (0)], \tag{4.17}$$

$$\sigma[(k, k + l - 1), (0)] \otimes \sigma[(1), (0)]. \tag{4.18}$$

The representation (4.18) can only occur in the restriction to $W(C_{n-1}) \times W(C_1)$ of $\sigma[(1, k, k + l - 1), (0)]$ or $\sigma[(k, k + l), (0)]$. If $k > 1$, the first one contains $\sigma[(1, k - 1, k + l - 1), (0)]$ in its restriction, which is not in the sum of (4.17) and (4.18). If $k = 1$ then (4.17) can only occur in the restriction of $\sigma[(0, l + 2), (0)]$, or $\sigma[(1, l + 1), (0)]$. But V^M cannot consist of $\sigma[(0, l + 2), (0)]$ alone, because (4.17) does not occur in its restriction. If it consists of both $\sigma[(0, l + 2), (0)]$ and $\sigma[(1, l), (0)]$, then the restriction is too large. The claim is proved in this case. □

4.3. Orthogonal groups

Because we are dealing with the spherical case, we can use the groups $O(a, b)$, $SO(a, b)$, or the connected component of the identity, $SO_e(a, b)$. The corresponding K s are $O(a) \times O(b)$, $S(O(a) \times O(b))$, and $SO(a) \times SO(b)$, respectively. We will use $O(a, b)$ for the calculation of relevant K -types. For $SO(a)$, an irreducible representation will be identified by its highest weight in coordinates, $\mu(x_1, \dots, x_{[a/2]})$, or if there are repetitions, $\mu(x_1^{n_1}, \dots, x_k^{n_k})$. For $O(a)$ we use the parametrization of Weyl [36]. Embed $O(a) \subset U(a)$ in the standard way. Then we denote by $\mu(x_1, \dots, x_k, 0^{[a/2]-k}; \epsilon)$ the irreducible $O(a)$ -component generated by the highest weight of the representation

$$\mu(x_1, \dots, x_k, 1^{(1-\epsilon)(a/2-k)}, 0^{a-k-(1-\epsilon)(a/2-k)})$$

of $U(a)$. In these formulae, $\epsilon = \pm$, is often written as plus for 1, and minus for -1 .

4.4. We describe the *relevant* K -types for the orthogonal groups $O(a, a)$.

Definition 4.4 (even orthogonal groups). The *relevant* K -types for $O(a, a)$

$$\mu_e([a/2] - r, r) := \mu(0^{[a/2]}; +) \otimes \mu(2^r, 0^l; +), \tag{4.19}$$

$$\mu_o(r, [a/2] - r) := \mu(1^r, 0^l; +) \otimes \mu(1^r, 0^l; +), \tag{4.20}$$

where $r + l = [a/2]$.

Proposition 4.5. *The relevant K -types are quasi-spherical. The representation of $W(D_a)$ of $O(a, a)$ on V^M is*

$$\sigma[(r, a - r), (0)] \longleftrightarrow \mu(0^{[a/2]}; +) \otimes \mu(2^r, 0^l; +), \tag{4.21}$$

$$\sigma[(a - k), (k)] \longleftrightarrow \mu(1^k, 0^l; +) \otimes \mu(1^k, 0^l; +). \tag{4.22}$$

When $l = 0$, and a is even,

$$\sigma[(a/2, a/2), (0)] \longleftrightarrow \mu(0^{a/2}) \otimes \mu(2^{a/2-1}, \pm 2), \tag{4.23}$$

$$\sigma[(a/2), (a/2)]_{I,II} \longleftrightarrow \mu(1^{a/2-1}, \pm 1) \otimes \mu(1^{a/2-1}, \pm 1). \tag{4.24}$$

We will prove this together with the corresponding proposition for $O(a + 1, a)$ in § 4.6.

4.5. We describe the relevant K -types for $O(a + 1, a)$.

Definition 4.6 (odd orthogonal groups). The relevant K -types for $O(a + 1, a)$ are

$$\mu_e(a - r, r) := \mu(0^{[(a+1)/2]}; +) \otimes \mu(2^r, 0^l; +), \tag{4.25}$$

$$\mu_o(a - k, k) := \mu(1^k, 0^l; +) \otimes \mu(1^k, 0^s; +), \tag{4.26}$$

$$\mu_o(k, a - k) := \mu(1^{k+1}, 0^l; +) \otimes \mu(1^k, 0^s; +), \tag{4.27}$$

where $r + l = [a/2]$ in (4.25), $k + l = [(a + 1)/2]$, $k + s = [a/2]$ in (4.26), and $k + 1 + l = [(a + 1)/2]$, $k + s = [a/2]$ in (4.27).

Proposition 4.7. *The representations of $W(B_a)$ on V^M for the relevant K -types are*

$$\sigma[(r, a - r), (0)] \longleftrightarrow \mu(0^{[(a+1)/2]}; +) \otimes \mu(2^r, 0^l; +), \tag{4.28}$$

$$\sigma[(a - k), (k)] \longleftrightarrow \mu(1^k, 0^{[(a+1)/2]-k}; +) \otimes \mu(1^k, 0^{[a/2]-l}; +), \tag{4.29}$$

$$\sigma[(k), (a - k)] \longleftrightarrow \mu(1^{k+1}, 0^{[(a+1)/2]-k-1}; +) \otimes \mu(1^k, 0^{[a/2]-k}; +). \tag{4.30}$$

When a is even,

$$\sigma[(a/2), (a/2)] \longleftrightarrow \mu(1^{a/2}) \otimes \mu(1^{a/2-1}, \pm 1). \tag{4.31}$$

When a is odd,

$$\sigma[((a - 1)/2), ((a - 1)/2)] \longleftrightarrow \mu(1^{(a-1)/2}, \pm 1) \otimes \mu(1^{(a-1)/2}). \tag{4.32}$$

The proof will be in § 4.6.

4.6. Proof of Propositions 4.5 and 4.7

We use the standard realization of the orthogonal groups $O(a + 1, a)$ and $O(a, a)$. Let

$$\tilde{M} := \left\{ (\eta_0, \eta_1, \dots, \eta_a, \epsilon_1, \dots, \epsilon_a) : \eta_i, \epsilon_j = \pm 1, \prod \eta_i = \prod \epsilon_j = 1 \right\}, \tag{4.33}$$

viewed as the subgroup of $O(a + 1) \times O(a)$ with the η_i, ϵ_j on the diagonal. With the appropriate choice of $\mathfrak{a} \cong \mathbb{R}^a$, $\tilde{M} \subset N_K(\mathfrak{a})$, and the action is

$$(\eta_i, \epsilon_j) \cdot (\dots, x_k, \dots) = (\dots, \eta_k \epsilon_k x_k, \dots). \tag{4.34}$$

Then $M := K \cap B$ is the subgroup of \tilde{M} determined by the relations $\eta_j = \epsilon_j, j = 1, \dots, a$. Similarly for $O(a) \times O(a)$ but there is no η_0 .

We do the case $O(a + 1, a)$; $O(a, a)$ is similar. The representations $\mu_o(a - k, k)$ and $\mu_o(k, a - k)$ can be realized as

$$\bigwedge^k \mathbb{C}^{a+1} \otimes \bigwedge^k \mathbb{C}^a, \quad \text{respectively} \quad \bigwedge^{k+1} \mathbb{C}^{a+1} \otimes \bigwedge^k (\mathbb{C}^a).$$

Let e_i be a basis of \mathbb{C}^{a+1} and f_j a basis of \mathbb{C}^a . The space V^M is the span of the vectors $e_{i_1} \wedge \dots \wedge e_{i_k} \otimes f_{i_1} \wedge \dots \wedge f_{i_k}$, and $e_0 \wedge e_{i_1} \wedge \dots \wedge e_{i_k} \otimes f_{i_1} \wedge \dots \wedge f_{i_k}$. The elements of W corresponding to short root reflections all have representatives of the form $\eta_0 = -1, \eta_j = -1$, the rest zero. The action of $S_a \subset W$ on the space V^M is by permuting the e_i, f_j diagonally. Claims (4.22), (4.23) and (4.29), (4.30) follow from these considerations, we omit further details.

For cases (4.21) and (4.28) we do an induction on r . We do the case $O(a, a)$ only. The claim is clear for $r = 0$. Since the first factor of $\mu_e([a/2] - r, r)$ is the trivial representation, we only concern ourselves with the second factor. Consider $\bigwedge^r \mathbb{C}^a \otimes \bigwedge^r \mathbb{C}^a$. The space of M -fixed vectors has dimension $\binom{a}{r}$, and a basis is

$$e_{i_1} \wedge \dots \wedge e_{i_r} \otimes e_{i_1} \wedge \dots \wedge e_{i_r}. \tag{4.35}$$

As a module of S_a , this is

$$\text{Ind}_{S_r \times S_{a-r}}^{S_a} [\text{triv} \otimes \text{triv}] = \sum_{1 \leq j \leq r} (j, a - j). \tag{4.36}$$

On the other hand, the tensor product $\bigwedge^r \mathbb{C}^r \otimes \bigwedge^r \mathbb{C}^a$ consists of representations with highest weight $\mu(2^\alpha, 1^\beta, 0^\gamma)$. From the explicit description of $\bigwedge^k \mathbb{C}^a$, and the action of M , we can infer that V^M for $\beta \neq 0$ is (0). This is because the representation occurs in $\bigwedge^{\alpha+\beta} \mathbb{C}^a \otimes \bigwedge^\alpha \mathbb{C}^a$, which has no M -fixed vectors. But $\mu(2^j, 0^l)$ for $j \leq r$ occurs (for example by the P-R-V (Parthasarathy–Ranga Rao–Varadarajan) conjecture). By the induction hypothesis, $(j, a - j)$ occurs in $\mu(2^j, 0^l)$, for $j < r$, and so only $(r, a - r)$ is unaccounted for. Thus V^M for $\mu_e([a/2] - r, r)$ be $(r, a - r)$. The claim now follows from the fact that the action of the short root reflections is trivial, and the description of the irreducible representations of $W(B_a)$.

4.7. General linear groups

The maximal compact subgroup of $GL(a, \mathbb{R})$ is $O(a)$, the Weyl group is $W(A_{a-1}) = S_a$ and

$$M \cong \underbrace{O(1) \times \dots \times O(1)}_a.$$

We list the case of the connected component $GL(a, \mathbb{R})^+$ (matrices with positive determinant) instead, because its maximal compact group is $K = SO(a)$ which is connected, and irreducible representations are parametrized by their highest weights.

Definition 4.8. The *relevant* K -types are the ones with highest weights

$$\mu(2^k, 0^l).$$

The corresponding Weyl group representations on V^M are $\sigma[(k, a - k)]$.

We omit the details, the proof is essentially the discussion about the representation of S_a on $\bigwedge \mathbb{C}^a \otimes \bigwedge \mathbb{C}^a$ for the orthogonal groups.

4.8. Relevant W -types

Definition 4.9. Let W be the Weyl group of types B, C and D. The following W -types will be called *relevant*:

$$\sigma_e(n - r, r) := \sigma[(n - r), (r)], \quad \sigma_o(k, n - k) := \sigma[(k, n - k), (0)]. \tag{4.37}$$

In type D for n even, and $r = \frac{1}{2}n$, there are two W -types, $\sigma_e[(\frac{1}{2}n), (\frac{1}{2}n)]_{I,II} := \sigma[(\frac{1}{2}n), (\frac{1}{2}n)]_{I,II}$. If the root system is not simple, the relevant W -types are tensor products of relevant W -types on each factor.

5. Intertwining operators

5.1. The notation from § 4 is in effect. Recall that $X(\nu)$ denotes the spherical principal series. Let $w \in W$. Then there is an intertwining operator

$$I(w, \nu) : X(\nu) \rightarrow X(w\nu). \tag{5.1}$$

Let \dot{w} be a representative of w . If $f \in X(\nu)$, then

$$I(w, \nu)f(g) := \int_{N/N \cap wNw^{-1}} f(g\dot{w}^{-1}n) dn.$$

These intertwining operators make sense in both the real and p -adic case. They have the properties that the integral converges for $\text{Re}\langle \nu, \alpha \rangle \gg 0$, and $I(w, \nu)$ has an analytic continuation for $\langle \text{Re } \nu, \alpha \rangle \geq 0$.

If (μ, V) is a K -type (K the maximal compact subgroup in the real case, $G(\mathcal{R})$ in the p -adic case), then $I(w, \nu)$ induces a map

$$I_V(w, \nu) : \text{Hom}_K[V, X(\nu)] \rightarrow \text{Hom}_K[V, X(w\nu)]. \tag{5.2}$$

By Frobenius reciprocity, we get a map

$$R_V(w, \nu) : (V^*)^{K \cap B} \rightarrow (V^*)^{K \cap B}. \tag{5.3}$$

In case (μ, V) is trivial the spaces are one dimensional and $R_V(w, \nu)$ is a scalar. We normalize $I(w, \nu)$ so that this scalar is 1. The $R_V(w, \nu)$ are meromorphic functions in ν , and the $I(w, \nu)$ have the following additional properties.

- (1) If $w = s_{\alpha_1} \cdots s_{\alpha_k}$ is a reduced decomposition (α_i simple roots and s_{α_i} the corresponding root reflections) let $\ell(w) := k$. If $w = w_1 \cdot w_2$ with $\ell(w) = \ell(w_1) + \ell(w_2)$, then $I(w, \nu) = I(w_1, w_2\nu) \circ I(w_2, \nu)$. In particular, $I(w, \nu)$ factors into a product of intertwining operators I_j , one for each s_{α_j} . These operators are

$$I_j : X(s_{\alpha_{j+1}} \cdots s_{\alpha_k} \cdot \nu) \rightarrow X(s_{\alpha_j} \cdots s_{\alpha_k} \cdot \nu). \tag{5.4}$$

- (2) Let $P = MN$ be a standard parabolic subgroup (so $A \subset M$) and $w \in W(M, A)$. Because $X(\nu) = \text{Ind}_P^G[X_M(\nu)]$, we can induce the intertwining operator $I_M(w, \nu) : X_M(\nu) \rightarrow X_M(w\nu)$ to form the induced intertwining operator $\text{Ind}_M^G[I_M(w, \nu)] : X(\nu) \rightarrow X(w\nu)$,

$$\text{Ind}_M^G[I_M(w, \nu)] : \text{Ind}_P^G[X_M(\nu)] \rightarrow \text{Ind}_P^G[X_M(w\nu)].$$

Then $I(w, \nu) = \text{Ind}_M^G[I_M(w, \nu)]$.

- (3) If $\text{Re}\langle \nu, \alpha \rangle \geq 0$ for all positive roots α , then $R_V(w_0, \nu)$ has no poles, and the image of $I(w_0, \nu)$ ($w_0 \in W$ is the long element) is $L(\nu)$.
- (4) If $-\bar{\nu}$ is in the same Weyl group orbit as ν , let w be the shortest element so that $w\nu = -\bar{\nu}$. Then the hermitian dual of $X(\nu)$ is $X(\nu)^h = X(w\nu)$. Let $\langle \cdot, \cdot \rangle_h$ be the hermitian pairing between $X(\nu)$ and $X(w\nu)$. It follows that $L(\nu)$ is hermitian with inner product

$$\langle v_1, v_2 \rangle := \langle v_1, I(w, \nu)v_2 \rangle_h.$$

For the remainder of the section we consider the real case only.

Let α be a simple root and $P_\alpha = M_\alpha N$ be the standard parabolic subgroup so that the derived part of the Lie algebra M_α is isomorphic to the $\mathfrak{sl}(2, \mathbb{R})$ generated by the root vectors $E_{\pm\alpha}$. We assume that $\theta E_\alpha = -E_{-\alpha}$, where θ is the Cartan involution corresponding to K . Let $D_\alpha = \sqrt{-1}(E_\alpha - E_{-\alpha})$ and $s_\alpha = e^{\sqrt{-1}\pi D_\alpha/2}$. Here by s_α , we actually mean the representative in $N_K(A)$ of the Weyl group reflection. Then $s_\alpha^2 = m_\alpha$ is in $K \cap B \cap M_\alpha$. Since the square of any element in $K \cap B$ is in the centre, and $K \cap B$ normalizes the the root vectors, $\text{Ad } m(D_\alpha) = \pm D_\alpha$. Grade $V^* = \oplus V_i^*$ according to the absolute values of the eigenvalues of D_α (which are integers). Then $K \cap B$ preserves this grading and

$$(V^*)^{K \cap B} = \bigoplus_{i \text{ even}} (V_i^*)^{K \cap B}.$$

The map $\psi_\alpha : \mathfrak{sl}(2, \mathbb{R}) \rightarrow \mathfrak{g}$ determined by

$$\psi_\alpha \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = E_\alpha, \quad \psi_\alpha \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = E_{-\alpha}$$

determines a map

$$\Psi_\alpha : \text{SL}(2, \mathbb{R}) \rightarrow G \tag{5.5}$$

with image G_α , a connected group with Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

Proposition 5.1. On $(V_{2m}^*)^{K \cap B}$,

$$R_V(s_\alpha, \nu) = \begin{cases} \text{Id} & \text{if } m = 0, \\ \prod_{0 \leq j < m} \frac{2j + 1 - \langle \nu, \check{\alpha} \rangle}{2j + 1 + \langle \nu, \check{\alpha} \rangle} \text{Id} & \text{if } m \neq 0. \end{cases}$$

In particular, $I(w, \nu)$ is an isomorphism unless $\langle \nu, \check{\alpha} \rangle \in 2\mathbb{Z} + 1$.

Proof. The formula is well known for $\text{SL}(2, \mathbb{R})$. The second assertion follows from this and the listed properties of intertwining operators. □

Corollary 5.2. For relevant K -types the formula is

$$R_V(s_\alpha, \nu) = \begin{cases} \text{Id} & \text{on the } +1 \text{ eigenspace of } s_\alpha, \\ \frac{1 - \langle \nu, \check{\alpha} \rangle}{1 + \langle \nu, \check{\alpha} \rangle} \text{Id} & \text{on the } -1 \text{ eigenspace of } s_\alpha. \end{cases}$$

When restricted to $(V^*)^{K \cap B}$, the long intertwining operator is the product of the $R_V(s_\alpha, *)$ corresponding to the reduced decomposition of w_0 and depends only on the Weyl group structure of $(V^*)^{K \cap B}$.

Proof. Relevant K -types have the property that the even eigenvalues of D_α are $0, \pm 2$ only. The element s_α acts by 1 on the zero eigenspace of D_α and by -1 on the ± 2 eigenspace. The claim follows from this. □

5.2. We now show that the formulae in the previous section coincide with corresponding ones in the p -adic case. In the split p -adic case, spherical representations are a subset of representations with \mathcal{I} -fixed vectors, where \mathcal{I} is an Iwahori subgroup. As explained in [18], the category of representations with \mathcal{I} fixed vectors is equivalent to the category of finite-dimensional representations of the Iwahori–Hecke algebra $\mathcal{H} := \mathcal{H}(\mathcal{I} \backslash G / \mathcal{I})$. The equivalence is

$$\mathcal{V} \rightarrow \mathcal{V}^{\mathcal{I}}. \tag{5.6}$$

The papers [10] and [11] show that the problem of the determination of the unitary dual of representations with \mathcal{I} fixed vectors, is equivalent to the problem of the determination of the unitary irreducible representations of \mathcal{H} with real infinitesimal character. In fact it is the affine graded Hecke algebras we will need to consider, and they are as follows.

Let $\mathbb{A} := S(\check{\mathfrak{a}})$, and define the affine graded Hecke algebra to be $\mathbb{H} := \mathbb{C}[W] \otimes \mathbb{A}$ as a vector space, and usual algebra structure for $\mathbb{C}[W]$ and \mathbb{A} .

The generators of $\mathbb{C}[W]$ are denoted by t_α corresponding to the simple reflections s_α , while the generators of \mathbb{A} are $\omega \in \check{\mathfrak{a}}$. Impose the additional relation

$$\omega t_\alpha = s_\alpha(\omega) t_\alpha + \langle \omega, \check{\alpha} \rangle, \quad \omega \in \check{\mathfrak{a}}, \tag{5.7}$$

where t_α is the element in $\mathbb{C}[W]$ corresponding to the simple root α . If $X(\chi)$ is the standard (principal series) module determined by χ , then

$$X(\chi)^{\mathcal{I}} = \mathbb{H} \otimes_{\mathbb{A}} \mathbb{C}_\chi. \tag{5.8}$$

The intertwining operator $I(w, \chi)$ is a product of operators I_{α_i} according to a reduced decomposition of $w = s_{\alpha_1} \cdots s_{\alpha_k}$. If α is a simple root,

$$r_\alpha := (t_\alpha \check{\alpha} - 1) \frac{1}{\check{\alpha} - 1}, \quad I_\alpha : x \otimes \mathbb{1}_\chi \mapsto xr_\alpha \otimes \mathbb{1}_{s_\alpha \chi}. \tag{5.9}$$

The $I(w, \chi)$ have the same properties as in the real case. The r_α are multiplied on the right, so we can replace $\check{\alpha}$ with $-\langle \nu, \check{\alpha} \rangle$ in the formulae. Furthermore,

$$\mathbb{C}[W] = \sum_{\sigma \in \check{W}} V_\sigma \otimes V_\sigma^*.$$

Since r_α acts as multiplication on the right, it gives rise to an operator

$$r_\sigma(s_\alpha, \nu) : V_\sigma^* \rightarrow V_\sigma^*.$$

Theorem 5.3. *The $R_V(s_\alpha, \nu)$ for the real case on relevant K -types coincide with the $r_\sigma(s_\alpha, \nu)$ on the $V_\sigma^* \cong (V^*)^{K \cap B}$.*

Proof. These operators act the same way:

$$r_\sigma(s_\alpha, \nu) = \begin{cases} \text{Id} & \text{on the } +1 \text{ eigenspace of } s_\alpha, \\ \frac{1 - \langle \nu, \check{\alpha} \rangle}{1 + \langle \nu, \alpha \rangle} \text{Id} & \text{on the } -1 \text{ eigenspace of } s_\alpha. \end{cases} \tag{5.10}$$

The assertion is now clear from Corollary 5.2 and formula (5.7). □

Remark 5.4. (1) The Hecke algebra for a p -adic group G is typically defined using the dual root system of the complex group \check{G} . For example the formulae for r_α in the literature, e.g. [10–12], have roots instead of coroots.

(2) The intertwining operator I_α is the standard formula for $\text{SL}(2)$ applied to \mathcal{I} -fixed vectors. For more details the reader may consult [12] and [19].

5.3. The main point of §5.2 is that for the real case, and a relevant K -type (V, μ) , the intertwining operator calculations coincide with the intertwining operator calculations for the affine graded Hecke algebra on the space $V^{K \cap B}$. Thus we will deal with the Hecke algebra calculations exclusively, but the conclusions hold for both the real and p -adic case. Recall from §2.3 that to each χ we have associated a nilpotent orbit $\check{\mathcal{O}}$, and Levi components $\check{\mathfrak{m}}_{\text{BC}}$ and $\check{\mathfrak{m}}_{\text{KL}}$. These are special instances of the following situation. Assume that $\check{\mathcal{O}}$ is written as in (2.12). Explicitly, $((a_1, a_1), \dots, (a_k, a_k); (d_i))$ satisfy the following.

g̃ of type B: (d_i) all odd; they are relabelled $(2x_0 + 1, \dots, 2x_{2m} + 1)$;

g̃ of type C: (d_i) all even; they are relabelled $(2x_0, \dots, 2x_{2m})$;

g̃ of type D: (d_i) all odd; they are relabelled $(2x_0 + 1, \dots, 2x_{2m-1} + 1)$.

Similar to (2.13), let

$$\check{m} := \mathfrak{gl}(a_1) \times \cdots \times \mathfrak{gl}(a_k) \times \check{\mathfrak{g}}(n_0), \quad n_0 = n - \sum a_i. \tag{5.11}$$

We consider parameters of the form $\chi = \frac{1}{2}\check{h} + \nu$.

Write χ_0 for the parameter $\frac{1}{2}\check{h}$, and $\chi_i := (-\frac{1}{2}(a_i - 1) + \nu_i, \dots, \frac{1}{2}(a_i - 1) + \nu_i)$. We focus on χ_0 as a parameter on $\check{\mathfrak{g}}(n_0)$. We attach two Levi components

$$\left. \begin{array}{l} \check{\mathfrak{g}}_e: \\ \text{B} \quad \mathfrak{gl}(x_{2m-1} + x_{2m-2} + 1) \times \cdots \times \mathfrak{gl}(x_1 + x_0 + 1) \times \check{\mathfrak{g}}(x_{2m}), \\ \text{C} \quad \mathfrak{gl}(x_{2m-1} + x_{2m-2}) \times \cdots \times \mathfrak{gl}(x_1 + x_0) \times \check{\mathfrak{g}}(x_{2m}), \\ \text{D} \quad \mathfrak{gl}(x_{2m-1} + x_{2m-2} + 1) \times \cdots \times \mathfrak{gl}(x_1 + x_0 + 1), \\ \\ \check{\mathfrak{g}}_o: \\ \text{B} \quad \mathfrak{gl}(x_{2m} + x_{2m-1} + 1) \times \cdots \times \mathfrak{gl}(x_2 + x_1 + 1) \times \check{\mathfrak{g}}(x_0), \\ \text{C} \quad \mathfrak{gl}(x_{2m} + x_{2m-1}) \times \cdots \times \mathfrak{gl}(x_2 + x_1) \times \check{\mathfrak{g}}(x_0), \\ \text{D} \quad \mathfrak{gl}(x_{2m-3} + x_{2m-4} + 1) \times \cdots \times \mathfrak{gl}(x_{2m-2}) \times \check{\mathfrak{g}}(x_{2m-1} + 1). \end{array} \right\} \tag{5.12}$$

There are one-dimensional representations $L(\chi_e)$ and $L(\chi_o)$ such that the spherical irreducible representation $L(\chi_0) = \bar{X}(\chi_0)$ with infinitesimal character χ_0 is the spherical irreducible subquotient of $X_e := \text{Ind}_{P_e}^G(L(\chi_e))$ and $X_o := \text{Ind}_{P_o}^G(L(\chi_o))$ respectively.

The parameters χ_e and χ_o are written in terms of strings as follows:

$$\left. \begin{array}{l} X_e: \\ \text{B:} \quad \cdots(-x_{2i-1}, \dots, x_{2i-2}) \cdots (-x_{2m}, \dots, -1), \\ \text{C:} \quad \cdots(-x_{2i-1} + \frac{1}{2}, \dots, x_{2i-2} - \frac{1}{2}) \cdots (-x_{2m} + \frac{1}{2}, \dots, -\frac{1}{2}), \\ \text{D:} \quad \cdots(-x_{2i-1}, \dots, x_{2i-2}) \cdots; \\ \\ X_o: \\ \text{B:} \quad \cdots(-x_{2i}, \dots, x_{2i-1}) \cdots (-x_0, \dots, -1), \\ \text{C:} \quad \cdots(-x_{2i} + \frac{1}{2}, \dots, x_{2i-1} - \frac{1}{2}) \cdots (-x_0 + \frac{1}{2}, \dots, -\frac{1}{2}), \\ \text{D:} \quad \cdots(-x_{2i}, \dots, x_{2i-1}) \cdots (-x_{2m-2}, \dots, -1)(-x_{2m-1} + 1, \dots, 0). \end{array} \right\} \tag{5.13}$$

Theorem 5.5. For the Hecke algebra, p -adic groups,

$$\begin{aligned} [\sigma[(n-r), (r)] : X_e] &= [\sigma[(n-r), (r)] : L(\chi_0)], \\ [\sigma[(k, n-k), (0)] : X_o] &= [\sigma[(k, n-k), (0)] : L(\chi_0)] \end{aligned}$$

hold.

The proof is in §6.9.

For a general parameter $\chi = \chi_0 + \nu$, the strings defined in §2 and the above construction define parabolic subgroups with Levi components $\mathfrak{gl}(a_1) \times \cdots \times \mathfrak{gl}(a_k) \times \mathfrak{g}_e$ and $\mathfrak{gl}(a_1) \times \cdots \times \mathfrak{gl}(a_r) \times \mathfrak{g}_o$, and corresponding $L_e(\chi)$ and $L_o(\chi)$. We denote these induced modules by X_e and X_o as well.

Corollary 5.6. *The relations*

$$\begin{aligned} [\sigma[(n-r), (r)] : X_e] &= [\sigma[(n-r), (r)] : L(\chi)], \\ [\sigma[(k, n-k), (0)] : X_o] &= [\sigma[(k, n-k), (0)] : L(\chi)] \end{aligned}$$

hold in general. For real groups, in the notation of §§4.2–4.4,

$$\begin{aligned} [\mu_e(r, n-r) : X_e] &= [\mu_e(r, n-r) : L(\chi)], \\ [\mu_o(k, n-k) : X_o] &= [\mu_o(k, n-k) : L(\chi)]. \end{aligned}$$

Proof. The results in §5.2 show that the intertwining operators on $\sigma_e(k, n-k)$ for the p -adic group equal the intertwining operators for $\mu_e(k, n-k)$ for the real group, and similarly for σ_o and μ_o . Thus the multiplicities of the σ_e/σ_o in $L(\chi)$ for the p -adic case equal the multiplicities of the corresponding μ_e/μ_o in $L(\chi)$ in the real case. We do the p -adic case first. Recall Theorem 2.9 which states that $I_{M_{KL}}(\chi) = L(\chi)$. The Levi subgroup M_{KL} is a product of GL factors, which we will denote M_A , with a factor $G(n_0)$. So for W -type multiplicities we can replace $I_{M_{KL}}(\chi)$ by $\text{Ind}_{M_A \times G(n_0)}[\otimes \text{triv} \otimes L(\chi_0)]$. We explain the case of $\sigma_e = \sigma_e(k, n-k)$, that of σ_o being identical. By Frobenius reciprocity,

$$\begin{aligned} \text{Hom}_W[\sigma_e : L(\chi)] &= \text{Hom}_W[\sigma_e : I_{M_{KL}}(\chi)] \\ &= \text{Hom}_{W(M_A) \times W(G(n_0))}[\sigma_e : \text{triv} \otimes L(\chi_0)]. \end{aligned} \tag{5.14}$$

Using the formulae for restrictions of representations for Weyl groups of classical types, it follows that

$$\begin{aligned} \dim \text{Hom}_W[\sigma_e : L(\chi)] &= \sum_{k'} \dim \text{Hom}_{W(G(n_0))}[\sigma_e(k', n_0 - k') : L(\chi_0)] \\ &= \sum_{k'} \dim \text{Hom}_{W(G(n_0))}[\sigma_e(k', n_0 - k') : X_{e, G(n_0)}], \end{aligned} \tag{5.15}$$

where the last step is Theorem 5.3. Since $X_e = \text{Ind}_{M_A \times G(n_0)}^G[\otimes L(\chi_i) \otimes X_{e, G(n_0)}(\chi_0)]$, again by Frobenius reciprocity, one can show that this is also equal to $\dim \text{Hom}_W[\sigma_e : X_e]$. This proves the claim for the p -adic case.

In the real case the proof is complete once we observe that in all the steps for the p -adic case, the multiplicity of $\text{triv} \otimes \mu_e(k', n_0 - k')$ in the restriction of $\mu_e(k, n-k)$ matches the multiplicity of $\text{triv} \otimes \sigma_e(k', n_0 - k')$ in the restriction of $\sigma_e(k, n-k)$. Similarly for μ_o and σ_o . □

6. Hecke algebra calculations

6.1. The proof of the results in §5.3 is by a computation of intertwining operators on the relevant K -types. It only depends on the W -type of $V^{K \cap B}$, so we work in the setting of the Hecke algebra. The fact that we can deal exclusively with W -types, is a big advantage. In particular we do not have to worry about disconnectedness of Levi components. We will write $\text{GL}(k)$ for the Hecke algebra of type A and $G(n)$ for the types B, C or D as the case may be. This is so as to emphasize that the results are about groups, real or p -adic.

The intertwining operators will be decomposed into products of simpler operators induced from operators coming from maximal Levi subgroups. We introduce these first.

Suppose M is a Levi component of the form

$$\text{GL}(a_1) \times \cdots \times \text{GL}(a_i) \times G(n_0). \tag{6.1}$$

Let χ_i be characters for $\text{GL}(a_i)$. We simplify the notation somewhat by writing

$$\chi_i \longleftrightarrow (\nu_i) := (-\frac{1}{2}(a_i - 1) + \nu_i, \dots, \frac{1}{2}(a_i - 1) + \nu_i). \tag{6.2}$$

The parameter is antidominant, and so $L(\chi_i)$ occurs as a submodule of the principal series $X((\nu_i))$. The module is spherical one dimensional, and the action of \mathfrak{a} is

$$\chi_i(\omega) = \langle \omega, (\frac{1}{2}(a_i - 1) + \nu_i, \dots, -\frac{1}{2}(a_i - 1) + \nu_i) \rangle, \quad \omega \in \mathfrak{a}, \tag{6.3}$$

while W acts trivially. The trivial representation χ_0 of $G(n_0)$ corresponds to the string $(-n_0 + \epsilon, \dots, -1 + \epsilon)$ where

$$\epsilon := \begin{cases} 0 & \mathbb{H} \text{ of type B,} \\ \frac{1}{2} & \mathbb{H} \text{ of type C,} \\ 1 & \mathbb{H} \text{ of type D.} \end{cases} \tag{6.4}$$

We abbreviate this as (ν_0) . Again $L(\chi_0)$ is the trivial representation, and because χ_0 is antidominant, it appears as a submodule of the principal series $X(\chi_0)$. We abbreviate

$$X_M(\cdots(\nu_i)\cdots) := \text{Ind}_{\prod \text{GL}(a_i) \times G(n_0)}^G \left[\bigotimes \chi_i \otimes \text{triv} \right]. \tag{6.5}$$

The module $X_M(\cdots(\nu_i)\cdots)$ is a submodule of the standard module $X(\chi)$ with parameter corresponding to the strings

$$\chi := (\dots, -\frac{1}{2}(a_i - 1) + \nu_i, \dots, \frac{1}{2}(a_i - 1) + \nu_i, \dots, -n_0 + \epsilon, \dots, -1 + \epsilon). \tag{6.6}$$

In the setting of the Hecke algebra, the induced modules (6.5) is really

$$X_M(\cdots(\nu_i)\cdots) = \mathbb{H} \otimes_{\mathbb{H}_M} \left[\bigotimes \chi_i \otimes \text{triv} \right].$$

Let $w_{i,i+1} \in W$ be the shortest Weyl group element which interchanges the strings (ν_i) and (ν_{i+1}) in ν , and fixes all other coordinates. The intertwining operator $I_{w_{i,i+1}} : X(\nu) \rightarrow X(w_{i,i+1}\nu)$ restricts to an intertwining operator

$$I_{M,i,i+1}(\cdots(\nu_i)(\nu_{i+1})\cdots) : X_M(\cdots(\nu_i)(\nu_{i+1})\cdots) \rightarrow X_{w_{i,i+1}M}(\cdots(\nu_{i+1})(\nu_i)\cdots). \tag{6.7}$$

This operator is induced from the same kind for $GL(a_i + a_{i+1})$, where

$$M_{i,i+1} = GL(a_i) \times GL(a_{i+1}) \subset GL(a_i + a_{i+1})$$

is the Levi component of a maximal parabolic subgroup.

Let $w_l \in W$ be the shortest element which changes (ν_l) to $(-\nu_l)$, and fixes all other coordinates. It induces an intertwining operator

$$I_{M,l}(\cdots(\nu_l)(\nu_0)) : X_M(\cdots(\nu_l), (\nu_0)) \rightarrow X_{w_l M}(\cdots(-\nu_l), (\nu_0)). \tag{6.8}$$

In this case, $w_l M = M$, so we will not always include it in the notation. In type D, if $n_0 = 0$, the last entry of the resulting string might have to stay $-\frac{1}{2}(a_l - 1) + \nu_l$ instead of $\frac{1}{2}(a_l - 1) - \nu_l$. This operator is induced from the same kind on $G(a_l + n_0)$ with $M_l = GL(a_l) \times G(n_0) \subset G(a_l + n_0)$ the Levi component of a maximal parabolic subgroup.

Lemma 6.1. *The operators $I_{M,i,i+1}$ and $I_{M,l}$ are meromorphic in ν_i in both the real and p -adic case.*

- (1) $I_{M,i,i+1}$ has poles only if $\frac{1}{2}(a_i - 1) + \nu_i - \frac{1}{2}(a_{i+1} - 1) - \nu_{i+1} \in \mathbb{Z}$. If so, a pole only occurs if

$$-\frac{1}{2}(a_i - 1) + \nu_i < -\frac{1}{2}(a_{i+1} - 1) - \nu_{i+1}, \quad \frac{1}{2}(a_i - 1) + \nu_i < \frac{1}{2}(a_{i+1} - 1) + \nu_{i+1}.$$

- (2) $I_{M,l}$ has a pole only if $\frac{1}{2}(a_l - 1) + \nu_l \equiv \epsilon \pmod{\mathbb{Z}}$. In that case, a pole only occurs if

$$-\frac{1}{2}(a_l - 1) + \nu_l < 0.$$

Proof. We prove the assertion for $I_{M,i,i+1}$, the other one is similar. The fact that the integrality condition is necessary is clear. For the second condition, it is sufficient to consider the case $M = GL(a_1) \times GL(a_2) \subset GL(a_1 + a_2)$. If the strings are strongly nested, then the operator cannot have any pole because X_M is irreducible. It remains to show there is no pole in the case when $-\frac{1}{2}(a_2 - 1) + \nu_2 \leq -\frac{1}{2}(a_1 - 1) + \nu_1$, and $\frac{1}{2}(a_1 - 1) + \nu_1 > \frac{1}{2}(a_2 - 1) + \nu_2$. Let

$$\left. \begin{aligned} M' &:= GL(\frac{1}{2}(a_1 + a_2) + \nu_2 + \nu_1) \times GL(\frac{1}{2}(a_1 - a_2) + \nu_1 - \nu_2) \times GL(a_2), \\ (\nu'_1) &= (-\frac{1}{2}(a_1 - 1) + \nu_1, \dots, \frac{1}{2}(a_2 - 1) + \nu_2), \\ (\nu'_2) &= (\frac{1}{2}(a_2 - 1) + 1 + \nu_2, \dots, \frac{1}{2}(a_1 - 1) + \nu_1), \\ (\nu'_3) &= (\nu_2) = (-\frac{1}{2}(a_2 - 1) + \nu_2, \dots, \frac{1}{2}(a_2 - 1) + \nu_2). \end{aligned} \right\} \tag{6.9}$$

Then $X_M((\nu_1)(\nu_2)) \subset X_{M'}((\nu'_1)(\nu'_2)(\nu'_3))$, and $I_{M,1,2}$ is the restriction of

$$I_{w_{2,3}M',1,2}((\nu'_1)(\nu'_3)(\nu'_2)) \circ I_{M',2,3}((\nu'_1)(\nu'_2)(\nu'_3))$$

to X_M . Because the strings $(\nu'_1)(\nu'_3)$ are strongly nested, $I_{w_{2,3}M',1,2}$ has no pole, and $I_{M',2,3}$ has no pole because it is a restriction of operators coming from $SL(2)$ s which do not have poles. The claim follows. □

Let σ be a W -type. We are interested in computing $r_\sigma(w, \dots(\nu_i) \dots)$, where w changes all the ν_i for $1 \leq i$ to $-\nu_i$. The operator can be factored into a product of $r_\sigma(w_{i,i+1}, *)$ of the type (6.7) and $r_\sigma(w_l, *)$ of the type (6.8). These operators are more tractable. Here is a more precise explanation. Let M be the Levi component

$$\text{GL}(a_1) \times \dots \times \text{GL}(a_i + a_{i+1}) \times \dots \quad \text{in case (6.7),} \tag{6.10}$$

$$\text{GL}(a_1) \times \dots \times G(a_l + n_0) \quad \text{in case (6.8).} \tag{6.11}$$

Since X_M is induced from the trivial $W(M)$ module,

$$\begin{aligned} \text{Hom}_W[\sigma, X_M((\nu_i))] &= \text{Hom}_{W(M)}[\sigma|_{W(M)} : \text{triv} \otimes X_{M_{i,i+1}}((\nu_i), (\nu_{i+1})) \otimes \text{triv}] \\ &\qquad \qquad \qquad \text{in case (6.7),} \end{aligned} \tag{6.12}$$

$$\text{Hom}_W[\sigma, X((\nu_i))] = \text{Hom}_{W(M)}[\sigma|_{W(M)} : \text{triv} \otimes X_{M_l}((\nu_l), (\nu_0))] \quad \text{in case (6.8),} \tag{6.13}$$

where $M_{i,i+1} = \text{GL}(a_i) \times \text{GL}(a_{i+1})$ is a maximal Levi component of $\text{GL}(a_i + a_{i+1})$ and $M_l = \text{GL}(a_l) \times G(n_0)$ is a maximal Levi component of $G(a_l + n_0)$. To compute the $r_\sigma(w_{i,i+1}, *)$ and $r_\sigma(w_l, *)$, it is enough to compute the corresponding r_{σ_j} for the σ_j occurring in the restriction $\sigma|_{W(M)}$ in the cases $\text{GL}(a_i) \times \text{GL}(a_{i+1}) \subset \text{GL}(a_i + a_{i+1})$ and $\text{GL}(a_l) \times G(n_0) \subset G(a_l + n_0)$. The restrictions of relevant W -types to Levi components consist of relevant W -types of the same kind, i.e. $\sigma[(n-r), (r)]$ restricts to a sum of representations of the kind σ_e , and $\sigma[(k, n-k), (0)]$ restricts to a sum of σ_o . Typically the multiplicities of the factors are 1.

We also note that

$$X_M|_W = \sum_{\sigma \in \hat{W}} V_\sigma \otimes (V_\sigma^*)^{W(M)}. \tag{6.14}$$

So the $r_\sigma(w, *)$ map $(V_\sigma^*)^{W(M)}$ to $(V_\sigma^*)^{W(wM)}$.

6.2. Maximal Levi components, summary of results

In the next sections we will compute the cases of Levi components of maximal parabolic subgroups. We summarize the results. A typical Levi component will be denoted $\text{GL}(k) \times G(n) \subset G(k+n)$. The type refers to the type of $G(n)$.

Theorem 6.2.

Type A. The intertwining operator $I_{M,1,2}((\nu_1)(\nu_2))$ restricted to $\sigma(m, k+n-m)$ is multiplication by the scalar

$$r_{\sigma(m, k+n-m)}(k, n, \nu_1, \nu_2) = \prod_{0 \leq j \leq m-1} \frac{(\nu_1 - \frac{1}{2}(k-1)) - (\frac{1}{2}(n-1) + \nu_2 + 1) + j}{(\nu_1 + \frac{1}{2}(k-1)) - (-\frac{1}{2}(n-1) + \nu_2 - 1) - j}. \tag{6.15}$$

Type B. For $\sigma = \sigma_e(m, k + n - m)$,

$$r_{\sigma_e(m, k+n-m)}((\nu)) = \prod_{0 \leq j \leq m-1} \frac{n + \frac{1}{2} - (-\frac{1}{2}(k-1) + \nu) - j}{n + \frac{1}{2} + (\frac{1}{2}(k-1) + \nu) - j}. \tag{6.16}$$

Type C. For $\sigma = \sigma_e(m, k + n - m)$,

$$r_{\sigma_e(m, k+n-m)}((\nu)) = \prod_{0 \leq j \leq m-1} \frac{n + \frac{1}{2} - (-\frac{1}{2}(k-1) + \nu) - j}{n + \frac{1}{2} + (\frac{1}{2}(k-1) + \nu) - j}. \tag{6.17}$$

Type D. For $\sigma = \sigma_e(m, k + n - m)$,

$$r_{\sigma_e(m, k+n-m)}((\nu)) = \prod_{0 \leq j \leq m-1} \frac{n - (-\frac{1}{2}(k-1) + \nu) - j}{n + (\frac{1}{2}(k-1) + \nu) - j}. \tag{6.18}$$

For types B and C the formulae hold for $n = 0$ as well. In type D when $n = 0$, there are two cases $GL(k) \subset G(k)$ and $GL(k)' \subset G(n)$. The formula is the same for $r_{\sigma_e(n-m, m)}((\nu))$ and $r_{\sigma_e(n-m, m)}((\nu)')$:

$$\prod_{0 \leq j < m} \frac{(\frac{1}{2}(k-1) - \nu) - j}{(\frac{1}{2}(k-1) + \nu) - j}. \tag{6.19}$$

For $2m = n$ there are two representations with subscript I and II; the formula above is the same.

For $\sigma = \sigma_o(m, k + n - m)$ and types B, C and D, the scalar $r_{\sigma_o(m, k+n-m)}((\nu)(\nu_0))$ equals

$$\prod_{0 \leq j \leq m-1} \frac{(\nu - \frac{1}{2}(k-1)) - (1 - \epsilon) + j}{(\nu + \frac{1}{2}(k-1)) - (-n - \epsilon) - j} \frac{(-n - \epsilon) - (-\nu + \frac{1}{2}(k-1)) + j}{(1 - \epsilon) - (-\nu - \frac{1}{2}(k-1)) - j}. \tag{6.20}$$

In this case $n > 0$ for σ_o to occur in the induced module.

6.3. $GL(k) \times GL(n) \subset GL(k + n)$

This is the case of $I_{i, i+1}$ with $i < l$. The module $X_M((\nu_1), (\nu_2))$ induced from the characters corresponding to

$$(-\frac{1}{2}(k-1) + \nu_1, \dots, \frac{1}{2}(k-1) + \nu_1), (-\frac{1}{2}(n-1) + \nu_2, \dots, \frac{1}{2}(n-1) + \nu_2) \tag{6.21}$$

has the following S_{k+n} structure. Let $s := \min(k, n)$ and write $\sigma(m, k + n - m)$ for the module corresponding to the partition $(m, k + n - m)$, $0 \leq m \leq s$. Then

$$X_M((\nu_1), (\nu_2))|_W = \bigoplus_{0 \leq m \leq s} \sigma(m, k + n - m). \tag{6.22}$$

Lemma 6.3. For $1 \leq m \leq s$, the intertwining operator $I_{M,1,2}((\nu_1)(\nu_2))$ restricted to σ gives

$$r_{\sigma(m,k+n-m)}(\nu_1, \nu_2) = \prod_{0 \leq j \leq m-1} \frac{(\nu_1 - \frac{1}{2}(k-1)) - (\frac{1}{2}(n-1) + \nu_2 + 1) + j}{(\nu_1 + \frac{1}{2}(k-1)) - (-\frac{1}{2}(n-1) + \nu_2 - 1) - j}.$$

Proof. The proof is an induction on k, n and m . We omit most details but give the general idea. Assume $0 < m < s$, the case $m = s$ is simpler. Embed $X_M((\nu_1), (\nu_2))$ into $X_{M'}((\nu'), (\nu''), (\nu_2))$, where $M' = GL(k-1) \times GL(1) \times GL(n)$, corresponding to the strings

$$(-\frac{1}{2}(k-1) + \nu_1, \dots, \frac{1}{2}(k-3) + \nu_1), (\frac{1}{2}(k-1) + \nu_1), (-\frac{1}{2}(n-1) + \nu_2, \dots, \frac{1}{2}(n-1) + \nu_2). \tag{6.23}$$

The intertwining operator $I_{M,1,2}(\nu_1, \nu_2)$ is the restriction of

$$I_{M',1,2}(\nu', \nu_2, \nu'') \circ I_{M',2,3}(\nu'; \nu'', \nu_2) \tag{6.24}$$

to $X_M((\nu_1), (\nu_2)) \subset X_{M'}((\nu'), (\nu''), (\nu_2))$. By an induction on $k+n$ we can assume that these operators are known. The W -type $\sigma(m, k+n-m)$ occurs with multiplicity 1 in $X_M((\nu_1), (\nu_2))$ and with multiplicity 2 in $X_{M'}((\nu'), (\nu''), (\nu_2))$. The restrictions are

$$\sigma(m, k+n-m)|_{W(M')} = \text{triv} \otimes \sigma(m-1, n+1-m) + \text{triv} \otimes \sigma(m, n-m) \quad \text{for } I_{M',1,2}, \tag{6.25}$$

$$\sigma(m, k+n-m)|_{W(M')} = \sigma(1, n) + \sigma(0, n+1) \quad \text{for } I_{M',2,3}. \tag{6.26}$$

The representation $\sigma(m, k+n-m)$ has a realization as harmonic polynomials in $S(\mathfrak{a})$ spanned by

$$\prod_{1 \leq \ell \leq m} (\epsilon_{i_\ell} - \epsilon_{j_\ell}), \tag{6.27}$$

where $(i_1, j_1), \dots, (i_m, j_m)$ are m pairs of integers satisfying $1 \leq i_\ell, j_\ell \leq k+n$, and $i_\ell \neq j_\ell$. We apply the intertwining operator to the $S_k \times S_n$ -fixed vector

$$e := \sum_{x \in S_k \times S_n} x \cdot [(\epsilon_1 - \epsilon_{k+1}) \times \dots \times (\epsilon_m - \epsilon_{k+m})]. \tag{6.28}$$

The intertwining operator $I_{M',2,3}$, has a simple form on the vectors

$$e_1 := \sum_{x \in S_{k-1} \times S_{n+1}} x \cdot [(\epsilon_1 - \epsilon_{k+1}) \times \dots \times (\epsilon_m - \epsilon_{k+m})] \quad \text{in } \sigma(0, n+1), \tag{6.29}$$

$$e_2 := \sum_{x \in S_{k-1} \times S_1 \times S_n} x \cdot [(\epsilon_1 - \epsilon_{k+1}) \times \dots \times (\epsilon_{m-1} - \epsilon_{k+m-1})(\epsilon_k - \epsilon_{k+m})] \quad \text{in } \sigma(1, n), \tag{6.30}$$

which appear in (6.26). They are mapped into scalar multiples (given by the lemma) of the vectors e'_1, e'_2 which are invariant under $S_{k-1} \times S_n \times S_1$, and transform according to

$\text{triv} \otimes \sigma(0, n + 1)$ and $\text{triv} \otimes \sigma(1, n)$. We choose

$$\left. \begin{aligned} e'_1 &= e_1, \\ e'_2 &:= \sum_{x \in S_{k-1} \times S_n \times S_1} x \cdot [(\epsilon_1 - \epsilon_k) \times \cdots \times (\epsilon_{m-1} - \epsilon_{k+m-2})(\epsilon_{k+n} - \epsilon_{k+m-1})]. \end{aligned} \right\} \quad (6.31)$$

The intertwining operator $I_{M',1,2}$ has a simple form on the vectors invariant under $S_{k-1} \times S_n \times S_1$ transforming according to $\sigma(m, k + n - m - 1)$ and $\sigma(m - 1, k + n - m)$. We can choose multiples of

$$f_1 := \sum_{x \in S_{k-1} \times S_n \times S_1} x \cdot [(\epsilon_1 - \epsilon_k) \times \cdots \times (\epsilon_{m-1} - \epsilon_{k+m-2})(\epsilon_m - \epsilon_{k+m-1})], \quad \text{in } \sigma(m - 1, k + n - m), \quad (6.32)$$

$$f_2 := \sum_{x \in S_{k-1} \times S_n \times S_1} x \cdot [(\epsilon_1 - \epsilon_k) \times \cdots \times (\epsilon_{m-1} - \epsilon_{k+m-2}) \cdot (e_m + \cdots + \epsilon_{k-1} + \epsilon_k + \epsilon_{k+m} + \cdots + \epsilon_{k+n-1} - (k + n - 2m + 1)\epsilon_n)], \quad \text{in } \sigma(m, k + n - m - 1). \quad (6.33)$$

The fact that f_1 transforms according to $\sigma(m, k + n - 1)$ follows from (6.27). The fact that f_2 transforms according to $\sigma(m - 1, k + n)$ is slightly more complicated. The product $\prod(\epsilon_1 - \epsilon_k) \times \cdots \times (\epsilon_{m-1} - \epsilon_{k+m-2})$ transforms according to $\sigma(m - 1, m - 1)$ under S_{2m-2} . The vector

$$[e_m + \cdots + \epsilon_{k-1} + \epsilon_k + \epsilon_{k+m} + \cdots + \epsilon_{k+n-1} - (k + n - 2m + 1)\epsilon_{k+n}]$$

is invariant under the $S_{k+n-2m-1}$ acting on the coordinates $\epsilon_m, \dots, \epsilon_k, \epsilon_{k+m}, \dots, \epsilon_{k+n-1}$. Since $\sigma(m, k + n - m - 1)$ does not have such invariant vectors, the product inside the sum in (6.33) must transform according to $\sigma(m - 1, k + n - m)$. The average under x in (6.33) is non-zero. The operator $I_{M',2,3}$ maps f_1 and f_2 into multiples (using the induction hypothesis) of the vectors f'_1, f'_2 which are the $S_n \times S_{k-1} \times S_1$ invariant vectors transforming according to $\sigma(m, k + n - 1)$ and $\sigma(m - 1, k + n - m)$. The composition $I_{M',1,2} \circ I_{M',2,3}$ maps e into a multiple of

$$e' := \sum_{\sigma \in S_n \times S_k} \sigma \cdot [(\epsilon_1 - \epsilon_{n+1}) \times \cdots \times (\epsilon_m - \epsilon_{n+m})]. \quad (6.34)$$

The multiple is computable by using the induction hypothesis and the expression of

- e in terms of e_1, e_2 ,
- e'_1, e'_2 in terms of f_1, f_2 , and
- e' in terms of f'_1, f'_2 .

For example for the case $k = 1$, we get the following formulae:

$$\left. \begin{aligned}
 e &= n(\epsilon_1 + \dots + \epsilon_k) - k(\epsilon_{k+1} + \dots + \epsilon_{k+n}), \\
 e_1 &= (n + 1)(\epsilon_1 + \dots + \epsilon_{k-1}) - (k - 1)(\epsilon_k + \dots + \epsilon_{k+n}), \\
 e_2 &= n\epsilon_k - (\epsilon_{k+1} + \dots + \epsilon_{k+n}), \\
 f_1 &= n(\epsilon_1 + \dots + \epsilon_{k-1}) - (k - 1)(\epsilon_k + \dots + \epsilon_{k+n-1}), \\
 f_2 &= (\epsilon_1 + \dots + \epsilon_{k-1}) + (\epsilon_k + \dots + \epsilon_{k+n-1}) - (k + n - 1)\epsilon_{k+n}, \\
 e' &= -k(\epsilon_1 + \dots + \epsilon_n) - n(\epsilon_{n+1} + \dots + \epsilon_{k+n}), \\
 e'_1 &= (n + 1)(\epsilon_1 + \dots + \epsilon_{k-1}) - (k - 1)(\epsilon_k + \dots + \epsilon_{k+n}), \\
 e'_2 &= -(\epsilon_k + \dots + \epsilon_{k+n-1}) + n(\epsilon_{k+n}), \\
 f'_1 &= -(k + 1)(\epsilon_1 + \dots + \epsilon_n) + n(\epsilon_{n+1} + \dots + \epsilon_{k+n-1}), \\
 f'_2 &= (\epsilon_1 + \dots + \epsilon_n) + (\epsilon_{n+1} + \dots + \epsilon_{k+n-1}) - (k + n - 1)\epsilon_{k+n}.
 \end{aligned} \right\} \tag{6.35}$$

Then

$$\left. \begin{aligned}
 e &= \frac{k - 1}{n + 1}e_1 - \frac{k + n}{n + 1}e_2, \\
 e'_1 &= \frac{k + n}{k + n - 1}f_1 + \frac{k - 1}{k + n - 1}f_2, \\
 e'_2 &= \frac{1}{k + n - 1}f_1 - \frac{n}{k + n - 1}f_2, \\
 e' &= \frac{k + n}{k + n - 1}f'_1 - \frac{n}{k + n - 1}f'_2.
 \end{aligned} \right\} \tag{6.36}$$

□

6.4. $GL(k) \times G(n) \subset G(n + k)$

In the next sections we prove Theorem 5.3 in the case of a parabolic subgroup with Levi component $GL(k) \times G(n)$ for the induced module

$$X_M((\nu_1)(\nu_0)) = \text{Ind}_M^G[L(\chi_1) \otimes L(\chi_0)]. \tag{6.37}$$

The notation is as in §6.1.

The strings are

$$(-\frac{1}{2}(k - 1) + \nu, \dots, \frac{1}{2}(k - 1) + \nu)(-n + 1 + \epsilon, \dots, -1 + \epsilon). \tag{6.38}$$

Recall that $\epsilon = 0$ when the Hecke algebra is type B, $\epsilon = \frac{1}{2}$ for type C, and $\epsilon = 1$ for type D, and

$$r_\sigma(\nu) : (V_\sigma^*)^{W(M)} \rightarrow (V_\sigma^*)^{W(M)}. \tag{6.39}$$

We will compute $r_\sigma(w_{1,}(\nu)(\nu_0))$ by induction on k . In this case the relevant W -types have multiplicity less than or equal to 1 so r_σ is a scalar.

6.5. We start with the special case $k = 1$ when the maximal parabolic subgroup P has Levi component $M = \text{GL}(1) \times G(n) \subset G(n + 1)$. In type D we assume $n \geq 1$. Then

$$X_M|_W = \sigma[(n + 1), (0)] + \sigma[(1, n), (0)] + \sigma[(n), (1)], \tag{6.40}$$

and all the W -types occurring are relevant. In types B and C the operator $r_\sigma(\nu)$ is the restriction to $(V_\sigma^*)^{W(M)}$ of the product

$$r_{1,2} \circ \cdots \circ r_{n,n+1} \circ r_{n+1} \circ r_{n,n+1} \circ \cdots \circ r_{1,2} \tag{6.41}$$

as an operator on V_σ . Here $r_{i,j}$ is the $r_\sigma(w, *)$ corresponding to the root $\epsilon_i - \epsilon_j$ and r_{n+1} is the r_σ corresponding to ϵ_{n+1} or $2\epsilon_{n+1}$ in types B and C. In type D, the operator is

$$r_{1,2} \circ \cdots \circ r_{n,n+1} \circ \tilde{r}_{n,n+1} \circ \cdots \circ r_{1,2}, \tag{6.42}$$

where $r_{i,i+1}$ are as before, and $\tilde{r}_{n,n+1}$ corresponds to $\epsilon_n + \epsilon_{n+1}$. Since the multiplicities are 1, this is a scalar.

Proposition 6.4. *The scalar $r_\sigma(w_1, ((\nu)(\nu_0)))$ is*

	$\sigma_e(1, n) = \sigma[(n), (1)]$	$\sigma_o(1, n) = \sigma[(1, n), (0)]$
B	$\frac{n + 1 - \nu}{n + 1 + \nu}$	$-\frac{n + 1 - \nu}{n + 1 + \nu}$
C	$\frac{\frac{1}{2} + n - \nu}{\frac{1}{2} + n + \nu}$	$\frac{\frac{1}{2} + n - \nu}{\frac{1}{2} + n + \nu} \frac{\frac{1}{2} - \nu}{\frac{1}{2} + \nu}$
D	$\frac{n - \nu}{n + \nu}$	$\frac{n - \nu}{n + \nu} \frac{1 - \nu}{1 + \nu}$

(6.43)

Proof. We perform an induction on n .

The reflection representation $\sigma[(n), (1)]$ has dimension $n + 1$ and the usual basis $\{\epsilon_i\}$. The $W(M)$ -fixed vector is ϵ_1 . The representation $\sigma[(1, n), (0)]$ has a basis $\epsilon_i^2 - \epsilon_j^2$ with the symmetric square action. The $W(M)$ -fixed vector is

$$\epsilon_1^2 - \frac{1}{n}(\epsilon_2^2 + \cdots + \epsilon_{n+1}^2).$$

The case $n = 0$ for type C is clear; the intertwining operator is 1 on $\mu_o(1, 0) = \text{triv}$ and $(\frac{1}{2} - \nu)/(\frac{1}{2} + \nu)$ on $\mu_e(0, 1) = \text{sgn}$. We omit the details for type B. In type for $n = 1$, i.e. D₂, the middle W -type in (6.40) decomposes further

$$\sigma[(2), (0)] + \sigma[(1), (1)]_I + \sigma[(1), (1)]_{II} + \sigma[(0), (2)]. \tag{6.44}$$

The representations $\sigma[(1), (1)]_{I,II}$ are one dimensional with bases $\epsilon_1 \pm \epsilon_2$. The result is straightforward in this case as well.

We now do the induction step. We give details for type B. In the case $\sigma_e(1, n)$, embed X_M in the induced module from the characters corresponding to

$$(\nu)(-n)(-n + 1, \dots, -1). \tag{6.45}$$

Write $M' = \text{GL}(1) \times \text{GL}(1) \times G(n - 1)$ for the Levi component corresponding to these three strings. Then the intertwining operator $I : X_M((\nu)(\nu_0)) \rightarrow X_M((- \nu)(\nu_0))$ is the restriction of

$$I_{M',1,2}((-n), (-\nu)(\nu_0)) \circ I_{M',2}((-n)(\nu)(\nu_0)) \circ I_{M',1,2}(\nu, (-n), (\nu_0)). \tag{6.46}$$

The r_σ have a corresponding decomposition

$$(r_\sigma)_{M',1,2}((- \nu), (-n)(\nu_0)) \circ (r_\sigma)_{M',2}((-n)(\nu)(\nu_0)) \circ (r_\sigma)_{M',1,2}(\nu)(-n)(\nu_0). \tag{6.47}$$

We need the restrictions of $\mu_e(1, n)$ and $\mu_o(1, n)$ to $W(M')$. We have

$$\begin{aligned} \text{Ind}_{W(B_{n-1})}^{W(B_{n+1})}[\sigma[(n - 1), (0)]] &= \sigma[(n + 1), (0)] + 2\sigma[(n), (1)] + 2\sigma[(1, n), (0)] \\ &\quad + 2\sigma[(1, n - 1), (1)] + \sigma[(n - 1), (2)] + \sigma[(n - 1), (1, 1)] \\ &\quad + \sigma[(2, n - 1), (0)] + \sigma[(1, 1, n - 1), (0)], \end{aligned} \tag{6.48 a}$$

$$\text{Ind}_{W(B_n)}^{W(B_{n+1})}[\sigma[(n), (0)]] = \sigma[(n + 1), (0)] + \sigma[(n), (1)] + \sigma[(1, n), (0)], \tag{6.48 b}$$

$$\text{Ind}_{W(B_1)W(B_n)}^{W(B_{n+1})}[\sigma[(1), (0)] \otimes \sigma[(n), (0)]] = \sigma[(n + 1), (0)] + \sigma[(1, n), (0)], \tag{6.48 c}$$

$$\text{Ind}_{W(B_1)W(B_n)}^{W(B_{n+1})}[\sigma[(0), (1)] \otimes \sigma[(n), (0)]] = \sigma[(n - 1), (1)]. \tag{6.48 d}$$

Thus $\mu_e(1, n)$ occurs with multiplicity 2 in $X_{M'}$. The $W(M')$ fixed vectors are the linear span of ϵ_1, ϵ_2 . The intertwining operators $I_{M',1,2}$ and $I_{M',2}$ are induced from maximal parabolic subgroups whose Levi components we label M_1 and M_2 . Then $\epsilon_1 + \epsilon_2$ transforms like $\text{triv} \otimes \text{triv}$ under $W(M_1)$ and $\epsilon_1 - \epsilon_2$ transforms like $\text{sgn} \otimes \text{triv}$. The vector ϵ_1 is fixed under $W(B_n)$ (which corresponds to M_2) and the vector ϵ_2 is fixed under $W(B_{n-1})$ and transforms like $\mu_o(1, n)$ under $W(B_n)$. The matrix r_σ is, according to (6.47),

$$\begin{bmatrix} \frac{1}{2 + \nu - n} & \frac{\nu - n + 1}{2 + \nu - n} \\ \frac{\nu - n + 1}{1 + \nu - n + 1} & \frac{1}{2 + \nu - n} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \frac{n - \nu}{n + \nu} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{1 + \nu + n} & \frac{\nu + n}{1 + \nu + n} \\ \frac{\nu + n}{1 + \nu + n} & \frac{1}{2 + \nu + n} \end{bmatrix}. \tag{6.49}$$

So the vector ϵ_1 is mapped into

$$\frac{n + 1 - \nu}{n + 1 + \nu} \epsilon_1$$

as claimed.

For $\sigma_o(1, n)$ we apply the same method. In this case the operator $I_{M',2}$ is the identity because in the representation $\mu_o(1, n)$ the element t_n corresponding to the short simple root acts by 1.

The calculation for type D is analogous, we sketch some details. We decompose the strings into

$$(\nu)(-n + 1, \dots, -1)(0), \tag{6.50}$$

and $M' = \text{GL}(1) \times \text{GL}(n - 1) \times \text{GL}(1)$. Then

$$\begin{aligned} I_{M,1}((\nu)(\nu_0)) &= I_{M',1,2}((-n + 1, \dots, -1)(-\nu)(0)) \circ I_{M',1}((-n + 1, \dots, -1)(\nu)(0)) \\ &\quad \circ I_{M',1,2}((\nu)(-n + 1, \dots, -1)(0)). \end{aligned} \tag{6.51}$$

□

6.6. In this section we consider (6.38) for $k > 1, n \geq 1$ and the W -types $\sigma_e(m, n+k-m)$ for $0 \leq m \leq k$ (notation as in Definition 4.9). These are the W -types which occur in X_M , with $M = \text{GL}(k) \times G(n) \subset G(k+n)$.

Proposition 6.5. *The $r_\sigma(w_1, ((\nu)(\nu_0)))$ for $\sigma = \sigma_e(m, n+k-m)$ are scalars. They equal type B,*

$$\prod_{0 \leq j \leq m-1} \frac{n + 1 - (-\frac{1}{2}(k-1) + \nu) - j}{n + 1 + (\frac{1}{2}(k-1) + \nu) - j}, \tag{6.52}$$

type C,

$$\prod_{0 \leq j \leq m-1} \frac{n + \frac{1}{2} - (-\frac{1}{2}(k-1) + \nu) - j}{n + \frac{1}{2} + (\frac{1}{2}(k-1) + \nu) - j}, \tag{6.53}$$

type D,

$$\prod_{0 \leq j \leq m-1} \frac{n - (-\frac{1}{2}(k-1) + \nu) - j}{n + (\frac{1}{2}(k-1) + \nu) - j}. \tag{6.54}$$

Proof. The proof is by induction on k . The case $k = 1$ was done in § 6.5 so we only need to do the induction step. For types B and C factor the intertwining operator as follows. Decompose the string

$$((\nu')(\frac{1}{2}(k-1) + \nu)(\nu_0)) := ((-\frac{1}{2}(k-1) + \nu, \dots, \frac{1}{2}(k-3) + \nu)(\frac{1}{2}(k-1) + \nu)(\nu_0)) \tag{6.55}$$

and let $M' := \text{GL}(k-1) \times \text{GL}(1) \times G(n)$, and $M'' = \text{GL}(1) \times \text{GL}(k-1) \times G(n)$. Thus

$$\begin{aligned} I_{M,1} &= I_{M'',2}((-\frac{1}{2}(k-1) - \nu)(\nu')(\nu_0)) \circ I_{M',1,2}((\nu')(-\frac{1}{2}(k-1) - \nu)(\nu_0)) \\ &\quad \circ I_{M',2}((\nu')(\frac{1}{2}(k-1) + \nu)(\nu_0)). \end{aligned} \tag{6.56}$$

$I_{M',1,2}$ and $I_{M',2}$ were computed earlier, while $I_{M'',2}$ is known by induction. Then

$$\begin{aligned} \sigma_e(m, n+k-m)|_{W(\text{GL}(k-1)) \times W(G(n+1))} &= \text{triv} \otimes [\sigma_e(1, n) + \sigma_e(0, n+1)] + \dots, \end{aligned} \tag{6.57}$$

$$\begin{aligned} \sigma_e(m, n+k-m)|_{W(\text{GL}(k)) \times W(G(n+k-1))} &= [(k) \otimes \text{triv} + (1, k-1) \otimes \text{triv}] + \dots, \end{aligned} \tag{6.58}$$

$$\begin{aligned} \sigma_e(m, n+k-m)|_{W(\text{GL}(1)) \times W(G(n+k-1))} &= \text{triv} \otimes [\sigma_e(m-1, n+k-m) + \sigma_e(m, n+k-1-m)] + \dots, \end{aligned} \tag{6.59}$$

where \dots denote W -types which are not spherical for $W(M)$, so do not matter for the computations.

The W -type

$$\sigma_e(m, n + k - m) \cong \bigwedge^m \sigma_e(1, n + k - 1).$$

It occurs with multiplicity 2 in $X_{M'}$ for $0 < m < \min(k, n)$ and multiplicity 1 for $m = \min(k, n)$. We will write out an explicit basis for the invariant $S_1 \times S_{k-1} \times W(B_n)$ vectors. Formulae (6.53)–(6.55) then come down to a computation with 2×2 matrices as in the case $k = 1$. Let

$$e := \frac{1}{m!(k-m)!} \sum_{x \in S_k} x \cdot [\epsilon_1 \wedge \dots \wedge \epsilon_m]. \tag{6.60}$$

This is the $S_k \times W(B_n)$ fixed vector of $\sigma_e(m, n + k - m)$. It decomposes as

$$e = e_0 + e_1 = f_0 + f_1, \tag{6.61}$$

where

$$\left. \begin{aligned} e_0 &= \frac{1}{m!(k-1-m)!} \sum_{x \in S_{k-1} \times S_1} x \cdot [\epsilon_1 \wedge \dots \wedge \epsilon_m], \\ e_1 &= \frac{1}{(m-1)!(k-m)!} \sum_{x \in S_{k-1} \times S_1} x \cdot [\epsilon_1 \wedge \dots \wedge \epsilon_{m-1}] \wedge \epsilon_k, \\ f_0 &= \frac{1}{m!(k-1-m)!} \sum_{x \in S_1 \times S_{k-1}} x \cdot [\epsilon_2 \wedge \dots \wedge \epsilon_{m+1}], \\ f_1 &= \frac{1}{(m-1)!(k-m)!} \sum_{x \in S_1 \times S_{k-1}} \epsilon_1 \wedge x \cdot [\epsilon_2 \wedge \dots \wedge \epsilon_m]. \end{aligned} \right\} \tag{6.62}$$

Also let

$$\left. \begin{aligned} e'_0 = e''_0 &= \frac{1}{(m-1)!(k-m)!} \sum_{x \in S_k} x \cdot [\epsilon_1 \wedge \dots \wedge \epsilon_m], \\ e'_1 &= \sum_{x \in S_{k-1} \times S_1} x \cdot [\epsilon_1 \wedge \dots \wedge \epsilon_{m-1} \wedge (\epsilon_m - \epsilon_k)], \\ e''_1 &= \sum_{x \in S_1 \times S_{k-1}} x \cdot [(-\epsilon_1 + \epsilon_{m+1}) \wedge \epsilon_2 \wedge \dots \wedge \epsilon_{m+1}]. \end{aligned} \right\} \tag{6.63}$$

Then

$$\left. \begin{aligned} e_0 &= \frac{k-m}{k} e'_0 + \frac{m}{k} e'_1, & e_1 &= \frac{m}{k} e'_0 - \frac{m}{k} e'_1, \\ e''_0 &= f_0 + f_1, & e''_1 &= f_0 - \frac{k-m}{m} f_1. \end{aligned} \right\} \tag{6.64}$$

We now compute the action of the intertwining operators. The following relations hold:

$$\left. \begin{aligned} I_{M',2}(e_0) &= e_0, & I_{M',2}(e_1) &= \frac{n + \epsilon - (\frac{1}{2}(k - 1) + \nu)}{n + \epsilon + (\frac{1}{2}(k - 1) + \nu)} e_1, \\ I_{M',12}(e'_0) &= e''_0, & I_{M',12}(e'_1) &= \frac{2\nu - 1}{2\nu + k - 1} e''_1, \\ I'_{M'',2}(f_0) &= \prod_{0 \leq j \leq m-2} \frac{n + \epsilon - (-\frac{1}{2}(k - 1) + \nu) - j}{n + \epsilon + (\frac{1}{2}(k - 3) + \nu) - j} f_0, \\ I_{M'',2}(f_1) &= \prod_{0 \leq j \leq m-1} \frac{n + \epsilon - (-\frac{1}{2}(k - 1) + \nu) - j}{n + \epsilon + (\frac{1}{2}(k - 3) + \nu) - j} f_1, \end{aligned} \right\} \tag{6.65}$$

where $\epsilon = 1$ in type B, $\epsilon = \frac{1}{2}$ in type C, and $\epsilon = 0$ in type D. Then

$$I_{M',2}(e_0 + e_1) = e_0 + \frac{n + \epsilon - (\frac{1}{2}(k - 1) + \nu)}{n + \epsilon + (\frac{1}{2}(k - 1) + \nu)} e_1. \tag{6.66}$$

Substituting the expressions of e_0, e_1 in terms of e'_0, e'_1 , we get

$$\left[\frac{k - m}{k} + \frac{m}{k} \frac{n + \epsilon - (\frac{1}{2}(k - 1) + \nu)}{n + \epsilon + (\frac{1}{2}(k - 1) + \nu)} \right] e'_0 + \frac{m}{k} \left[1 - \frac{n + \epsilon - (\frac{1}{2}(k - 1) + \nu)}{n + \epsilon + (\frac{1}{2}(k - 1) + \nu)} \right] e'_1. \tag{6.67}$$

Applying $I_{M,2}$ to this has the effect that e'_0 is sent to e''_0 and the term in e'_1 is multiplied by $(2\nu - 1)/(2\nu + k - 1)$ and e'_1 is replaced by e''_1 . Substituting the formulae for e''_0 and e''_1 in terms of f_0, f_1 , and applying $I_{M'',2}$, we get the claim of the proposition. \square

6.7. We now treat the case $\sigma = \sigma_o(m, n + k - m)$. We assume $n > 0$ or else these W -types do not occur in the induced module X_M .

Proposition 6.6. *The $r_\sigma(w_1, ((\nu)(\nu_0)))$ are scalars. They equal*

$$\prod_{0 \leq j \leq m-1} \frac{(\nu - \frac{1}{2}(k - 1)) - (1 - \epsilon) + j}{(\nu + \frac{1}{2}(k - 1)) - (-n - \epsilon) - j} \cdot \frac{(-n - \epsilon) - (-\nu + \frac{1}{2}(k - 1)) + j}{(1 - \epsilon) - (-\nu - \frac{1}{2}(k - 1)) - j}. \tag{6.68}$$

Proof. The intertwining operator $I_M(\nu)$ decomposes in the same way as (6.56). Furthermore, $\sigma_o(m, n + k - m) = \wedge^m \sigma_o(1, n + k - 1)$. The difference from the cases σ_e is that while $\sigma_e(1, n + k - 1)$ is the reflection representation, and therefore realized as the natural action on $\epsilon_1, \dots, \epsilon_{n+k}$, $\sigma_o(1, n + k - 1)$ occurs in $S^2\sigma_e(1, n + k - 1)$, generated by $\epsilon_i^2 - \epsilon_j^2$ with $i \neq j$. We can apply the same technique as for $\sigma_e(m, n + k - m)$, and omit the details. \square

GL(k) \subset G(k) in types B and C

The formulae in Propositions 6.5 and 6.6 hold with $n = 0$. The proof is the same, but because $n = 0$, (ν_0) is not present. The operator $I_{M',2}$ is an intertwining operator in $SL(2)$ and therefore simpler.

6.8. $GL(k) \subset G(k)$ in type D

In this subsection we consider the maximal Levi components $M := GL(k) \subset G(k)$ and $M' := GL(k)' \subset G(k)$ for type D_n . The parameter corresponds to the string

$$(\nu) := (-\frac{1}{2}(k-1) + \nu, \dots, \frac{1}{2}(k-1) + \nu)$$

or

$$(\nu') := (-\frac{1}{2}(k-1) + \nu, \dots, -\frac{1}{2}(k-1) - \nu).$$

k even. The W -structure of $X_M((\nu))$ and $X_{M'}((\nu)')$ is $\sigma_e[(n-r), (r)]$ for $0 \leq r < \frac{1}{2}k$, and $\sigma_e[(\frac{1}{2}k), (\frac{1}{2}k)]_I$, or $\sigma_e[(\frac{1}{2}k), (\frac{1}{2}k)]_{II}$ respectively, with multiplicity 1. There are intertwining operators

$$\left. \begin{aligned} I_M((\nu)) : X_M((\nu)) &\rightarrow X_M((-\nu)), \\ I_{M'}((\nu)') : X_{M'}((\nu)') &\rightarrow X_{M'}((-\nu)') \end{aligned} \right\} \tag{6.69}$$

corresponding to the shortest Weyl group element changing $((\nu))$ to $((-\nu))$. They determine scalars $r_\sigma((\nu))$ and $r_\sigma((\nu)')$.

k odd. The W -structure in this case is $\sigma_e[(n-r), (r)]$ with $0 \leq r \leq [\frac{1}{2}k]$ for both X_M and $X_{M'}$, again with multiplicity 1. In this case there is a shortest Weyl group element which changes $((\nu))$ to $((-\nu)')$, and one which changes $((\nu)')$ to $((-\nu))$. These elements give rise to intertwining operators

$$\left. \begin{aligned} I_M((\nu)) : X_M((\nu)) &\rightarrow X_{M'}((-\nu)'), \\ I_{M'}((\nu)') : X_{M'}((\nu)') &\rightarrow X_M((-\nu)). \end{aligned} \right\} \tag{6.70}$$

Because the W -structure of X_M and $X_{M'}$ is the same, and W -types occur with multiplicity 1, these intertwining operators define scalars $r_\sigma(\nu)$ and $r_\sigma((\nu)')$.

Proposition 6.7. *The scalars $r_\sigma((\nu))$ and $r_\sigma((\nu)')$ are*

$$r_{\sigma_e[(n-r), (r)]}((\nu)) = \prod_{0 \leq j < r} \frac{(\frac{1}{2}(k-1) - \nu) - j}{(\frac{1}{2}(k-1) + \nu) - j}. \tag{6.71}$$

These numbers are the same for $((\nu))$ and $((\nu)')$ and representations with subscripts I, II; they depend only on r .

6.9. Proof of Theorem 5.3

We use the results in the previous sections to prove the theorem in general. We give the details in the case of the group of type B and W -types σ_e . Thus the Hecke algebra is type C. There are no significant changes in the proof for the other cases. Recall the notation from § 2.3. Conjugate ν the middle element for the nilpotent orbit with partition $(2x_0, \dots, 2x_{2m})$ so that it is dominant (i.e. the coordinates are in decreasing order),

$$\nu = (x_{2m} - \frac{1}{2}, \dots, x_{2m} - \frac{1}{2}, \dots, x_0 - \frac{1}{2}, \dots, x_0 - \frac{1}{2}, \dots, \frac{1}{2}, \dots, \frac{1}{2}).$$

Then ν is dominant, so $X(\nu)$ has a unique irreducible quotient $L(\nu)$. We factor the long intertwining operator so that

$$X(\nu) \xrightarrow{I_1} X_e(\nu) \xrightarrow{I_2} X(-\nu), \tag{6.72}$$

where X_e was defined in §5.3. The claim will follow if the decomposition has the property that the operator I_1 is onto, and I_2 is into, when restricted to the σ_e isotypic component.

Proof that I_1 is onto. The operator I_1 is a composition of several operators. First take the long intertwining operator induced from the Levi component $GL(n)$,

$$X(x_{2m} - \frac{1}{2}, \dots, \frac{1}{2}) \rightarrow X(\frac{1}{2}, \dots, x_{2m} - \frac{1}{2}), \tag{6.73}$$

corresponding to the shortest Weyl group element that permutes the entries of the parameter from decreasing order to increasing order. The image is the induced from the corresponding irreducible spherical module $L(\frac{1}{2}, \dots, x_{2m} - \frac{1}{2})$ on $GL(n)$. In turn this is induced irreducible from one-dimensional spherical characters on a $GL(x_0) \times \dots \times GL(x_{2m})$ Levi component corresponding to the strings

$$(\frac{1}{2}, \dots, x_0 - \frac{1}{2}) \cdots (\frac{1}{2}, \dots, x_{2m} - \frac{1}{2})$$

or any permutation thereof. This is well known by results of Bernstein–Zelevinski in the p -adic case, [34] for the real case.

Compose with the intertwining operator

$$X(\cdots(\frac{1}{2}, \dots, x_{2m} - \frac{1}{2})) \rightarrow X(\cdots(-x_{2m} + \frac{1}{2}, \dots, -\frac{1}{2})), \tag{6.74}$$

all other entries unchanged. This intertwining operator is induced from the standard long intertwining operator on $G(x_{2m})$ which has image equal to the trivial representation. The image is an induced module from characters on $GL(x_0) \times \dots \times GL(x_{2m-1}) \times G(x_{2m})$. Now compose with the intertwining operator

$$\begin{aligned} X(\cdots(\frac{1}{2}, \dots, x_{2m-1} - \frac{1}{2})(-x_{2m} + \frac{1}{2}, \dots, -\frac{1}{2})) \\ \rightarrow X(\cdots(-x_{2m-1} + \frac{1}{2}, \dots, -\frac{1}{2})(-x_{2m} + \frac{1}{2}, \dots, -\frac{1}{2})) \end{aligned} \tag{6.75}$$

(again all other entries unchanged). This is $I_{M,2m-1}$ defined in (6.8), so its restriction of (6.75) to the σ_e isotypic component is an isomorphism. Now compose this operator with the one corresponding to

$$\begin{aligned} X(\cdots(\frac{1}{2}, \dots, x_{2m-2} - \frac{1}{2})(-x_{2m-1} + \frac{1}{2}, \dots, \frac{1}{2})\cdots) \\ \rightarrow X(\cdots(-x_{2m-1} + \frac{1}{2}, \dots, x_{2m-2} - \frac{1}{2})\cdots) \end{aligned} \tag{6.76}$$

with all other entries unchanged. This is induced from

$$GL(x_0) \times \cdots \times GL(x_{2m-3}) \times GL(x_{2m-2} + x_{2m-1}) \times G(x_{2m})$$

and the image is the representation induced from the character corresponding to the string

$$(-x_{2m-1} - \frac{1}{2}, \dots, -\frac{1}{2}, \frac{1}{2}, \dots, x_{2m-2}) \text{ on } \text{GL}(x_{2m-2} + x_{2m-1}).$$

Now compose further with the intertwining operator

$$\begin{aligned} X(\dots(-x_{2m-1} + \frac{1}{2}, \dots, x_{2m-2} - \frac{1}{2})(-x_{2m} - \frac{1}{2}, \dots, -\frac{1}{2})) \\ \rightarrow X((-\dots(-x_{2m-1} + \frac{1}{2}, \dots, x_{2m-2} - \frac{1}{2}) \dots(-x_{2m} - \frac{1}{2}, \dots, -\frac{1}{2})) \end{aligned} \tag{6.77}$$

from the representation induced from

$$\text{GL}(x_0) \times \dots \times \text{GL}(x_{2m-3}) \times \text{GL}(x_{2m-2} + x_{2m-1}) \times G(x_{2m})$$

to the induced from

$$\text{GL}(x_{2m-2} + x_{2m-1}) \times \text{GL}(x_0) \times \dots \times \text{GL}(x_{2m-3}) \times G(x_{2m}).$$

By Lemma 6.3, this intertwining operator is an isomorphism on any σ_e isotypic component. In fact, because the strings are strongly nested, the irreducibility results for $\text{GL}(n)$, Theorem 2.2 and Proposition 3.8 for type A, imply that the induced modules are isomorphic.

We have constructed a composition of intertwining operators from the standard module $X(\nu)$ where the coordinates of ν are positive and in decreasing order (i.e. dominant) to a module induced from

$$\text{GL}(x_{2m-2} + x_{2m-1}) \times \text{GL}(x_0) \times \dots \times \text{GL}(x_{2m-3}) \times G(x_{2m})$$

corresponding to the strings

$$\begin{aligned} ((-x_{2m-1} + \frac{1}{2}, \dots, x_{2m-2} - \frac{1}{2})(\frac{1}{2}, \dots, x_0 - \frac{1}{2}), \dots, \\ (\frac{1}{2}, \dots, x_{2m-3} - \frac{1}{2})(-x_{2m} + \frac{1}{2}, \dots, -\frac{1}{2})) \end{aligned}$$

so that the restriction to any σ_e isotypic component is onto. We can repeat the procedure with x_{2m-4} , x_{2m-3} and so on to get an intertwining operator from $X(\nu)$ to the induced from

$$\text{GL}(x_{2m-1} + x_{2m-2}) \times \dots \times \text{GL}(x_1 + x_0) \times G(x_{2m})$$

corresponding to the strings

$$((-x_{2m-1} + \frac{1}{2}, \dots, x_{2m-2} - \frac{1}{2}) \dots (-x_1 + \frac{1}{2}, \dots, x_0 - \frac{1}{2}), (-x_{2m} + \frac{1}{2}, \dots, -\frac{1}{2})).$$

This is the operator I_1 , and it is onto on the $\sigma_e(*)$ isotypic components.

Proof that I_2 is into. We now deal with I_2 . Consider the group $G(x_1 + x_0 + x_{2m})$ and the Levi component $M = \text{GL}(x_1 + x_0) \times G(x_{2m})$. Let M' be the Levi component

$$M' := \text{GL}(x_{2m-1} + x_{2m-2}) \times \dots \times \text{GL}(x_3 + x_2) \times \text{GL}(x_1) \times \text{GL}(x_0) \times G(x_{2m}). \tag{6.78}$$

Then X_e embeds in

$$X_{M'}(\cdots(-x_1 + \frac{1}{2}, \dots, -\frac{1}{2})(\frac{1}{2}, \dots, x_0 - \frac{1}{2})(-x_{2m} + \frac{1}{2}, \dots, -\frac{1}{2})).$$

The intertwining operator $I_{M',m+1}$ which changes the string $(x_0 - \frac{1}{2}, \dots, \frac{1}{2})$ to $(-x_0 + \frac{1}{2}, \dots, -\frac{1}{2})$ is an isomorphism on the σ_e W -types, by the results in §§6.1–6.6. Since the strings are strongly nested, the operators $I_{M,i,i+1}$ are all isomorphisms, so we can construct an intertwining operator to an induced module $X_{M''}(\nu'')$ where

$$\left. \begin{aligned} M'' &= \text{GL}(x_1) \times \text{GL}(x_0) \times \text{GL}(x_{2m-1} + x_{2m-2}) \times \cdots \times G(x_{2m}), \\ \nu'' &= (-x_1 + \frac{1}{2}, \cdots - \frac{1}{2})(-x_0 + \frac{1}{2}, \dots, -\frac{1}{2}) \cdots, \end{aligned} \right\} \tag{6.79}$$

which is an isomorphism on the σ_e isotypic components. Repeating this argument for x_3, x_2 up to x_{2m-1}, x_{2m-2} we get an intertwining operator from X_e to an induced module $X_{M^{(3)}}(\nu^{(3)})$ where

$$\left. \begin{aligned} M^{(3)} &:= \text{GL}(x_1) \times \text{GL}(x_0) \times \cdots \times \text{GL}(x_{2m-1}) \times \text{GL}(x_{2m-2}) \times G(x_{2m}), \\ \nu^{(3)} &:= (-x_1 + \frac{1}{2}, \dots, \frac{1}{2})(-x_0 + \frac{1}{2}, \dots, -\frac{1}{2}) \\ &\quad \cdots (-x_{2m-1} + \frac{1}{2}, \dots, -\frac{1}{2})(-x_{2m-2} + \frac{1}{2}, \dots, -\frac{1}{2})(-x_{2m} + \frac{1}{2}, \dots, -\frac{1}{2}), \end{aligned} \right\} \tag{6.80}$$

which is an isomorphism on the σ_e isotypic components. Let

$$M^{(4)} := \text{GL}(x_1) \times \text{GL}(x_0) \times \cdots \times \text{GL}(x_{2m-1}) \times G(x_{2m})$$

and let $\nu^{(4)}$ be the same as $\nu^{(3)}$ but the last string is viewed as giving a parameter of a one-dimensional representation on $\text{GL}(x_{2m})$. Then $M^{(4)} \subset M^{(3)}$, and $X_{M^{(3)}}(\nu^{(3)})$ is a submodule of $X_{M^{(4)}}(\nu^{(4)})$. The induced module from $M^{(4)}$ to $M^{(5)} := \text{GL}(x_{2m} + \cdots + x_0)$ is irreducible because the strings are strongly nested on the GL factors. Thus the intertwining operator which takes $\nu^{(4)}$ to $-\nu$ is an isomorphism on $X_{M^{(4)}}(\nu^{(4)})$. So the induced intertwining operator to G is therefore injective and maps to $X(-\nu)$. The composition of all these operators is I_1 , and is therefore injective on the σ_e -isotypic components. The proof is complete in this case.

The case of σ_o is similar, and we omit the details.

7. Necessary conditions for unitarity

7.1. We will need the following notions.

Definition 7.1. We will say a spherical irreducible module $L(\chi)$ is *r-unitary* if the form is positive on all the relevant W -types. Similarly, an induced module $I_M(\chi) := \text{Ind}_M^G[L_M(\chi)]$ is *r-irreducible* if all relevant W -types occur with the same multiplicity in $I_M(\chi)$ as in $L(\chi)$.

7.2. We recall (6.4),

$$\epsilon = \begin{cases} \frac{1}{2} & G \text{ of type B (}\mathbb{H} \text{ of type C),} \\ 0 & G \text{ of type C (}\mathbb{H} \text{ of type B),} \\ 1 & G \text{ of type D.} \end{cases}$$

Definition 7.2. A string of the form $(f + \nu, \dots, F + \nu)$ with $f, F \in \epsilon + \mathbb{Z}$ is called *adapted*, if it is

- of *even* length for G of type B,
- of *odd* length for G of types C and D.

Otherwise we say the string is *not* adapted.

We will consider the following case. Let $\check{O} \subset \check{\mathfrak{g}}$ correspond to the partition

$$\check{O} \longleftrightarrow ((a_1, a_1), \dots, (a_r, a_r); d_1, \dots, d_l) \tag{7.1}$$

so that \check{O} meets the Levi component $\check{\mathfrak{m}} = \mathfrak{gl}(a_1) \times \dots \times \mathfrak{gl}(a_r) \times \check{\mathfrak{g}}(n_0)$, with $2n_0 + [1 - \epsilon] = d_1 + \dots + d_l$, where $[x]$ is the integer part of x . The intersection of \check{O} with each $\mathfrak{gl}(a_i)$ is the principal nilpotent, and the intersection with $\check{\mathfrak{g}}(n_0)$ is the *even* nilpotent orbit \check{O}_0 with partition (d_1, \dots, d_l) . Let

$$\left. \begin{aligned} \chi_i &= (f_i + \nu_i, \dots, F_i + \nu_i), \quad 1 \leq i \leq r, \\ \chi_0 &= \frac{1}{2}\check{h}_0, \quad \text{where } \check{h}_0 \text{ is a neutral element for } (d_1, \dots, d_l), \end{aligned} \right\} \tag{7.2}$$

and χ be the parameter obtained by concatenating the χ_i . Then $L(\chi)$ is the spherical subquotient of

$$\text{Ind}_M^G \left[\bigotimes_{1 \leq i \leq r} L(\chi_i) \otimes L(\chi_0) \right]. \tag{7.3}$$

The next theorem gives necessary conditions for the unitarity of $L(\chi)$.

Theorem 7.3. *In types B and C, the nilpotent orbit \check{O}_0 is arbitrary. In type D assume that either $\check{O}_0 \neq (0)$, or else that the rank is even. The representation $L(\chi)$ is unitary only if*

- (1) any string that is not adapted can be written in the form

$$(-E + \tau, \dots, E - 1 + \tau), \quad 0 < \tau \leq \frac{1}{2}, \quad E \equiv \epsilon \pmod{\mathbb{Z}}; \tag{7.4}$$

- (2) any string that is adapted can be written in the form

$$\left. \begin{aligned} &(-E + \tau, \dots, E + \tau), \quad 0 < \tau \leq \frac{1}{2}, \quad E \equiv \epsilon \pmod{\mathbb{Z}}, \\ \text{or} \\ &(-E - 1 + \tau, \dots, E - 1 + \tau), \quad 0 < \tau \leq \frac{1}{2}, \quad E \equiv \epsilon \pmod{\mathbb{Z}}. \end{aligned} \right\} \tag{7.5}$$

This is simply the fact that the ν_j satisfy $0 < \nu_j < \frac{1}{2}$ or $\frac{1}{2} < \nu_j < 1$ in Theorem 3.2. The proof will be given in the next sections. It is by induction on the dimension of $\check{\mathfrak{g}}$, the number of strings with coordinates in an A_τ (Definition 3.7) with $\tau \neq 0$, and by downward induction on the dimension of \check{O} . The unitarity of the representation when there are no coordinates in any A_τ with $\tau \neq 0$ is done in §9.

7.3. The proposition in this section is a restatement of Theorem 7.3 for the case of a parameter of the form (7.6). It is the first case in the initial step of the induction proof of Theorem 7.3. I have combined the two cases in (2) into a single string $(-E + \nu, \dots, E + \nu)$ with $0 < \nu < 1$ by changing $(-E - 1 + \tau, \dots, E - 1 + \tau)$ into $(-E + (1 - \tau), \dots, E + (1 - \tau))$. This notation seemed more convenient for the case when there is a single such string present.

Consider the representation $L(\chi)$ corresponding to the strings

$$(a + \epsilon + \nu, \dots, A + \epsilon + \nu)(-x_0 + \epsilon, \dots, -1 + \epsilon), \quad |a| \leq A, \quad 0 < \nu < 1, \tag{7.6}$$

where $a, A \in \mathbb{Z}$, and ϵ is as in 7.2. This is the case when $L(\chi)$ is the spherical subquotient of an induced from a character on a maximal parabolic subalgebra of the form $\mathfrak{gl}(A - a + 1) \times \mathfrak{g}(x_0)$. The second string may not be present; these are the cases $x_0 = -1$ for \mathfrak{g} of types B and D, $x_0 = 0$ for type C.

Proposition 7.4. *In type D, assume that if there is no string $(-x_0 + \epsilon, \dots, -1 + \epsilon)$, then $A - a + 1$ is even. Let $L(\chi)$ correspond to (7.6). Then $L(\chi)$ is r -unitary if and only if $a + \epsilon = -A - \epsilon$, and the following hold.*

- (1) *Assume that $(a + \epsilon + \nu, \dots, A + \epsilon + \nu)$ is adapted. If $x_0 = A - a + 1$, then $0 \leq \nu < 1$, otherwise $\nu = 0$.*
- (2) *If $(a + \epsilon + \nu, \dots, A + \epsilon + \nu)$ is not adapted, then $0 \leq \nu < \frac{1}{2}$.*

Proof. This is a corollary of the formulae in §6.3. □

7.4. Initial step

We do the case when there is a single A_τ with $0 < \tau < \frac{1}{2}$, and the coordinates form a single string. We write the string as in (7.6), $(a + \epsilon + \nu, \dots, A + \epsilon + \nu)$ with $0 < \nu < 1$. We let \check{O} be the nilpotent orbit with partition $((A - a + 1, A - a + 1); d_1, \dots, d_l)$. Let $\check{\mathfrak{m}} := \mathfrak{gl}(A - a + 1) \times \check{\mathfrak{g}}(n_0)$, and let \check{O}_0 be the intersection of \check{O} with $\check{\mathfrak{g}}(n_0)$. In type D, either $\check{O}_0 \neq (0)$ or else $A - a + 1$ is even. The statement of Theorem 7.3 is equivalent to the following proposition.

Proposition 7.5. *Assume \check{O}_0 is even, and χ is attached to \check{O} . Then $L(\chi)$ is r -unitary only if $a + \epsilon = -A - \epsilon$, and the following hold.*

- (1) *If $(a + \epsilon + \nu, \dots, A + \epsilon + \nu)$ is adapted, then $\nu = 0$, unless there is $d_j = A - a + 1$, in which case $0 \leq \nu < 1$.*
- (2) *If $(a + \epsilon + \nu, \dots, A + \epsilon + \nu)$ is not adapted, then $0 \leq \nu < \frac{1}{2}$.*

Proof. We do the case of G of type C only, the others are similar. So $\epsilon = 0$, and adapted means the length of the string is odd, not adapted means the length of the string is even. The nilpotent orbit \check{O}_0 corresponds to the partition $(2x_0 + 1, \dots, 2x_{2m} + 1)$ and the parameter has strings

$$(1, \dots, x_0)(0, 1, \dots, x_1) \cdots (1, \dots, x_{2m}).$$

The partition of \check{O} is $(A - a + 1, A - a + 1, 2x_0 + 1, \dots, 2x_{2m} + 1)$.

We want to show that if $A + a > 0$, or if $A + a = 0$ and there is no $x_i = A$, then $L(\chi)$ is not r -unitary. We do an upward induction on the rank of $\check{\mathfrak{g}}$ and a downward induction on the dimension of $\check{\mathcal{O}}$. In the argument below with deformations of strings, we use implicitly the irreducibility results from § 2.6. So the first case is when $\check{\mathcal{O}}$ is maximal, i.e. the principal nilpotent ($m = 0$). The claim follows from Proposition 7.4. So we assume that m is strictly greater than 0.

Assume $x_{2i} < A \leq x_{2i+1}$ for some i . This case includes the possibility $x_{2m} < A$. We will show by induction on rank of $\check{\mathfrak{g}}$ and dimension of $\check{\mathcal{O}}$ that the form is negative on a W -type of the form $\sigma[(n-r), (r)]$. So we use the module X_e (notation as in § 5.3). If there is any pair $x_{2j} = x_{2j+1}$, the module X_e is unitarily induced from $\text{GL}(2x_{2j} + 1) \times G(n - 2x_{2j} - 1)$ and all W -types $\sigma[(n-r), (r)]$ have the same multiplicity in $L(\chi)$ as in X_e . We can remove the string corresponding to $(x_{2j}x_{2j+1})$ in X_e as justified by Lemma 3.6. By induction on rank we are done. Similarly we can remove any pair (x_{2j}, x_{2j+1}) such that either $x_{2j+1} \leq |a|$ or $A \leq x_{2j}$ as follows. Let $M := \text{GL}(x_{2j} + x_{2j+1} + 1) \times G(n - x_{2j} - x_{2j+1} - 1)$. There is χ_M such that $L(\chi)$ is the spherical subquotient of

$$\text{Ind}_M^G[L(-x_{2j+1}, \dots, x_{2j}) \otimes L(\chi_M)]. \tag{7.7}$$

Precisely, χ_M is obtained from χ by removing the entries $(1, \dots, x_{2j}), (0, 1, \dots, x_{2j+1})$. Write

$$\chi_t := (-x_{2j+1} + t, \dots, x_{2j} + t; \chi_M). \tag{7.8}$$

The induced module

$$X_e(\chi_t) := \text{Ind}_M^G[L(-x_{2j+1} + t, \dots, x_{2j} + t) \otimes L(\chi_M)] \tag{7.9}$$

has $L(\chi_t)$ as its irreducible spherical subquotient. For $0 \leq t \leq \frac{1}{2}(x_{2j+1} - x_{2j})$, the multiplicities of $\sigma[(n-r), (r)]$ in $L(\chi_t)$ and $X_e(\chi_t)$ coincide. Thus the signatures on the $\sigma[(n-r), (r)]$ in $L(\chi_t)$ are constant for t in the above interval. At $t = \frac{1}{2}(x_{2j+1} - x_{2j})$, $X_e(\chi_t)$ is unitarily induced from $\text{triv} \otimes X'_e$ on $\text{GL}(x_{2j} + x_{2j+1} + 1) \times G(n - x_{2j} - x_{2j+1} - 1)$ and we can remove the string corresponding to $(x_{2j}x_{2j+1})$. The induction hypothesis applies to X'_e .

When $A + a = 0$, by the above argument, we are reduced to the case

$$\check{\mathcal{O}}_0 \longleftrightarrow (2x_0 + 1, 2x_1 + 1, 2x_2 + 1), \quad x_0 < A < x_1 \leq x_2. \tag{7.10}$$

We reduce to (7.10) when $A + a > 0$ as well. We assume $2m = 2i + 2$, since pairs (x_{2j}, x_{2j+1}) with $A \leq x_{2j}$ can be removed. Suppose there is a pair (x_{2j}, x_{2j+1}) such that $|a| < x_{2j+1}$, and $j \neq i$. The assumption is that $x_{2i} < A \leq x_{2i+1}$ so $x_{2j+1} \leq x_{2i} < A$.

We consider the deformation χ_t in (7.8) with

$$\begin{aligned} 0 \leq t < \nu, \quad a < 0, \\ -\nu < t \leq 0, \quad a \geq 0. \end{aligned}$$

In either case $X_e(\chi_t) = X_e(\chi)$, so the multiplicities of the $\sigma[(n-r), (r)]$ do not change until t reaches ν in the first case, $-\nu$ in the second case. If the signature on some

$\sigma[(n - r), (r)]$ isotypic component is positive semidefinite on $L(\chi)$, the same has to hold when $t = \nu$ or $-\nu$ respectively. The corresponding nilpotent orbit for this parameter is strictly larger, but it has two strings with coordinates which are not integers (so the induction hypothesis does not apply yet). For example, if $a < 0$, the strings for $X_e(\chi_\nu)$ are (aside from the ones that were unchanged)

$$(-x_{2j+1} + \nu, \dots, A + \nu), \quad (a + \nu, \dots, x_{2j} + \nu). \tag{7.11}$$

We can deform the parameter further by replacing the second string by $(a + \nu - t', \dots, x_{2j} + \nu - t')$ with $0 \leq t' < \nu$. The strings of the corresponding X_e do not change until t' reaches ν . At $t' = \nu$ the corresponding nilpotent orbit \check{O}' has partition

$$(\dots, 2|a| + 1, \dots, (2x_{2j+1} + 1)^\wedge, \dots, A + x_{2j+1} + 1, A + x_{2j+1} + 1, \dots) \tag{7.12}$$

which contains \check{O} in its closure. Since $x_{2j+1} < A$, the induction hypothesis applies. The form is indefinite on a W -type $\sigma[(n - r), (r)]$, so this holds for the original χ as well.

We have reduced to case (7.10), i.e. the partition of \check{O}_0 has just three terms $(2x_0 + 1, 2x_1 + 1, 2x_2 + 1)$. We now reduce further to the case

$$\check{O}_0 \longleftrightarrow (2x_0 + 1), \quad x_0 < A, \tag{7.13}$$

which is the initial step.

Let $I(t)$ be the induced module corresponding to the strings

$$(-x_2 + t, \dots, x_1 + t)(a + \nu, \dots, A + \nu)(-x_0, \dots, -1), \tag{7.14}$$

i.e. induced from

$$\mathrm{GL}(x_1 + x_2) \times \mathrm{GL}(-a + A + 1) \times G(x_0). \tag{7.15}$$

Consider the irreducible spherical module for the last two strings in (7.14), inside the induced module from the Levi component $\mathrm{GL}(-a + A + 1) \times G(x_0) \subset G(-a + A + 1 + x_0)$. By § 7.1, the form is negative on $\sigma[(x_0 - a + A), (1)]$ if $x_0 < a$, negative on $\sigma[(A), (x_0 + 1 - a)]$ if $a \leq x_0$. In the second case the form is positive on all $\sigma[(A + r), (x_0 + 1 - a - r)]$ for $1 < r < x_0 + 1 - a$. So let $r_0 := 1$ or $x_0 + 1 - a$ depending on these two cases. The multiplicity formulae from § 6.3 imply that

$$[\sigma[(n - r_0), (r_0)] : I(t)] = [\sigma[(n - r_0), (r_0)] : L(\chi)] \quad \text{for } 0 \leq t \leq \frac{1}{2}(x_2 - x_1).$$

Thus signatures do not change when we deform t to $\frac{1}{2}(x_2 - x_1)$, where $I(t)$ is unitarily induced. We conclude that the form on $L(\chi)$ is negative on $\sigma[(n - r_0), (r_0)]$.

Assume $x_{2i-1} < A \leq x_{2i}$. In this case we can do the same arguments using X_o and $\sigma[(k, n - k), (0)]$. We omit the details. □

7.5. Induction step

The case when the parameter has a single string with coordinates in an A_τ with $0 < \tau < \frac{1}{2}$ was done in § 7.4. So we assume there is more than one string. Again we do the case G of type C, and omit the details for the other ones.

Write the two strings as in (2.6),

$$(e + \tau_1, \dots, E + \tau_1), \quad (f + \tau_2, \dots, F + \tau_2), \tag{7.16}$$

where $0 < \tau_1 \leq \frac{1}{2}$ and $0 < \tau_2 \leq \frac{1}{2}$. Recall that because we are in type C, $e, E, f, F \in \mathbb{Z}$, and $\epsilon = 0$.

We need to show that if $F + f > 0$ or $F + f < -2$ when $F + f$ is even, or $F + f < -1$ when $F + f$ is odd, then the form is negative on a relevant W -type. Because $\tau_1, \tau_2 > 0$, and since r -reducibility and r -unitarity are not affected by small deformations, we may as well assume that $(f + \tau_2, \dots, F + \tau_2)$ is the only string with coordinates in A_{τ_2} , and $(e + \tau_1, \dots, E + \tau_1)$ the only one with coordinates in A_{τ_1} .

The strategy is as follows. Assume that $L(\chi)$ is r -unitary. We deform (one of the strings of) χ to a χ_t in such a way that the corresponding induced module is r -irreducible over a finite interval, but is no longer so at the endpoint, say t_0 . Because of the continuity in t , the module $L(\chi_{t_0})$ is still r -unitary. The deformation is such that $L(\chi_{t_0})$ belongs to a larger nilpotent orbit than $L(\chi)$, so the induction hypothesis applies, and we get a contradiction. Sometimes we have to repeat the procedure before we arrive at a contradiction.

So replace the first string by

$$(e + \tau_1 + t, \dots, E + \tau_1 + t). \tag{7.17}$$

If $\chi = (e + \tau_1, \dots, E + \tau_1; \chi_M)$, then

$$\begin{aligned} \chi_t &= (e + \tau_1 + t, \dots, E + \tau_1 + t; \chi_M), \\ X(\chi_t) &:= \text{Ind}_M^G [L(e + \tau_1 + t, \dots, E + \tau_1 + t) \otimes L(\chi_M)], \end{aligned}$$

where the Lie algebra of M is $\mathfrak{m} = \mathfrak{gl}(E - e + 1) \times \mathfrak{g}(n - E + e - 1)$ (so $\check{\mathfrak{m}} = \mathfrak{gl}(E - e + 1) \times \check{\mathfrak{g}}(n - E + e - 1)$).

If $E < |e|$, we deform t in the negative direction, otherwise in the positive direction. If $t + \tau_1$ reaches 0 or $\frac{1}{2}$, before the nilpotent orbit changes, we should rewrite the string to conform to the conventions (2.42) and (2.43). This means that we rewrite the string as $(e' + \tau'_1, \dots, E' + \tau'_1)$ with $0 \leq \tau'_1 \leq \frac{1}{2}$, and continue the deformation with a t going in the direction $t < 0$ if $E' < |e'|$, and $t > 0$ if $E' \geq |e'|$. This is not essential for the argument. We may as well assume that the following cases occur.

- (1) The nilpotent orbit changes at $t_0 = -\tau_1$.
- (2) the nilpotent orbit does not change, and at $t_0 = -\tau_1$, either $e, E > x_{2m} + 1$ or $-e, -E > x_{2m} + 1$. This is the *easy* case when t can be deformed to ∞ without any r -reducibility occurring.
- (3) The nilpotent orbit changes at a t_0 such that $0 < \tau_1 + t_0 \leq \frac{1}{2}$.

In the first case, the induction hypothesis applies, and since the string $(f + \tau_2, \dots, F + \tau_2)$ is unaffected, we conclude that the signature is negative on a relevant W -type. In the second case we can deform the string so that either $e + \tau_1 + t = x_{2m} + 1$ or $E + \tau_1 + t = -x_{2m} - 1$. The induction hypothesis applies, and the form is negative definite on a W -relevant type. In the third case, the only way the nilpotent orbit can change is if the string $(e + \tau_1 + t_0, \dots, E + \tau_1 + t_0)$ can be combined with another string to form a strictly longer string. If $\tau_1 + t_0 \neq \tau_2$, the induction hypothesis applies, and since the string $(f + \tau_2, \dots, F + \tau_2)$ is unaffected, the form is negative on a relevant W -type. If the nilpotent does not change at $t = \tau_2 - \tau_1$, continue the deformation in the same direction. Eventually either (1) or (2) are satisfied, or else we are in case (3), and the strings in (7.16) combine to give a larger nilpotent. There are four cases:

$$\left. \begin{aligned} (1) \quad & e < f \leq E \leq F, e \leq f \leq E < F, \\ (2) \quad & f \leq e \leq F < E, f < e \leq F \leq E, \\ (3) \quad & e \leq E = f - 1 < F, \\ (4) \quad & f \leq F = e - 1 < E. \end{aligned} \right\} \tag{7.18}$$

Assume $|e| \leq E$. Then t is deformed in the positive direction so $\tau_1 < \tau_2$. If $e \leq 0$, we look at the deformation (7.17) for $-\tau_1 \leq t \leq 0$. If the nilpotent changes for some $-\tau_1 < t < 0$, the string $(f + \tau_2, \dots, F + \tau_2)$ is not involved, the induction hypothesis applies, so the parameter is not r -unitary. Otherwise at $t = -\tau_1$ there is one less string with coordinates in an A_τ with $\tau \neq 0$, and again the induction hypothesis applies so the original parameter is not r -unitary. Thus we are reduced to the case $0 < e < E$. Then consider the nilpotent orbit for the parameter with $t = -\tau_1 + \tau_2$. In cases (1), (2) and (3) of (7.18), the new nilpotent is larger, and one of the strings is

$$(e + \tau_2, \dots, F + \tau_2), \tag{7.19}$$

instead of (7.16), and $e + F > 0$. The induction hypothesis applies, so the parameter is not r -unitary, nor is the original one.

In case (4) of (7.18), the new nilpotent corresponds to the strings

$$(f + \tau_2, \dots, E + \tau_2). \tag{7.20}$$

The induction hypothesis applies, so $f + E = 0, -2$ if $f + E$ is even or $f + E = -1$ if it is odd. If this is the case, consider a new deformation in (7.17), this time $-1 + \tau_2 < t \leq 0$. We may as well assume that the parameter is r -irreducible in this interval, or else the argument from before gives the desired conclusion. So we arrive at the case when $t = -1 + \tau_2$. The new nilpotent corresponds to the strings

$$(f + \tau_2, \dots, E - 1 + \tau_2), \quad (e - 1 + \tau_1), \quad (F + \tau_2). \tag{7.21}$$

Write the parameter as $(\chi'; e - 1 + \tau_2, F + \tau_2)$. Since $e - 1 = F$, the induced module

$$I = \text{Ind}_{\text{GL}(2) \times G(n-2)}^G [L(e - 1 + \tau_2, F + \tau_2) \otimes L(\chi')] \tag{7.22}$$

is unitarily induced from a module which is hermitian and r -irreducible. But the parameter on $GL(2)$ is not unitary unless $e - 1 = F = 0$. Furthermore, $f + E - 1 = 0, -2$ if $f + E$ is odd or $f + E - 1 = -1$ if $f + E$ is even. So the original parameter (7.16) is

$$\left. \begin{aligned} (1 + \tau_1, \dots, E + \tau_1), & \quad (1 - E + \tau_2, \dots, \tau_2), & \quad f + E = 0, \\ (1 + \tau_1, \dots, E + \tau_1), & \quad (-E + \tau_2, \dots, \tau_2), & \quad f + E = -2, \\ (1 + \tau_1, \dots, E + \tau_1), & \quad (-E - 1 + \tau_2, \dots, \tau_2), & \quad f + E = -1. \end{aligned} \right\} \tag{7.23}$$

Apply the deformation $t + \tau_2$ in the second string with $-\tau_2 < t \leq 0$. We may as well assume that the parameter stays r -irreducible in this interval. But then the induction hypothesis applies at $t = -\tau_2$ because there is one less string with coordinates in A_τ with $\tau \neq 0$. However, the first string does not satisfy the induction hypothesis.

Assume $|e| > E$. The same argument applies, but this time it is $e < E < 0$ that requires extra arguments, and in case (3) instead of case (4) of (7.18) we have to consider several deformations.

7.6. Proof of the necessary condition for unitarity in Theorem 3.2

We first reduce to the case of Theorem 7.3. The difference is that the coordinates in A_0 may not form a $\frac{1}{2}\check{h}$ for an even nilpotent orbit. However, because of Theorem 2.9, and properties of petite K -types, r -reducibility and r -unitary are unaffected by small deformations of the χ'_1, \dots, χ'_r (notation as in (2.59)). So we can deform the strings corresponding to χ'_1, \dots, χ'_r with coordinates in A_0 , so that their coordinates are no longer in A_0 . Then the assumptions in Theorem 7.3 are satisfied.

The argument now proceeds by analysing each size of strings separately. In the deformations that we will consider, strings of different sizes cannot combine so that the nilpotent orbit attached to the parameter changes.

Fix a size of strings with coordinates not in A_0 . If the strings are not adapted, they can be written in the form

$$(-E - 1 + \tau_i, \dots, E + \tau_i), \quad 0 < \tau_i \leq \frac{1}{2}, \quad E \equiv \epsilon \pmod{\mathbb{Z}}. \tag{7.24}$$

So there is nothing to prove. Now consider a size of strings that are adapted. Suppose there are *two* strings of the form

$$(-E - 1 + \tau_i, \dots, E - 1 + \tau_i), \quad 0 < \tau_i \leq \frac{1}{2}, \quad E \equiv \epsilon \pmod{\mathbb{Z}}. \tag{7.25}$$

Let $\mathfrak{m} := \mathfrak{gl}(2E + 1) \times \mathfrak{g}(n - 2E - 1)$ (recall that $\mathfrak{g}(a)$ means a subalgebra/Levi component of the same type as \mathfrak{g} of rank a) and write

$$\chi := ((-E - 1 + \tau_i, \dots, E - 1 + \tau_i; -E - 1 + \tau_i, \dots, E - 1 + \tau_i); \chi_M). \tag{7.26}$$

The module

$$\text{Ind}_M^G [L(-E - 1 + \tau_i, \dots, E - 1 + \tau_i; -E - 1 + \tau_i, \dots, E - 1 + \tau_i) \otimes L(\chi_M)] \tag{7.27}$$

is r -irreducible, and unitarily induced from a hermitian module on M where the module on $GL(2E + 1)$ is not unitary. Thus $L(\chi)$ is not unitary either. So $L(\chi)$ is unitary only if for each τ_i there is at most one string of the form $(-E - 1 + \tau_i, \dots, E - 1 + \tau_i)$.

Suppose there are two strings as in (7.24) with $\tau_1 < \tau_2$. If there is no string $(-E + \tau_3, \dots, E + \tau_3)$ with $\tau_1 < \tau_3 < \tau_2$, then when we deform $(-E - 1 + \tau_1 + t, \dots, E - 1 + \tau_1 + t)$ for $0 \leq t \leq \tau_2 - \tau_1$, $X(\chi_t)$ stays r -irreducible. At $t = \tau_2 - \tau_1$ we are in case (7.25), so the parameter is not unitary.

On the other hand, suppose that there are two strings of the form

$$(-E + \tau_i, \dots, E + \tau_i), \quad \text{same } \tau_i. \tag{7.28}$$

Let \mathfrak{m} be as before, and write

$$\chi := ((-E + \tau_i, \dots, E + \tau_i; -E + \tau_i, \dots, E + \tau_i); \chi_M). \tag{7.29}$$

The module

$$\text{Ind}_M^G [L(-E + \tau_i, \dots, E + \tau_i; -E + \tau_i, \dots, E + \tau_i) \otimes L(\chi_M)] \tag{7.30}$$

is irreducible, and unitarily induced from a hermitian module on M where the module on $GL(2E + 1)$ is unitary. Thus $L(\chi)$ is unitary if and only if $L(\chi_M)$ is unitary.

So we may assume that for each τ_i there is at most one string of the form $(-E + \tau_i, \dots, E + \tau_i)$.

Similarly if there are two strings of the form $(-E + \tau_1, \dots, E + \tau_1)$ and $(-E + \tau_2, \dots, E + \tau_2)$, such that there is no string of the form $(-E - 1 + \tau_3, \dots, E - 1 + \tau_3)$ with $\tau_1 < \tau_3 < \tau_2$ we reduce to the case (7.28).

Let τ_k be the largest such that a string of the form $(-E + \tau_k, \dots, E + \tau_k)$ occurs, and τ_{k+1} the smallest such that a string $(-E - 1 + \tau_{k+1}, \dots, E - 1 + \tau_{k+1})$ occurs. If $\tau_k > \tau_{k+1}$, we can deform $(-E + \tau_k + t, \dots, E + \tau_k + t)$ with $0 \leq t \leq 1 - \tau_k - \tau_{k+1}$. No r -reducibility occurs, and we are again in case (7.25). The module is not unitary. If on the other hand $\tau_k < \tau_{k+1}$, the deformation $(-E - 1 + \tau_{k+1} + t, \dots, E - 1 + \tau_{k+1} - t)$ for $0 \leq t \leq 1 - \tau_k - \tau_{k+1}$ brings us to the case (7.28).

Together the above arguments show that conditions (1) and (2) of Theorem 3.2 in types C and D must be satisfied. It remains to check that for the case of adapted strings, if there is an odd number of a given size $2E + 1$, then there is a $d_j = 2E + 1$. This is condition (3) in Theorem 3.2.

The arguments above (also the unitarity proof in the case $\check{O} = (0)$) show that an $L(\chi)$ is unitary only if it is of the following form. There is a Levi component $\mathfrak{m} = \mathfrak{gl}(a_1) \times \dots \times \mathfrak{gl}(a_r) \times \mathfrak{g}(n - \sum a_i)$, and parameters $\chi_1, \dots, \chi_r, \chi_0$ such that,

$$L(\chi) = \text{Ind}_M^G \left[\bigotimes L(\chi_i) \otimes L(\chi_0) \right], \tag{7.31}$$

with the following additional properties:

- (1) the χ_i for $i > 0$ are as in Lemma 3.4, with $0 < \nu < \frac{1}{2}$, in particular unitary;
- (2) χ_0 is such that there is at most one string for every A_τ with $\tau \neq 0$, and the strings are of different sizes.

In addition, conditions (1) and (2) of Theorem 3.2 are satisfied for the strings. To complete the proof we therefore only need to consider the case of $L(\chi_0)$. We can deform the parameters of the strings in the A_τ with $\tau \neq 0$ to zero without r -reducibility occurring. If $L(\chi)$ is unitary, then so is the parameter where we deform all but one $\tau \neq 0$ to zero. But for a parameter with a single string belonging to an A_τ with $\tau \neq 0$, the necessary conditions for unitarity are given in § 7.4.

8. Real nilpotent orbits

In this chapter we review some well-known results for real nilpotent orbits. Some additional details and references can be found in [21]. The notation in this section differs from the previous sections, and is spelled out in § 8.2.

8.1. To motivate the need for these results we start with a review of properties the asymptotic associated support/cycle, and wavefront set.

Let π be an admissible (\mathfrak{g}_c, K) module. We review some facts from [14]. The distribution character Θ_π lifts to an invariant eigendistribution θ_π in a neighbourhood of the identity in the Lie algebra. For $f \in C_c^\infty(U)$, where $U \subset \mathfrak{g}$ is a small enough neighbourhood of 0, let $f_t(X) := t^{-\dim \mathfrak{g}_c} f(t^{-1}X)$. Then

$$\theta_\pi(f_t) = t^{-d} \sum_j c_j \hat{\mu}_{\mathcal{O}_j(\mathbb{R})}(f) + \sum_{i>0} t^{d+i} D_{d+i}(f). \tag{8.1}$$

The D_i are homogeneous invariant distributions (each D_i is tempered and the support of its Fourier transform is contained in the nilpotent cone). The $\mu_{\mathcal{O}_j}$ are invariant measures supported on real forms \mathcal{O}_j of a single complex orbit \mathcal{O}_c , and $\mu_{\mathcal{O}_j(\mathbb{R})}$ is the Liouville measure on the nilpotent orbit associated to the symplectic form induced by the Cartan–Killing form. Furthermore, $d = \dim_{\mathbb{C}} \mathcal{O}_c/2$, and the number c_j is called the multiplicity of $\mathcal{O}_j(\mathbb{R})$ in the leading term of the expansion. The closure of the union of the supports of the Fourier transforms of all the terms occurring in (8.1) is called the *asymptotic support*, denoted $\text{AS}(\pi)$. The leading term in (8.1) will be called $\text{AC}(\pi)$. We will use the fact that the nilpotent orbits in the leading term are contained in the *wavefront set* of θ_π at the origin, denoted $\text{WF}(\pi)$.

Alternatively, [32] attaches to each π a combination of θ -stable orbits with integer coefficients

$$\text{AV}(\pi) = \sum a_j \mathcal{O}_j, \tag{8.2}$$

where \mathcal{O}_j are nilpotent K_c -orbits in \mathfrak{s}_c . The main result of [28] is that $\text{AC}(\pi)$ in (8.1) and $\text{AV}(\pi)$ in (8.2) correspond via Theorem 8.6. Precisely, the leading term in formula (8.7), and (8.2) are the same, when we identify real and θ stable nilpotent orbits via the Kostant–Sekiguchi correspondence. We will use this when we need to compare $\text{AC}(\pi)$ and $\text{AV}(\pi)$. The algorithms in § 8 compute the associated variety of an induced representation as a set, which we also denote by $\text{AV}(\pi)$ when there is no possibility of confusion. The multiplicities are computed in the real setting in [5, Theorem 5.0.7]. The formula is as follows. Let $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ be a real parabolic subalgebra. Suppose $\text{AC}(\pi) = \sum c_j \mathcal{O}_{j,\mathfrak{m}}$, for

a representation π of M . Let $v_j \in \mathcal{O}_{j,m}$ be representatives, and write $v_{ij} = v_j + X_{ij}$ for representatives of the induced orbits \mathcal{O}_{ij} from $\mathcal{O}_{j,m}$. Then

$$\text{AC}(\text{Ind}_P^G(\pi)) = \sum_{i,j} c_j \left| \frac{C_G(v_{ij})}{C_P(v_{ij})} \right| \mathcal{O}_{ij}. \tag{8.3}$$

We will need an analogue of (8.3) for θ -stable induction (derived functors construction or cohomological induction) and $\text{AV}(\pi)$. We present a special case which is well known, e.g. [31]. Fix a regular dominant integral infinitesimal character χ . The Beilinson–Bernstein localization functor Δ_χ provides an equivalence of categories between admissible (\mathfrak{g}_c, K) modules, and K_c -equivariant holonomic \mathcal{D}_χ -modules on the flag variety $\mathcal{B} := G_c/B_c$. Fix χ to be the infinitesimal character of the trivial representation, and let $\text{Ch}(\pi)$ be the *characteristic cycle* of $\Delta_\chi(\pi)$. A review of its definition and properties can be found for example in [1, Chapter 19]. Let $\mathfrak{p}_c = \mathfrak{l}_c + \mathfrak{u}_c$ be a θ -stable parabolic subalgebra, and \mathcal{P} be the generalized flag variety of parabolic subalgebras of type \mathfrak{p}_c , and $f : \mathcal{B} \rightarrow \mathcal{P}$ the canonical projection map. Then $\bar{Q} := K_c \cdot \mathfrak{p}_c \subset \mathcal{P}$ is closed, and let Q be the K_c -orbit which is open in $f^{-1}(\bar{Q})$. If $\bar{\mathcal{S}}$ is the \mathcal{D} -module on \mathcal{P} corresponding to the trivial local system on \bar{Q} , let \mathcal{S} be the irreducible module corresponding to the trivial local system on Q , and π be the irreducible (\mathfrak{g}_c, K) module corresponding to \mathcal{S} by localization. Then $\text{Ch}(\mathcal{S}) = T_{\bar{Q}}^* \bar{\mathcal{S}}$, the conormal bundle of \bar{Q} . By Theorem 20.1 of [1], $\text{Ch}(\mathcal{S}) = T_Q^* \mathcal{B}$.

Let

$$\mu : T^* \mathcal{B} \rightarrow \mathfrak{g}^*$$

be the moment map. Then $\mu(T_Q^* \mathcal{B}) = \overline{K_c \cdot (\mathfrak{u}_c \cap \mathfrak{s}_c)}$ is a union of finitely many nilpotent K_c -orbits in \mathfrak{s}_c . The unique orbit which intersects $\mu(T_Q^* \mathcal{B})$ in a dense open set is called $\text{ind}_{\mathfrak{u}_c \cap \mathfrak{s}_c}^{\mathfrak{s}_c}(0)$. Similarly write $\text{ind}_{\mathfrak{p}_c}^{\mathfrak{g}_c}(0)$ for the unique nilpotent G_c -orbit which intersects \mathfrak{u}_c in an open set.

Proposition 8.1. *Assume that $\mu|_{T_Q^* \mathcal{B}}$ is birational. Then*

$$\text{Ad } G_c \text{ind}_{\mathfrak{u}_c \cap \mathfrak{s}_c}^{\mathfrak{s}_c}(0) = \text{ind}_{\mathfrak{p}_c}^{\mathfrak{g}_c}(0),$$

and

$$\text{AC}(\pi) = \text{ind}_{\mathfrak{u}_c \cap \mathfrak{s}_c}^{\mathfrak{s}_c}(0).$$

Proof. The first assertion is a dimension count, recall that for $e \in \mathfrak{s}_c$, $\dim \text{Ad } G_c e = 2 \dim \text{Ad } K_c e$. The second assertion follow from the results in [20], particularly Corollary 2.5.6. Under the assumptions of the proposition, and the notation in [20] preceding Corollary 2.5.6, F_ν is a single point, so $c_{F_\nu} = 1$. □

Let ξ be a unitary character of the Levi component of a θ -stable parabolic subalgebra $\mathfrak{q}_c = \mathfrak{l}_c + \mathfrak{u}_c$. Cohomological induction (detailed in, for example, [23]) associates to ξ a family of (\mathfrak{g}_c, K) modules $\mathcal{R}_{\mathfrak{q}_c}^i(\xi)$. Extending the notion of AC or AV by linearity to the Grothendieck group, we can talk about $\text{AC}(\sum (-1)^i \mathcal{R}_{\mathfrak{q}_c}^i(\xi))$ or $\text{AV}(\sum (-1)^i \mathcal{R}_{\mathfrak{q}_c}^i(\xi))$.

We first consider the case when ξ is dominant for $\Delta(\mathfrak{u}_c)$, and the infinitesimal character is regular integral, in particular equal to the one of the trivial representation. Then all $\mathcal{R}_{\mathfrak{q}_c}^i(\xi) = 0$ except for $i = S := \dim(\mathfrak{u}_c \cap \mathfrak{k}_c)$, and case $\mathcal{R}_{\mathfrak{q}_c}^i(\xi)$ is π corresponding to \mathcal{S} defined earlier.

Corollary 8.2.

$$\text{AC} \left(\sum (-1)^i \mathcal{R}_{\mathfrak{q}_c}^i(\xi) \right) = (-1)^S.$$

In particular if ξ is such that $\mathcal{R}_{\mathfrak{q}_c}^i(\xi) = 0$ for $i \neq S$, then

$$\text{AC}(\mathcal{R}_{\mathfrak{q}_c}^S(\xi)) = \text{ind}_{\mathfrak{q}_c \cap \mathfrak{s}_c}^{\mathfrak{s}_c}(0)$$

(in particular $\mathcal{R}_{\mathfrak{q}_c}^S(\xi)$ is also non-zero).

Proof. Embed π into a coherent family as in [20]. The polynomial c_{F_ν} is a constant multiple of

$$\text{Ch}(\mathcal{S}) \prod_{\alpha \in \Delta^+(\mathfrak{l}_c)} \langle \nu, \check{\alpha} \rangle \Big/ \prod_{\alpha \in \Delta^+(\mathfrak{l}_c)} \langle \rho_{\mathfrak{l}_c}, \check{\alpha} \rangle. \tag{8.4}$$

The result follows from Proposition 8.1 combined with the fact that $\text{Ch}(\mathcal{S}) = 1$ and the other factor is 1 when evaluated at $\nu = \xi + \rho_{\mathfrak{l}_c}$. □

We will use such results in the case of highest weight modules as well. Fix a Borel subgroup $B_c = H_c N_c$ with Lie algebra $\mathfrak{b}_c = \mathfrak{h}_c + \mathfrak{n}_c \subset \mathfrak{p}_c$. The nilradical N_c acts on \mathcal{B} and on \mathcal{S} with finitely many orbits. All the notions described earlier for (\mathfrak{g}_c, K) orbits have a counterpart in this setting (but simpler: for example, we do not need to use cohomological induction). For example, again fixing a regular integral dominant infinitesimal character χ , the localization functor provides an equivalence between the category of highest weight modules and N_c -equivariant \mathcal{D}_χ -modules. The orbit \bar{Q} is a single point, and Q is the set of Borel subalgebras contained in \mathfrak{p}_c . The analogue of the representation π is $U(\mathfrak{g}_c) \otimes_{\mathfrak{p}_c} \mathbb{C}_{\rho(\mathfrak{n}_c)}$.

Corollary 8.3. For any character ξ of \mathfrak{l}_c ,

$$\text{AC}(U(\mathfrak{g}_c) \otimes_{\mathfrak{p}_c} \mathbb{C}_\xi) = \text{ind}_{\mathfrak{p}_c}^{\mathfrak{g}_c}(0).$$

8.2. The results in this section are well known, an account with details is given for example in [21]. Fix a real form \mathfrak{g} of a complex semisimple Lie algebra \mathfrak{g}_c . Let θ_c be the complexification of the Cartan involution θ of \mathfrak{g} , and write $\bar{}$ for the conjugation. Let G_c be the adjoint group with Lie algebra \mathfrak{g}_c , and let

$$\mathfrak{g}_c = \mathfrak{k}_c + \mathfrak{s}_c, \quad \mathfrak{g} = \mathfrak{k} + \mathfrak{s} \tag{8.5}$$

be the Cartan decomposition. Write $K_c \subset G_c$ for the subgroup corresponding to \mathfrak{k}_c , and G and K for the real Lie groups corresponding to \mathfrak{g} and \mathfrak{k} .

Theorem 8.4 (Jacobson–Morozov).

- (1) There is a one-to-one correspondence between G_c -orbits $\mathcal{O}_c \subset \mathfrak{g}_c$ of nilpotent elements and G_c -orbits of Lie triples $\{e, h, f\}$, i.e. elements satisfying

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

This correspondence is realized by completing a nilpotent element $e \in \mathcal{O}$ to a Lie triple.

- (2) Two Lie triples $\{e, h, f\}$ and $\{e', h', f'\}$ are conjugate if and only if the elements h and h' are conjugate.

8.3. Suppose $e \in \mathfrak{g}$ is nilpotent. Then one can still complete it to a Lie triple $e, h, f \in \mathfrak{g}$. A Lie triple where all the elements are in \mathfrak{g} is called *real* and denoted as ρ -stable. ρ -stable is a definition. A Lie triple is called *Cayley* if in addition $\theta(h) = -h, \theta(e) = -f$. Every real Lie triple is conjugate by G to one which is Cayley.

Theorem 8.5 (Kostant–Rao). Two real Lie triples are conjugate if and only if the elements $e - f$ and $e' - f'$ are conjugate under G . Equivalently, two Cayley triples are conjugate if and only if $e - f$ and $e' - f'$ are conjugate under K .

8.4. Suppose $e \in \mathfrak{s}_c$ is nilpotent. Then e can be completed to a Lie triple satisfying

$$\theta_c(e) = -e, \quad \theta_c(h) = h, \quad \theta_c(f) = -f. \tag{8.6}$$

We call such a triple θ -stable. To any Cayley triple $\{e, h, f\}$ one can associate a θ -stable triple $\{\tilde{e}, \tilde{h}, \tilde{f}\}$ as in (8.6), by the formulae

$$\tilde{e} := \frac{1}{2}(e + f + ih), \quad \tilde{h} := i(e - f), \quad \tilde{f} := \frac{1}{2}(e + f - ih). \tag{8.7}$$

A Lie triple is called *normal* if in addition to (8.6) it satisfies $\bar{e} = f, \bar{h} = -h$.

Theorem 8.6 (Kostant–Sekiguchi).

- (1) Any θ -stable triple is conjugate via K_c to a normal one.
- (2) Let $\tilde{e}, \tilde{e}' \in \mathfrak{s}_c$ be nilpotent elements, and complete them to normal triples $\{\tilde{e}, \tilde{h}, \tilde{f}\}$ and $\{\tilde{e}', \tilde{h}', \tilde{f}'\}$. Then \tilde{e} and \tilde{e}' are conjugate by K_c , if and only if the corresponding Lie normal triples are conjugate by K_c , if and only if \tilde{h} and \tilde{h}' are conjugate under K_c .
- (3) Two θ -stable triples are conjugate under K if and only if the elements \tilde{h}, \tilde{h}' are conjugate under K_c .
- (4) The correspondence (8.7) is a bijection between G orbits of nilpotent elements in \mathfrak{g} and K_c orbits of nilpotent elements in \mathfrak{s}_c .

Proposition 8.7. The correspondence between real and θ -stable orbits preserves closure relations.

Proof. This is the main result in [13]. □

8.5. Let $\mathfrak{p}_c = \mathfrak{m}_c + \mathfrak{n}_c$ be a parabolic subalgebra of \mathfrak{g}_c . Let $\mathfrak{c}_c := \text{Ad } M_c \cdot e$ be the orbit of a nilpotent element $e \in \mathfrak{m}_c$. According to [25], the induced orbit from \mathfrak{c}_c is the unique G_c orbit \mathfrak{C}_c which has the property that $\mathfrak{C}_c \cap [\mathfrak{c}_c + \mathfrak{n}_c]$ is dense (and open) in $\mathfrak{c}_c + \mathfrak{n}_c$.

Proposition 8.8. Let $E = e + n \in e + \mathfrak{n}_c \subset \mathfrak{c}_c + \mathfrak{n}_c$.

- (1) $\dim Z_{M_c}(e) = \dim Z_{G_c}(E)$.
- (2) $\mathfrak{C}_c \cap [\mathfrak{c}_c + \mathfrak{n}_c]$ is a single P_c orbit.

This is Theorem 1.3 in [25]. In particular, an element $E' = e' + n' \in \mathfrak{c}_c + \mathfrak{n}_c$ is in \mathfrak{C}_c if and only if the map

$$\text{ad } E' : \mathfrak{p}_c \rightarrow T_{e'}\mathfrak{c}_c + \mathfrak{n}_c, \quad \text{ad } E'(y) = [E', y] \tag{8.8}$$

is onto.

Another characterization of the induced orbit is the following.

Proposition 8.9. The orbit \mathfrak{C}_c is the unique open orbit in $\text{Ad } G_c(e + \mathfrak{n}_c) = \text{Ad } G_c(\mathfrak{c}_c + \mathfrak{n}_c)$, as well as in the closure $\overline{\text{Ad } G_c(e + \mathfrak{n}_c)} = \text{Ad } G_c(\overline{\mathfrak{c}_c} + \mathfrak{n}_c)$.

We omit the proof, but note that the statements about the closures follow from the fact that G_c/P_c is compact.

Proposition 8.10. The orbit \mathfrak{C}_c depends on $\mathfrak{c}_c \subset \mathfrak{m}_c$, but not on \mathfrak{n}_c .

Proof. This is proved in §2 of [25]. We give a different proof which generalizes to the real case. Let $\xi \in \mathfrak{h}_c \subset \mathfrak{m}_c$ be an element in the centre of \mathfrak{m}_c such that $\langle \xi, \alpha \rangle \neq 0$ for all roots $\alpha \in \Delta(\mathfrak{n}_c, \mathfrak{h}_c)$. Then by a standard argument,

$$\text{Ad } P_c(\xi + e) = \xi + \mathfrak{c}_c + \mathfrak{n}_c. \tag{8.9}$$

Again because G_c/P_c is compact,

$$\overline{\bigcup_{t>0} \text{Ad } G_c(t\xi + e)} \setminus \bigcup_{t>0} \text{Ad } G_c(t\xi + e) = \text{Ad } G_c(\overline{\mathfrak{c}_c} + \mathfrak{n}_c). \tag{8.10}$$

Formula (8.10) is valid for any parabolic subgroup with Levi component M_c . The claim follows because the left-hand side of (8.10) only depends on M_c and the orbit \mathfrak{c}_c . \square

We now consider the case of real induction. Let $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ be a real parabolic subalgebra, $e \in \mathfrak{m}$ a nilpotent element, and $\mathfrak{c} := \text{Ad } M e$.

Definition 8.11. The ρ -induced set from \mathfrak{c} to \mathfrak{g} is the finite union of orbits $\mathfrak{C}_i := \text{Ad } G E_i$ such that one of the following equivalent conditions hold.

- (1) \mathfrak{C}_i is open in $\text{Ad } G(\mathfrak{c} + \mathfrak{n})$ and $\overline{\bigcup \mathfrak{C}_i} = \overline{\text{Ad } G(e + \mathfrak{n})}$.
- (2) The intersection $\mathfrak{C}_i \cap [\mathfrak{c} + \mathfrak{n}]$ is open in $\mathfrak{c} + \mathfrak{n}$, and the union of the intersections is dense in $\mathfrak{c} + \mathfrak{n}$.

We write

$$\text{ind}_{\mathfrak{p}}^{\mathfrak{g}}(\mathfrak{c}) = \bigcup \mathfrak{c}_i \tag{8.11}$$

and we say that each E_i is real or ρ -induced from e . Sometimes we will write $\text{ind}_{\mathfrak{p}}^{\mathfrak{g}}(e)$.

We omit the details of the proof of the equivalence of the two statements.

Proposition 8.12. *The ρ -induced set depends on the orbit \mathfrak{c} of e and the Levi component \mathfrak{m} , but not on \mathfrak{n} .*

Proof. The proof is essentially identical to the one in the complex case. We omit the details. □

In terms of the θ -stable versions \tilde{e} of e , and \tilde{E}_i of E_i , ρ -induction is computed in [7]. This is as follows. Let $\mathfrak{h}_c \subset \mathfrak{m}_c$ be the complexification of a maximally split real Cartan subalgebra \mathfrak{h} , and $\xi \in \mathcal{Z}(\mathfrak{m}_c) \cap \mathfrak{s}_c$ an element of \mathfrak{h} such that

$$\alpha \in \Delta(\mathfrak{n}, \mathfrak{h}) \text{ if and only if } \alpha(\xi) > 0.$$

Then

$$\overline{\bigcup \text{Ad } K_c(\tilde{E}_i)} = \overline{\bigcup_{t>0} \text{Ad } K_c(t\xi + \tilde{e})} \setminus \bigcup_{t>0} \text{Ad } K_c(t\xi + \tilde{e}). \tag{8.12}$$

8.6. Let $\mathfrak{q}_c = \mathfrak{l}_c + \mathfrak{u}_c$ be a θ -stable parabolic subgroup, and write $\bar{\mathfrak{q}}_c = \mathfrak{l}_c + \bar{\mathfrak{u}}_c$ for its complex conjugate. Let $e \in \mathfrak{l}_c \cap \mathfrak{s}_c$ be a nilpotent element.

Proposition 8.13. *There is a unique K_c -orbit $\mathcal{O}_{K_c}(E)$ so that its intersection with $\mathcal{O}_{L_c \cap K_c}(e) + (\mathfrak{u}_c \cap \mathfrak{s}_c)$ is open and dense.*

Proof. This follows from the fact that $e + (\mathfrak{u}_c \cap \mathfrak{s}_c)$ is formed of nilpotent orbits, there are a finite number of nilpotent orbits, and being complex, the K_c -orbits have even real dimension. □

Definition 8.14. The orbit $\mathcal{O}_{K_c}(E)$ as in the proposition above is called θ -induced from e , and we write

$$\text{ind}_{\mathfrak{q}_c}^{\mathfrak{g}_c}(\mathcal{O}_{\mathfrak{l}_c}(e)) = \mathcal{O}_{K_c}(E),$$

and say that E is θ -induced from e .

Remark. The induced orbit is characterized by the property that it is the (unique) largest dimensional one which meets $e + \mathfrak{u}_c \cap \mathfrak{s}_c$. It depends on e as well as \mathfrak{q}_c , not just e and \mathfrak{l}_c .

8.7. $u(p, q)$

Let V be a complex finite dimensional vector space of dimension n . There are two inner classes of real forms of $GL(V)$. One is such that θ is an outer automorphism. It consists of the real form $GL(n, \mathbb{R})$, and when n is even, also $U^*(n)$. The other one is such that θ is inner, and consists of the real forms $U(p, q)$ with $p + q = n$. In §§ 8.7–8.10, we investigate ρ and θ induction for the forms $u(p, q)$, and then derive the corresponding results for $so(p, q)$ and $sp(n, \mathbb{R})$ from them in §§ 8.11–8.12. The corresponding results for the other real forms are easier, the case of $GL(n, \mathbb{R})$ is well known, and we will not need $u(n)^*$. The usual description of $u(p, q)$ is that V is endowed with a hermitian form (\cdot, \cdot) of signature (p, q) , and $u(p, q)$ is the Lie algebra of skew hermitian matrices with respect to this form. Fix a positive definite hermitian form $\langle \cdot, \cdot \rangle$. We will identify the complexification of $\mathfrak{g} := u(p, q)$ with $\mathfrak{g}_c := \mathfrak{gl}(V)$, and the complexification of $U(p, q)$ with $GL(V)$. Up to conjugacy by $GL(V)$,

$$(v, w) = \langle \theta v, w \rangle, \quad \theta^2 = 1. \tag{8.13}$$

The eigenspaces of θ on V will be denoted V^\pm . The Cartan decomposition is $\mathfrak{g}_c = \mathfrak{k}_c + \mathfrak{s}_c$, where \mathfrak{k}_c is the $+1$ eigenspace, and \mathfrak{s}_c the -1 eigenspace of $\text{Ad } \theta$.

The classification of nilpotent orbits of $u(p, q)$ is by signed tableaux as in Theorem 9.3.3 of [21]. The same parametrization applies to θ stable orbits under K_c .

8.8. A parabolic subalgebra of $\mathfrak{gl}(V)$ is the stabilizer of a generalized flag

$$(0) = W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_k = V. \tag{8.14}$$

Fix complementary spaces V_i ,

$$W_i = W_{i-1} + V_i, \quad i > 0. \tag{8.15}$$

They determine a Levi component

$$\mathfrak{l} \cong \mathfrak{gl}(V_1) \times \cdots \times \mathfrak{gl}(V_k). \tag{8.16}$$

Conjugacy classes under K_c of θ -stable parabolic subalgebras are parametrized by ordered pairs $(p_1, q_1), \dots, (p_k, q_k)$ such that the sum of the p_i is p , and the sum of the q_i is q . A realization in terms of flags is as follows. Choose the W_i to be stable under θ , or equivalently that the restriction of the hermitian form to each W_i is non-degenerate. In this case we may assume that the V_i are θ -stable as well, and let $\mathfrak{q}_c = \mathfrak{l}_c + \mathfrak{u}_c$ be the corresponding parabolic subalgebra of $\mathfrak{gl}(V)$. The signature of the form restricted to V_i is (p_i, q_i) , so that

$$\mathfrak{l}_c \cap \mathfrak{g} \cong u(p_1, q_1) \times \cdots \times u(p_k, q_k). \tag{8.17}$$

The following algorithm for computing the induced orbit in the case $\mathfrak{g} \cong u(p, q)$ and a maximal parabolic subalgebra holds.

Suppose the signature of V_1 is (a_+, a_-) . Then add a_+ pluses to the beginning of the largest possible rows of e starting with minus and a_- minuses to the largest possible rows of e starting with plus. If a_+ is larger than the number of rows starting with minus, add a new row of size 1 starting with plus. A similar rule applies to a_- .

If $e \in \mathfrak{gl}(V_1)$, the analogous procedure applies, but the a_+ pluses are added at the end of the largest possible rows finishing in minus and a_- minuses to the end of the largest possible rows finishing in plus.

Because induction is transitive, the above algorithm can be generalized to compute the θ -induced of any nilpotent orbit. We omit the details. See [17] and [31] for related results.

8.9. Conjugacy classes under G of real parabolic subalgebras are given by ordered subsequences n_1, \dots, n_k and a pair (p_0, q_0) such that $\sum n_i + p_0 = p$ and $\sum n_i + q_0 = q$. The complexification of the corresponding real parabolic subalgebra is given as follows. Start with a partial flag

$$(0) = W_0 \subsetneq \dots \subsetneq W_k \tag{8.18}$$

such that the hermitian form is trivial when restricted to W_k , and complete it to

$$(0) = W_0 \subsetneq \dots \subsetneq W_k \subsetneq W_k^* \subsetneq \dots \subsetneq W_0^* = V. \tag{8.19}$$

Choose transverse spaces

$$W_i = W_{i-1} + V_i, \quad W_i^* = W_{i-1}^* + V_i^*, \quad W_k^* = W_k + V_0. \tag{8.20}$$

They determine a Levi component

$$\mathfrak{l}_c = \mathfrak{gl}(V_1) \times \dots \times \mathfrak{gl}(V_k) \times \mathfrak{gl}(V_0) \times \mathfrak{gl}(V_k^*) \times \dots \times \mathfrak{gl}(V_1^*), \tag{8.21}$$

so that

$$\mathfrak{l}_c \cap \mathfrak{g} = \mathfrak{gl}(V_1, \mathbb{C}) \times \dots \times \mathfrak{gl}(V_k, \mathbb{C}) \times u(p_0, q_0). \tag{8.22}$$

Then $n_i = \dim V_i$, and (p_0, q_0) is the signature of V_0 .

8.10. Suppose $\mathfrak{p}_c = \mathfrak{m}_c + \mathfrak{n}_c$ is the complexification of a real parabolic subalgebra corresponding to the flag $(0) \subset V_1 \subset V_1 + V_0 \subset V_1 + V_0 + V_1^*$, and let $e \in \mathfrak{gl}(V_0)$ be a real nilpotent element. The rest of the notation is as in § 8.5.

Theorem 8.15. *The tableau of an orbit $\text{Ad } G(E_i)$ which is in the ρ -induced set $\text{ind}_{\mathfrak{p}}^{\mathfrak{g}}(\mathfrak{c})$, is obtained from the tableau of e as follows.*

Add two boxes to the end of each of $\dim V_1$ of the largest rows such that the result is still a signed tableau.

Proof. We use (8.9) and (8.10). Let $\alpha \in \text{Hom}[V_1, V_1^*] \oplus \text{Hom}[V_1^*, V_1]$ be non-degenerate such that $\alpha^2 = \text{Id} \oplus \text{Id}$, and extend it to an endomorphism $\xi \in \mathfrak{gl}(V)$ so that its restriction

to V_0 is zero. This is an element such that the centralizer of $\text{ad } \xi$ is \mathfrak{m} , in particular, $[\xi, e] = 0$. Let

$$P(X) = X^m + a_{m-1}X^{m-1} + \dots + a_0 \tag{8.23}$$

be any polynomial in $X \in \mathfrak{gl}(V)$. Suppose $t_i \in \mathbb{R}$ are such that $t_i \rightarrow 0$, and assume there are $g_i \in K$ such that $t_i g_i(\xi + e)g_i^{-1} \rightarrow E$. Then

$$\ker t_i^m P(g_i(\xi + e)g_i^{-1}) \cong \ker P(\xi + e). \tag{8.24}$$

On the other hand,

$$t_i^m P(g_i(\xi + e)g_i^{-1}) = [t_i g_i(\xi + e)g_i^{-1}]^m + a_{m-1}t_i[t_i g_i(\xi + e)g_i^{-1}]^{m-1} + \dots + t_i^m \text{Id} \rightarrow E^m, \tag{8.25}$$

as $t_i \rightarrow 0$. Thus

$$\dim \ker E^m|_{V_{\pm}} \geq \dim \ker P(\xi + e)|_{V_{\pm}}. \tag{8.26}$$

Choosing $P(X) = (X^2 - 1)X^n$, we conclude that E must be nilpotent. Choosing $P(X) = X^m, (X \pm 1)X^{m-1}$ or $P(X) = (X^2 - 1)X^{m-2}$, we can bound the dimensions of $\ker E^m|_{V_{\pm}}$ to conclude that it must be in the closure of one of the nilpotent orbits given by the algorithm of the theorem. The fact that these nilpotent orbits are in (8.10) follows by a direct calculation which we omit. \square

8.11. $\mathfrak{sp}(V)$

Suppose $\mathfrak{g}_c \cong \mathfrak{sp}(V_0)$, where $(V_0, \langle \cdot, \cdot \rangle)$ is a real symplectic vector space of dimension n . The complexification $(V, \langle \cdot, \cdot \rangle)$ admits a complex conjugation $\bar{}$, and we define a non-degenerate hermitian form

$$(v, w) := \langle v, \bar{w} \rangle \tag{8.27}$$

which is of signature (n, n) . Denote by $u(n, n)$ the corresponding unitary group. Since $\mathfrak{sp}(V_0)$ stabilizes (\cdot, \cdot) , it embeds in $u(n, n)$, and the Cartan involutions are compatible. The results of §§ 8.2–8.4 together with § 8.7 imply the following classification of nilpotent orbits of $\mathfrak{sp}(V_0)$ or equivalently θ -stable nilpotent orbits. See Chapter 9 of [21] for a more detailed explanation.

Each orbit corresponds to a signed tableau so that every odd part occurs an even number of times. Odd sized rows occur in pairs, one starting with plus the other with minus.

A real parabolic subalgebra of $\mathfrak{sp}(V)$ is the stabilizer of a flag of isotropic subspaces

$$(0) = \mathcal{W}_0 \subset \dots \subset \mathcal{W}_k, \tag{8.28}$$

so that the symplectic form restricts to 0 on \mathcal{W}_k . As before, complete this to a flag

$$(0) = \mathcal{W}_0 \subset \dots \subset \mathcal{W}_k \subset \mathcal{W}_k^* \subset \dots \subset \mathcal{W}_0^* = V. \tag{8.29}$$

We choose transverse spaces

$$\mathcal{W}_i = \mathcal{W}_{i-1} + V_i, \quad \mathcal{W}_k^* = \mathcal{W}_k + \mathcal{W}, \quad \mathcal{W}_{i-1}^* = \mathcal{W}_i^* + V_i^* \tag{8.30}$$

in order to fix a Levi component. We get

$$\mathfrak{l} \cong \mathfrak{gl}(V_1) \times \cdots \times \mathfrak{gl}(V_k) \times \mathfrak{sp}(\mathcal{W}). \tag{8.31}$$

If we assume that V_i, \mathcal{W} are θ -stable, then the corresponding parabolic subalgebra is θ -stable as well, and the real points of the Levi component are

$$\mathfrak{l}_0 \cong u(p_1, q_1) \times \cdots \times u(p_k, q_k) \times \mathfrak{sp}(\mathcal{W}), \tag{8.32}$$

where (p_i, q_i) is the signature of V_i . The Levi component of the parabolic subalgebra corresponding to (8.30) in $\mathfrak{gl}(V)$ satisfies

$$\mathfrak{l}' \cong u(p_1, q_1) \times \cdots \times u(p_k, q_k) \times u(n_0, n_0) \times u(q_k, p_k) \times \cdots \times u(q_1, p_1). \tag{8.33}$$

For a maximal θ -stable parabolic subalgebra, the Levi component \mathfrak{l} satisfies $\mathfrak{l} \cong u(p_1, q_1) \times \mathfrak{sp}(\mathcal{W})$. Let $e \in \mathfrak{sp}(\mathcal{W})$ be a θ -stable nilpotent element. The algorithm for induced nilpotent orbits in § 8.8 implies the following algorithm for $\text{ind}_{\mathfrak{q}_c}^{\mathfrak{g}_c}(e)$.

- (1) Add p boxes labelled with pluses to the beginning of the longest rows starting with minuses, and q minuses to the beginning of the longest rows starting with pluses.
- (2) Add q pluses to the ending of the longest possible rows starting with minuses, and p minuses to the beginning of the longest possible rows starting with pluses.

Unlike in the complex case, the result is automatically the signed tableau corresponding to a nilpotent element in $\mathfrak{sp}(V)$. Again see [17] and [31] for related results.

For a maximal real parabolic subalgebra, we must assume that $\bar{V}_1 = V_1, \bar{\mathcal{W}} = \mathcal{W}$. Let $V_{1,0}$ and \mathcal{W}_0 be their real points. The Levi component satisfies

$$\mathfrak{l} \cong \mathfrak{gl}(V_{1,0}) \times \mathfrak{sp}(\mathcal{W}_0). \tag{8.34}$$

The results in § 8.10 imply the following algorithm for real induction.

- (1) Add two boxes to the largest $\dim V_1$ rows of e so that the result is still a signed tableau for a nilpotent orbit.
- (2) Suppose $\dim V_1$ is odd and the last row that would be increased by 2 is odd size as well. In this case there is a pair of rows of this size, one starting with plus the other with minus. In this case increase these two rows by one each.

8.12. $\mathfrak{so}(p, q)$

Suppose $\mathfrak{g} \cong \mathfrak{so}(V_0)$, where $(V_0, \langle \cdot, \cdot \rangle)$ is a real non-degenerate quadratic space of signature (p, q) . The complexification admits a hermitian form $\langle \cdot, \cdot \rangle$ with signature (p, q) as well as a complex non-degenerate quadratic form (\cdot, \cdot) , which restrict to $\langle \cdot, \cdot \rangle$ on V_0 . The form $\langle \cdot, \cdot \rangle$ gives an embedding of $\mathfrak{o}(p, q)$ into $\mathfrak{u}(p, q)$ with compatible Cartan involutions. The results of §§ 8.2–8.4 together with § 8.7 imply the following classification of nilpotent orbits of $\mathfrak{so}(V_0)$ or equivalently θ -stable nilpotent orbits. See Chapter 9 of [21] for more details.

Orbits correspond to signed tableaux so that every even part occurs an even number of times. Even sized rows occur in pairs, one starting with plus the other with minus. When all the rows have even sizes, there are two nilpotent orbits denoted I and II.

A parabolic subalgebra of $\mathfrak{so}(V)$ is the stabilizer of a flag of isotropic subspaces

$$(0) = \mathcal{W}_0 \subset \dots \subset \mathcal{W}_k, \tag{8.35}$$

so that the quadratic form restricts to 0 on \mathcal{W}_k . As before, complete this to a flag

$$(0) = \mathcal{W}_0 \subset \dots \subset \mathcal{W}_k \subset \mathcal{W}_k^* \subset \dots \subset \mathcal{W}_0^* = V. \tag{8.36}$$

We choose transverse spaces

$$\mathcal{W}_i = \mathcal{W}_{i-1} + V_i, \quad \mathcal{W}_k^* = \mathcal{W}_k + \mathcal{W}, \quad \mathcal{W}_{i-1}^* = \mathcal{W}_i^* + V_i^* \tag{8.37}$$

in order to fix a Levi component,

$$\mathfrak{l} \cong \mathfrak{gl}(V_1) \times \dots \times \mathfrak{gl}(V_k) \times \mathfrak{so}(\mathcal{W}). \tag{8.38}$$

To get a θ -stable parabolic subalgebra we must assume V_i, \mathcal{W} are θ -stable. If the signature of V_i with respect to $\langle \cdot, \cdot \rangle$ is (p_i, q_i) , and that of \mathcal{W} is (p_0, q_0) , then

$$\mathfrak{l}_0 \cong u(p_1, q_1) \times \dots \times u(p_k, q_k) \times \mathfrak{so}(p_0, q_0). \tag{8.39}$$

The parabolic subalgebra corresponding to (8.36) in $\mathfrak{gl}(V)$ satisfies

$$\mathfrak{l}' \cong u(p_1, q_1) \times \dots \times u(p_k, q_k) \times u(p_0, q_0) \times u(p_k, q_k) \times \dots \times u(p_1, q_1). \tag{8.40}$$

For a maximal θ -stable parabolic subalgebra, the Levi component \mathfrak{l} satisfies $\mathfrak{l} \cong u(p_1, q_1) \times \mathfrak{so}(\mathcal{W})$. Let $e \in \mathfrak{so}(\mathcal{W})$ be a θ -stable nilpotent element. The algorithm for induced nilpotent orbits in § 8.8 implies the following algorithm for $\text{ind}_{\mathfrak{q}_c}^{\mathfrak{g}_c}(e)$.

- (1) Add p_1 pluses to the beginning of the longest possible rows starting with minuses, and q_1 minuses to the beginning of the longest possible rows starting with pluses.
- (2) Add p_1 pluses to the ending of the longest possible rows starting with minuses, and q_1 minuses to the beginning of the longest possible rows starting with pluses.

Unlike in the complex case, the result is automatically a signed tableau for a nilpotent element in $\mathfrak{so}(V)$.

For a maximal real parabolic subalgebra, we must assume that $\bar{V}_1 = V_1, \bar{\mathcal{W}} = \mathcal{W}$. Let $V_{1,0}$ and \mathcal{W}_0 be their real points. The Levi component satisfies

$$\mathfrak{l} \cong \mathfrak{gl}(V_{1,0}) \times \mathfrak{so}(W_0). \tag{8.41}$$

The results in § 8.10 imply the following algorithm for real induction.

Add two boxes to $\dim V_1$ of the largest possible rows so that the result is still a signed tableau for a nilpotent orbit. Suppose $\dim V_1$ is even and the last row that would be increased by 2 is even size as well. In this case there is a pair of rows of this size, one starting with plus the other with minus. Increase these two rows by one each so that the result is still a signed tableau. When there are only even sized rows and $\dim V_1$ is even as well, type I goes to type I and type II goes to type II.

9. Unitarity

In this section we prove the unitarity of the representations of the form $L(\chi)$ where $\chi = \frac{1}{2}\hbar$. As already mentioned, in the p -adic case this is done in [10]. It amounts to the observation that the Iwahori–Matsumoto involution preserves unitarity, and takes such an $L(\chi)$ into a tempered representation.

The idea of the proof in the real case is described in [3]. We will do an induction on rank. We rely heavily on the properties of the wavefront set, asymptotic support and associated variety, and their relations to primitive ideal cells and Harish-Chandra cells. The facts about asymptotic support and associated variety are in § 8.1, and those about cells are in §§ 9.1–9.3. Details are in [14], [16] and [4]. In §§ 9.4–9.6 we give details of the proof of the unitarity in the case of $SO(2n + 1)$. The proof is simpler than in [3].

9.1. We will make heavy use of the results in [33]. Fix a regular integral infinitesimal character χ_{reg} . Let $\mathcal{G}^G(\chi_{\text{reg}})$ be the set of parameters of irreducible admissible (\mathfrak{g}_c, K) modules with infinitesimal character χ_{reg} . We will suppress the superscript G whenever there is no danger of confusion. The explicit description of $\mathcal{G}^G(\chi_{\text{reg}})$ is well known, called the Langlands or Vogan classification. We will use the description in [33]. Denote by $\mathbb{Z}\mathcal{G}(\chi_{\text{reg}})$ the corresponding Grothendieck group of characters. Recall from [33] (and references therein) that there is an action of the Weyl group on $\mathbb{Z}\mathcal{G}(\chi_{\text{reg}})$, called the *coherent continuation action*. As a set, $\mathcal{G}(\chi_{\text{reg}})$ decomposes into a disjoint union of blocks \mathcal{B} , and so $\mathbb{Z}\mathcal{G}$ decomposes into a direct sum

$$\mathbb{Z}\mathcal{G}(\chi_{\text{reg}}) = \bigoplus \mathbb{Z}\mathcal{G}_{\mathcal{B}}(\chi_{\text{reg}}). \tag{9.1}$$

Each $\mathbb{Z}\mathcal{G}_{\mathcal{B}}(\chi_{\text{reg}})$ is preserved by the coherent continuation action. We give the explicit description of the $\mathbb{Z}\mathcal{G}_{\mathcal{B}}$ in all classical cases. Many of the details, and complementary explanations can be found in [26].

Type B. In order to conform to the duality between type B and type C in [33], we only consider the real forms with $p > q$. The Weyl group acts on the linear space $\sum \mathbb{Z}\mathcal{G}^{\text{SO}(p,q)}(\chi_{\text{reg}})$, and the representation is

$$\sum_{p>q} \mathbb{Z}\mathcal{G}^{\text{SO}(p,q)}(\chi_{\text{reg}}) = \sum_{a,b,\tau} \text{Ind}_{W_a \times W_b \times W_{2s} \times S_t}^{W_n} [\text{sgn} \otimes \text{sgn} \otimes \sigma[\tau, \tau] \otimes \text{triv}], \tag{9.2}$$

where τ is a partition of s , and $a + b + 2s + t = n$. The multiplicity of a $\sigma[\tau_L, \tau_R]$ in one of the induced modules in (9.2) is as follows. Choose a τ that fits inside both τ_L and τ_R , and label it with a bullet. Add the *number* ‘a’ of r and the number ‘b’ of r' to τ_R , at most one to each row for inducing from the sign representations of W_a and W_b . Add ‘t’ c , at most one to each column, to τ_L or τ_R for inducing from the trivial representation on S_t . The multiplicity of σ in the induced module for a given (τ, a, b) is then the number of ways that τ_L, τ_R can be filled in this way. This procedure uses induction in stages, and the well-known formula

$$\text{Ind}_{S_n}^{W_n}(\text{triv}) = \sum_{k+l=n} \sigma[(k), (l)]. \tag{9.3}$$

Example. Let $\mathfrak{g}_c = \mathfrak{so}(5)$. The real forms are $\mathfrak{so}(3, 2)$, $\mathfrak{so}(4, 1)$, $\mathfrak{so}(5)$. The choices of (τ, a, b, t) are

$$\left. \begin{array}{l} (1, 1, 0, 0), \quad (1, 0, 1, 0), \quad (1, 0, 0, 1), \\ (0, 2, 0, 0), \quad (0, 1, 1, 0), \quad (0, 1, 0, 1), \quad (0, 0, 2, 0), \quad (0, 0, 1, 1), \quad (0, 0, 0, 2). \end{array} \right\} \quad (9.4)$$

Let $\sigma = \sigma[(1), (1)]$. Then its multiplicity is given by the number of labellings

$$\left. \begin{array}{l} (\bullet, \bullet), \quad \emptyset, \quad \emptyset, \\ \emptyset, \quad \emptyset, \quad (c, r), \quad \emptyset, \quad (c, r'), \quad (c, c). \end{array} \right\} \quad (9.5)$$

For $\sigma = \sigma[(0), (2)]$ we get

$$\left. \begin{array}{l} \emptyset, \quad \emptyset, \quad \emptyset, \\ \emptyset, \quad (0, rr'), \quad (0, rc), \quad \emptyset, \quad (0, r'c), \quad (0, cc). \end{array} \right\} \quad (9.6)$$

□

The following formula sorts the representations according to the various real forms $\mathrm{SO}(p, q)$ with $p + q = 2n + 1$. Each real form gives a single block. A representation occurring in \mathcal{G} , labelled as above, occurs in $\mathbb{Z}\mathcal{G}^{\mathrm{SO}(p, q)}$ with

$$p = n + 1 + |\#r' - \#r| - \epsilon, \quad \text{where } \epsilon = \begin{cases} 0 & \text{if } \#r' \geq \#r, \\ 1 & \text{otherwise.} \end{cases} \quad (9.7)$$

In the above example, (\bullet, \bullet) , (c, c) , $(0, rr')$, $(0, rc)$ and $(0, cc)$ belong to $\mathfrak{so}(3, 2)$ while (c, r') and $(0, r'c)$ belong to $\mathfrak{so}(4, 1)$.

To each pair of partitions parametrizing a representation of W ,

$$\tau_L = (r_0, \dots, r_{2m}), \quad \tau_R = (r_1, \dots, r_{2m-1}), \quad r_i \leq r_{i+2}, \quad (9.8)$$

Lusztig attaches a *symbol*

$$\begin{pmatrix} r_0 & r_2 + 1 & & \cdots & & r_{2m} + m \\ & r_1 & r_3 + 1 & \cdots & r_{2m-1} + m - 1 & \end{pmatrix}. \quad (9.9)$$

The symbol is called special if

$$r_0 \leq r_1 \leq r_2 + 1 \leq r_3 + 1 \leq \cdots \leq r_{2m} + m. \quad (9.10)$$

Two representations belong to the same double cell if and only if their symbols have the same entries. Given a special symbol of the form (9.9), the corresponding nilpotent orbit \mathcal{O}_c has partition obtained as follows. Form the set

$$\{2r_{2i} + 2i + 1, 2r_{2j-1} + 2j - 2 : 0 \leq i \leq m, 1 \leq j \leq m\}, \quad (9.11)$$

and order the numbers in increasing order, $x_0 \leq \cdots \leq x_{2m}$. The partition of \mathcal{O}_c is

$$(x_0, x_1 - 1, \dots, x_i - i, \dots, x_{2m} - 2m). \quad (9.12)$$

Type C. The representation $\mathcal{G}(\chi_{\text{reg}})$ is obtained from the one in type B by tensoring with sgn . Thus

$$\mathbb{Z}\mathcal{G}(\chi_{\text{reg}}) = \sum_{a,b,\tau} \text{Ind}_{S_t \times W_{2s} \times W_a \times W_b}^{W_n} [\text{sgn} \otimes \sigma[\tau, \tau] \otimes \text{triv} \otimes \text{triv}], \tag{9.13}$$

where τ is a partition of s , and $a + b + 2s + t = n$. This takes into account the duality in [33] of types B and C. We write r for the sign representation of S_t , and c and c' for the trivial representations of W_a, W_b . A representation of W is parametrized by a pair of partitions (τ_L, τ_R) , with

$$\tau_L = (r_0, \dots, r_{2m}), \quad \tau_R = (r_1, \dots, r_{2m-1}), \quad r_i \leq r_{i+2}. \tag{9.14}$$

The associated symbol is

$$\begin{pmatrix} r_0 & r_2 + 1 & & \dots & & r_{2m} + m \\ & r_1 & r_3 + 1 & \dots & r_{2m-1} + m - 1 & \end{pmatrix}, \tag{9.15}$$

and it is called special if

$$r_0 \leq r_1 \leq r_2 + 1 \leq r_3 + 1 \leq \dots \leq r_{2m} + m. \tag{9.16}$$

Two representations belong to the same double cell if their symbols have the same entries. Given a special symbol as in (9.15), the nilpotent orbit \mathcal{O}_c attached to the double cell has partition obtained as follows. Order the set

$$\{2r_{2i} + 2i, 2r_{2j-1} + 2j - 1 : 0 \leq i \leq m, 1 \leq j \leq m\} \tag{9.17}$$

in increasing order, $x_0 \leq \dots \leq x_{2m}$. Then the partition of \mathcal{O}_c is

$$(x_0, \dots, x_j - j, \dots, x_{2m} - 2m). \tag{9.18}$$

The decomposition into blocks is obtained from the one for type B by tensoring with sgn .

Type D. Since in this case $\sigma[\tau_L, \tau_R]$ and $\sigma[\tau_R, \tau_L]$ parametrize the same representation (except of course when $\tau_L = \tau_R$ which corresponds to two non-isomorphic representations), we assume that the size of τ_L is the larger one. The Cartan subgroups are parametrized by integers $(t, u, 2s, p, q)$, $p + q + 2s + t + u = n$. There are actually two Cartan subgroups for each $s > 0$, related by the outer automorphism of order 2. Then

$$\sum \mathbb{Z}\mathcal{G}^{\text{SO}(p,q)}(\chi_{\text{reg}}) = \sum \text{Ind}_{W_a \times W_b \times W'_{2s} \times W_t \times W_u}^{W'_n} [\text{sgn} \otimes \text{sgn} \otimes \sigma[\tau, \tau]_{\text{I,II}} \otimes \text{triv} \otimes \text{triv}], \tag{9.19}$$

where $a + b + 2s + t + u = n$. The sum is also over τ which is a partition of s . We label the σ by \bullet s, trivial representations by c and c' and the sgn representations by r and r' . These are added to τ_L when inducing. In this case the sum on the left is over the real forms

$SO(p, q)$ with all $p + q = 2n$. Each real form gives rise to a single block. A representation labelled as above belongs to the block $\mathbb{Z}\mathcal{G}^{SO(p,q)}$ with $p = n + \#r' - \#r$.

If

$$\tau_L = (r_0, \dots, r_{2m-2}), \quad \tau_R = (r_1, \dots, r_{2m-1}), \tag{9.20}$$

then the associated symbol is

$$\begin{pmatrix} r_0 & r_2 + 1 & \cdots & r_{2m-2} + m - 1 \\ r_1 & r_3 + 1 & \cdots & r_{2m-1} + m - 1 \end{pmatrix}. \tag{9.21}$$

A representation is called special if the symbol satisfies

$$r_0 \leq r_1 \leq r_2 + 1 \leq r_3 + 1 \leq \cdots \leq r_{2m-1} + m - 1. \tag{9.22}$$

Two representations belong to the same double cell if their symbols have the same entries. The nilpotent orbit \mathcal{O}_c attached to the special symbol is given by the same procedure as for type B.

9.2. We follow §14 of [33]. We say that $\pi' \leq \pi$ if π' is a factor of $\pi \otimes F$ with F a finite-dimensional representation with highest weight equal to an integer sum of roots. Two irreducible representations π, π' are said to be in the same Harish-Chandra cell if $\pi' \leq \pi$ and $\pi \leq \pi'$. The Harish-Chandra cell of π is denoted $\mathcal{C}(\pi)$.

Recall the relation $<$ from Definition 14.6 of [33]. The cone above π is defined to be

$$\bar{\mathcal{C}}^{LR}(\pi) := \{\pi' : \pi' \overset{LR}{<} \pi\}. \tag{9.23}$$

The subspace in $\mathbb{Z}\mathcal{G}$ generated by the elements in $s\bar{\mathcal{C}}^{LR}(\pi)$ is a representation of W denoted $\bar{\mathcal{V}}^{LR}(\pi)$. The equivalence $\overset{LR}{\approx}$ is defined by

$$\gamma \overset{LR}{\approx} \phi \iff \gamma \overset{LR}{<} \phi \overset{LR}{<} \gamma. \tag{9.24}$$

The Harish-Chandra cell $\mathcal{C}(\pi)$ is then

$$\mathcal{C}(\pi) = \{\pi' : \pi' \overset{LR}{\approx} \pi\}. \tag{9.25}$$

Define

$$\mathcal{C}_+^{LR}(\pi) = \bar{\mathcal{C}}^{LR}(\pi) \setminus \mathcal{C}(\pi) \tag{9.26}$$

and $\mathcal{V}(\pi)$ and $\bar{\mathcal{V}}^{LR}(\pi)_+$ in analogy with $\bar{\mathcal{V}}^{LR}(\pi)$. Thus there is a representation of W on $\mathcal{V}(\pi)$ by the natural isomorphism

$$\mathcal{V}(\pi) \cong \bar{\mathcal{V}}^{LR}(\pi) / \bar{\mathcal{V}}^{LR}(\pi)_+. \tag{9.27}$$

Let $\mathcal{O}_c \subset \mathfrak{g}_c$ be a nilpotent orbit. We say that a Harish-Chandra cell $\mathcal{C}(\pi)$ is attached to a complex orbit \mathcal{O}_c if

$$\overline{\text{Ad } G_c(\text{AS}(\pi))} = \bar{\mathcal{O}}_c.$$

The sum of the Harish-Chandra cells attached to \mathcal{O}_c is denoted $\mathcal{V}(\mathcal{O}_c)$.

Let $\mathfrak{h}_a \subset \mathfrak{g}_c$ be an abstract Cartan subalgebra and let Π_a be a set of (abstract) simple roots. Let γ be a Langlands parameter of an irreducible representation $\mathcal{L}(\gamma)$ with infinitesimal character the same as the trivial representation (or some fixed finite-dimensional representation). Denote by $\tau(\gamma)$ the τ -invariant as defined in [33]. Given a block \mathcal{B} and disjoint orthogonal sets $S_1, S_2 \subset \Pi_a$, define

$$\mathcal{B}(S_1, S_2) = \{\gamma \in \mathcal{B} \mid S_1 \subset \tau(\gamma), S_2 \cap \tau(\gamma) = \emptyset\}. \tag{9.28}$$

If in addition we are given a nilpotent orbit $\mathcal{O}_c \subset \mathfrak{g}_c$, we can also define

$$\mathcal{B}(S_1, S_2, \mathcal{O}_c) = \{\gamma \in \mathcal{B}(S_1, S_2) \mid \text{AS}(\mathcal{L}(\gamma)) \subset \overline{\mathcal{O}_c}\}. \tag{9.29}$$

If an S_i is absent from the notation, assume $S_i = \emptyset$.

Let $W_i = W(S_i)$, and define

$$\left. \begin{aligned} m_S(\sigma) &= [\sigma : \text{Ind}_{W_1 \times W_2}^W(\text{sgn} \otimes \text{triv})], \\ m_{\mathcal{B}}(\sigma) &= [\sigma : \mathcal{G}_{\mathcal{B}}(\chi_{\text{reg}})]. \end{aligned} \right\} \tag{9.30}$$

In the case of G_c viewed as a real group, the cones defined by (9.23) are parametrized by nilpotent orbits in \mathfrak{g}_c . In other words, $\pi' \stackrel{\text{LR}}{<} \pi$ if and only if $\text{AS}(\pi') \subset \overline{\text{AS}(\pi)}$. So let $\mathcal{C}(\mathcal{O}_c)$ be the cone corresponding to \mathcal{O}_c . Note that in this case $W_c \cong W \times W$, and the representations are of the form $\sigma \otimes \sigma$.

Theorem 9.1. *Let G be the real points of a connected linear reductive group G_c . Then*

$$|\mathcal{B}(S_1, S_2, \mathcal{O}_c)| = \sum_{\sigma \otimes \sigma \in \mathcal{C}(\mathcal{O}_c)} m_{\mathcal{B}}(\sigma)m_S(\sigma).$$

Proof. Consider $\mathbb{Z}\mathcal{B}(S_1, S_2, \mathcal{O}_c) \subset \mathbb{Z}\mathcal{B}(\emptyset, \emptyset, \mathcal{O}_c)$. Then $\mathcal{B}(\emptyset, \emptyset, \mathcal{O}_c)$ is a representation of W which consists of the representations in $\mathcal{V}(\pi)$ with $\text{AS}(\pi) \subset \overline{\mathcal{O}_c}$. The fact that the representation $\mathcal{V}(\pi)$ is formed of σ with $\sigma \otimes \sigma \in \mathcal{C}(\mathcal{O}_c)$ follows from the argument before Theorem 1 of [26]. This accounts for $m_{\mathcal{B}}(\sigma)$ in the sum. The expressions of the action of W given by Lemma 14.7 in [33] and Frobenius reciprocity imply that the dimension of $\mathbb{Z}\mathcal{B}(S_1, S_2, \mathcal{O}_c)$ equals the left-hand side of the formula in the theorem, and it equals the cardinality of $\mathcal{B}(S_1, S_2, \mathcal{O}_c)$. □

9.3. Assume that $\check{\mathcal{O}}$ is even. Then $\lambda := \frac{1}{2}\check{h}$ is integral, and it defines a set S_2 by

$$S_2 = S(\lambda) = \{\alpha \in \Pi_a \mid (\alpha, \lambda) = 0\}. \tag{9.31}$$

Let \mathcal{O}_c be the nilpotent orbit attached to $\check{\mathcal{O}}$ by the duality in [15]. Then the *special unipotent representations* $\text{Unip}(\check{\mathcal{O}})$ attached to $\check{\mathcal{O}}$, are defined to be the representations π with infinitesimal character λ and $\text{AS}(\pi) \subset \mathcal{O}_c$. The translation principle in Chapter 8 Definition 8.6, and discussed in greater detail in Chapter 16, particularly Lemma 16.2 and Theorem 16.4, of [1], establish a bijection between the irreducible admissible representations in $\bigcup \mathcal{B}(\emptyset, S(\lambda))$ at regular integral infinitesimal character λ' , and irreducible

admissible representations at infinitesimal character λ . Because this bijection is realized as tensoring with a finite-dimensional representation and projecting onto the appropriate infinitesimal character, $\text{AS}(\pi)$ is preserved. Thus there is a bijection

$$\text{Unip}(\check{\mathcal{O}}) \longleftrightarrow \bigcup_{\mathcal{B}} \mathcal{B}(\emptyset, S(\lambda), \mathcal{O}_c). \tag{9.32}$$

So we can use Theorem 9.3 to count the number of unipotent representations. In the classical groups case, $m_{\mathcal{B}}(\sigma)$ is straightforward to compute. For the special unipotent case, $m_S(\sigma)$ equals 0 except for the representations occurring in a particular left cell sometimes also called the *Lusztig cell*, which we denote $\bar{\mathcal{C}}^L(\mathcal{O}_c)$. The multiplicities of the representations occurring in $\bar{\mathcal{C}}^L(\mathcal{O}_c)$ are all 1. These representations are in 1–1 correspondence with the conjugacy classes in Lusztig’s quotient of the component group $\bar{A}(\check{\mathcal{O}})$. See [16] for details.

Theorem 9.2.

$$|\text{Unip}(\check{\mathcal{O}})| = \sum_{\mathcal{B}} \sum_{\sigma \in \bar{\mathcal{C}}^L(\mathcal{O}_c)} m_{\mathcal{B}}(\sigma).$$

Theorem 9.3 (McGovern [26]). *In the classical groups $\text{Sp}(n)$, $\text{SO}(p, q)$, each Harish-Chandra cell is of the form $\bar{\mathcal{C}}^L(\mathcal{O}_c)$.*

Definition 9.4. We say that a nilpotent orbit \mathcal{O}_c is *smoothly cuspidal* if it satisfies

- type B**, all odd sizes except for the largest one occur an even number of times;
- type C**, all even sizes occur an even number of times;
- type D**, all odd sizes occur an even number of times,

For $\mathcal{O}(\mathbb{R})$, a real form of \mathcal{O}_c , write $A(\mathcal{O}(\mathbb{R}))$ for its (real) component group.

Proposition 9.5. *Let \mathcal{O}_c be a smoothly cuspidal orbit with dual $\check{\mathcal{O}}$. Then $A(\check{\mathcal{O}}) = \bar{A}(\check{\mathcal{O}})$, and in particular, $|\mathcal{C}^L(\mathcal{O}_c)| = |A(\check{\mathcal{O}})|$. Furthermore,*

$$|\text{Unip}(\check{\mathcal{O}})| = \sum_{\mathcal{O}(\mathbb{R})} |A(\mathcal{O}(\mathbb{R}))|,$$

where the sum is over all real forms $\mathcal{O}(\mathbb{R})$ of \mathcal{O}_c .

Proof. This is Theorem 5.3 in [3]. The proof consists of a direct calculation of multiplicities in the coherent continuation representation using the results developed earlier in this section. □

Remark. This proposition does not hold for orbits that are not smoothly cuspidal. An example is when \mathcal{O}_c is the principal nilpotent orbit. □

We omit the proof of the following proposition. It involves results of Borho, MacPherson, Joseph and Kashiwara which say that the AS-set of an irreducible highest weight module is the closure of a single nilpotent orbit. See references in [15], and also another proof in [35].

Proposition 9.6. *Assume \check{O} is even. Then the AS-set of $L(\chi_{\check{O}})$ satisfies the property that $\overline{\text{Ad } G_c(\text{AS}(L(\chi_{\check{O}})))}$ is the closure of the special orbit \mathcal{O}_c dual (in the sense of [15]) to \check{O} .*

9.4. We now return to type $G = \text{SO}(2n + 1)$. Consider the spherical irreducible representation $L(\chi_{\check{O}})$ with $\chi_{\check{O}} = \frac{1}{2}\check{h}$ corresponding to a nilpotent orbit \check{O} in $\mathfrak{sp}(n)$. If the orbit \check{O} meets a proper Levi component $\check{\mathfrak{m}}$, then $L(\check{O})$ is a subquotient of a representation which is unitarily induced from a unipotent representation on \mathfrak{m} (compare with (1.3)). By induction, $L(\chi_{\check{O}})$ is unitary. Thus we only consider the cases when \check{O} does not meet any proper Levi component. This means

$$\check{O} = (2x_0, \dots, 2x_{2m}), \quad 0 \leq x_0 < \dots < x_i < x_{i+1} < \dots < x_{2m}, \tag{9.33}$$

so these orbits are even.

Proposition 9.7. *Let \check{O} be as in (9.33). The Harish-Chandra cell of $L(\chi_{\check{O}})$ corresponds to \mathcal{O}_c with partition*

$$\underbrace{(1, \dots, 1)}_{r_1}, \underbrace{(2, \dots, 2)}_{r_2}, \dots, \underbrace{(2m, \dots, 2m)}_{r_{2m}}, \underbrace{(2m + 1, \dots, 2m + 1)}_{r_{2m+1}}, \tag{9.34}$$

where

$$\begin{aligned} r_{2i+1} &= 2(x_{2m-2i} - x_{2m-2i-1} + 1), \\ r_{2i} &= 2(x_{2m-2i+1} - x_{2m-2i} - 1), \\ r_{2m+1} &= 2x_0 + 1. \end{aligned}$$

The columns of \mathcal{O}_c are $(2x_{2m} + 1, 2x_{2m-1} - 1, \dots, 2x_0 + 1)$.

Proof. The proof is a calculation involving Lusztig cells and their duals. We omit the details. □

Remark. In general if \check{O} is distinguished, then \mathcal{O}_c is smoothly cuspidal. But there are many more smoothly cuspidal orbits. For example the orbit with partition $(2, 2)$ in $\mathfrak{sp}(4)$ is smoothly cuspidal, but its dual is \check{O} with partition $(1, 1, 3)$ which is not distinguished.

Definition 9.8. Given an orbit \mathcal{O}_c with partition (9.34) or more generally a smoothly cuspidal orbit, we call the *split real form* \mathcal{O}_{spl} the one which satisfies for each row size,

types C and D, the number of rows starting with plus and minus is equal;

type B, in addition to the condition in types C and D for rows of size less than $2m + 1$, for size $2m + 1$, the number of starting with plus is one more than those starting with minus.

Theorem 9.9. *The WF-set of the spherical representation $L(\chi_{\check{O}})$ with \check{O} satisfying (9.33) is the closure of the split real form \mathcal{O}_{spl} of the (complex) orbit \mathcal{O}_c given by (9.34).*

Proof. The main idea is outlined in [3]. We use the fact that if π is a factor of π' , then $\text{WF}(\pi) \subset \text{WF}(\pi')$. We do an induction on m , the number of parts in the partition of $\check{\mathcal{O}}$. The claim amounts to showing that if E occurs in $\text{WF}(L(\chi_{\check{\mathcal{O}}}))$, then the signatures of E, E^2, \dots are greater than the pairs

$$(x_{2m} + 1, x_{2m}), (x_{2m} + x_{2m-1}, x_{2m} + x_{2m-1}), \dots, (x_{2m} + \dots + x_1, x_{2m} + \dots + x_1), (x_{2m} + \dots + x_1 + x_0 + 1, x_{2m} + \dots + x_1 + x_0). \tag{9.35}$$

The first pair is the signature of E , the second the signature of E^2 and so on. The first pair of numbers is obtained by counting the pluses and minuses in the first column of the nilpotent orbit, the second pair by counting the same in the first two columns and so on.

The statement is clear when $m = 0$; $L(\chi_{\check{\mathcal{O}}})$ is the trivial representation. Let $\check{\mathcal{O}}_1$ be the nilpotent orbit corresponding to

$$(2x_0, \dots, 2x_{2m-2}). \tag{9.36}$$

By induction, $\text{WF}(L(\chi_{\check{\mathcal{O}}_1}))$ is the split real form of the nilpotent orbit corresponding to the partition

$$(\underbrace{1, \dots, 1}_{r'_1}, \underbrace{2, \dots, 2}_{r'_2}, \dots, \underbrace{2m-2, \dots, 2m-2}_{r'_{2m-2}}, \underbrace{2m-1, \dots, 2m-1}_{r'_{2m-1}}), \tag{9.37}$$

where the columns are $(2x_{2m-2} + 1, 2x_{2m-3} - 1, \dots, 2x_0 + 1)$. Let \mathfrak{p} be the real parabolic subalgebra with Levi component $\mathfrak{g}(n - x_{2m} - x_{2m-1}) \times \mathfrak{gl}(x_{2m} + x_{2m-1})$. There is a character χ of $\mathfrak{gl}(x_{2m} + x_{2m-1})$ such that $\pi := L(\chi_{\check{\mathcal{O}}})$ is a factor of $\pi' := \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}[L(\chi_{\check{\mathcal{O}}_1}) \otimes \chi]$. But by § 8, $\text{WF}(\pi')$ is in the closure of nilpotent orbits corresponding to partitions

$$(\underbrace{2, \dots, 2}_{r_1/2+r_2}, \dots, \underbrace{2m, \dots, 2m}_{r_{2m}}, \underbrace{2m+1, \dots, 2m+1}_{r_{2m+1}}) \text{ if } r_1 \text{ even}, \tag{9.38}$$

or

$$(1, 1, \underbrace{2, \dots, 2}_{(r_1-1)/2+r_2}, \dots, \underbrace{2m, \dots, 2m}_{r_{2m}}, \underbrace{2m+1, \dots, 2m+1}_{r_{2m+1}}) \text{ if } r_1 \text{ odd}, \tag{9.39}$$

and the dots refer to sizes as in (9.34). It follows that the signatures for E^k in $\text{WF}(L(\chi_{\check{\mathcal{O}}}))$ are greater than the pairs

$$(a_+, a_-), (x_{2m} + x_{2m-1}, x_{2m} + x_{2m-1}), \dots, \tag{9.40}$$

for some $a_+ + a_- = x_{2m} + 1$, and the dots are pairs as in (9.35). Also, each row size greater than two and less than $2m + 1$ has an equal number that start with plus and minus. For size $2m + 1$ there is one more row starting with plus than minus.

We now do the same argument with $\check{\mathcal{O}}_2$ corresponding to

$$(2x_0, \dots, 2x_{2m-3}, (2x_{2m-2})^\wedge, (2x_{2m-1})^\wedge, 2x_{2m}).$$

The WF-set corresponding to $L(\chi_{\check{\mathcal{O}}_2})$ is the split real form of the nilpotent orbit with partition

$$\underbrace{(1, \dots, 1)}_{r'_1}, \underbrace{(2, \dots, 2)}_{r'_2}, \dots, \underbrace{(2m-2, \dots, 2m-2)}_{r''_{2m-2}}, \underbrace{(2m-1, \dots, 2m-1)}_{r''_{2m-1}}, \tag{9.41}$$

with columns $(2x_{2m} + 1, 2x_{2m-3} - 1, \dots, 2x_0 + 1)$. Let \mathfrak{p} be the real parabolic subalgebra with Levi component $\mathfrak{g}(n - x_{2m-1} - x_{2m-2}) \times \mathfrak{gl}(x_{2m-1} + x_{2m-2})$. There is a character χ of $\mathfrak{gl}(x_{2m-1} + x_{2m-2})$ such that $\pi := L(\chi_{\check{\mathcal{O}}})$ is a factor of $\pi'' := \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}[L(\chi_{\check{\mathcal{O}}_2}) \otimes \chi]$. But by §8, $\text{WF}(\pi'')$ is in the closure of nilpotent orbits corresponding to partitions

$$\underbrace{(1, \dots, 1)}_{r_1+r_2/2}, \underbrace{(3, \dots, 3)}_{r_2/2+r_3}, \dots, \underbrace{(2m, \dots, 2m)}_{r_{2m}}, \underbrace{(2m+1, \dots, 2m+1)}_{r_{2m+1}} \text{ if } r_2 \text{ even,} \tag{9.42}$$

or

$$\underbrace{(1, \dots, 1)}_{r_1+(r_2-1)/2}, \underbrace{(2, 2)}_{(r_2-1)/2+r_3}, \underbrace{(3, \dots, 3)}_{\dots}, \underbrace{(2m, \dots, 2m)}_{r_{2m}}, \underbrace{(2m+1, \dots, 2m+1)}_{r_{2m+1}} \text{ if } r_2 \text{ odd,} \tag{9.43}$$

where the dots refer to pairs as in (9.34). Thus $\text{WF}(L(\chi_{\check{\mathcal{O}}}))$ is also contained in the closure of the nilpotent orbits with signatures

$$(x_{2m} + 1, x_{2m}), (x_{2m} + 1 + a_+, x_{2m} + a_-), (x_{2m} + 1 + x_{2m-1} + x_{2m-2}, x_{2m} + 1 + x_{2m-1} + x_{2m-2}), \dots, \tag{9.44}$$

for some $a_+ + a_- = x_{2m-1}$, and the dots refer to pairs as in (9.35).

The claim follows. □

9.5. Consider the special case when

$$x_0 = x_1 - 1 \leq x_2 = x_3 - 1 \leq \dots \leq x_{2m-2} = x_{2m-1} - 1 \leq x_{2m}. \tag{9.45}$$

This makes \mathcal{O}_c of the form (9.34) with $r_{2i} = 0$, i.e. the partition of \mathcal{O}_c has only odd terms. The (Lusztig) cell $\check{\mathcal{C}}^L(\mathcal{O}_c)$ has size 2^m (which is true for any \mathcal{O}_c with $\check{\mathcal{O}}$ distinguished, not just for the particular form (9.45)). We produce 2^m distinct irreducible representations with AC-set equal to \mathcal{O}_{sp1} . So \mathfrak{g} is $\text{so}(2p+1, 2p)$. Let \mathfrak{h} be the compact Cartan subalgebra. We write the coordinates

$$(a_1, \dots, a_p \mid b_1, \dots, b_p), \tag{9.46}$$

where the first p coordinates before the $|$ are in the Cartan subalgebra of $\text{so}(2p+1)$ and the last p coordinates are in $\text{so}(2p)$. The roots $\epsilon_i \pm \epsilon_j$, ϵ_i with $1 \leq i, j \leq p$ are all compact and so are $\epsilon_{p+k} \pm \epsilon_{p+l}$ with $1 \leq k, l \leq p$. The roots $\epsilon_i \pm \epsilon_{p+k}$, ϵ_{p+k} are non-compact. Let

$\mathfrak{q}_c = \mathfrak{l}_c + \mathfrak{u}_c$ be a θ -stable parabolic subalgebra with Levi component

$$\left. \begin{aligned} \mathfrak{l} &:= \mathfrak{l}_c \cap \mathfrak{g} \\ &= u(x_{2i_1+1}, x_{2i_1}) \times u(x_{2i_2}, x_{2i_2+1}) \times \cdots \times u(x_{2i_{m-1}}, x_{2i_{m-1}+1}) \times \text{so}(x_{2m} + 1, x_{2m}) \\ \text{or} \\ \mathfrak{l} &:= \mathfrak{l}_c \cap \mathfrak{g} \\ &= u(x_{2i_1+1}, x_{2i_1}) \times u(x_{2i_2}, x_{2i_2+1}) \times \cdots \times u(x_{2i_{m-1}+1}, x_{2i_{m-1}}) \times \text{so}(x_{2m} + 1, x_{2m}) \end{aligned} \right\} \tag{9.47}$$

(depending on the parity of m), where the i_j are the numbers $0, \dots, m - 1$ in some order. The parabolic subalgebra \mathfrak{q}_c corresponds to the weight

$$\left. \begin{aligned} \xi &= (m^{x_{2i_1+1}}, \dots, 1^{x_{2i_{m-1}+1}}, 0^{x_{2m}} \mid m^{x_{2i_1}}, \dots, 1^{x_{2i_{m-1}}}, 0^{x_{2m}}) \\ \text{or} \\ \xi &= (m^{x_{2i_1+1}}, \dots, 1^{x_{2i_{m-1}}}, 0^{x_{2m}} \mid m^{x_{2i_1}}, \dots, 1^{x_{2i_{m-1}+1}}, 0^{x_{2m}}), \end{aligned} \right\} \tag{9.48}$$

depending whether m is odd or even.

The derived functor modules $\mathcal{R}_{\mathfrak{q}_c}^i(\xi)$ from characters on \mathfrak{l}_c have AC-set contained in \mathcal{O}_{spl} . To get infinitesimal character χ_{\emptyset} , these characters can only be

$$\xi_{i_j}^{\pm} := \pm(\frac{1}{2}, \dots, \frac{1}{2}), \tag{9.49}$$

on the unitary factors $u(x_{2i_j+1}, x_{2i_j})$ or $u(x_{2i_j}, x_{2i_j+1})$, and trivial on $\mathfrak{g}(x_{2m})$. We need to show that there are choices of parabolic subalgebras \mathfrak{q}_c as in (9.47) and characters as in (9.49) so that we get 2^m non-zero and distinct representations. For this we have to specify the Langlands parameters.

For each subset $A := \{k_1, \dots, k_r\} \subset \{0, \dots, m - 1\}$, k_j in decreasing order, label the complement $A^c := \{\ell_1, \dots, \ell_t\}$, A^c also labelled in decreasing order, and consider the θ -stable parabolic subalgebra $\mathfrak{q}_{c,A}$ as in (9.47) and (9.48) corresponding to

$$\{i_1, \dots, i_{m-1}\} = \{k_1, \dots, k_r, \ell_1, \dots, \ell_t\}. \tag{9.50}$$

We will consider the representations $\mathcal{R}_{\mathfrak{q}_{c,A}}(\xi_A)$, where ξ_A is the concatenation of the $\xi_{i_j}^{\pm}$ with plus for the first r , and minus for the last t .

Lemma 9.10.

$$\mathcal{R}_{\mathfrak{q}_{c,A}}^i(\xi_A) = \begin{cases} 0 & \text{if } i \neq \dim(\mathfrak{u}_{c,A} \cap \mathfrak{k}_c), \\ \text{non-zero irreducible} & \text{if } i = \dim(\mathfrak{u}_{c,A} \cap \mathfrak{k}_c). \end{cases}$$

Proof. For the vanishing part we check that the conditions in Proposition 5.93 in [23, Chapter V, § 7] are satisfied. It is sufficient to show that

$$\text{ind}_{\mathfrak{q}_{c,A}, L \cap K}^{\mathfrak{g}, K}(Z_{\mathfrak{q}_{c,A}}^{\#}) := \text{U}(\mathfrak{g}) \otimes_{\mathfrak{q}_{c,A}} Z_{\mathfrak{q}_{c,A}}^{\#} \tag{9.51}$$

is irreducible. Here $Z_{\mathfrak{q}_{c,A}}^{\#}$ is the one-dimensional module corresponding to $\xi_A - \rho(\mathfrak{u}_{c,A})$, with

$$\rho(\mathfrak{u}_{c,A}) := \frac{1}{2} \sum_{\alpha \in \Delta(\mathfrak{u}_{A,c})} \alpha.$$

The derived functors are normalized so that if W has infinitesimal character χ , then so do the $\mathcal{R}_{\mathfrak{q}_c}^i(W)$.

The notions and results about associated cycles in § 8.1 apply. The associated cycle of (9.51) is \mathcal{O}_c , the dual of $\tilde{\mathcal{O}}$ from (9.34), and the multiplicity is 1. Any composition factor cannot have associated cycle formed of nilpotent orbits of strictly smaller dimension than \mathcal{O}_c because the results in [16] apply. So if there is more than one factor, the multiplicity of \mathcal{O}_c would be strictly larger than 1. Thus (9.51) is irreducible.

By the same reasoning, $\mathcal{R}_{\mathfrak{q}_{c,A}}^{\dim(\mathfrak{u}_{c,A} \cap \mathfrak{k}_c)}(\xi_A)$ is irreducible if non-zero because the multiplicity of the corresponding nilpotent orbit in \mathfrak{s}_c in AV is 1. To show that $\mathcal{R}_{\mathfrak{q}_{c,A}}^{\dim(\mathfrak{u}_{c,A} \cap \mathfrak{k}_c)}(\xi_A) \neq 0$, we use the bottom layer K -types defined in [23, Chapter V, § 6]. To simplify the notation slightly, we write

$$\left. \begin{aligned} a_1 = x_{2k_1+1}, \quad b_1 = x_{2k_1}, \quad \dots, \quad a_r = x_{2k_r}, \quad b_r = x_{2k_r+1}, \quad r \text{ even,} \\ a_1 = x_{2k_1+1}, \quad b_1 = x_{2k_1}, \quad \dots, \quad a_r = x_{2k_r+1}, \quad b_r = x_{2k_r}, \quad r \text{ odd,} \end{aligned} \right\} \quad (9.52)$$

and \mathfrak{u}_c for $\mathfrak{u}_{c,A}$. Let also $a := \sum a_j, b := \sum b_j$. Note that $|a_j - b_j| = 1$, and also $|a - b| = 1$. Then

$$\mu := \xi + 2\rho(\mathfrak{u} \cap \mathfrak{s}) - \rho(\mathfrak{u}) = (1^a, 0^{p-a} \mid 1^b, 0^{p-b}) \quad (9.53)$$

is dominant, therefore bottom layer. The aforementioned results then imply the non-vanishing. □

We now show that there are 2^m distinct representations. To do this we will compute Langlands parameters. We rely on the notation and conventions in Chapter 11 of [1]. Many details are in the references to earlier work given there. In [1] a Langlands parameter is a triple $A = (A^{\text{can}}, \mathbb{R}_{\mathbb{R}}^+, \mathbb{R}_{\mathbb{R}}^+)$, where A^{can} is an irreducible representation of a θ -stable Cartan subgroup $H_{\mathbb{R}}$ (denoted $T_{\mathbb{R}}$ in [1]). Attached to such data is a *standard module* $X(A)$, with a canonical irreducible subquotient $M(A)$. We will determine $\tilde{\lambda} := dA^{\text{can}}$ and the lowest K -types of the derived functor modules constructed earlier. The Lie algebra of H decomposes according to the ± 1 eigenvalues of θ into $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$. Then $\tilde{\lambda} = (\lambda, \nu)$. Recall from [32, Chapter 5] that the centralizer of \mathfrak{a} is a quasi-split θ -stable Levi component \mathfrak{l} , and that $M(A)$ is determined by the fact that it contains a *lowest K -type* μ related to λ by the formula $\lambda = \mu + 2\rho_c + \rho + \nu$ where $\nu = \sum c_i \alpha_i$ is a combination of strongly orthogonal imaginary roots with coefficients $0 < c_i \leq \frac{1}{2}$.

To compute λ and μ , we will need to use the intermediate parabolic subalgebras

$$\mathfrak{q}_{c,A} \subset \mathfrak{q}'_{c,A} \subset \mathfrak{q}''_{c,A} \subset \mathfrak{g}_c \quad (9.54)$$

with Levi components

$$\left. \begin{aligned} \mathfrak{l}'_A &= u(a_1, b_1) \times \dots \times u(a_r, b_r) \times \mathfrak{g}(n - a - b), \\ \mathfrak{l}''_A &= u(a, b) \times \mathfrak{g}(n - a - b). \end{aligned} \right\} \quad (9.55)$$

Apply induction in stages from $\mathfrak{q}_{c,A}$ to $\mathfrak{q}'_{c,A}$ first. On the factor $\mathfrak{g}(n - a - b)$ the K -type μ in (9.53) is trivial, so the Langlands parameter is that of the spherical principal series. Similarly on the $u(a_j, b_j)$ assume the infinitesimal character is $\chi_j :=$

$(\max(a_j, b_j), \dots, \min(a_j, b_j))$ with the coordinates going down by 1, and the Langlands parameter is that of a principal series with the appropriate one-dimensional Langlands subquotient. Let $\mathfrak{h}_A \subset \mathfrak{l}'_{c,A}$ be the the most split Cartan subalgebra. In particular the real roots are

$$\alpha_d := \epsilon_d + \epsilon_{d+p}, \quad \sum_{j \leq s} a_j < d < \sum_{j \leq s} a_j + \min(a_j, b_j), \quad 0 \leq s \leq r - 1. \tag{9.56}$$

For each factor $u(a_j, b_j)$ the Langlands parameter is of the form (λ_j, ν_j) where $\lambda_j \in \mathfrak{h}_A \cap \mathfrak{k}_c$, and $\nu_j \in \mathfrak{h}_A \cap \mathfrak{s}_c$. Then

$$\lambda_j = \left(\frac{1}{2} a_j \mid \frac{1}{2} b_j \right), \tag{9.57}$$

while

$$\langle \nu_j, \alpha_d \rangle = \max(a_j, b_j) - \left(d - \sum_{j \leq s} a_j \right). \tag{9.58}$$

Proposition 9.11. *The representations $\mathcal{R}_{\mathfrak{q}_{c,A}}^{\dim(\mathfrak{u}_{c,A} \cap \mathfrak{k}_c)}(\xi_A)$ have Langlands parameters (λ^G, ν) , where λ^G is obtained by concatenating the λ_j in (9.57) and ν satisfies (9.58).*

Proof. We write $\mathfrak{l}'_c = \mathfrak{l}'_{c,A}$ and $\mathfrak{l}' = \mathfrak{l}'_{c,A} \cap \mathfrak{g}$. There is a non-zero map $X_{\mathfrak{l}'_c}(\lambda^G, -\nu) \rightarrow L_{\mathfrak{l}'}(\lambda^G, -\nu)$ given by the Langlands classification. Thus there is a map

$$\mathcal{R}_{\mathfrak{q}'_{c,L' \cap K}}^{\dim \mathfrak{k}_c \cap \mathfrak{u}'_c} [X_{\mathfrak{l}'_c}(\lambda^G, -\nu)] \rightarrow \mathcal{R}_{\mathfrak{q}'_{c,L' \cap K}}^{\dim \mathfrak{k}_c \cap \mathfrak{u}'_c} (L_{\mathfrak{l}'_c}(\lambda^G, -\nu)) = \mathcal{R}_{\mathfrak{q}_c, L \cap K}^{\dim \mathfrak{k}_c \cap \mathfrak{u}_c} (\xi_A), \tag{9.59}$$

which is non-zero on the bottom layer K -type (9.53). On the other hand, because these are standard modules,

$$\mathcal{R}_{\mathfrak{q}}^i(X_{\mathfrak{l}'}(\lambda^G, \nu)) = \begin{cases} X(\lambda^G, \nu) & \text{if } i = \dim \mathfrak{k}_c \cap \mathfrak{u}_c, \\ 0 & \text{otherwise.} \end{cases} \tag{9.60}$$

The proof follows. □

9.6.

Theorem 9.12. *The spherical unipotent representations $L(\chi_{\check{O}})$ with \check{O} even are unitary.*

Proof. We present the details for $G = \text{SO}(n + 1, n)$. Write $\mathfrak{g}(n)$ for the Lie algebra containing \mathcal{O}_{spl} the split real form of \mathcal{O}_c , same as the support $\text{AC}(L(\chi_{\check{O}}))$. There is a (real) parabolic subalgebra \mathfrak{p}^+ with Levi component $\mathfrak{m}^+ := \mathfrak{gl}(n_1) \times \dots \times \mathfrak{gl}(n_k) \times \mathfrak{g}(n)$ in \mathfrak{g}^+ of rank $n_1 + \dots + n_k + n$, such that the split form $\mathcal{O}_{\text{spl}}^+$ of

$$\mathcal{O}_c^+ := (1, 1, 3, 3, \dots, 2a - 1, 2a - 1, 2a + 1)$$

for some a depending on $(2x_0, \dots, 2x_{2m})$ is induced from \mathcal{O} on $\mathfrak{g}(n)$, trivial on the \mathfrak{gl} factors. We will consider the representation

$$I(\pi) := \text{Ind}_{\mathfrak{m}^+}^{\mathfrak{g}^+} [\text{triv} \otimes \dots \otimes \text{triv} \otimes \pi]. \tag{9.61}$$

We show that the form on $I(\pi)$ induced from π is positive definite; this implies that the form on π is definite. We do this by showing that the possible factors of $I(\pi)$ have to be unitary, and the forms on their lowest K -types are positive definite.

By the results in §§ 8 and 9, more specifically §§ 8.1 and 9.3, and particularly Proposition § 9.5, we conclude that there are $3^a \cdot 2^a$ unipotent representations in the block of the spherical irreducible representation; all the factors of $I(\pi)$ are in this block. Here 3^a is the number of real forms of \mathcal{O}_c^+ , and 2^a is the number of representations in the corresponding Lusztig cell. We describe how to get $3^a \cdot 2^a$ distinct representations. For each real form \mathcal{O}_j^+ of \mathcal{O}_c^+ we produce one representation π such that $\text{AC}(\pi) = \bar{\mathcal{O}}_j^+$. Then Theorem 9.3 implies that there is a Harish-Chandra cell with 2^a representations with this property, since $\bar{A}(\check{\mathcal{O}}) = A(\check{\mathcal{O}}) = (\mathbb{Z}/2)^a$. Since these cells must be disjoint, this gives the required number.

From § 8.1, each such form \mathcal{O}_j^+ is θ -stable induced from the trivial nilpotent orbit on a parabolic subalgebra with Levi component a real form of

$$\mathfrak{gl}(1) \times \mathfrak{gl}(3) \times \cdots \times \mathfrak{gl}(2a - 1) \times \mathfrak{g}_c(m).$$

Using the results in [23], for each such parabolic subalgebra, we can find a derived functor induced module from an appropriate one-dimensional character, that is non-zero and has associated variety equal to the closure of the given real form. Actually it is enough to construct this derived functor module at regular infinitesimal character where the fact that it is non-zero irreducible is considerably easier.

So in this block, there is a cell for each real form of \mathcal{O}_c^+ , and each cell has 2^a irreducible representations with infinitesimal character $\chi_{\check{\mathcal{O}}}$. In particular for \mathcal{O}_{spl} , the Levi component is $u(1, 0) \times u(1, 2) \times u(3, 2) \times \cdots \times \mathfrak{so}(a, a + 1)$. For this case, § 9.5 produced exactly 2^a parameters; their lowest K -types are of the form $\mu_e(n - k, k)$. These are the only possible constituents of the induced from $L(\chi_{\check{\mathcal{O}}})$. Since the constituents of the restriction of a $\mu_e(n - k, k)$ to a Levi component are again $\mu_e(m - l, l)$ s, the only way $L(\chi_{\check{\mathcal{O}}})$ can fail to be unitary is if the form is negative on one of the K -types $\mu_e(n - k, k)$. But §§ 5 and 6.3 show that the form is positive on the K -types μ_e of $L(\chi_{\check{\mathcal{O}}})$. □

10. Irreducibility

10.1. To complete the classification of the unitary dual we also need to prove the following irreducibility theorem. It is needed to show that the regions in Theorem 3.2 are indeed unitary in the real case.

Theorem 10.1. *Assume $\check{\mathcal{O}}$ is even, and such that $x_{i-1} = x_i = x_{i+1}$ for some i . Let $\mathfrak{m} = \mathfrak{gl}(x_i) \times \mathfrak{g}(n - x_i)$, and $\check{\mathcal{O}}_1 \subset \mathfrak{g}(n - x_i)$ be the nilpotent orbit obtained from $\check{\mathcal{O}}$ by removing two rows of size x_i . Then*

$$L(\chi_{\check{\mathcal{O}}}) = \text{Ind}_{\text{GL}(x_i) \times G(n-x_i)}^{G(n)} [\text{triv} \otimes L(\chi_{\check{\mathcal{O}}_1})].$$

In the p -adic case this follows from the work of Kazhdan–Lusztig [10]. In the real case, it follows from the following proposition.

Proposition 10.2. *The associated variety of a spherical representation $L(\chi_{\check{\mathcal{O}}})$ is given by the sum with multiplicity one of the following real forms of the complex orbit \mathcal{O}_c .*

Types B and D. *On the odd sized rows, the difference between the number of pluses and number of minuses is 1, 0 or -1 .*

Type C. *On the even sized rows, the difference between the number of pluses and number of minuses is 1, 0 or -1 .*

The proof of the proposition is lengthy, and follows from more general results which are unpublished. We will give a different proof of Theorem 10.1 in the next sections.

Remark. When $\check{\mathcal{O}}_1$ is even, but $\check{\mathcal{O}}$ is not, and just $x_i = x_{i+1}$, the proof follows from [10] in the p -adic case, and the Kazhdan–Lusztig conjectures for non-integral infinitesimal character in the real case. We have already used these results in the course of the paper. □

The outline of the proof is as follows. In § 10.2, we prove some auxiliary reducibility results in the case when $\check{\mathcal{O}}$ is induced from the trivial nilpotent orbit of a maximal Levi component. In § 10.3, we combine these results with intertwining operator techniques to complete the proof of Theorem 10.1.

10.2. We need to study the ρ -induced modules from the trivial module on $\mathfrak{m} \subset \mathfrak{g}(n)$ where $\mathfrak{m} \cong \mathfrak{gl}(n)$, or in some cases $\mathfrak{m} \cong \mathfrak{gl}(a) \times \mathfrak{gl}(b)$ with $a + b = n$.

Type B

Assume $\check{\mathcal{O}}$ corresponds to the partition $2x_0 = 2x_1 = 2a$ in $\mathfrak{sp}(n, \mathbb{C})$. The infinitesimal character is $(-a + \frac{1}{2}, \dots, a - \frac{1}{2})$ and the nilpotent orbit \mathcal{O}_c corresponds to

$$(1, 1, \underbrace{2, \dots, 2}_{2a-2}, 3).$$

We are interested in the composition series of

$$\text{Ind}_{\text{GL}(2a)}^{G(2a)}[\text{triv}]. \tag{10.1}$$

There are three real forms of \mathcal{O}_c in $\mathfrak{so}(2a + 1, 2a)$:

$$\begin{array}{ccc}
 + & - & + \\
 + & - & \\
 - & + & \\
 \vdots & \vdots & \\
 + & - & \\
 - & + & \\
 + & & \\
 + & &
 \end{array}
 \qquad
 \begin{array}{ccc}
 + & - & + \\
 + & - & \\
 - & + & \\
 \vdots & \vdots & \\
 + & - & \\
 - & + & \\
 + & & \\
 - & &
 \end{array}
 \qquad
 \begin{array}{ccc}
 - & + & - \\
 + & - & \\
 - & + & \\
 \vdots & \vdots & \\
 + & - & \\
 - & + & \\
 + & & \\
 + & &
 \end{array}
 \tag{10.2}$$

The associated cycle of (10.1) is the middle nilpotent orbit in (10.2) with multiplicity 2. Section 6 shows that there are at least two factors characterized by the fact that they contain the petite K -types which are the restrictions to $S[O(2a + 1) \times O(2a)]$ of

$$\left. \begin{aligned} &\mu(0^a; +) \otimes \mu(0^a; +), \\ &\mu(1, 0^{a-1}; -) \otimes \mu(0^a; -). \end{aligned} \right\} \tag{10.3}$$

Thus because of multiplicity 2, there are exactly two factors. One of the factors is spherical. The non-spherical factor has Langlands parameter

$$\left. \begin{aligned} &\lambda^G = (\frac{1}{2}, 0, \dots, 0 \mid 0, \dots, 0), \\ &\nu = (0, a - \frac{1}{2}, a - \frac{1}{2}, \dots, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}). \end{aligned} \right\} \tag{10.4}$$

The Cartan subalgebra for the non-spherical parameter is such that the root ϵ_1 is non-compact imaginary, $\epsilon_i, \epsilon_i \pm \epsilon_j$ with $j > i \geq 2$, are real. The standard module $X(\lambda^G, \nu)$ which has $\bar{X}(\lambda^G, \nu)$ as quotient is the one for which ν is dominant. Thus we conjugate the Cartan subalgebra such that ϵ_{2a} is non-compact imaginary, $\epsilon_i, \epsilon_i \pm \epsilon_j$ with $i < j < 2a$ are real, and the usual positive system $\Delta^+ = \{\epsilon_i, \epsilon_i \pm \epsilon_j\}_{i < j}$.

Type C

Consider \tilde{O} which corresponds to the partition $2x_0 = 2x_1 = 2a + 1 < 2x_2 = 2b + 1$ in $\mathfrak{so}(n, \mathbb{C})$. The infinitesimal character is

$$(-a, \dots, a)(-b, \dots, -1). \tag{10.5}$$

The nilpotent orbit \mathcal{O}_c is induced from the trivial one on $\mathfrak{gl}(2a + 1) \times \mathfrak{g}(b)$ and corresponds to

$$\underbrace{(1, \dots, 1)}_{2b-2a-2}, 2, 2, \underbrace{3, \dots, 3}_{2a}. \tag{10.6}$$

We are interested in the composition series of

$$\text{Ind}_{\text{GL}(2a+1) \times G(b)}^{G(2a+b+1)}[\text{triv}]. \tag{10.7}$$

There are three real forms of (10.6):

$$\begin{array}{cccccc} + & - & + & & + & - & + & & + & - & + \\ - & + & - & & - & + & - & & - & + & - \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ + & - & + & & + & - & + & & + & - & + \\ - & + & - & & - & + & - & & - & + & - \\ + & - & & & + & - & & & - & + & \\ + & - & & & - & + & & & - & + & \\ + & & & & + & & & & + & & \\ - & & & & - & & & & - & & \\ \vdots & & & & \vdots & & & & \vdots & & \\ + & & & & + & & & & + & & \\ - & & & & - & & & & - & & \end{array} \tag{10.8}$$

The AC cycle of (10.7) consists of the middle nilpotent orbit in (10.8) with multiplicity 2. By a similar argument as for type B, we conclude that the composition series consists of two representations containing the petite K -types

$$\left. \begin{array}{l} \mu(0^n), \\ \mu(1^{a+1}, 0^{b-1}, (-1)^{a+1}). \end{array} \right\} \tag{10.9}$$

These are also the lowest K -types of the representations. The non-spherical representation has parameter

$$\left. \begin{array}{l} \lambda^G = (\frac{1}{2}^a, 0^b, (-\frac{1}{2})^a), \\ \nu = (\frac{1}{2}^a, 0^b, \frac{1}{2}^a). \end{array} \right\} \tag{10.10}$$

Type D

Let $\check{\mathcal{O}}$ correspond to the partition $2x_0 = 2x_1 = 2a + 1$ in $\mathfrak{so}(n, \mathbb{C})$. The infinitesimal character is $(-a, \dots, a)$. The real forms of the nilpotent orbit \mathcal{O} are

$$\begin{array}{cc} + & - \\ - & + \\ \vdots & \vdots \\ + & - \\ - & + \end{array} \tag{10.11}$$

There are two nilpotent orbits with this partition labelled I and II. Each of them is induced from $\mathfrak{m} \cong \mathfrak{gl}(2a)$, there are two such Levi components. We are interested in the induced modules

$$\text{Ind}_{\text{GL}(2a)}^{G(2a)}[\text{triv}]. \tag{10.12}$$

The multiplicity of the nilpotent orbit (10.11) in the AC cycle of (10.12) is 1, so the representations are irreducible.

We summarize these calculations in a proposition.

Proposition 10.3. *The composition factors of the induced module from the trivial representation on \mathfrak{m} , where $\mathfrak{m} = \mathfrak{gl}(2a)$ for $\text{SO}(2a + 1, 2a)$ and $\text{SO}(2a, 2a)$, or $\mathfrak{gl}(2a + 1) \times \mathfrak{g}(b)$ in type $\text{Sp}(4a + 2b + 2)$, all have relevant lowest K -types. In particular, the induced module is generated by spherically relevant K -types. Precisely,*

type B, *the representation is generated by the K -types of the form μ_e ;*

type C, *the representation is generated by the K -types of the form μ_o ;*

type D, *the representation is generated by the spherical K -type $\mu_e(0) = \mu_o(0)$.*

10.3. We now prove the irreducibility result mentioned at the beginning of the section in the case of \mathfrak{g} of type B; the other cases are similar. Recall that we denote $\check{\mathcal{O}} = (2x_0, 2x_1, \dots, 2x_{2m})$. Let $\check{\mathcal{O}}_1$ be the nilpotent orbit where we have removed two entries equal to $2a$ from $\check{\mathcal{O}}$. Let $\mathfrak{m} := \mathfrak{gl}(2a) \times \mathfrak{g}(n - 2a)$. Then $L(\chi_{\check{\mathcal{O}}})$ is the spherical subquotient of the induced representation

$$I(a, L(\chi_{\check{\mathcal{O}}_1})) := \text{Ind}_{\mathfrak{m}}^{\mathfrak{g}} [(-a + \frac{1}{2}, \dots, a - \frac{1}{2}) \otimes L(\chi_{\check{\mathcal{O}}_1})]. \tag{10.13}$$

It is enough to show that if a parameter is unipotent, and satisfies $x_{i-1} = x_i = x_{i+1} = a$, then $I(a, L(\chi_{\check{\mathcal{O}}_1}))$ is generated by its K -types μ_e . This is because by Theorem 5.3, the K -types of type μ_e in (10.13) occur with full multiplicity in the spherical irreducible subquotient, and the module is unitary. We will use various irreducibility results stated in §2 without mention.

First, we reduce to the case when there are no $0 < x_j < a$. Let ν be the dominant parameter of $L(\chi_{\check{\mathcal{O}}})$, and assume i is the smallest index so that $x_{i-1} = a$. There is an intertwining operator

$$X(\nu) \rightarrow I(\frac{1}{2}, \dots, x_0 - \frac{1}{2}; \dots; \frac{1}{2}, \dots, x_{i-2} - \frac{1}{2}; \nu'), \tag{10.14}$$

where I is induced from

$$\mathfrak{gl}(x_0) \times \dots \times \mathfrak{gl}(x_{i-2}) \times \mathfrak{g} \left(n - \sum_{j < i-1} x_j \right)$$

with characters on the \mathfrak{gl} factors corresponding to the strings in (10.14), and the irreducible module $L(\nu')$ on $\mathfrak{g}(n - \sum_{j < i-1} x_j)$. The intertwining operator is onto, and thus the induced module is generated by its spherical vector. By the induction hypothesis, the induced module from $(-a + \frac{1}{2}, \dots, a - \frac{1}{2}) \otimes L(\nu'')$ on $\mathfrak{gl}(2a) \times \mathfrak{g}(n - \sum_{j \leq i} x_j)$ is irreducible. The parameter ν'' is such that the induced module has infinitesimal character ν' , and contains $L(\nu')$ as its spherical subquotient. But

$$\begin{aligned} & I(\frac{1}{2}, \dots, x_0 - \frac{1}{2}; \dots; \frac{1}{2}, \dots, x_{i-2} - \frac{1}{2}; -a + \frac{1}{2}, \dots, a - \frac{1}{2}; \nu'') \\ & \cong I(-a + \frac{1}{2}, \dots, a - \frac{1}{2}; \frac{1}{2}, \dots, x_0 - \frac{1}{2}; \dots; \frac{1}{2}, \dots, x_{i-2} - \frac{1}{2}, \nu''). \end{aligned} \tag{10.15}$$

This module maps by an intertwining operator onto $I(a, L(\chi_{\check{\mathcal{O}}_1}))$, so $I(a, L(\chi_{\check{\mathcal{O}}_1}))$ is generated by its spherical vector.

So we have reduced to the case when

$$\left. \begin{aligned} & x_0 = x_1 = x_2 = a \\ & \text{or} \\ & x_0 = 0 < x_1 = x_2 = x_3 = a. \end{aligned} \right\} \tag{10.16}$$

Suppose we are in the first case of (10.16) and $m = 1$. The infinitesimal character is

$$(a - \frac{1}{2}, a - \frac{1}{2}, a - \frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}),$$

each coordinate occurring three times. The induced module

$$I(-a + \frac{1}{2}, \dots, a - \frac{1}{2}) \tag{10.17}$$

of $\mathfrak{g}(2a)$ is a direct sum of irreducible factors computed in § 10.2; in particular it is generated by K -types of the form $\mu_e(2a - k, k)$ (with $k = 0, 1$). Consider the module

$$I(a - \frac{1}{2}; \dots; \frac{1}{2}; -a + \frac{1}{2}, \dots, a - \frac{1}{2}), \tag{10.18}$$

induced from characters on $GL(1) \times \dots \times GL(1) \times GL(2a)$. It is a direct sum of induced modules from the two factors of (10.17). Each such induced module is a homomorphic image of the corresponding standard module with dominant parameter. So (10.18) is also generated by its μ_e isotypic components. But then

$$I(a - \frac{1}{2}; \dots; \frac{1}{2}; -a + \frac{1}{2}, \dots, a - \frac{1}{2}) \cong I(-a + \frac{1}{2}, \dots, a - \frac{1}{2}; a - \frac{1}{2}; \dots; \frac{1}{2}) \tag{10.19}$$

so the latter is also generated by its μ_e isotypic components. Finally, the intertwining operator

$$I(a - \frac{1}{2}; \dots; \frac{1}{2}) \rightarrow I(\frac{1}{2}, \dots, a - \frac{1}{2}) \tag{10.20}$$

is onto, and the image of the intertwining operator

$$I(\frac{1}{2}, \dots, a - \frac{1}{2}) \rightarrow I(-a + \frac{1}{2}, \dots, -\frac{1}{2}) \tag{10.21}$$

is onto $L(-a + \frac{1}{2}, \dots, -\frac{1}{2})$. Thus the module induced from $\mathfrak{gl}(2a) \times \mathfrak{g}(a)$,

$$I(-a + \frac{1}{2}, \dots, a - \frac{1}{2}; L(-a + \frac{1}{2}, \dots, -\frac{1}{2})), \tag{10.22}$$

is generated by its μ_e isotypic components. Since the multiplicity of these K -types in (10.22) is the same as in the irreducible spherical module, it follows that they must be equal.

Now suppose that we are in the first case of (10.16) and $m > 1$, or in the second case, and $m > 2$. The parameter has another $x_{2m-1} \leq x_{2m}$. We use an argument similar to the one above to show that the module

$$I(-x_{2m-1} + \frac{1}{2}, \dots, x_{2m} - \frac{1}{2}, L(\chi_{\check{\mathcal{O}}_2})), \tag{10.23}$$

induced from $\mathfrak{gl}(x_{2m-1} + x_{2m}) \times \mathfrak{g}(n - x_{2m-1} - x_{2m})$, where $\check{\mathcal{O}}_2$ is the nilpotent orbit with partition obtained from $\check{\mathcal{O}}$ by removing $2x_{2m-1}, 2x_{2m}$, is generated by its μ_e isotypic components. The claim then follows because the induced module is a homomorphic image of (10.23). Precisely, $X(\nu)$ maps onto

$$I(x_{2m-1} + \frac{1}{2}, \dots, x_{2m} - \frac{1}{2}; \frac{1}{2}, \dots, x_0 - \frac{1}{2}; \dots; \frac{1}{2}, \dots, x_{2m-2} - \frac{1}{2}; L(-x_{2m-1} + \frac{1}{2}, -x_{2m-1} + \frac{1}{2}, \dots, -\frac{1}{2}, -\frac{1}{2})). \tag{10.24}$$

So this module is generated by its spherical vector. Replace $L(-x_{2m-1} + \frac{1}{2}, -x_{2m-1} + \frac{1}{2}, \dots, -\frac{1}{2}, -\frac{1}{2})$ by $I(-x_{2m-1} + \frac{1}{2}, \dots, x_{2m-1} - \frac{1}{2})$. The ensuing module is a direct sum of

two induced modules by § 10.2. They are both homomorphic images of standard modules, so generated by their lowest K -types, which are of type μ_e . Next observe that the map

$$\begin{aligned}
 &I(x_{2m-1} + \frac{1}{2}, \dots, x_{2m} - \frac{1}{2}; \frac{1}{2}, \dots, x_0 - \frac{1}{2}; \dots; \\
 &\frac{1}{2}, \dots, x_{2m-2} - \frac{1}{2}; -x_{2m-1} + \frac{1}{2}, \dots, x_{2m-1} - \frac{1}{2}) \\
 &\rightarrow I(-x_{2m-1} + \frac{1}{2}, \dots, x_{2m} - \frac{1}{2}; \frac{1}{2}, \dots, x_0 - \frac{1}{2}; \dots; \frac{1}{2}, \dots, x_{2m-2} - \frac{1}{2}) \quad (10.25)
 \end{aligned}$$

is onto. So the target module is generated by its μ_e isotypic components. The module

$$I(\frac{1}{2}, \dots, x_0 - \frac{1}{2}; \dots; \frac{1}{2}, \dots, x_{2m-2} - \frac{1}{2}) \quad (10.26)$$

(the string $(-x_{2m-1} + \frac{1}{2}, \dots, x_{2m} - \frac{1}{2})$ has been removed) has $L(-x_{2m-2} + \frac{1}{2}, \dots, \frac{1}{2})$ as its unique irreducible quotient, because it is the homomorphic image of an $X(\tilde{\nu})$ with $\tilde{\nu}$ dominant. Therefore, it is generated by its spherical vector. Combining this with the induction assumption, we conclude that

$$I(-x_{2m-1} + \frac{1}{2}, \dots, x_{2m} - \frac{1}{2}; -a + \frac{1}{2}, \dots, a - \frac{1}{2}; L(\check{\mathcal{O}}_3)) \quad (10.27)$$

is generated by its μ_e isotypic components. Here $\check{\mathcal{O}}_3$ is obtained from $\check{\mathcal{O}}_2$ by removing two entries equal to $2a$. It is isomorphic to

$$I(-a + \frac{1}{2}, \dots, a - \frac{1}{2}; -x_{2m-1} + \frac{1}{2}, \dots, x_{2m} - \frac{1}{2}; L(\check{\mathcal{O}}_3)). \quad (10.28)$$

Finally, the multiplicities of the μ_e isotypic components of $I(-x_{2m-1} + \frac{1}{2}, \dots, x_{2m} - \frac{1}{2}; L(\check{\mathcal{O}}_3))$ are the same as for the irreducible subquotient $L(\check{\mathcal{O}}_1)$. This completes the proof of the claim in this case.

It remains to consider the case when $m = 2$ and $x_0 = 0 < x_1 = x_2 = x_3 = a \leq x_4$. In this case, the module

$$I(a + \frac{1}{2}, \dots, x_4 - \frac{1}{2}; -a + \frac{1}{2}, \dots, a - \frac{1}{2}; -a + \frac{1}{2}, \dots, a - \frac{1}{2}) \quad (10.29)$$

is generated by its μ_e isotypic components because of Proposition 10.3, and arguments similar to the above. Therefore, the same holds for

$$I(-a + \frac{1}{2}, \dots, x_4 - \frac{1}{2}; -a + \frac{1}{2}, \dots, a - \frac{1}{2}), \quad (10.30)$$

which is a homomorphic image via the intertwining operator which interchanges the first two strings. But this is isomorphic to

$$I(-a + \frac{1}{2}, \dots, a - \frac{1}{2}, -a + \frac{1}{2}, \dots, x_4 - \frac{1}{2}). \quad (10.31)$$

Then $I(-a + \frac{1}{2}, \dots, a - \frac{1}{2}, L(-x_4 + \frac{1}{2}, \dots, -\frac{1}{2}, -\frac{1}{2}))$ is a homomorphic image of (10.31) so it is generated by its μ_e isotypic components. By § 5.3, the multiplicities of the μ_e isotypic components are the same in $I(-a + \frac{1}{2}, \dots, a - \frac{1}{2}, L(-x_4 + \frac{1}{2}, \dots, -\frac{1}{2}, -\frac{1}{2}))$ as in $L(\chi_{\check{\mathcal{O}}})$.

This completes the proof of Theorem 10.1. □

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