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Mixing operators with prescribed unimodular eigenvalues

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Abstract. For arbitrary closed countable subsets Z of the unit circle examples of topologically mixing operators on Hilbert spaces are given which have a densely spanning set of eigenvectors with unimodular eigenvalues restricted to Z. In particular, these operators cannot be ergodic in the Gaussian sense.

Key words: Taylor shift, unimodular eigenvalues, mixing operator, rational approximation
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1. *Introduction and main result*

The dynamical behaviour of linear operators acting on Fréchet spaces X has been investigated intensively in the last years. Recommended introductions are the textbooks [3] and [8], and also the recent article [11]. It turns out that the richness of eigenvectors corresponding to unimodular eigenvalues strongly influences the metric dynamical properties of linear operators. In particular, a linear operator on a Hilbert space admits a Gaussian invariant measure of full support if and only if it has spanning unimodular eigenvectors and is ergodic in the Gaussian sense (that is, ergodic with respect to a Gaussian measure of full support) if and only if it has perfectly spanning unimodular eigenvectors. For these and corresponding results we refer in particular to [1, 2, 4, 9] and again to [11].

Due to recent deep results of Menet [13] and Grivaux, Matheron and Menet [11], in the situation of Hilbert spaces X the picture has become quite complete for the case of chaotic operators, that is, for hypercyclic operators having eigenvectors corresponding to roots of unity (that is, periodic vectors) which span a dense subspace of X. Less is known, however, in the case of absence of periodic or almost periodic vectors (cf. [11, §1.3]). In [10, Question 3] (see also [11, Question 7.7]) it is asked whether a hypercyclic operator with a densely spanning set of eigenvectors corresponding to rationally independent eigenvalues is already ergodic.

In this paper, we give examples of topologically mixing operators on Hilbert spaces which have a densely spanning set of eigenvectors with unimodular eigenvalues restricted to an arbitrary prescribed closed, countable subset Z of the unit circle $\mathbb{T}$. In particular, such operators cannot be ergodic in the Gaussian sense. Choosing Z as a rationally independent set, Question 3 from [10] can be answered in the negative, at least in the weak form that ergodicity in the Gaussian sense does not always follow from the existence of a spanning set of eigenvectors corresponding to a rationally independent set of unimodular eigenvalues.

For an open set Ω in the extended plane $\mathbb{C}_\infty$, we denote by $H(\Omega)$ the Fréchet space of functions holomorphic in Ω and vanishing at ∞ endowed with the topology of locally uniform convergence, where, as usual, via stereographic projection we identify $\mathbb{C}_\infty$ and the sphere $\mathbb{S}^2$ endowed with the spherical metric. We consider Bergman spaces on general open sets $\Omega \subset \mathbb{C}_\infty$. For $0 \le p < \infty$, let $A^p(\Omega)$ be the space of all functions f holomorphic in Ω that satisfy

$$\|f\|_p := \|f\|_{\Omega,p} := \left(\int_\Omega |f|^p \, dm_2 \right)^{1/p} < \infty,$$

where m_2 denotes the spherical measure on $\mathbb{C}_\infty$. Then $A^0(\Omega) = H(\Omega)$ and for $p \ge 1$ the spaces $(A^p(\Omega), \|\cdot\|_p)$ are Banach spaces. In the case $p = 2$, the norm is induced by the inner product $(f, g) \mapsto \int_\Omega f\overline{g} \, dm_2$.

In what follows we always consider open sets Ω with $0 \in \Omega$, in which case we define $T = T_{A^p(\Omega)} : A^p(\Omega) \to A^p(\Omega)$ by

$$Tf(z) := (f(z) - f(0))/z \quad (z \ne 0), Tf(0) := f'(0).$$

If $f(z) = \sum_{\nu=0}^\infty a_\nu z^\nu$, then

$$Tf(z) = \sum_{\nu=0}^\infty a_{\nu+1} z^\nu$$

for $|z|$ sufficiently small. We call T the Taylor (backward) shift on $A^p(\Omega)$. Writing $S_n f(z) := \sum_{\nu=0}^n a_\nu z^\nu$ for the nth partial sum of the Taylor expansion $\sum_{\nu=0}^\infty a_\nu z^\nu$ of f about 0, by induction it is easily seen that, for $n \in \mathbb{N}_0$,

$$T^{n+1} f(z) = (f - S_n f(z))/z^{n+1} \quad (z \ne 0) \text{ and } T^{n+1} f(0) = a_{n+1}. \tag{1}$$

In [5] it is shown that for open sets Ω with $0 \in \Omega$ the Taylor shift T is topologically mixing on $H(\Omega)$ if and only if each connected component of $\mathbb{C}_\infty \setminus \Omega$ meets $\mathbb{T}$. Results concerning topological and metric dynamics of the Taylor shift on Bergman spaces are proved in [6] and [16].

We write $M^* := (\mathbb{C}_\infty \setminus M)^{-1}$ for $M \subset \mathbb{C}_\infty$. Then Ω^* is a compact plane set (note that $0 \in \Omega$) and the spectrum of T is contained in Ω^*. In the case $1 \le p < 2$ it is easily seen that $f \in A^p(\Omega)$ is an eigenfunction for T if and only if, for some $\alpha \in \Omega^*$, the function f is a scalar multiple of γ_α, where γ_α is defined by

$$\gamma_\alpha(z) = 1/(1 - \alpha z)$$

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for $z \in \Omega$ (with $0 \cdot \infty := 0$). In this case, α is the corresponding eigenvalue and the spectrum as well as the point spectrum both equal Ω^*. If $p \geq 2$, then the functions γ_α still belong to $A^p(\Omega)$ for all α in the interior of Ω^*, but in general not for α belonging to the boundary of Ω^*. If Ω has small spherical measure near a boundary point $1/\alpha$ of Ω, it may, however, happen that γ_α is again an eigenfunction of T (that is, γ_α belongs to $A^p(\Omega)$). For example, $\gamma_1 \in A^2(\Omega)$ in the case of the crescent-shaped region $\Omega = \mathbb{D} \setminus \{z : |z - 1/2| \leq 1/2\}$ where $\mathbb{D}$ denotes the open unit disc in $\mathbb{C}$. This opens up the possibility of placing eigenvalues at certain points of the boundary of Ω^*. A corresponding construction leads to our main result. We write $\mathcal{E}(T)$ for the set of unimodular eigenvalues of T, which for $T = T_{A^2(\Omega)}$ equals the set of $\lambda \in \mathbb{T}$ such that $\gamma_\lambda \in A^2(\Omega)$.

THEOREM 1.1. *Let $Z \subset \mathbb{T}$ be an infinite closed set. Then there is a open set $\Omega \subset \mathbb{D}$ so that the Taylor shift $T = T_{A^2(\Omega)}$ is topologically mixing, $\mathcal{E}(T) \subset Z$ and $\{\gamma_\lambda : \lambda \in \mathcal{E}(T)\}$ spans a dense subspace of $A^2(\Omega)$.*

Remark 1.2. Since Hilbert spaces are of cotype 2, the main theorem from [4] implies that, in the situation of Theorem 1.1, for countable Z the Taylor shift T is not ergodic in the Gaussian sense. So, as already mentioned above, Question 3 from [10] can be answered in the negative in so far as the existence of a spanning set of eigenvectors corresponding to a rationally independent set of unimodular eigenvalues does not always imply ergodicity in the Gaussian sense. We are left with the open question whether T is ergodic with respect to some measure of full support or (upper) frequently hypercyclic.

If in the situation of Theorem 1.1 the set Z consists of roots of unity, then the Taylor shift is chaotic with unimodular eigenvalues only in Z and not ergodic in the Gaussian sense. So we have an alternative construction for a chaotic operator on a Hilbert space that is not ergodic in the Gaussian sense. The first construction of such an operator on $\ell_2(\mathbb{N})$ given by Menet [13] even leads to an operator which is not (upper) frequently hypercyclic. The approach via Taylor shift is, however, quite different and gives more flexibility in prescribing unimodular eigenvectors.

2. *Proof of Theorem 1.1*

As our main tool for the proof of Theorem 1.1 we seek results on rational approximation in the mean. In what follows we restrict to open sets Ω which are bounded in $\mathbb{C}$ or contain the point ∞.

Remark 2.1. Let Ω be an open and bounded set in $\mathbb{C}$ and suppose that each point on the boundary of Ω^* belongs to the boundary of some component of the interior $(\Omega^*)^\circ$ of Ω^*. Then [12, Theorem 4] implies that the span of $\{\gamma_\alpha : \alpha \in (\Omega^*)^\circ\}$ is dense in $A^2(\Omega)$. If Ω is open in the extended plane with $\infty \in \Omega$, the same also holds (see [16, Remark 2.6]).

As indicated in the introduction, the function γ_α belongs to $A^2(\Omega)$ if α belongs to the boundary of Ω^* and Ω has small spherical measure near $1/\alpha$. For $k \in \mathbb{N}_0$, $z \in \Omega$ and $\alpha \in \mathbb{C}$ we write

$$\gamma_{\alpha,k}(z) := z^k/(1 - \alpha z)^{k+1} = z^k \gamma_\alpha^{k+1}(z).$$

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We show that in the case of sufficiently small measure near $1/\alpha$ all functions $\gamma_{\alpha,k}$ belong to $A^2(\Omega)$ and that under appropriate conditions they span a dense subspace of $A^2(\Omega)$. The approach is strongly influenced by the proof of a result on the completeness of polynomials in $A^2(\Omega)$ (see [14, Theorem 12.1]; cf. also [7, Ch. I, §3]).

Let $\delta > 0$ and let the 'cup' C_δ be defined as the interior of the convex hull of $\{t + is(t) : -\delta \leq t \leq \delta\}$, where

$$s(t) := \exp(-\exp(1/|t|))$$

(with $s(0) = 0$). With that, we say that two components A, B of an open set $U \subset S := \mathbb{R} + i(-\pi/2, \pi/2)$ are directly bridged at $w \in S$, if $\omega \in \mathbb{T}$ and $\delta > 0$ exist with $w + \omega C_\delta \subset A$ and $w - \omega C_\delta \subset B$. If $\varphi : S \to \mathbb{S}^2$ is the standard parametrization of $\mathbb{S}^2 \setminus \{\pm(0, 0, 1)\}$, that is,

$$\varphi(t + is) = (\cos(s)\cos(t),\ \cos(s)\sin(t),\ \sin(s))$$

for $t \in \mathbb{R}$ and $-\pi/2 < s < \pi/2$, we say that two components C, D of an open set $V \subset \mathbb{C}_\infty$ with $0, \infty \notin \partial V$ are directly bridged, if the corresponding inverse images under φ in S are bridged at some w. In this case $\zeta = \varphi(w)$ is said to be a bridge point for (C, D). We say that C, D are bridged if finitely many components $C_0, C_1, \ldots, C_m$ exist with $C_0 = C$, $C_m = D$ and so that C_j, C_{j-1} are directly bridged. If $\mathcal{C}$ denotes the set of components of V, then bridging induces an equivalence relation $\sim$ on $\mathcal{C}$. If a system $\mathcal{D} \subset \mathcal{C}$ is a complete system of representatives for $\sim$, we briefly say that the system is complete for V.

THEOREM 2.2. *Let Ω be an open set in $\mathbb{C}_\infty$ which is bounded in $\mathbb{C}$ or contains ∞, and suppose that each point on the boundary of Ω^* belongs to the boundary of some component of the interior $(\Omega^*)^\circ$ of Ω^*. Moreover, suppose $\mathcal{D}$ to be complete for $(\Omega^*)^\circ$, and for $D \in \mathcal{D}$ let α_D be either in D or a bridge point of (C, D) for some $C \in \mathcal{C}$. Then both $\{\gamma_\alpha : \alpha \in \bigcup_{D\in\mathcal{D}} D\}$ and $\{\gamma_{\alpha_D,k} : D \in \mathcal{D},\ k \in \mathbb{N}_0\}$ have dense span in $A^2(\Omega)$.*

Proof. 1. Let $g \in A^2(\Omega) = A^2(\Omega)'$. Then the Cauchy transform $Vg : (\Omega^*)^\circ \to \mathbb{C}$ of g, defined by

$$(Vg)(\alpha) = \int_\Omega \gamma_\alpha(z)\overline{g}(z)\, dm_2(z)$$

for $\alpha \in (\Omega^*)^\circ$, is holomorphic with

$$(Vg)^{(k)}(\alpha) = k! \int_\Omega \gamma_{\alpha,k}(z)\overline{g}(z)\, dm_2(z)$$

for all $\alpha \in (\Omega^*)^\circ$ and $k \in \mathbb{N}_0$. According to the Hahn–Banach theorem and Remark 2.1, it suffices to show that $Vg|_C = 0$ for all $C \in \mathcal{C}$ under each of the two conditions stated in the theorem.

The crucial point is that Vg extends at bridge points ζ to a function which belongs to a quasi-analytic subclass of $C^\infty(I)$ for some line I with $I \setminus \{\zeta\} \subset (\Omega^*)^\circ$. Let ζ be a bridge point for (C, D). In order to reduce notational complexity we assume that $\zeta = 1$ and that C lies in $\mathbb{D}$ and D outside $\overline{\mathbb{D}}$. Considering the fact that 1 is a bridge point, we may fix $0 < r < 1$ in such a way that the corresponding 'cup' sets $\pm C_r$ satisfy $\varphi^{-1}(\Omega) \cap \pm C_r = \varnothing$. Let

$$I = \varphi(i[-r/2, r/2])$$

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(note that I is a compact interval in $\mathbb{R}$ with 1 in its interior). To show that $Vg|_{I\setminus\{1\}}$ extends at 1 to a C^∞-function on I and that the extension (again denoted by Vg) belongs to a quasi-analytic subclass of $C^\infty(I)$, we estimate the derivatives of Vg on $I \setminus \{1\}$. By the Cauchy–Schwarz inequality we have

$$|(Vg)^{(k)}(x)| \leq k! \int_\Omega |\gamma_{x,k}\overline{g}|\, dm_2 \leq k!\ \|g\|_2 \left(\int_\Omega |\gamma_{x,k}|^2 dm_2 \right)^{1/2} \tag{2}$$

for $k \in \mathbb{N}_0$ and $1 \neq x \in I$. So it suffices to estimate the latter integrals. We define

$$W_A(k, x) := \int_A |\gamma_{x,k}|^2 dm_2 = \int_A \frac{|z|^{2k} dm_2(z)}{|1 - xz|^{2k+2}}$$

for $k \in \mathbb{N}_0$, $x \in I$ and measurable $A \subset \Omega$. With $Q := [-r, r] + i[-r, r]$, we have

$$\sup_{x \in I} W_{\Omega\setminus\varphi(Q)}(k, x) = \mathcal{O}(q_1^k)$$

for some positive q_1. To estimate $W_{\varphi(Q)\cap\Omega}$, we observe that the shape of Ω in $\varphi(Q)$ allows that with some constant $c > 0$ we have $|1 - xz| \geq c|1 - z|$ for all $x \in I$ and all $z \in \Omega \cap \varphi(Q)$. Thus, for $x \in I$ we obtain (by substituting $u = e^{1/t}$ in the last step)

$$\begin{aligned} W_{\Omega\cap\varphi(Q)}(k, x) &\leq q_2^k W_{\Omega\cap\varphi(Q)}(k, 1) \leq q_2^k \int_{-r}^{r} \int_{-s(t)}^{s(t)} \frac{\cos s}{|t + is|^{2k+2}}\, ds\, dt \\ &\leq 4q_2^k \int_0^r \frac{s(t)}{t^{2k+2}}\, dt = 4q_2^k \int_{e^{1/r}}^\infty e^{-u} u^{-1} \log^{2k}(u)\, du, \end{aligned}$$

with a positive constant q_2. For k sufficiently large and $u \geq k^2$ we have $\log^{2k}(u) \leq e^{u/2}$. Hence, by splitting the integral at k^2, one can see that

$$\int_{e^{1/r}}^\infty e^{-u} u^{-1} \log^{2k}(u)\, du = \mathcal{O}(k^2 \log^{2k}(k^2)) = \mathcal{O}(5^k \log^{2k}(k)).$$

Putting everything together, we find that

$$\sup_{x \in I} W_\Omega(k, x) = \mathcal{O}(q_3^k \log^{2k}(k)) \tag{3}$$

for some positive constant q_3. Since $|\gamma_{x,k}(z)| \leq |\gamma_{1,k}(z)|$ for $z \in \varphi(Q)$ and $x \in I$, and since there is a constant c_k with $|\gamma_{x,k}(z)| \leq c_k$ for $z \notin \varphi(Q)$ and $x \in I$, the m_2-integrability of g on Ω and of $|\gamma_{1,k}\overline{g}|$ on $\varphi(Q) \cap \Omega$ implies, by differentiation of parameter integrals, that Vg extends to a function in $C^\infty(I)$ (which we again denote by Vg). Moreover, combining (2) and (3), we obtain

$$\sup_{x \in I} |(Vg)^{(k)}(x)| = \mathcal{O}(k!\ \sqrt{q_3}^{\,k} \log^k(k)).$$

Hence, the Denjoy–Carleman theorem shows that Vg belongs to a quasi-analytic subclass of $C^\infty(I)$.

2. Suppose that $g \perp \gamma_\alpha$ for all $\alpha \in \bigcup_{D\in\mathcal{D}} D$, that is, $(Vg)|_D = 0$ for all $D \in \mathcal{D}$. If $\mathcal{D} = \mathcal{C}$ then $Vg = 0$. If $C \in \mathcal{C} \setminus \mathcal{D}$ then C, D are bridged for some $D \in \mathcal{D}$. We can assume that C, D are directly bridged with bridge point $\zeta = 1$ as in part 1 above. From the

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assumption $(Vg)|_D = 0$ we have $(Vg)^{(k)}(1) = 0$ for all k, and then also $(Vg)|_C = 0$ by quasi-analyticity an the identity theorem for holomorphic functions.

3. Suppose that $g \perp \{\gamma_{\alpha_D,k} : D \in \mathcal{D},\ k \in \mathbb{N}_0\}$, that is, $\int_\Omega \gamma_{\alpha_D,k}\overline{g}\, dm_2 = 0$ for all $D \in \mathcal{D}$ and $k \in \mathbb{N}_0$, and let $C \in \mathcal{C}$. If $C \in \mathcal{D}$ then $\alpha_C \in C$ or α_C is a bridge point of C, E for some $E \in \mathcal{C}$. In both cases, according to part 1 of the proof (with $\alpha_C = 1$ without loss of generality) we have $(Vg)|_C = 0$. If $C \notin \mathcal{D}$, then there is $D \in \mathcal{D}$ so that C, D are bridged. Again, we can assume that C, D are directly bridged with bridge point $\zeta = 1$ as above. If $\alpha_D \in D$, then $(Vg)|_D = 0$. Hence $(Vg)^{(k)}(1) = 0$ for all k and then $(Vg)|_C = 0$. If α_D is a bridge point of (D, E) for some $E \in \mathcal{C}$, then by the same argument as above we can now conclude $(Vg)|_D = 0$. As in the first case, $(Vg)|_C = 0$. □

Remark 2.3. If, for some component $D \in \mathcal{C}$, the single set system $\{D\}$ is complete for $(\Omega^*)^\circ$ and if α is a point in D or a bridge point of (C, D) for some component C, then the span of $\{\gamma_{\alpha,k} : k \in \mathbb{N}_0\}$ is dense in $A^2(\Omega)$. In particular, if $\alpha = 0$ we conclude that the polynomials form a dense set in $A^2(\Omega)$ (cf. [14, Theorem 12.1], where actually a weaker condition on the sharpness of Ω near $1/\alpha$ is proved to be sufficient).

From the Godefroy–Shapiro criterion (see, for example, [8, Theorem 3.1]) and Theorem 2.2 we obtain the following corollary.

COROLLARY 2.4. *Let Ω be an open set in $\mathbb{C}_\infty$ which is bounded in $\mathbb{C}$ or contains ∞, and suppose that each point in the boundary of Ω^* belongs to the boundary of some component of $(\Omega^*)^\circ$. If complete systems $\mathcal{D}$ and $\mathcal{D}'$ for $(\Omega^*)^\circ$ exist with $D \subset \mathbb{D}$ for all $D \in \mathcal{D}$ and $D' \subset \mathbb{C}_\infty \setminus \overline{\mathbb{D}}$ for all $D' \in \mathcal{D}'$, then $T_{A^2(\Omega)}$ is topologically mixing.*

Example 2.5. Let $\Omega = \mathbb{C}_\infty \setminus (\overline{U} \cup \overline{G}) = ((1/\overline{U}) \cup (1/\overline{G}))^*$, where $G, 1/U \subset \mathbb{D}$ are domains with $0 \notin \overline{G}$ and so that U, G (or, equivalently, $1/G, 1/U$) are bridged. In this case, $\mathcal{D} = \{1/U\}$ and $\mathcal{D}' = \{1/G\}$ are complete systems for $(\Omega^*)^\circ$. By Corollary 2.4, the Taylor shift $T_{A^2(\Omega)}$ is mixing. If $0 \in 1/U$ (for example, for $1/U = \mathbb{D}$, in which case $\Omega = \mathbb{D} \setminus \overline{G}$), the polynomials are dense in $A^2(\Omega)$ and the Taylor shift on $A^2(\mathbb{D})$ is a quasi-factor of the Taylor shift on $A^2(\Omega)$.

THEOREM 2.6. *For each infinite closed set $Z \subset \mathbb{T}$, a domain $G \subset \mathbb{D}$ bridged to $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}$ exists with $\overline{G} \cap \mathbb{T} \subset Z$ and so that the span of $\{\gamma_\zeta : \zeta \in \overline{G} \cap \mathbb{T}\}$ is dense in $A^2(\mathbb{D} \setminus \overline{G})$.*

Proof. Let

$$U_r := \{z \in \mathbb{C} : |z - 1| < r\} \cap \mathbb{D}.$$

In our construction we will cut out sets from $\mathbb{D}$, essentially of the form $\zeta\varphi(C_\delta)$, that are 'flat' at points $\zeta \in \mathbb{T}$ in such a way that these ζ become bridge points to $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}$. We have to control the integrals of γ_ζ over the remaining areas next to such a bridge point. To this end, throughout the proof we fix a $\delta > 0$ and a corresponding $r_0 > 0$ that is sufficiently small such that for all $\zeta \in \mathbb{T} \cap \overline{U_{r_0}}$ the set $U_{r_0} \setminus \zeta\varphi(C_\delta)$ consists of the (one or two) component(s) bounded only by $\{\varphi(t + is(t)) : t \in [-\delta, \delta]\}$ and $\overline{U_{r_0}} \cap \mathbb{T}$, that is, for every z in this set we find $t' \in [-\delta, \delta] \setminus \{0\}$ and $\zeta' \in \overline{U_{r_0}} \cap \mathbb{T}$ such that $\varphi(t' + is(t'))$ and ζ' define a line through the origin and the line segment between those points contains z

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and does not intersect with $\varphi(C_\delta)$. The definition of C_δ is extended to $C_{\delta,\rho}$, with $\rho > 0$, defined as the interior of

$$\mathrm{conv}\{t + i\, ds(t) : -\delta \le t \le \delta,\, \rho \le d \le 1\},$$

with conv denoting the convex hull. Together, for $\zeta \in \mathbb{T} \cap \overline{U_{r_0}}$, this allows us to decrease the area of $U_r \setminus \zeta\varphi(C_{\delta,\rho})$ arbitrarily close to zero by decreasing ρ.

We may suppose that Z has an accumulation at 1. In the initial step of our construction we set $Z_0 := \varnothing$ and $G_0 = \varphi(C_{\delta,1})$. This implies $\gamma_{1,k} \in A^2(\mathbb{D} \setminus \overline{G_0})$ for all k. For a subsequent $n \in \mathbb{N}$, the iterative procedure goes as follows. We choose $0 < r_n < r_{n-1}$ such that

$$\int_{U_{r_n} \setminus \overline{G_{n-1}}} |\gamma_{1,j}|^2 \, dm_2 < 1/n \tag{4}$$

for $j = 1, \ldots, n$. By a variant of Runge's theorem on approximation by rational functions with first-order poles, as can be found in [15], we find a finite set

$$Z_n \subset \overline{U_{r_n/2}} \cap \left(Z \setminus \left(\bigcup_{j=0}^{n-1} Z_j \cup \{1\} \right) \right)$$

and an $R_j = R_{n,j} \in \mathrm{span}\{\gamma_\zeta : \zeta \in Z_n\}$ such that

$$\max_{z \in \overline{\mathbb{D}} \setminus U_{r_n}} |\gamma_{1,j}(z) - R_j(z)| < 1/n \tag{5}$$

for $j = 1, \ldots, n$. Then we set

$$G_n := \bigcup_{\zeta \in Z_n} \zeta\varphi(C_{\delta,\rho_n})$$

where $\rho_n > 0$ is chosen in a way that the area of $U_{r_n} \setminus \overline{G_n}$ is sufficiently small to yield

$$\int_{U_{r_n} \setminus \overline{G_n}} |R_j|^2 \, dm_2 < 1/n. \tag{6}$$

This implies $\gamma_\zeta \in A^2(\mathbb{D} \setminus \overline{G_n})$ for all $\zeta \in Z_n$, and inequalities (4)–(6) give

$$\int_{\mathbb{D} \setminus \overline{G_n}} |R_j(z) - \gamma_{1,j}(z)|^2 \, dm_2(z) < 3/n$$

for $j = 1, \ldots, n$. For $G := \bigcup_{n \in \mathbb{N}} G_n$ we finally have $\gamma_\zeta \in A^2(\mathbb{D} \setminus \overline{G})$ for all $\zeta \in \mathbb{T} \cap \overline{G}$, and for every non-negative integer k the function $\gamma_{1,k}$ belongs to the closure of the span of $\{\gamma_\zeta : \zeta \in \overline{G} \cap \mathbb{T}\}$. According to Example 2.5 with $1/U = \mathbb{D}$, the latter implies the denseness of $\mathrm{span}\{\gamma_\zeta : \zeta \in \overline{G} \cap \mathbb{T}\}$ in $A^2(\mathbb{D} \setminus \overline{G})$. □

Proof of Theorem 1.1. Let $\Omega = \mathbb{D} \setminus \overline{G}$ where G is as in Theorem 2.6. Then $\overline{G} \cap \mathbb{T} \subset \mathcal{E}(T)$. According to Example 2.5 and Theorem 2.6, T is topologically mixing. Since each point in $\mathbb{T} \setminus \overline{G}$ is an interior point of $\overline{\mathbb{D}}$, no point in $\mathbb{T} \setminus \overline{G}$ belongs to the point spectrum, that is, $\mathcal{E}(T) \subset Z$. □

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Remark 2.7.

(1) By deleting sufficiently small parts from G it is possible to modify G to an open set W in such a way that $\Omega = \mathbb{D} \setminus \overline{W}$ is connected and the Taylor shift $T_{A^2(\Omega)}$ satisfies the same conditions as in Theorem 1.1.

(2) The statement (and proof) of Theorem 2.6 can be modified in such a way that $\Omega^* \cap \mathbb{T} \subset Z$, that is, the spectrum intersects $\mathbb{T}$ only in Z (cf. [10, Question 3]).

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