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On the terminology and geometric aspects of redundant parallel manipulators

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SUMMARY
Parallel kinematics machines (PKMs) can exhibit kinematics as well as actuation redundancy. While the meaning of kinematic redundancy has been already clarified for serial manipulators, actuation redundancy, which is only possible in PKMs, is differently classified in the literature. In this paper a consistent terminology for general redundant PKM is proposed. A kinematic model is introduced with the configuration space (c-space) as central part. The notion of kinematic redundancy is recalled for PKM. C-space, output, and input singularities are distinguished. The significance of the c-space geometry is emphasized, and it is pointed out geometrically that input singularities can be avoided by redundant actuation schemes. In order to distinguish different actuation schemes of PKM, a nonlinear control system is introduced whose dynamics evolves on c-space. The degree of actuation (DOA) is introduced as the number of independent control vector fields, and PKMs are classified as full-actuated and underactuated machines. Relating this DOA to degree of freedom allows to classify the actuation redundancy.

KEYWORDS: Parallel manipulator; Terminology; Kinematic redundancy; Actuation redundancy; Singularities.

1. Introduction

Redundancy has been introduced to provide the dexterity needed for manipulation tasks and to overcome the kinematic and dynamic limitations of parallel kinematics machines (PKMs). Kinematically redundant PKMs were first proposed as variable geometry trusses (VGT),[35,50,52] where special attention was payed to the elastic properties.[23,54] Although several design concepts, such as tetrahedron-based VGT,[21] 2-degree of freedom (DOF) planar PKM,[14] and hyper-redundant PKM,[8] were proposed, kinematically redundant PKM did not draw as much attention as did the redundantly actuated PKM (RA-PKM). Moreover, redundant actuation schemes were developed to increase and homogenize kinematic manipulability and stiffness, to increase the achievable acceleration, and to eliminate singularities and thus enlarge the usable workspace, as reported for a number of prototypes in refs. [1, 6, 11, 18, 27, 53, 60, 61, 66, 67], and references given in ref. [34]. Actuation redundancy can be achieved by actuation of passive joints or by additional kinematic chains connecting the moving and fixed platform, which is the standard approach. It is also beneficial for the calibration as shown in refs. [22, 58, 65], since it provides additional sensor data. It was shown in refs. [49, 64] that actuation redundancy improves the kinematic manipulability and eventually avoids singularities. The latter were investigated in refs. [16, 32]. Stiffness and force capabilities were studied in refs. [7, 13, 48]. Investigations of optimal force distribution and the related design issues have been reported in refs. [26, 29, 30, 31, 44]]. The advantages of RA-PKM are however due to a more complex control, in particular if model-based controlled is envisioned, as pointed out in refs. [11, 17, 20, 36, 41, 42, 53, 66]. One critical point is the redundancy resolution within the feed-forward part of the controller. Another challenge is the presence of undesired antagonistic control forces that are due to geometric model uncertainties[40] but are also inherent in decentralized control schemes.[41] These forces cause elastic deformations that must be taken into account for the calibration, in particular, of heavy payload RA-PKM.[15]

Despite the advances in control and design of redundant PKM, there is yet no consistent terminology. Whereas for redundant serial manipulators the terminology was already clarified in ref. [12], there is no established classification of redundant PKM. In particular, actuation redundancy is frequently referred to as overactuation, and occasionally RA-PKMs are confused with overconstrained mechanisms. Clear definitions and concepts are essential for the systematic design and analysis of novel RA-PKM.

This paper aims on clarifying the terminology for redundant PKM in general and RA-PKM in particular. As a basis for such a classification, a kinematic model for PKM is first introduced in Section 2.1, and the PKM motion equations are recalled in Section 2.2, which are used in the model-based control allowing to represent PKMs as nonlinear control systems in Section 3. As preparation, singularities of PKMs are classified as input, output, and c-space singularities in Section 4. This is used in Section 5.1 to distinguish different actuation modes. Section 5.2 recalls the definition of kinematic redundancy adopted to PKM. The main contribution of this paper is the definition of actuation in Section 5.3 upon the nonlinear control system. PKMs are further classified as full-actuated and underactuated.

The discussion explicitly refers to the geometric aspects of redundancy. The central object in the kinematics of PKM is the c-space in which the manipulator motion is encoded. In contrast to serial manipulators, where the c-space is a

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smooth manifold, its geometry is usually rather complicated for PKM, and it is only locally a smooth manifold. The abundance of singularities in the c-space is reflected by non-smooth motions impairing the PKM control. On the other hand, it is this c-space geometry, and its dimension relative to the number of actuators, that gives rise to different actuation schemes. From the geometric point of view, the actuation corresponds to a parameterization of the c-space in terms of actuator coordinates, and the c-space of a PKM being embedded in the joint space allows for redundant actuation of PKM by combining different parameterizations. It is pointed out geometrically that input singularities can be avoided by redundant actuation.

2. Parallel Manipulator Modeling

2.1. Manipulator kinematics

A PKM consists of a moving platform, carrying an end-effector (EE) that is connected to the base platform by several (possibly identical) kinematic chains (limbs, struts, legs) containing actuated joints. PKMs are thus characterized by closed kinematic loops imposing certain constraints. Denoted by $\mathbf{q} \in \mathbb{V}^n$, the vector of joint variables $q^a, a = 1, \ldots, n$ (higher DOF joints are split into 1-DOF joints), where $\mathbb{V}^n := \mathbb{T}^{n_R} \times \mathbb{R}^{n_P}$ is the joint space of a PKM comprising n_R revolute and n_P prismatic/screw joints. The vector $\mathbf{q} \in \mathbb{V}^n$ is referred to as the configuration of the PKM. A configuration is clearly admissible only if it satisfies loop constraints. The r geometric constraints are summarized in the system as

$$\mathbf{0} = \mathbf{h}(\mathbf{q}), \ \ \mathbf{h}(\mathbf{q}) \in \mathbb{R}^r. \tag{1}$$

Time differentiation yields the kinematic constraints,

$$\mathbf{0} = \mathbf{J}(\mathbf{q})\,\dot{\mathbf{q}}, \ \ \mathbf{J}(\mathbf{q}) \in \mathbb{R}^{r,n} \tag{2}$$

with the constraint Jacobian $\mathbf{J}$.

In any practical implementation there are further constraints due to joint limits, or to prevent collisions. This is expressed as a system of c inequality constraints,

$$\mathbf{0} \leq \mathbf{g}(\mathbf{q}), \ \ \mathbf{g}(\mathbf{q}) \in \mathbb{R}^c. \tag{3}$$

Then the configuration space of the PKM, being the set of all admissible configurations, is

$$V := \left\{ \mathbf{q} \in \mathbb{V}^n | \mathbf{h}(\mathbf{q}) = \mathbf{0}, \mathbf{g}(\mathbf{q}) \geq \mathbf{0} \right\}. \tag{4}$$

It is instructive to consider the c-space geometry. Neglecting inequality constraints (3), V is a variety and only locally (i.e. closed to a given configuration $\mathbf{q}$) a smooth manifold. These manifolds are separated by the singular points of V where the rank of $\mathbf{J}$ changes (Section 4.1). If in the neighborhood of $\mathbf{q}$ in V the number of locally independent constraints is constant $\operatorname{rank}\mathbf{J} \leq r$, the local DOF of the PKM is $\delta_{\mathrm{loc}} := n - \operatorname{rank}\mathbf{J}$. It is important to note that the DOF is in fact a local property of the mechanism, and the PKM might even attain different mobilities without disassembling it as for the so-called kinematotropic mechanisms.[63] The maximal local DOF is referred to as the global DOF denoted by δ.

The PKM interacts with its environment via an EE – the mechanical output. This EE is represented by an EE-frame that is rigidly attached to it. The configuration of this EE-frame relative to a world-fixed (inertial) frame is represented by a matrix $\mathbf{C} \in SE(3)$.[43] The **output mapping** $f_{\mathrm{O}} : V \to SE(3)$, yields the EE-configuration $\mathbf{C} = f_{\mathrm{O}}(\mathbf{q})$ in terms of configuration $\mathbf{q}$. Then the **workspace** of the PKM is the set of attainable EE-configurations:

$$\mathcal{W} := \{ f_{\mathrm{O}}(\mathbf{q}) \, | \mathbf{q} \in V \} \subset SE(3). \tag{5}$$

Usually only a part of this $\mathcal{W}$ is used depending on the application and the presence of singularities.

The EE-velocity is represented by a twist coordinate vector $\mathbf{V} \in \mathbb{R}^6$, respectively $\widehat{\mathbf{V}} \in se(3)$. The instantaneous EE-kinematics is determined by

$$\mathbf{V} = \mathbf{J}_{\mathrm{O}}(\mathbf{q})\,\dot{\mathbf{q}} \tag{6}$$

relating the EE-velocities to the state of the PKM, where $\mathbf{J}_{\mathrm{O}}(\mathbf{q}) : T_{\mathbf{q}}V \to se(3)$ is the **output Jacobian**.

The PKM motion is determined by the motion of its actuators – the mechanical inputs. The relation of actuator and PKM motion is expressed by the **input mapping** $f_{\mathrm{I}} : V \to \mathcal{I}$ that assigns to any PKM configuration the admissible inputs. This relation may not be unique, as there may be different inputs corresponding to the same configuration. If the PKM is equipped with m actuator inputs, $\mathcal{I}$ is m-dimensional. The inputs are not necessarily the motion of some joints of the PKM, since the actuators may act at arbitrary locations of the PKM, e.g. via tendons or pulleys. Generally arbitrary exogenous inputs could be considered. In the following it is assumed that m joints are directly actuated. Then the joint coordinate vector can be split into coordinates of m active and n–m passive joints, $\mathbf{q}_{\mathrm{a}}$ and $\mathbf{q}_{\mathrm{p}}$, respectively. That is, $\mathbf{q}_{\mathrm{a}}$ are the kinematic inputs, and the input mapping is simply the projection of $\mathbb{V}^n$ to the corresponding m-dimensional subspace. With this splitting, the kinematic constraints (2) become

$$\mathbf{0} = \mathbf{J}_{\mathrm{p}}(\mathbf{q})\,\dot{\mathbf{q}}_{\mathrm{p}} + \mathbf{J}_{\mathrm{a}}(\mathbf{q})\,\dot{\mathbf{q}}_{\mathrm{a}}, \tag{7}$$

where $\mathbf{J}_{\mathrm{p}}(\mathbf{q}) \in \mathbb{R}^{r,n-m}$, $\mathbf{J}_{\mathrm{a}}(\mathbf{q}) \in \mathbb{R}^{r,m}$.

The kinematic PKM model can be schematically represented as

$$\boxed{\mathcal{W} \xleftarrow{f_{\mathrm{O}}} V \xrightarrow{f_{\mathrm{I}}} \mathcal{I}} \tag{8}$$

Clearly the central object is the c-space V geometrically representing the mechanism. The input and output mapping yields the input and output, respectively, that corresponds to a given configuration. They are not one to one for PKM in general and for redundant PKM in particular.

Figure 1 shows a schematic representation of this model for a 2-dimensional (2D) c-space in a 3D joint space that can be locally parameterized by q^2 and q^3, or by q^1 and q^2, for instance. The latter may be used as inputs, and the projection

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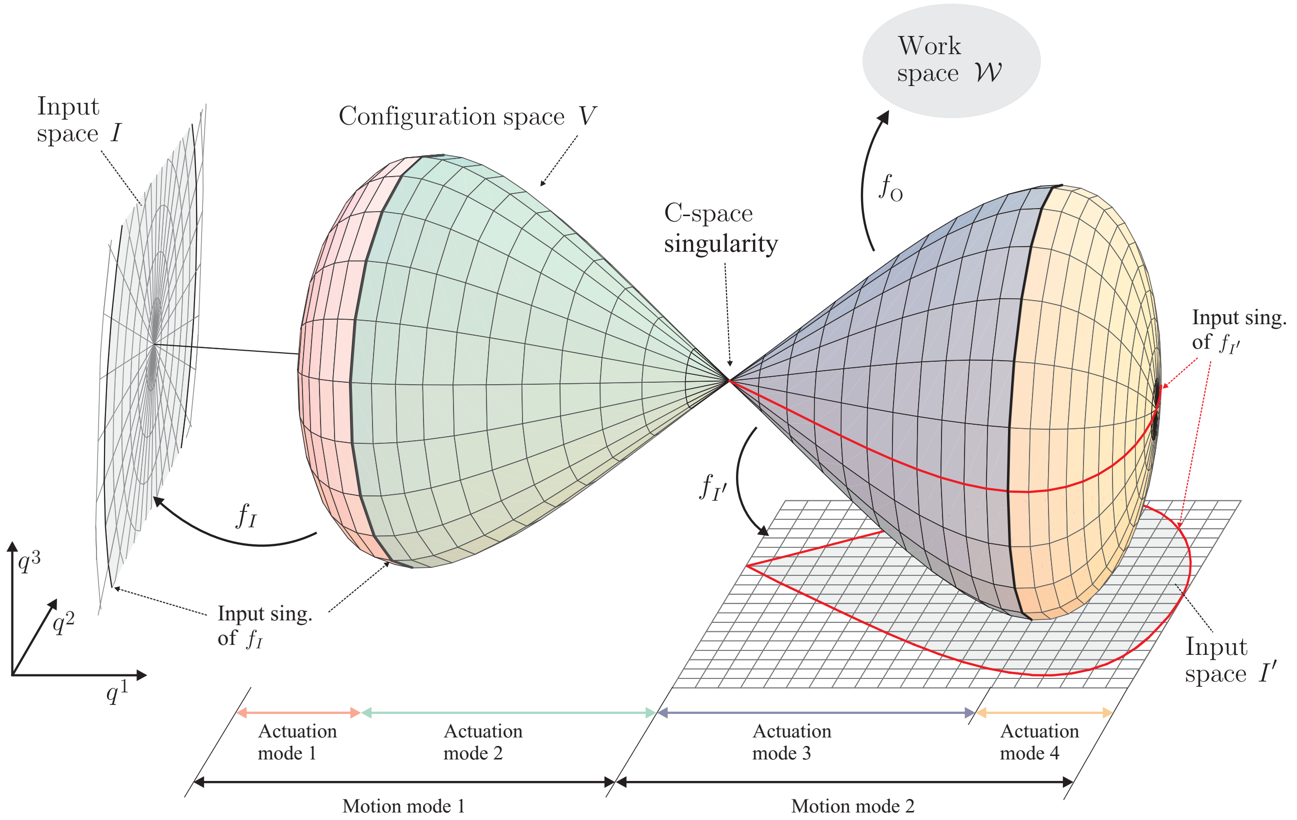

Fig. 1. (Colour online) Geometric interpretation of the PKM control system. The operation modes refer to input space $\mathcal{I}$.

of V on the respective coordinate subspaces gives the input spaces $\mathcal{I}$ and $\mathcal{I}'$.

Model (8) is rather abstract. For the analysis of a PKM in a given configuration the following instantaneous kinematics model is used

$$\boxed{\begin{aligned} \mathbf{0} &= \mathbf{J}_{\mathrm{p}}(\mathbf{q})\,\dot{\mathbf{q}}_{\mathrm{p}} + \mathbf{J}_{\mathrm{a}}(\mathbf{q})\,\dot{\mathbf{q}}_{\mathrm{a}} \\ \mathbf{V} &= \mathbf{J}_{\mathrm{O}}(\mathbf{q})\,\dot{\mathbf{q}} \\ \dot{\mathbf{q}}_{\mathrm{a}} &= \mathbf{J}_{\mathrm{I}}\dot{\mathbf{q}} \end{aligned} \qquad (9)}$$

The first implicit equation locally describes the c-space V. The second equation is the differential output mapping relating instantaneous PKM and EE motion. The third equation is the differential input mapping that yields the instantaneous input motion. Since the input mapping is the projection from V to the $\mathbf{q}_{\mathrm{a}}$ coordinate subspace I, the Jacobian $\mathbf{J}_{\mathrm{I}}$ is constant with entries 1 and 0.

Remark 1. The instantaneous Model (9) applies to general PKM. It admits to separately investigate the mechanism's kinematics and its interaction with the environment via inputs and outputs. It further allows to exhaustively analyze and classify the corresponding critical phenomena as it is presented in ref. [68]. On the other hand, the input–output kinematics of the PKM is often represented in the form $\mathbf{M}_2\mathbf{V} = \mathbf{M}_1\dot{\mathbf{q}}_{\mathrm{a}}$,[19,55] which is easily obtained with the reciprocal screw approach. If $\mathbf{M}_2$ is square, the forward Jacobian is $\mathbf{J}_{\mathrm{F}} = \mathbf{M}_2^{-1}\mathbf{M}_1$. That is, the three mappings in Model (9) are resolved. However, the internal state of the PKM is hidden.

2.2. *Motion equations in minimal coordinates*

A dynamical model is indispensable for the model-based control of PKM. A PKM is a force-controlled multibody system (MBS) subject to geometric constraints due to kinematic loops. In applications where the manipulator interacts with its environment, the PKM is subjected to additional, possibly non-holonomic, constraints. The latter will not be taken into account here. There are several approaches for deriving motion equations of constrained MBS that have different numerical efficiencies. Independent from the applied principle, a basic fundamental difference is the choice of independent generalized coordinates. It is crucial that the dynamic PKM model, which is eventually used for the control, is given in terms of a minimal set of δ generalized coordinates. The solution of the geometric constraints (1) can (locally) be expressed in terms of a subset $\mathbf{q}_2$ comprising δ_{loc} joint variables so that the configuration is given as $\mathbf{q} = \psi(\mathbf{q}_2)$. In other words, $\mathbf{q}_2$ are local coordinates on V for a parameterization ψ that admits to describe the internal kinematics of the mechanism. It is usually impossible to explicitly derive the relation of $\mathbf{q}$ on some chosen $\mathbf{q}_2$. Therefore, a standard method for MBS with kinematic loops is to introduce a relation on velocity and acceleration levels after a coordinate partitioning in dependent and independent joint variables.[2,46] With this approach the computationally efficient forms of PKM motion equations can be derived as it was pursued in refs. [11, 24, 33, 36, 45].

Denote with $\mathbf{c} \equiv (c_1, \ldots, c_m)$ the vector of m generalized control forces in the actuated joints, and with $\tau \in se^*(3)$ the EE wrench due to external loads at EE. Then the PKM dynamics is governed by the motion equations[36]

$$\overline{\mathbf{G}}(\mathbf{q})\ddot{\mathbf{q}}_2 + \overline{\mathbf{C}}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}}_2 + \overline{\mathbf{Q}}(\mathbf{q},\dot{\mathbf{q}},t) + \overline{\mathbf{J}}_{\mathrm{E}}^{T}(\mathbf{q})\tau = \mathbf{A}^{T}(\mathbf{q})\mathbf{c}, \tag{10}$$

where $\overline{\mathbf{G}}$ is the generalized mass matrix, $\overline{\mathbf{C}}\dot{\mathbf{q}}_2$ represents the generalized Coriolis and centrifugal forces, and $\overline{\mathbf{Q}}$ represents all remaining generalized forces. The $m \times \delta_{\mathrm{loc}}$ control matrix $\mathbf{A}$ is such that $\dot{\mathbf{q}}_{\mathrm{a}} = \mathbf{A}\dot{\mathbf{q}}_2$, and the right-hand side $\overline{\mathbf{Q}}_{\mathrm{a}} = \mathbf{A}^T\mathbf{c}$ in Eq. (10) are the generalized control forces because of actuator forces $\mathbf{c}$. Equation (10) is a system of δ_{loc} ordinary differential equations in $\mathbf{q} \in \mathbb{V}^n$ that, together with the r kinematic constraints, completely determines the PKM dynamics.

If the number m of actuated joints exceeds the PKM DOF (i.e. the PKM is redundantly actuated), then $\mathbf{A}^T$ is not square and has a null-space of dimension $m - \delta_{\mathrm{loc}}$. Only the actuator forces $\mathbf{c}$ not in the null-space of $\mathbf{A}^T$ are effective control forces. It is a peculiarity of RA-PKM that actuator forces can be generated in the null-space of $\mathbf{A}^T$. Such null-space forces, giving rise to internal prestress, can be exploited for second-level tasks such as backlash avoidance or stiffness control.[28,36,37,56,57]

3. The Associated Nonlinear Control Systems

A PKM is a force-controlled (holonomically) constrained dynamical system whose dynamics is governed by Eq. (10). The control purpose is to manipulate the EE that embodies the system's mechanical output. This makes PKM the second-order control-affine control system on the configuration space V that is represented as the first-order control system,[5,47]

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{i=1}^{m} \mathbf{g}_i(\mathbf{x})\, c^i, \tag{11}$$
$$\mathbf{C} = f_{\mathrm{O}}(\mathbf{x}),$$

with state vector $\mathbf{x} := (\mathbf{q}_2, \dot{\mathbf{q}}_2)$. Therein

$$\mathbf{f} := \begin{pmatrix} \dot{\mathbf{q}}_2 \\ -\overline{\mathbf{G}}^{-1}(\overline{\mathbf{C}}\dot{\mathbf{q}}_2 + \overline{\mathbf{Q}} + \overline{\mathbf{J}}_{\mathrm{E}}^{T}\tau) \end{pmatrix} \tag{12}$$

is the drift vector field, and the columns $\mathbf{g}_i$, $i = 1, \ldots, m \leq n$ of

$$\mathbf{g} := \begin{pmatrix} \mathbf{0} \\ \overline{\mathbf{G}}^{-1}\mathbf{A}^T \end{pmatrix} \tag{13}$$

are the control vector fields through which the control forces affect the system. The actuation of the PKM determines the immediate effect of control forces in a given pose of a PKM. Apparently the DOA has to do with the number of independent control vector fields, as well as with the vector space spanned by $\mathbf{g}_i$ (see Section 5).

4. Critical Configurations

Singularities are manifested in the qualitative change of the manipulator's kinematic and static properties. A PKM may become structurally unstable, lose the ability to properly interact with its environment, or become non-manipulable. This must be taken into account for the proper definition of actuation. For this purpose it is sufficient to distinguish c-space, output, and input singularities. These singularity types can occur simultaneously, and their combination, if possible, may lead to phenomena such as instantaneously impossible input motions, or instantaneously redundant inputs. An exhaustive study is presented in ref. [68], where six different types are identified and all possible combinations are listed. In the following c-space, input, and output singularities are classified as far as necessary for introducing a sensible notion of redundancy. It is instructive to explicitly refer to the c-space geometry, since this allows interpreting the different singularities and the actuation redundancy geometrically. Moreover, the c-space topology reveals all motion characteristics as shown in refs. [33] and [38]. In ref. [33] the input, output, and c-space singularities were analyzed using differential forms. The local structure of the c-space was addressed in ref. [38].

At this point a remark is in order. Singularities are identified upon the rank of certain Jacobians. It should be stressed that a singular point is one in which the rank of the considered Jacobian changes, but frequently any situation is referred to as singular whenever the rank is lower than expected.

4.1. C-space singularities

The configuration space V is a variety in $\mathbb{V}^n$. V comprises several connected smooth manifolds (subspaces like smooth curves or surfaces) that are separated by singular points, indicating "non-smoothness" of V at these points. Points of V that belong to a smooth manifold are called regular. The attribute "singular," meaning solitary or unique, reflects the fact that they are special in the sense that almost all points are regular. Clearly the mechanism's mobility has to do with the dimension of V. A motion of the PKM corresponds to a curve in V, and in points where V is not a smooth manifold, the motion can be non-smooth.

The PKM mobility can be clearly defined upon the c-space topology. The **differential DOF** (or instantaneous DOF) of the mechanism is defined as $\delta_{\mathrm{diff}}(\mathbf{q}) := n - \operatorname{rank} \mathbf{J}(\mathbf{q})$. A point $\mathbf{q} \in V$ is **regular** if and only if it belongs to a submanifold of V, i.e. there is a neighborhood $U(\mathbf{q})$ such that δ_{diff} is constant in $U(\mathbf{q}) \cap V$, otherwise it is **singular**. The **local DOF** in $\mathbf{q}$, denoted by $\delta_{\mathrm{loc}}(\mathbf{q})$, is the local dimension of V. This is the highest dimension of manifolds passing through $\mathbf{q}$. If $\mathbf{q}$ is regular, then V is locally a $\delta_{\mathrm{loc}}(\mathbf{q})$-dimensional manifold. In case of kinematotropic mechanisms[63] there are different local DOF in a connected component of V. The **global DOF** δ is the highest local dimension of V. If V is not connected, i.e. there are different assemblies of the mechanism, which can not be attained via an admissible finite motion, the global DOF needs to be restricted to the relevant assembly. A detailed discussion of the geometric mobility concept can be found in refs. [38, 39].

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Since V is locally a smooth manifold if and only if the constraint Jacobian $\mathbf{J}$ has constant (not necessarily full) rank, one can introduce the following.

Definition 1. A point $\mathbf{q} \in V$ is called a **c-space singularity** if and only if rank $\mathbf{J}$ is not constant in any neighborhood of $\mathbf{q}$ in V.

As a practical consequence, even if the c-space has locally a dimension δ_{loc}, in a c-space singularity no subset of δ_{loc} joint variables can be used to parameterize V, and the generalized mass matrix $\overline{\mathbf{G}}$ in Eq. (10) is not regular.

For example, the c-space $V \in \mathbb{R}^3$ in Fig. 1 is a 2D smooth manifold except at the indicated c-space singularity. At any other point it is smooth with $\mathbf{J}$ having constant rank, and a unique tangent plane can be attached to V. At the c-space singularity there is no unique tangent plane, and the differential DOF of the PKM increases.

4.2. Input singularities

Naturally, a configuration is regarded as an input singularity if the interdependence of actuator motion and the motion of the PKM undergoes a qualitative change. To state this more precisely, it is necessary to separately consider the motion of actuated and passive joints.

Definition 2. The configuration $\mathbf{q} \in V$ is called **passive singularity** (**actuator singularity**) if rank $\mathbf{J}_{\text{p}}$ (rank $\mathbf{J}_{\text{a}}$) is not constant in any neighborhood of $\mathbf{q}$ in V. If $\mathbf{q}$ is either a passive or an actuator singularity, it is called **input singularity**.

Active and passive singularities can occur simultaneously without being c-space singularities.

If in the model in Fig. 1 the projection of V onto the q^2–q^3 coordinate plane is used as input space $\mathcal{I}$, its boundary points are passive (input) singularities, since there V is normal to $\mathcal{I}$ so that instantaneous motions q^1 are possible independently from the inputs. To cope with this problem, in such configurations, joints 1 and 2 could be used as actuators with input space $\mathcal{I}'$. Also, this actuation scheme exhibits input singularities at the boundary of $\mathcal{I}'$.

The constraints (7) can be solved for the velocities of passive and actuator joints. The general solution is respectively

$$\dot{\mathbf{q}}_{\text{p}} = -\mathbf{J}_{\text{p}}^{+}\mathbf{J}_{\text{a}}\dot{\mathbf{q}}_{\text{a}} + \dot{\mathbf{q}}_{\text{p0}}, \quad \text{with} \quad \dot{\mathbf{q}}_{\text{p0}} \in \mathrm{N}(\mathbf{J}_{\text{p}}), \qquad (14)$$
$$\dot{\mathbf{q}}_{\text{a}} = -\mathbf{J}_{\text{a}}^{+}\mathbf{J}_{\text{p}}\dot{\mathbf{q}}_{\text{p}} + \dot{\mathbf{q}}_{\text{a0}}, \quad \text{with} \quad \dot{\mathbf{q}}_{\text{a0}} \in \mathrm{N}(\mathbf{J}_{\text{a}}),$$

where $N(\mathbf{J})$ is the null-space of $\mathbf{J}$. The left pseudoinverse $\mathbf{J}^{+}$ always exists. If rank $\mathbf{J}_{\text{a}} < m$, the null-space $N(\mathbf{J}_{\text{a}})$ is non-empty, and there exist instantaneous motions of actuator joints even if all passive joints are locked. If rank $\mathbf{J}_{\text{p}} < n - m$, the null-space $N\left(\mathbf{J}_{\text{p}}\right)$ is non-empty, and there exist instantaneous motions of passive joints for locked actuator joints. Whether these instantaneous motions correspond to finite motions depends on whether the considered configuration is regular, that is whether the rank of the respective Jacobian is constant in the neighborhood of $\mathbf{q}$. In input singularities the rank of $\mathbf{J}_{\text{p}}$ or $\mathbf{J}_{\text{a}}$ drops and the respective null-space increases.

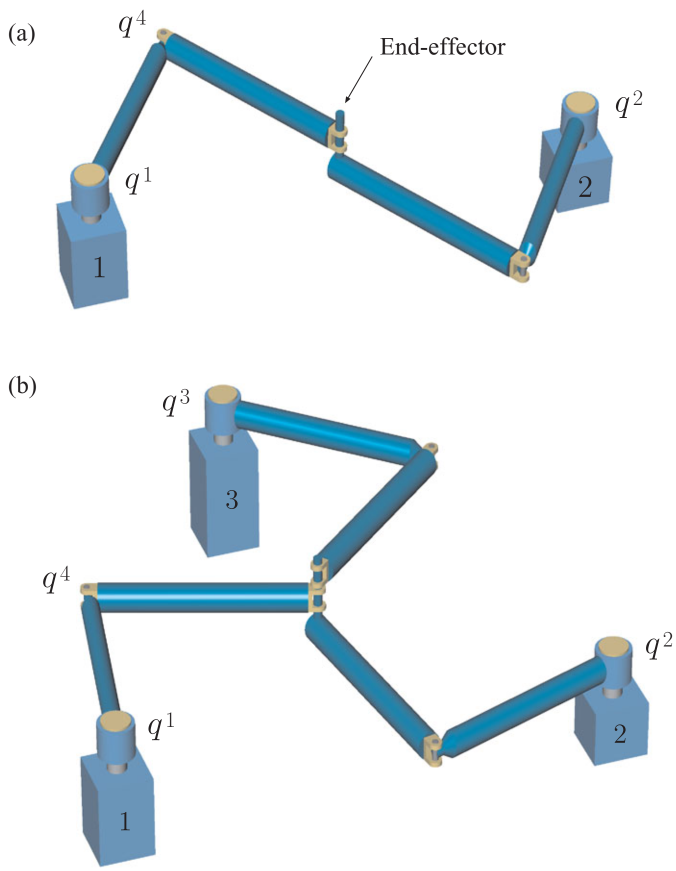

Fig. 2. (Colour online) (a) The 5-bar mechanism, and (b) its redundant extension, the <u>R</u>R/2<u>R</u>RR PKM.

Remark 2. In ref. [19] a classification of input and output singularities of PKM was proposed upon a formulation of the form $\mathbf{M}_2\mathbf{V} = \mathbf{M}_1\dot{\mathbf{q}}_{\text{a}}$ (see also Remark 1). Accordingly, singularities of types I and II are identified when, respectively, $\mathbf{M}_1$ or $\mathbf{M}_2$ is not full rank. Types I and II singularities are also called serial and parallel singularities, respectively, as type II only occurs for PKM. Type II singularities are also termed force singularities, since certain EE-wrenches can not be equilibrated by control forces. This classification is useful when considering the PKM as a transmission device, relating input and output motions. The internal state of the PKM is hidden, however; for example, the PKM may be in a passive singularity even if both matrices are regular.

Example 1. The 5-bar mechanism in Fig. 2(a) and the mechanism in Fig. 2(b), which is obtained by adding a third kinematic chain between the EE and the base, are examples showing the avoidance of input singularities by means of redundant actuation. The EE of both mechanisms can be positioned in the plane, and the two translation components are the mechanical outputs. The 5-bar mechanism is controlled by two drive units at the base. Adding an identical actuated chain to the 5-bar mechanism does not change the DOF nor the EE mobility so that both mechanisms have the DOF 2. In the pose shown in Fig. 2(a) the 5-bar mechanism exhibits a passive input singularity. This is revealed by the manipulability measure defined as the

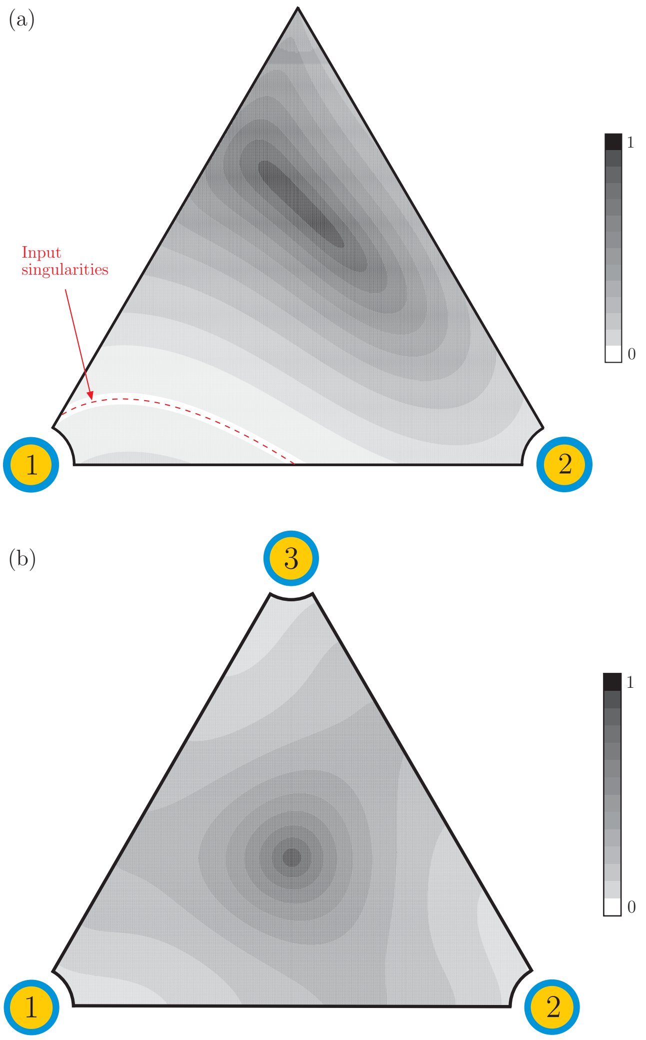

Fig. 3. (Colour online) Manipulability measure $1/\kappa$ of (a) the 5-bar mechanism, and (b) the $\underline{\text{R}}$R/2$\underline{\text{R}}$RR PKM.

inverse of the condition number κ of $\mathbf{J}_{\text{F}}\mathbf{J}_{\text{F}}^{T}$,[43,51,59] where the forward kinematic Jacobian $\mathbf{J}_{\text{F}}$ relates the EE-velocity $\mathbf{V}$ and actuator velocities $\dot{\mathbf{q}}_{\text{a}}$ according to $\mathbf{V} = \mathbf{J}_{\text{F}}\dot{\mathbf{q}}_{\text{a}}$ in Remark 1 (notice that for this positioning device the measure does not depend on the scaling of translations and rotations, which is problematic in general for spatial manipulators). Figure 3 shows the distribution of $1/\kappa$ for the EE positions in the workspace. It is clearly visible that the kinematic dexterity measures of two manipulators differ significantly. Moreover, the 5-bar mechanism exhibits singularities, where the EE-motion is not controllable by the actuators, reflected by a drop of manipulability. The configuration in Fig. 2(a) is such an input singularity. This indeterminacy is removed with the redundant third actuated chain in Fig. 2(b). Following the conventional notation, the mechanism in Fig. 2(b) will be denoted as $\underline{\text{R}}$R/2$\underline{\text{R}}$RR, indicating that the EE is connected to the fixed base by one kinematic chain (chain 2) comprising two revolute joints and two chains (chains 1 and 3) with three revolute joints, where underlines specify the actuated joints.

Geometrically the occurrence of an input singularity means that, in the configuration in Fig. 2(a), the 2D c-space cannot be parameterized by the two actuator joint angles q^1 and q^2. Figure 4 shows the q^1–q^2–q^4 section of the c-space,

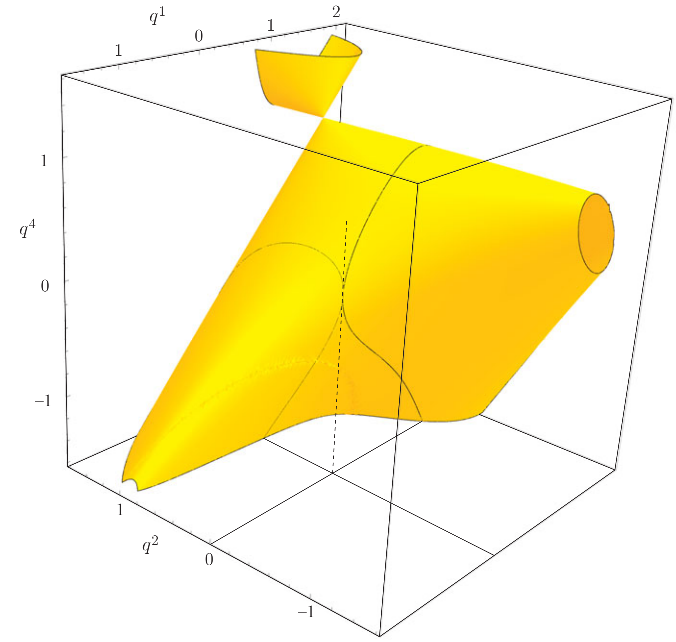

Fig. 4. (Colour online) q^1–q^2–q^4 c-space section of the 5-bar mechanism.

where the origin $\mathbf{q}_0 = \mathbf{0}$ is assigned to the input singularity. Apparently the projection of V onto the q^1–q^2 coordinate plane is not unique, and at $\mathbf{q}_0$ the joint angle q^4 does not depend uniquely on the input coordinates q^1 and q^2. These input singularities occur whenever the two middle links are parallel, i.e. when the mechanism resembles a 4-bar mechanism, and the EE position for these input singularities lies on the coupler curve of this 4-bar mechanism, restricted to the work space as indicated in Fig. 3(a). The non-uniqueness problem, and thus the input singularities, are removed by adding a third actuated chain that yields the redundantly actuated $\underline{\text{R}}$R/2$\underline{\text{R}}$RR PKM in Fig. 2(b). Then the mobility is unaltered but the number of parameters used to prescribe the motion is increased. Beside the increased and homogenized manipulability, the apparent advantage of this redundant actuation is the elimination of input singularities. The dimension of the joint space is increased without increasing the dimension of the c-space (i.e. DOF), and the c-space of the $\underline{\text{R}}$R/2$\underline{\text{R}}$RR is embedded in an 8D joint space (while that of the 5-bar is embedded in a 5D space), which gives more freedom for choosing actuator coordinates, or even using a redundant set. In other words, two 5-bar mechanisms are connected and two copies of the c-pace in Fig. 4 are glued together by identifying the respective q^1–q^4 subspaces.

4.3. Output singularities

The output mapping f_{O} assigns to any configuration an EE-pose, and the output Jacobian the EE-twist to the PKM state. Output singularities are situations where the number of instantaneous motions that are determined by the PKM motion changes.

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Definition 3. The configuration $\mathbf{q} \in V$ is called **output singularity** if rank $\mathbf{J}_O$ is not constant in any neighborhood of $\mathbf{q}$ in V.

Apparently the occurrence of output singularities depends on how the outputs are assigned to the PKM according to f_O. They indicate a change of the way the PKM interacts with its environment, but not critical configurations of the PKM itself.

5. Proposal for a Terminology

5.1. *Operation modes*

Critical configurations impair the integrity of the PKM and the stability of its dynamics model. A reliable operation is only ensured in regular configurations. The submanifolds of regular points constitute modes of operation of the PKM. In this regard only those are relevant that can be attained by a motion starting from the initial assembly, but not those that could be attained by opening kinematic loops and assembling it differently.

Definition 4. The connected subvarieties of V are called the **assembly modes** of the PKM. The connected submanifolds of regular points of V are called **motion modes** of the PKM. The submanifolds of the motion modes consisting of configurations that are not input singularities are called **actuation modes** of the PKM. The submanifolds of the motion modes consisting of configurations that are neither input nor output singularities are called **operation modes** of the PKM.

This is a refinement of the c-space according to critical configurations. The motion modes consist of all regular configurations in which the PKM is stable in the sense that it does not exhibit c-space singularities. The actuation modes are the submanifolds where, in addition, a continuous control of the PKM is ensured. Finally, a further restriction to the configurations where a continuous interaction with the environment is ensured, so that the PKM can be operated as transmission device, yields the operation modes. The motion and actuation modes are indicated for the geometric model in Fig. 1, which has only one assembly mode.

Remark 3 (aspects). The motion modes consist of all configurations in which the PKM performs smooth motions. The actuation modes are submanifolds of the motion modes that in addition ensure uninterrupted actuation. For serial manipulators exists an equivalent to operation modes, called aspects, which refer to the manifolds of regular points of forward kinematics.[3] The so-called "generalized aspects" were introduced for a fully parallel PKM in ref. [9] as the connected submanifolds where both Jacobians $\mathbf{M}_1$ and $\mathbf{M}_2$ in Remark 1 were regular. These submanifolds may, however, comprise other (e.g. c-space) singularities, since, as already mentioned, the internal state of a PKM is ignored.

It is sensible to classify the tasks of a PKM according to critical configurations. To this end consider a task with corresponding task space $T \subset SE(3)$, i.e. a set of poses the EE has to attain. It is assumed that T is connected.

Definition 5. A task with task space T is a **regular task** if there is an operation mode $M \subset V$ such that $T \subseteq f_O(M)$.

Remark 4. A regular task can be accomplished without passing through any critical configuration. An interesting concept in this regard is the notion of cuspidal serial manipulators[62] that is being adopted for PKM.[9,10]

5.2. *Kinematic redundancy*

Traditionally, kinematic redundancy refers to situations were the DOF of a manipulator exceeds the required EE-mobility. This notion can be directly adopted for PKM.

Definition 6. Consider a PKM with global mobility δ. The PKM is called **kinematically redundant** if $\dim \mathcal{W} < \delta$. The **degree of kinematic redundancy** is $\rho_k := \delta - \dim \mathcal{W}$. The motion that the PKM can perform with fixed EE-configuration $\mathbf{C} \in SE(3)$ is called the **self-motion** with $\mathbf{C}$. The submanifolds of $S_\mathbf{C} := \{\mathbf{q} | \mathbf{C} = f_E(\mathbf{q})\} \subset V$ are called the self-motion manifolds for this EE-pose. For serial manipulators, this was studied in ref. [4].

Remark 5. This definition appears similar to the definition of kinematic redundancy of serial manipulators. It is important, however, to notice the specifics of PKM. While the c-space of serial manipulators is a smooth manifold, the c-space V of a PKM comprises manifolds (possibly of different dimensions, as for kinematotropic mechanisms) that are separated by c-space singularities. For this reason the global DOF appears in the above definition, and it should be noticed that possibly $\dim \mathcal{W} < \delta_{\text{loc}}$ in one motion mode, while $\dim \mathcal{W} \geq \delta_{\text{loc}}$ in another mode.

This notion of kinematic redundancy is solely based on the local dimensions of c-space and workspace, but does not take into account how (or even if) the task motion is embedded in the workspace. Even if the PKM is kinematically redundant, it may not be able to accomplish a particular task. Moreover, the motion characteristics may be different in different motion modes. For instance, the EE may perform planar motions in one mode and spherical motions in another. Therefore, it is appropriate to introduce the notion of task redundancy.

Consider a task with corresponding task space $T \subset SE(3)$, assumed to be connected.

Definition 7. The PKM is called **task-redundant** if $\dim T < \dim \mathcal{W}$ and $T \subseteq \mathcal{W}$, and **task-deficient** if $T \subseteq \mathcal{W}$.

Remark 6. Notice that a manipulator may be kinematically redundant as well as task-deficient.

5.3. *Types of actuation*

The above preliminaries admit to introduce a stringent terminology for RA-PKM. The following definitions refer to a configuration $\mathbf{q}$ in a certain actuation mode where the PKM has local DOF δ_{loc}.

Definition 8. In the considered actuation mode, the DOA is the number of independent input vector fields in the control system (11):

$$\alpha(\mathbf{q}) := \operatorname{rank} \mathbf{g}(\mathbf{q}) = \operatorname{rank} \mathbf{A}(\mathbf{q}). \tag{15}$$

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If $\alpha < \delta_{\mathrm{loc}}$, the PKM is called **underactuated**, and if $\alpha = \delta_{\mathrm{loc}}$, the PKM is **full-actuated**. The **degree of redundancy** of the actuation is $\rho_\alpha := m - \alpha$. The PKM is called **redundantly actuated** if $\rho_\alpha > 0$ and **non-redundantly actuated** if $\rho_\alpha = 0$.

Remark 7. Redundantly actuated PKMs are occasionally termed "overactuated." Notwithstanding that RA-PKM can be underactuated, a full-actuated PKM is completely actuated, and an improvement is impossible. Hence, the term "overactuation" should not be used. Geometrically, underactuation refers to situations where the active joint variables do not constitute local coordinates on V, i.e. $\mathbf{q}_{\mathrm{a}}$ does not fully determine the PKM configuration.

Remark 8. Actuation refers to the immediate effect of control forces on the state of the system. It is a pointwise property. The effect of the actuation on the PKM motion is described by the controllability of the system. This is a local property, i.e. considering the effect of actuation over a small time.[5,47] Clearly, an underactuated PKM can be controllable (but it is questionable whether such a PKM offers sufficient stability).

Remark 9. The above DOA definition makes explicit use of the expression of Eq. (13) and thus of Eq. (10). This is only valid at regular configurations (i.e. in actuation modes), since in c-space or input singularities the projected mass matrix $\overline{\mathbf{G}}$ or the control matrix $\mathbf{A}$ in Eq. (10) may be singular, or does not exist. Nevertheless, the DOA is a general concept, and input vector fields can indeed be assigned to any (possible singular) configuration. Then the DOA would change at input singularities. For instance, the Gough–Stewart platform is non-redundantly full-actuated as long as it does not enter the well-known input singularities where the prismatic joint screws become dependent and form a 5-system. In these input singularities the control matrix $\mathbf{A}^T$ has rank 5. Hence, the DOA reduces to 5, and the PKM is redundantly underactuated in this input singularity.

The geometric meaning of redundancy can vividly be explained for the 2D c-space $V \in \mathbb{R}^3$ in Fig. 1. Clearly, at the indicated c-space singularity, V loses its manifold property, as one can not assign a 2D tangent plane at this point. The two manifolds, separated by the c-space singularity, are the motion modes of this fictitious PKM. The system has global and local DOF $\delta = 2$, and can be locally controlled using joint variables q^2 and q^3 as inputs. Then the PKM is non-redundantly full-actuated. The input space $\mathcal{I}$ is the q^2–q^3 section of V, and q^2 and q^3 are local coordinates on V. This parameterization fails when the PKM attains a configuration that projects to the boundary of $\mathcal{I}$. The corresponding points in V are the input singularities as depicted in Fig. 1. The latter separate the c-space V into the indicated actuation modes. However, the PKM cannot be steered from actuation mode 1 to 2 using the inputs q^2 and q^3. This would be possible using q^1 and q^2 with input space $\mathcal{I}'$. Obviously, also for $\mathcal{I}'$, there are input singularities, as indicated. Neither $\mathcal{I}$ nor $\mathcal{I}'$ alone is a globally feasible input space, but the combination of these (non-redundant) actuation schemes gives rise to one with input space $\mathcal{I} \cup \mathcal{I}' \equiv V$ and (redundant) inputs q^1, q^2, and q^3 that is free of input singularities. Then the PKM is always redundantly full-actuated.

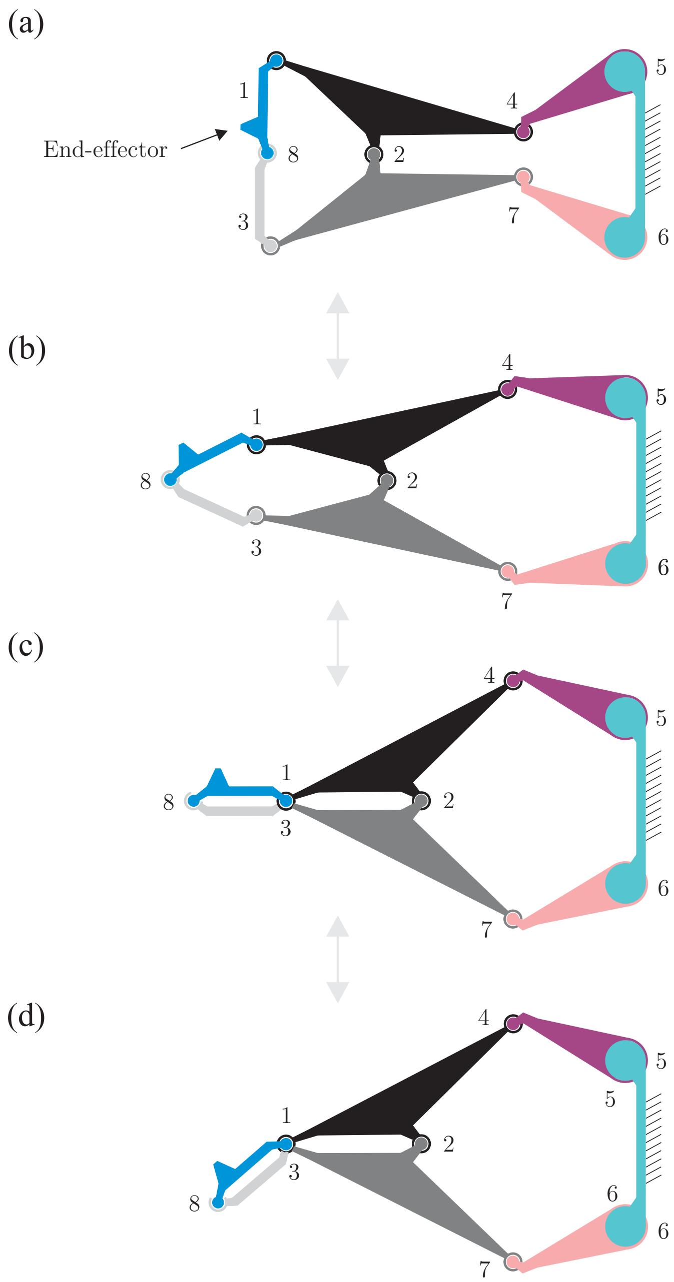

Fig. 5. (Colour online) A 2-DOF manipulator in different actuation modes.

Example 2. Consider two actuation schemes of the planar mechanism with DOF $\delta = 2$ in Fig. 5. First assume that joints 5 and 6 are actuated. The configuration shown in Fig. 5(a) is an input and passive singularity, as the motion of joints 1, 3, and 8 are instantaneously independent from the input motion. From there the mechanism can enter an actuation mode, whose configuration is shown in Fig. 5(b). In this mode the DOA is $\alpha = 2$, the system is non-redundantly full-actuated. It can leave this actuation (and motion) mode when steered into the c-space singularity (Fig. 5(c)), where two branches (motion modes) of the configuration space intersect. Figure 5(d) shows a configuration in one of the possible actuation modes with DOA $\alpha = 1$ so that the mechanism is

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redundantly underactuated with actuation redundancy $\rho = 1$. In fact, the motion of joints 1, 3, and 8 cannot be controlled by the actuated joints 5 and 6.

Now assume that in addition to joints 5 and 6 joint 1 is also actuated. Then configurations (a) and (b) belong to a single actuation mode, where the system is redundantly full-actuated. That is, configuration (a) is not an input singularity for this redundant actuation scheme. After passing through the c-space singularity, in the actuation mode (d), the mechanism is again redundantly full-actuated.

Example 1 (cont). In the configuration $\mathbf{q}_0$ of the 5-bar mechanism in Fig. 2(a) the constraint and actuator Jacobian have full ranks, rank $\mathbf{J}(\mathbf{q}_0) = 3$ and rank $\mathbf{J}_a(\mathbf{q}_0) = 2$, respectively, but rank $\mathbf{J}_p(\mathbf{q}_0) = 2$. Moreover, since $\mathbf{J}_p$ has full rank 3 outside $\mathbf{q}_0$, this is a passive singularity. There are possible instantaneous motions, $\dot{\mathbf{q}}_p \in \ker \mathbf{J}_p$, of passive joints without actuator motions. Because of rank $\mathbf{J}(\mathbf{q}_0) >$ rank $\mathbf{J}_p(\mathbf{q}_0)$ not all actuator motions are feasible. In fact, only coupled actuator motions are possible in this configuration. This situation is classified in ref. [68] as a singularity of redundant passive motion (RPM) and impossible input (II) type. If one would assign a DOA to this configuration, it would be $\alpha(\mathbf{q}_0) = 0$. Otherwise the 5-bar mechanism has DOA $\alpha = 2$. The 5-bar mechanism can be regarded as instantaneously redundantly underactuated in $\mathbf{q}_0$, as the control forces cannot fully actuate the mechanism. Moreover, if the mechanism is at rest in $\mathbf{q}_0$, and if there are no external or inertia forces, it is not controllable. The images of actuation modes in workspace are visible in Fig. 3(a). The two actuation modes are separated by input singularities.

The redundantly actuated $\underline{R}$R/2$\underline{R}$RR PKM does not possess such input singularities, and it has a single uninterrupted actuation mode.

Remark 10. Apparently actuation redundancy allows for elimination of input singularities. A fundamental question is what degree of redundancy is sufficient for achieving full actuation in all regular configurations. A preliminary answer follows from the embedding theorem by Whitney[25] that, adapted to this problem, states that any δ-dimensional c-space can be embedded in an Euclidean space of dimension $2\delta + 1$. It is not sure, however, that $2\delta + 1$ actuators are sufficient for full actuation. In case of the 5-bar linkage, the mechanism could always be fully actuated using five inputs. But one must be careful, since the theorem only says that V can be embedded in some Euclidean space. It does not say that this space is the (non-Euclidean) joint space.

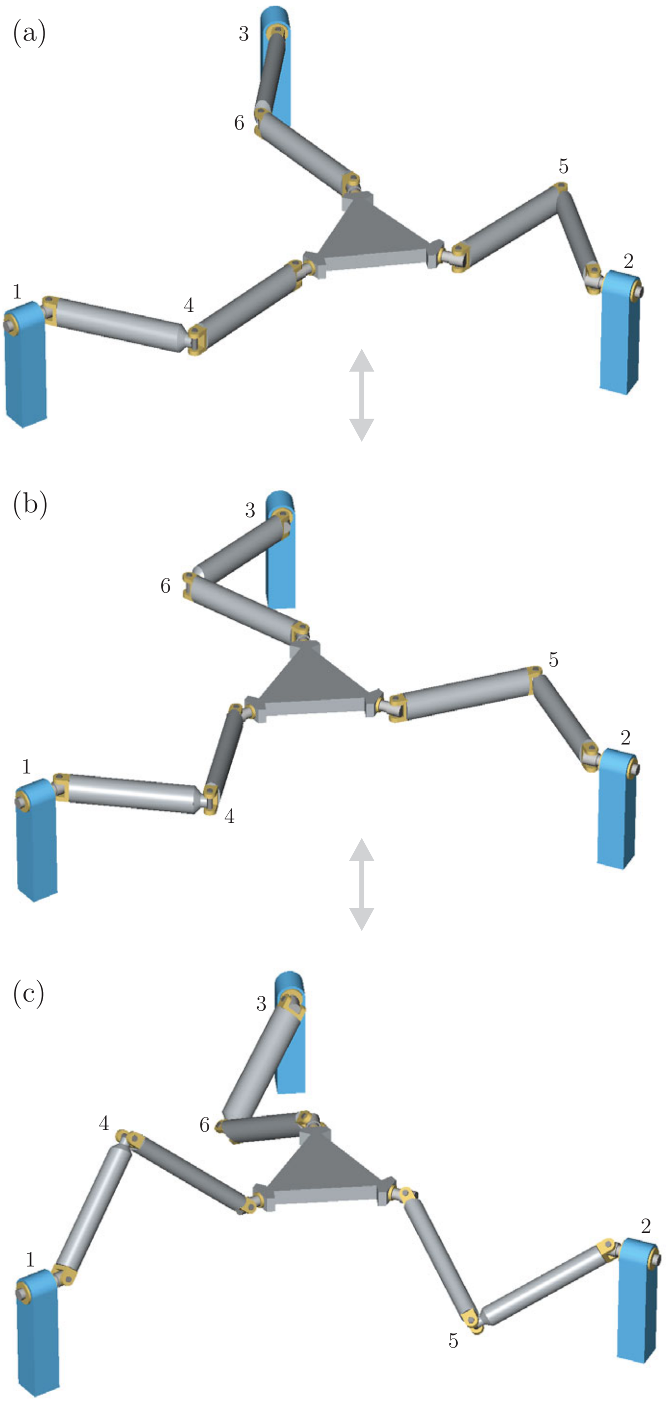

Fig. 6. (Colour online) The 3URU DYMO PKM in its (a) planar operation mode, (b) c-space singularity, and (c) lockup mode.

Example 3. Consider the 3-URU Double-Y Multi-Operational (DYMO) PKM in Fig. 6(a) that was reported in ref. [69]. This PKM, with mobility $\delta = 3$, exhibits operation modes with different degrees of actuation. First consider the actuation scheme with joints 4–6 are actuated. In this mode the PKM acts as a planar manipulator, since the platform can only move in the horizontal plane. The PKM configuration is completely determined by the active joints so that in this mode the PKM is non-redundantly full-actuated with DOA $\alpha = 3$. It can leave this operation mode via the c-space singularity in Fig. 6(b), where different motion modes (branches of the configuration space) intersect. In this singularity the platform center is above the center of the base triangle. Figure 6(c) shows a configuration in one of the possible operation modes. This mode was called the lockup mode,[69] since the platform is immobile. In fact, the platform motion is independent of the motion of the limbs. With joints 4–6 actuated, the PKM is redundantly underactuated with DOA $\alpha = 0$ and DOA redundancy, $\rho_\alpha = 3$. That is, not only is the platform fixed but the PKM motion cannot be even controlled, as the limbs can spin freely. In order to fully actuate the PKM in this mode, one needs to actuate joints 1–3. Now, if one intends to take advantage of the lockup mode, one would need additional (possibly low-powered)

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actuators in joints 1–3. Then with joints 1–6 actuated, the PKM is redundantly full-actuated with DOF $\alpha = 3$ and $\rho_\alpha = 3$. Note that this redundancy is not achieved by the addition of kinematic chains connecting ground and moving platform but by the activation of passive joints.

This redundant actuation would also be required to operate in the further modes of this PKM that exhibits a spherical and mixed mode of the platform motion.[69] Using the redundant actuation scheme with joints 1–6 the PKM is full-actuated in any mode.

6. Summary

In this paper a terminology for redundant PKM has been proposed. The aim of this contribution was to provide consistent definitions upon a general model, and to highlight the geometric aspects of redundancy. To this end, a kinematic model is introduced with the c-space as central part. Input, output, and c-space singularities are distinguished and used to introduce motion, actuation, and operation modes. The notion of kinematic redundancy is recalled and task redundancy is introduced.

A dynamic model was introduced that enables to treat PKM as nonlinear control systems and to define actuation. The DOA was introduced as the number of independent control vector fields, and PKM are classified as full-actuated and underactuated. Further, actuation redundancy is defined as the difference of the number of actuators and the DOA. It is pointed out geometrically that input singularities can be avoided by redundant actuation.

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