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JOSEPH IDEALS AND LISSE MINIMAL W-ALGEBRAS

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Abstract We consider a lifting of Joseph ideals for the minimal nilpotent orbit closure to the setting of affine Kac–Moody algebras and find new examples of affine vertex algebras whose associated varieties are minimal nilpotent orbit closures. As an application we obtain a new family of lisse (C_2-cofinite) W-algebras that are not coming from admissible representations of affine Kac–Moody algebras.

Keywords: Joseph ideal; associated variety; Deligne exceptional series; affine Kac–Moody algebra; affine W-algebra

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1. Introduction

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$, $(\ |\ )$ be a normalized invariant inner product, i.e., $\frac{1}{2h^\vee}\times$ Killing form.

Let $\widehat{\mathfrak{g}}=\mathfrak{g}[t,t^{-1}]\oplus\mathbb{C}K\oplus\mathbb{C}D$ be the Kac–Moody Lie algebra associated with $\mathfrak{g}$ and $(\ |\ )$, with the commutation relations

$$[x(m),y(n)]=[x,y](m+n)+m(x\mid y)\delta_{m+n,0}K,$$
$$[D,x(m)]=mx(m),\quad [K,\widehat{\mathfrak{g}}]=0,$$

where $x(m)=x\otimes t^m$. For $k\in\mathbb{C}$, set

$$V^k(\mathfrak{g})=U(\widehat{\mathfrak{g}})\otimes_{U(\mathfrak{g}[t]\oplus\mathbb{C}K\oplus\mathbb{C}D)}\mathbb{C}_k,$$

where $\mathbb{C}_k$ is the one-dimensional representation of $\mathfrak{g}[t]\oplus\mathbb{C}K\oplus\mathbb{C}D$ on which $\mathfrak{g}[t]\oplus D$ acts trivially and K acts as multiplication by k. The space $V^k(\mathfrak{g})$ is naturally a vertex algebra, and it is called the *universal affine vertex algebra associated with* $\mathfrak{g}$ *at level* k. By the PBW theorem, $V^k(\mathfrak{g})\cong U(\mathfrak{g}[t^{-1}]t^{-1})$ as $\mathbb{C}$-vector spaces.

Let $V_k(\mathfrak{g})$ be the unique simple graded quotient of $V^k(\mathfrak{g})$. As a $\widehat{\mathfrak{g}}$-module, $V_k(\mathfrak{g})$ is isomorphic to the irreducible highest weight representation of $\widehat{\mathfrak{g}}$ with highest weight $k\Lambda_0$, where Λ_0 is the dual element of K.

Let X_V be the *associated variety* [5] of a vertex algebra V, which is the maximum spectrum of *Zhu's C_2-algebra*,

$$R_V := V/C_2(V).$$

In the case V is a quotient of $V^k(\mathfrak{g})$, $V/C_2(V) = V/\mathfrak{g}[t^{-1}]t^{-2}V$ and we have a surjective Poisson algebra homomorphism

$$\mathbb{C}[\mathfrak{g}^*] = S(\mathfrak{g}) \twoheadrightarrow V/\mathfrak{g}[t^{-1}]t^{-2}V, \quad x \mapsto \overline{x(-1)} + \mathfrak{g}[t^{-1}]t^{-2}V,$$

where $\overline{x(-1)}$ denotes the image of $x(-1)$ in the quotient V. Then X_V is just the zero locus of the kernel of the above map in $\mathfrak{g}^*$. It is G-invariant and conic, where G is the adjoint group of $\mathfrak{g}$. Note that on the contrary to the associated variety of a primitive ideal of $U(\mathfrak{g})$, the variety $X_{V_k(\mathfrak{g})}$ is not necessarily contained in the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$. In fact, $X_{V_k(\mathfrak{g})} = \mathfrak{g}^*$ for a generic k since $V_k(\mathfrak{g}) = V^k(\mathfrak{g})$ in this case.

A conjecture of Feigin and Frenkel, proved in [7], states that $X_{V_k(\mathfrak{g})} \subset \mathcal{N}$ if $V_k(\mathfrak{g})$ is admissible [31]. In fact, it is also believed that the converse is true, that is, $X_{V_k(\mathfrak{g})} \subset \mathcal{N}$ only if $V_k(\mathfrak{g})$ is admissible, so that the condition $X_{V_k(\mathfrak{g})} \subset \mathcal{N}$ gives a geometric characterization of admissible affine vertex algebras. One of the aims of this paper is to provide a counterexample to this fact, that is, there exist non-admissible affine vertex algebras $V_k(\mathfrak{g})$ such that $X_{V_k(\mathfrak{g})} \subset \mathcal{N}$.

Let $(e_\theta, h_\theta, f_\theta)$ be an $\mathfrak{sl}_2$-triple associated with the highest positive root θ of $\mathfrak{g}$. Let $\mathbb{O}_{min} = G \cdot f_\theta$ be the unique minimal non-trivial nilpotent orbit of $\mathfrak{g}$ which is of dimension $2h^\vee - 2$ [41], where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$.

Consider the *Deligne exceptional series*

$$A_1 \subset A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8$$

discussed in [17, 18].

Theorem 1.1. (1) *Assume that $\mathfrak{g}$ belongs to the Deligne exceptional series and that*

$$k = -\frac{h^\vee}{6} - 1.$$

Then $X_{V_k(\mathfrak{g})} = \overline{\mathbb{O}_{min}}$.

(2) *Assume that $\mathfrak{g}$ is of type D_4, E_6, E_7, E_8 and that k is an integer such that*

$$-\frac{h^\vee}{6} - 1 \leqslant k \leqslant -1.$$

Then $X_{V_k(\mathfrak{g})} = \overline{\mathbb{O}_{min}}$.

(3) *Assume that $\mathfrak{g}$ is of type D_l, $l \geqslant 5$. Then $X_{V_k(\mathfrak{g})} = \overline{\mathbb{O}_{min}}$ for $k = -2, -1$.*

Note that for $\mathfrak{g}$ of type A_1, A_2, G_2, F_4, the rational number $-h^\vee/6 - 1$ is admissible. However, for types D_4, E_6, E_7, E_8, the number $-h^\vee/6 - 1$ is a negative integer which is certainly non-admissible [33, Proposition 1.2].

A consequence of the fact $X_{V_k(\mathfrak{g})} \subset \mathcal{N}$ is that $V_k(\mathfrak{g})$ has only finitely many simple modules in the category $\mathcal{O}$ (cf. Corollary 5.3), as in case $V_k(\mathfrak{g})$ is admissible [1–3, 9, 11, 36, 37]. If $\mathfrak{g}$ belongs to the Deligne exceptional series outside the type A and $k = -h^\vee/6 - 1$, it is possible to derive the classification of simple $V_k(\mathfrak{g})$-modules that belong to $\mathcal{O}$ from Joseph's result [28] in the following manner.

If $\mathfrak{g}$ is not of type A, it is known [24, 27] that there exists a unique completely prime ideal $\mathcal{J}_0$ in $U(\mathfrak{g})$, called the *Joseph ideal*, whose associated variety is $\overline{\mathbb{O}_{min}}$, that is, $\overline{\mathbb{O}_{min}}$ is the zero locus in $\mathfrak{g}^*$ of $\operatorname{gr} \mathcal{J}_0$. As a by-product, we obtain a lifting to the Joseph ideal in the following sense. For a $\mathbb{Z}_{\geqslant 0}$-graded vertex algebra V, let $A(V)$ be its Zhu's algebra [42]. Such a vertex algebra V is called a *chiralization* of an algebra A if $A(V) \cong A$. We claim that if $\mathfrak{g}$ belongs to the Deligne exceptional series outside the type A and if $k = -h^\vee/6 - 1$, then $V_k(\mathfrak{g})$ is a chiralization of $\mathbb{C} \oplus U(\mathfrak{g})/\mathcal{J}_0$. Namely,

$$A(V_k(\mathfrak{g})) \cong U(\mathfrak{g})/\mathcal{J}_{\mathcal{W}} \cong \mathbb{C} \times U(\mathfrak{g})/\mathcal{J}_0,$$

for some ideal $\mathcal{J}_{\mathcal{W}}$ (cf. Proposition 2.2 and Theorem 3.1). Hence the classification of simple highest weight $U(\mathfrak{g})/\mathcal{J}_0$-modules obtained in [28] gives the classification of simple highest weight $V_k(\mathfrak{g})$-modules thanks to Zhu's theorem [42], which for types G_2, D_4, F_4 reproves the earlier results obtained in [11, 37, 38].

Another consequence of the fact $X_{V_k(\mathfrak{g})} \subset \mathcal{N}$ is that the D-module on the moduli stack of G-bundles on a curve obtained from $V_k(\mathfrak{g})$ by the Harish-Chandra localization [14, 22] has its micro-local support inside the global nilpotent cone. It would be very interesting to consider the associated modular functor (cf. [21]), or the corresponding conformal field theory (cf. [15, 16]). We hope to come back to this point in our future work.

In physics literature, the affine vertex algebras in Theorem 1.1(1) have been studied in the work [13] of Beem *et al.* in connection with *four-dimensional* superconformal field theory. The associated varieties of these vertex algebras seem to describe the Higgs branch of the corresponding four-dimensional theory. We also hope to come back to this point in our future work.

Theorem 1.1, or its proof, has the following important application:

Let $\mathcal{W}^k(\mathfrak{g}, f_\theta)$ be the *W-algebra associated with* $(\mathfrak{g}, f_\theta)$ at level k [29], which is a conformal vertex algebra with central charge

$$c(k) = \frac{k \dim \mathfrak{g}}{k + h^\vee} - 6k + h^\vee - 4$$

provided that $k \neq -h^\vee$. Note that if $\mathfrak{g}$ belongs to the Deligne exceptional series,

$$c(k) = -\frac{6(k + h^\vee/6 + 1)((h^\vee/6 + 1)k - (h^\vee - 4)h^\vee/6)}{(k + h^\vee)(h^\vee/6 + 1)},$$

so that $c(k) = 0$ for $k = -h^\vee/6 - 1$.

Denote by $\mathcal{W}_k(\mathfrak{g}, f_\theta)$ the unique simple quotient of $\mathcal{W}^k(\mathfrak{g}, f_\theta)$. Since $X_{\mathcal{W}^k(\mathfrak{g}, f_\theta)}$ is naturally isomorphic to the Slodowy slice $\mathcal{S}_{min}$ at f_θ [7, 19], with

$$\mathcal{S}_{min} := f_\theta + \mathfrak{g}^{e_\theta}, \quad \mathfrak{g}^{e_\theta} = \{x \in \mathfrak{g} \mid [x, e_\theta] = 0\},$$

the variety $X_{\mathcal{W}_k(\mathfrak{g}, f_\theta)}$ is a $\mathbb{C}^*$-invariant, Poisson subvariety of $\mathcal{S}_{min}$.

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It is known [19] that the (Ramond-twisted) Zhu's algebra of $\mathcal{W}^k(\mathfrak{g}, f_\theta)$ is naturally isomorphic to the finite W-algebra $U(\mathfrak{g}, f_\theta)$ associated with $(\mathfrak{g}, f_\theta)$ introduced by Premet [39].

Premet [40] has shown that the Joseph ideal is closely connected with one-dimensional representations of $U(\mathfrak{g}, f_\theta)$. The chiralization of $U(\mathfrak{g})/\mathcal{J}_W$ explained above is closely related with one-dimensional representations of $\mathcal{W}^k(\mathfrak{g}, f_\theta)$ as well. The significant difference in the affine setting is that $\mathcal{W}^k(\mathfrak{g}, f_\theta)$ does not necessarily admit one-dimensional representations. In fact, $\mathcal{W}^k(\mathfrak{g}, f_\theta)$, $\mathfrak{g} \neq \mathfrak{sl}_2$, admits one-dimensional representations if and only if $\mathcal{W}_k(\mathfrak{g}, f_\theta) = \mathbb{C}$, and this happens if and only if $\mathfrak{g}$ belongs to the Deligne exceptional series and $k = -h^\vee/6 - 1$, or $\mathfrak{g}$ is of type C_l and $k = -1/2$ (cf. Theorem 7.2).

Note that the trivial vertex algebra $\mathbb{C}$ is certainly a *lisse* vertex algebra. Here, recall that a vertex algebra V is called lisse, or *C_2-cofinite*, if $\dim X_V = 0$. Lisse vertex algebras may be regarded as an analogue of finite-dimensional algebras. One of the remarkable properties of a lisse vertex algebra V is the modular invariance of characters of modules [35, 42]. Further, if it is non-trivial and also *rational*, it is known [26] that under some mild assumptions the category of V-modules forms a modular tensor category, which for instance yields an invariant of 3-manifolds, see [12].

In [7], in order to approach the Kac–Wakimoto conjecture [33] on the rationality of *exceptional W-algebras*, the first author showed that each admissible affine vertex algebra produces exactly one lisse simple W-algebra. More precisely, the associated variety of an admissible affine vertex algebra $V_k(\mathfrak{g})$ is isomorphic to $\overline{\mathbb{O}}$ for some nilpotent orbit $\mathbb{O}$ of $\mathfrak{g}$, and if we take the nilpotent element f from this orbit $\mathbb{O}$, then $\mathcal{W}_k(\mathfrak{g}, f)$ is lisse. Until very recently it has been widely believed that these W-algebras are all the lisse W-algebras, cf. [33]. However, it turned out that there are a lot more.

Theorem 1.2. (1) *Let $\mathfrak{g}$ be of type D_4, E_6, E_7, E_8. For any integer k that is equal to or greater than $-h^\vee/6 - 1$, the simple W-algebra $\mathcal{W}_k(\mathfrak{g}, f_\theta)$ is lisse.*

(2) *Let $\mathfrak{g}$ be of type D_l with $l \geqslant 5$. For any integer k that is equal to or greater than -2, the simple W-algebra $\mathcal{W}_k(\mathfrak{g}, f_\theta)$ is lisse.*

In the case that $k = -h^\vee/6$, the first statement of Theorem 1.2 is a recent result of Kawasetsu [34]. Kawasetsu actually proved that $\mathcal{W}_{-h^\vee/6}(\mathfrak{g}, f_\theta)$ is rational and C_2-cofinite if $\mathfrak{g}$ belongs to the Deligne exceptional series, providing a first (surprising) example of rational and C_2-cofinite W-algebras that are not coming from admissible representations of $\widehat{\mathfrak{g}}$. Our present work is motivated by his result. It would be very interesting to know whether the lisse W-algebras appearing in Theorem 1.2 are rational or not. We hope to come back to this point in future work.

2. Minimal nilpotent orbit closures and Joseph ideals

Let J_0 be the prime ideal of $S(\mathfrak{g})$ corresponding to the minimal nilpotent orbit closure $\overline{\mathbb{O}_{min}}$ in $\mathfrak{g}^*$.

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Suppose that $\mathfrak{g}$ is not of type A. According to Kostant, J_0 is generated by a $\mathfrak{g}$-submodule $L_{\mathfrak{g}}(0) \oplus W$ in $S^2(\mathfrak{g})$ such that

$$S^2(\mathfrak{g}) = L_{\mathfrak{g}}(2\theta) \oplus L_{\mathfrak{g}}(0) \oplus W,$$

where $L_{\mathfrak{g}}(\lambda)$ is the irreducible representation of $\mathfrak{g}$ with highest weight λ and θ is the highest root of $\mathfrak{g}$.

Note that the above decomposition of $S^2(\mathfrak{g})$ still holds in type A, [25, Ch. IV, Proposition 2]. Also, note that $L_{\mathfrak{g}}(0) = \mathbb{C}\Omega$ where Ω is the Casimir element in $S(\mathfrak{g})$.

Lemma 2.1. *Suppose that $\mathfrak{g}$ is not of type A. The ideal J_W in $S(\mathfrak{g})$ generated by W contains Ω^2, and hence, $\sqrt{J_W} = J_0$.*

Proof. By the proof of [24, Theorem 3.1] J_W contains $\mathfrak{g} \cdot \Omega$, and the assertion follows. □

The structure of W was determined by Garfinkle [25]. Set

$$\mathfrak{g}(j) = \{x \in \mathfrak{g} \mid [h_\theta, x] = 2jx\}.$$

Then

$$\mathfrak{g} = \mathfrak{g}(-1) \oplus \mathfrak{g}(-1/2) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1/2) \oplus \mathfrak{g}(1),$$
$$\mathfrak{g}(-1) = \mathbb{C}f_\theta, \quad \mathfrak{g}(1) = \mathbb{C}e_\theta, \quad \mathfrak{g}(0) = \mathbb{C}h_\theta \oplus \mathfrak{g}^\natural, \quad \mathfrak{g}^\natural = \{x \in \mathfrak{g}(0) \mid (h_\theta|x) = 0\}.$$

The subalgebra $\mathfrak{g}^\natural$ is a reductive subalgebra of $\mathfrak{g}$ whose simple roots are the simple roots of $\mathfrak{g}$ perpendicular to θ. Write

$$[\mathfrak{g}^\natural, \mathfrak{g}^\natural] = \bigoplus_{i \geqslant 1} \mathfrak{g}_i$$

as a direct sum of simple summands, and let θ_i be the highest root of $\mathfrak{g}_i$.

If $\mathfrak{g}$ is neither of type A_l nor C_l,

$$W = \bigoplus_{i \geqslant 1} L_{\mathfrak{g}}(\theta + \theta_i).$$

If $\mathfrak{g}$ is of type C_l, then $\mathfrak{g}^\natural$ is simple of type C_{l-1}, so that there is a unique θ_1, and we have

$$W = L_{\mathfrak{g}}(\theta + \theta_1) \oplus L_{\mathfrak{g}}\left(\tfrac{1}{2}(\theta + \theta_1)\right).$$

If $\mathfrak{g}$ is not of type A, it is known [24, 27] that there exists a unique completely prime ideal $\mathcal{J}_0$ in $U(\mathfrak{g})$, called the *Joseph ideal*, whose associated variety is $\overline{\mathbb{O}_{min}}$. It is known that $\mathcal{J}_0$ is maximal and *primitive*. By [24, 25], $\mathcal{J}_0$ is generated by W and $\Omega - c_0$, where W is identified with a $\mathfrak{g}$-submodule of $U(\mathfrak{g})$ by the $\mathfrak{g}$-module isomorphism $S(\mathfrak{g}) \cong U(\mathfrak{g})$ and c_0 is the eigenvalue of Ω for the infinitesimal character that Joseph obtained in [27, Table p.15]. We have

$$\operatorname{gr} \mathcal{J}_0 = J_0 = \sqrt{J_W}$$

and this shows that $\mathcal{J}_0$ is indeed completely prime.

Let $\mathcal{J}_W$ be the two-sided ideal of $U(\mathfrak{g})$ generated by W.

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Proposition 2.2. *We have an algebra isomorphism*

$$U(\mathfrak{g})/\mathcal{J}_W \cong \mathbb{C} \times U(\mathfrak{g})/\mathcal{J}_0.$$

Proof. By the proof of [24, Theorem 3.1], $\mathcal{J}_W$ contains $(\Omega - c_0)\mathfrak{g}$. Hence it contains $(\Omega - c_0)\Omega$. Since $c_0 \neq 0$, we have an isomorphism of algebras

$$U(\mathfrak{g})/\mathcal{J}_W \xrightarrow{\sim} U(\mathfrak{g})/\langle \mathcal{J}_W, \Omega \rangle \times U(\mathfrak{g})/\langle \mathcal{J}_W, \Omega - c_0 \rangle.$$

As we have explained above, $\langle \mathcal{J}_W, \Omega - c_0 \rangle = \mathcal{J}_0$. Also, since $\mathcal{J}_W$ contains $(\Omega - c_0)\mathfrak{g}$, $\langle \mathcal{J}_W, \Omega \rangle$ contains $\mathfrak{g}$. Therefore $U(\mathfrak{g})/\langle \mathcal{J}_W, \Omega \rangle = \mathbb{C}$ as required. □

3. A lifting of Joseph ideals

For a $\mathbb{Z}_{\geqslant 0}$-graded vertex algebra $V = \bigoplus_d V_d$, let $A(V)$ be Zhu's algebra of V:

$$A(V) = V/V \circ V,$$

where $V \circ V$ is the $\mathbb{C}$-span of the vectors

$$a \circ b := \sum_{i \geqslant 0} \binom{\Delta}{i} a_{(i-2)} b$$

for $a \in V_\Delta$, $\Delta \in \mathbb{Z}_{\geqslant 0}$, $b \in V$, and $V \to (\mathrm{End}\, V)[[z, z^{-1}]]$, $a \mapsto \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$, denotes the state-field correspondence. The space $A(V)$ is a unital associative algebra with respect to the multiplication defined by

$$a * b := \sum_{i \geqslant 0} \binom{\Delta}{i} a_{(i-1)} b$$

for $a \in V_\Delta$, $\Delta \in \mathbb{Z}_{\geqslant 0}$, $b \in V$. More generally, for a V-module M, a bimodule $A(M)$ over $A(V)$ is defined similarly [23].

Zhu's algebra $A(V)$ naturally acts on the top degree component M_{top} of a $\mathbb{Z}_{\geqslant 0}$-graded V-module M, and $M \mapsto M_{top}$ gives [42] a one-to-one correspondence between simple graded V-modules and simple $A(V)$-modules.

The vertex algebra V is called a *chiralization* of an algebra A if $A(V) \cong A$.

For instance, consider the universal affine vertex algebra $V^k(\mathfrak{g})$. A $V^k(\mathfrak{g})$-module is the same as a smooth $\widehat{\mathfrak{g}}'$-module of level k, where $\widehat{\mathfrak{g}}' = [\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}] = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$. Zhu's algebra $A(V^k(\mathfrak{g}))$ is naturally isomorphic to $U(\mathfrak{g})$ ([23], see also [9, Lemma 2.3]), and hence, $V^k(\mathfrak{g})$ is a chiralization of $U(\mathfrak{g})$. The top degree component of the irreducible highest weight representation $L(\lambda)$ of $\widehat{\mathfrak{g}}$ with highest weight λ is $L_{\mathfrak{g}}(\bar{\lambda})$, where $\bar{\lambda}$ is the restriction of λ to the Cartan subalgebra of $\mathfrak{g}$.

Let $\widehat{\mathcal{J}_k}$ be the unique maximal ideal of $V^k(\mathfrak{g})$, so that

$$V_k(\mathfrak{g}) = V^k(\mathfrak{g})/\widehat{\mathcal{J}_k}.$$

We have the exact sequence $A(\widehat{\mathcal{J}_k}) \to U(\mathfrak{g}) \to A(V_k(\mathfrak{g})) \to 0$ since the functor $A(?)$ is right exact and thus $A(V_k(\mathfrak{g}))$ is the quotient of $U(\mathfrak{g})$ by the image $\mathcal{I}_k$ of $A(\widehat{\mathcal{J}_k})$ in $U(\mathfrak{g})$:

$$A(V_k(\mathfrak{g})) = U(\mathfrak{g})/\mathcal{I}_k.$$

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	$-\frac{h^\vee}{6}-1$	λ_0	W_0
G_2	$-\frac{5}{3}$	$\varpi_1+\frac{1}{3}\varpi_2$	$\{1, s_2\}$
D_4	-2	$\varpi_1+\varpi_3+\varpi_4$	$\{1, s_1, s_3, s_4\}$
F_4	$-\frac{5}{2}$	$\frac{1}{2}\varpi_1+\frac{1}{2}\varpi_2+\varpi_3+\varpi_4$	$\{1, s_1, s_2\}$
E_6	-3	$\varpi_1+\varpi_2+\varpi_3+\varpi_5+\varpi_6$	$\{1, s_2, s_3, s_1s_3, s_5, s_6s_5\}$
E_7	-4	$\varpi_1+\varpi_2+\varpi_3+\varpi_5+\varpi_6+\varpi_7$	$\{1, s_2, s_3, s_1s_3, s_5, s_6s_5, s_7s_6s_5\}$
E_8	-6	$\varpi_1+\varpi_2+\varpi_3+\varpi_5+\varpi_6+\varpi_7+\varpi_8$	$\{1, s_2, s_3, s_1s_3, s_5, s_6s_5, s_7s_6s_5, s_8s_7s_6s_5\}$

TABLE 1. $-h^\vee/6-1$, λ_0 and W_0.

One may ask whether $\mathcal{I}_k$ coincides with the Joseph ideal $\mathcal{J}_0$ for some $k \in \mathbb{C}$, so that $V_k(\mathfrak{g})$ is a chiralization of $U(\mathfrak{g})/\mathcal{J}_0$. But this can never happen. Indeed, $U(\mathfrak{g})/\mathcal{J}_0$ does not admit finite-dimensional representations while $\mathbb{C}$ is always an $A(V_k(\mathfrak{g}))$-module as $V_k(\mathfrak{g})$ is a module over itself and $V_k(\mathfrak{g})_{top} = \mathbb{C}$. However, by Proposition 2.2, it makes sense to ask the same question for the ideal $\mathcal{J}_W$.

Theorem 3.1. *Assume that $\mathfrak{g}$ belongs to the Deligne exceptional series outside the type A and that $k = -h^\vee/6-1$. Then $V_k(\mathfrak{g})$ is a chiralization of $U(\mathfrak{g})/\mathcal{J}_{\mathcal{W}}$, that is,*

$$A(V_k(\mathfrak{g})) \cong U(\mathfrak{g})/\mathcal{J}_{\mathcal{W}} \cong \mathbb{C} \times U(\mathfrak{g})/\mathcal{J}_0.$$

In particular, since $\mathcal{J}_0$ is maximal, the irreducible highest weight representation $L(\lambda)$ of $\widehat{\mathfrak{g}}$ is a $V_k(\mathfrak{g})$-module if and only if

$$\bar{\lambda} = 0 \quad or \quad \mathrm{Ann}_{U(\mathfrak{g})}\, L_{\mathfrak{g}}(\bar{\lambda}) = \mathcal{J}_0.$$

According to [28, 4.3], the weights μ such that $\mathrm{Ann}_{U(\mathfrak{g})} L_{\mathfrak{g}}(\mu) = \mathcal{J}_0$ are

$$w \circ (\lambda_0 - \rho) := w(\lambda_0) - \rho, \quad w \in W_0,$$

where the weight λ_0 and the subset W_0 of the Weyl group W of $\mathfrak{g}$ are described in Table 1. Here we adopt the standard Bourbaki numbering for the simple roots $\{\alpha_1, \ldots, \alpha_1\}$ of $\mathfrak{g}$, and we denote by $\varpi_1, \ldots, \varpi_l$ the corresponding fundamental weights.

Note that the last statement of Theorem 3.1 reproves the earlier results [11, Proposition 3.6(1)] for type G_2, [38, Theorem 4.3] for type D_4 and [37, Theorem 6.4] for type F_4.

For types G_2 and F_4, the level $k = -h^\vee/6-1$ is *admissible*, that is, $k\Lambda_0$ is an admissible weight [31] for $\widehat{\mathfrak{g}}$. Using [9, Proposition 3.3] one finds that

$$\{k\Lambda_0, w \circ (\lambda_0 - \rho) + k\Lambda_0 \mid w \in W_0\}$$

is exactly the set of admissible weights of level k whose integral Weyl group is isomorphic to that of $k\Lambda_0$, which agrees with [9, Main theorem].

Theorem 3.1 will be proved at the end of § 4.

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4. Singular vectors of affine vertex algebra of degree 2

By the PBW theorem, we have $V^k(\mathfrak{g}) \cong U(\mathfrak{g}[t^{-1}]t^{-1})$ as $\mathbb{C}$-vector spaces. Below we often identify $V^k(\mathfrak{g})$ with $U(\mathfrak{g}[t^{-1}]t^{-1})$.

The vertex algebra $V^k(\mathfrak{g})$ is naturally graded:

$$V^k(\mathfrak{g}) = \bigoplus_{d \in \mathbb{Z}_{\geqslant 0}} V^k(\mathfrak{g})_d, \quad V^k(\mathfrak{g})_d = \{v \in V^k(\mathfrak{g}) \mid Dv = -dv\}.$$

Note that each homogeneous component $V^k(\mathfrak{g})_d$ is a finite-dimensional $\mathfrak{g}$-submodule of $V^k(\mathfrak{g})$.

Lemma 4.1. *We have a $\mathfrak{g}$-module embedding*

$$\sigma_d : S^d(\mathfrak{g}) \hookrightarrow V^k(\mathfrak{g})_d, \quad x_1 \ldots x_d \mapsto \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} x_{\sigma(1)}(-1) \ldots x_{\sigma(d)}(-1).$$

Let v be a singular vector in $S^d(\mathfrak{g})$. Then $\sigma_d(v)$ is a singular vector of $V^k(\mathfrak{g})$ if and only if $f_\theta(1)\sigma_d(v) = 0$. For $d = 2$, we simply denote by σ the embedding σ_d.

Let $W = \bigoplus_i W_i$ be the decomposition of W into irreducible submodules, and let w_i be a highest weight vector of W_i.

Theorem 4.2. (1) *Assume that $\mathfrak{g}$ belongs to the Deligne exceptional series outside the type A.*

(a) *For any i, $\sigma(w_i)$ is a singular vector of $V^k(\mathfrak{g})$ if and only if*

$$k = -h^\vee/6 - 1.$$

(b) *Assume that $\mathfrak{g}$ is not of type G_2. For each $n \in \mathbb{Z}_{\geqslant 0}$ and each i, $\sigma(w_i)^{n+1}$ is a singular vector of $V^k(\mathfrak{g})$ if and only if*

$$k = n - h^\vee/6 - 1.$$

(2) *Let $\mathfrak{g}$ be of type B_l, $l \geqslant 3$, so that $W = W_1 \oplus W_2$ where $W_1 \cong L_\mathfrak{g}(\theta + \theta_1) = L_\mathfrak{g}(2\varpi_1)$ and $W_2 \cong L_\mathfrak{g}(\theta + \theta_2) = L_\mathfrak{g}(\varpi_4)$ if $l \geqslant 5$ (and $W_2 \cong L_\mathfrak{g}(\theta + \theta_2) = L_\mathfrak{g}(2\varpi_l)$ if $l = 3, 4$).*

(a) [36] *For each $n \in \mathbb{Z}_{\geqslant 0}$, $\sigma(w_1)^{n+1}$ is a singular vector of $V^k(\mathfrak{g})$ if and only if*

$$k = n - l + 3/2.$$

(b) *For each $n \in \mathbb{Z}_{\geqslant 0}$, $\sigma(w_2)^{n+1}$ is a singular vector of $V^k(\mathfrak{g})$ if and only if*

$$k = n - 2.$$

(3) [1] *Let $\mathfrak{g}$ be of type C_l, $l \geqslant 2$, so that $W = W_1 \oplus W_2$ where $W_1 \cong L_\mathfrak{g}(\theta + \theta_1) = L_\mathfrak{g}(2\varpi_2)$ and $W_2 \cong L_\mathfrak{g}(\frac{1}{2}\theta + \theta_1) = L_\mathfrak{g}(\varpi_2)$. For each $n \in \mathbb{Z}_{\geqslant 0}$, $\sigma(w_1)^{n+1}$ is a singular vector of $V^k(\mathfrak{g})$ if and only if*

$$k = n - 1/2.$$

Type	D_4	E_6	E_7	E_8
$h^\vee/6+1$	2	3	4	6
θ	(1211)	(122321)	(2234321)	(23465432)
θ_1	(1000)	(101111)	(0112221)	(22343210)
(β_j,δ_j),	(0100), (0111)	(010000), (011210)	(1000000), (1122100)	(00000001), (01122221)
$\beta_j+\delta_j=\theta-\theta_1$	(0101), (0110)	(010100), (011110)	(1010000), (1112100)	(00000011), (01122211)
		(010110), (010100)	(1011000), (1111100)	(00000111), (01122111)
			(1011100), (1111000)	(00001111), (01121111)
				(00011111), (01111111)
				(01011111), (00111111)

TABLE 2. Data for D_4, E_6, E_7, E_8.

(4) *Let $\mathfrak{g}$ be of type D_l, $l\geqslant 5$, so that $W=W_1\oplus W_2$ where $W_1\cong L_{\mathfrak{g}}(\theta+\theta_1)=L_{\mathfrak{g}}(2\varpi_1)$ and $W_2\cong L_{\mathfrak{g}}(\theta+\theta_2)=L_{\mathfrak{g}}(\varpi_4)$ if $l\geqslant 6$ (and $W_2\cong L_{\mathfrak{g}}(\theta+\theta_2)=L_{\mathfrak{g}}(\varpi_4+\varpi_5)$ if $l=5$).*

(a) [38] *For each $n\in\mathbb{Z}_{\geqslant 0}$, $\sigma(w_1)^{n+1}$ is a singular vector of $V^k(\mathfrak{g})$ if and only if*

$$k=n-l+2.$$

(b) *For each $n\in\mathbb{Z}_{\geqslant 0}$, $\sigma(w_2)^{n+1}$ is a singular vector of $V^k(\mathfrak{g})$ if and only if*

$$k=n-2.$$

Note that (1) for D_4 is also a particular case of [38], that (1)(a) for G_2 was proved in [11], and that (1) for F_4 was proved in [37].

Proof. (1) Assume that $\mathfrak{g}$ is of type D_4, E_6, E_7, E_8. Then it is enough to prove (b).

For E_6, E_7, E_8, $W=W_1$. For D_4, $W=W_1\oplus W_2\oplus W_3$. Using the Dynkin automorphism, we can assume that $i=1$, and that $W_1=L_{\mathfrak{g}}(2\varpi_1)$.

For types E_6 and E_7, $\mathfrak{g}$ is of *depth one*, [25, Ch. IV, Definition 1], and $(\theta-\theta_1)/2$ is not a root.

Then we apply [25, Ch. IV, Proposition 11] to construct a singular vector w_1 for W_1. Table 2 describes the pairs of positives roots (β_j,δ_j) such that

$$\beta_j+\delta_j=\theta-\theta_1.$$

The number of such pairs turns out to be equal to $h^\vee/6+1$. In this table, a positive root γ is represented by $(k_1,\dots,k_l)$ if $\gamma=\sum_{j=1}^{l}k_j\alpha_j$.

Choose a Chevalley basis $\{h_i\}_i\cup\{e_\alpha,\ f_\alpha\}_\alpha$ of $\mathfrak{g}$ so that the conditions of [25, Ch. IV, Definition 6] are fulfilled, that is

$$\forall j,\quad [e_{\delta_j},[e_{\beta_j},e_{\theta_1}]]=e_\theta,\quad [e_{\beta_j},e_{\theta_1}]=e_{\beta_j+\theta_1},\quad [e_{\delta_j},e_{\theta_1}]=e_{\delta_j+\theta_1}. \tag{1}$$

Then set

$$w_1:=e_\theta e_{\theta_1}-\sum_{k=1}^{\frac{h^\vee}{6}+1}e_{\beta_j+\theta_1}e_{\delta_j+\theta_1},$$

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so that

$$\sigma(w_1) = \frac{1}{2}(e_\theta(-1)e_{\theta_1}(-1) + e_{\theta_1}(-1)e_\theta(-1) \\ - \sum_{k=1}^{\frac{h^\vee}{6}+1}(e_{\beta_j+\theta_1}(-1)e_{\delta_j+\theta_1}(-1) + e_{\delta_j+\theta_1}(-1)e_{\beta_j+\theta_1}(-1))).$$

We observe using the relations (1) that for each j,

$$[[f_\theta, e_{\beta_j+\theta_1}], e_{\delta_j+\theta_1}] = [[f_\theta, e_{\delta_j+\theta_1}], e_{\beta_j+\theta_1}] = -e_{\theta_1}. \tag{2}$$

By (2), we get:

$$f_\theta(1)\cdot\sigma(w_1) = \left([f_\theta, e_\theta](0) + k + \frac{h^\vee}{6} + 1\right)e_{\theta_1}(-1) \\ - \sum_{k=1}^{\frac{h^\vee}{6}+1}(e_{\beta_j+\theta_1}(-1)[f_\theta, e_{\delta_j+\theta_1}](0) + e_{\delta_j+\theta_1}(-1)[f_\theta, e_{\beta_j+\theta_1}](0)).$$

Observe that

$$[f_\theta, e_\theta](0)\cdot\sigma(w_1) = -2\sigma(w_1)$$

since $\langle\theta+\theta_1, \theta^\vee\rangle = \langle\theta, \theta^\vee\rangle = 2$, and that

$$[f_\theta, e_{\delta_j+\theta_1}](0)\cdot\sigma(w_1) = [f_\theta, e_{\beta_j+\theta_1}](0)\cdot\sigma(w_1) = 0$$

since $-\theta+\delta_j+\theta_1$, $-\theta+\beta_j+\theta_1$ are perpendicular to $\theta+\theta_1$, the weight of $\sigma(w_1)$, for each j. In addition, since $\beta_j+2\theta_1$, $\delta_j+2\theta_1$ are not roots, $[e_{\theta_1}(-1), \sigma(w_1)] = 0$. So, for any $n \in \mathbb{Z}_{\geqslant 0}$ we get,

$$\begin{aligned} f_\theta(1)\cdot\sigma(w_1)^{n+1} &= \sigma(w_1)^n\left(k + \frac{h^\vee}{6} + 1\right)e_{\theta_1}(-1) \\ &\quad + \sum_{j=1}^{n}\left(\sigma(w_1)^{n-j}\left([f_\theta, e_\theta](0) + k + \frac{h^\vee}{6} + 1\right)\cdot\sigma(w_1)^j e_{\theta_1}(-1)\right) \\ &= \sum_{j=0}^{n}\left(-2j + k + \frac{h^\vee}{6} + 1\right)\sigma(w_1)^n e_{\theta_1}(-1) \\ &= (n+1)\left(-n + k + \frac{h^\vee}{6} + 1\right)\sigma(w_1)^n e_{\theta_1}(-1). \end{aligned}$$

Hence $\sigma(w_1)^{n+1}$ is a singular vector of $V^k(\mathfrak{g})$ for $k = n - h^\vee/6 - 1$.

Assume that $\mathfrak{g}$ has type E_8. Then $\mathfrak{g}$ is not of depth one and we follow the construction of [25, Ch. IV, § 4]. According to [25, Ch. IV, § 4], there is a positive root α such that the algebra $\tilde{\mathfrak{g}}$ generated by $e_\alpha, e_2, \ldots, e_8, f_\alpha, f_2, \ldots, f_8$ has type D_8, where $e_i, f_i, i = 1, \ldots, 8$ are the generators of a Chevalley basis of $\mathfrak{g}$ corresponding to the simple roots $\alpha_1, \ldots, \alpha_8$ in the Bourbaki numbering. Moreover, we have that $\alpha = \theta_1$. Then we apply the construction of [25, Ch. IV, § 1] to the algebra $\tilde{\mathfrak{g}}$ which is of depth one. One can choose our Chevalley

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basis $\{h_i\}_i \cup \{e_\alpha,\ f_\alpha\}_\alpha$ of $\mathfrak{g}$ so that the conditions of [25, Ch. IV, Definition 6] are fulfilled for $\tilde{\mathfrak{g}}$. Note that the highest root of $\tilde{\mathfrak{g}}$ is θ, that is, the same as for $\mathfrak{g}$.

Then we apply as in cases E_6, E_7 the construction of [25, Ch. IV, Proposition 11]. Table 2 describes the pairs of positives roots (β_j, δ_j) such that

$$\beta_j + \delta_j = \theta - \theta_1.$$

The number of such pairs is $h^\vee/6 + 1$ too.

Then we set

$$w_1 := e_\theta e_{\theta_1} - \sum_{k=1}^{\frac{h^\vee}{6}+1} e_{\beta_j+\theta_1} e_{\delta_j+\theta_1}.$$

We verify as for the types E_6, E_7 that $\sigma(w_1)^{n+1}$ is a singular vector of $V^k(\mathfrak{g})$ for $k = n - h^\vee/6 - 1$.

(2)(b) and (4)(b) Assume that $\mathfrak{g}$ is of type B_l, $l \geqslant 3$, or of type D_l, $l \geqslant 5$. Then in both cases, θ_2 is the highest root of the root system generated by $\alpha_3, \ldots, \alpha_l$, $(\theta - \theta_2)/2$ is not a root and there are precisely two pairs (β_j, δ_j) such that $\beta_j + \delta_j = \theta - \theta_2$. Namely, these pairs are:

$$(\beta_1, \delta_1) = (\alpha_2, \alpha_1 + \alpha_2 + \alpha_3) \quad \text{and} \quad (\beta_2, \delta_2) = (\alpha_2 + \alpha_3, \alpha_1 + \alpha_2).$$

According to [25, Ch. IV, Proposition 11],

$$w_2 := e_\theta e_{\theta_2} - \sum_{k=1}^{2} e_{\beta_j+\theta_2} e_{\delta_j+\theta_2}$$

is a singular vector for $\mathfrak{g}$. Moreover, all bracket relations (1) and (2) hold as in case (1)[1], with θ_2 in place of θ_1. Hence we get,

$$f_\theta(1) \cdot \sigma(w_2)^{n+1} = (-n + k + 2)\sigma(w_2)^n e_{\theta_2}(-1).$$

The statement follows. □

Remark 4.3. If $\mathfrak{g}$ is of type C_l, $l \geqslant 3$, we can construct a singular vector for $V^k(\mathfrak{g})$ of weight $\frac{1}{2}(\theta + \theta_1)$ with $k = -(l+2)/2$ as follows.

Set

$$\theta_0 := (\theta + \theta_1)/2 = \alpha_1 + 2(\alpha_2 + \cdots + \alpha_{l-1}) + \alpha_l.$$

For $j \in \{2, \ldots, l\}$, set

$$\beta_j := \alpha_1 + \alpha_2 + \cdots + \alpha_{j-1}, \quad \delta_j := \alpha_2 + \cdots + \alpha_{j-1} + 2(\alpha_j + \cdots + \alpha_{l-1}) + \alpha_l.$$

For $j \in \{3, \ldots, l\}$, set

$$\beta'_j := \alpha_2 + \cdots + \alpha_{j-1}, \quad \delta'_j := \alpha_1 + \cdots + \alpha_{j-1} + 2(\alpha_j + \cdots + \alpha_{l-1}) + \alpha_l.$$

[1]For B_3, a factor 2 appears in some brackets but this does not affect the final result.

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Then

$$\forall j \in \{3,\dots,l\}, \quad \beta_j+\delta_j=\beta_j'+\delta_j'=\theta_0=\tfrac{1}{2}(\theta+\theta_1) \quad \text{and} \quad \beta_2+\delta_2=\theta_0.$$

We can choose a Chevalley basis of $\mathfrak{g}$ such that the vector

$$\begin{aligned} v_2 \;:=\; & e_\theta(-1)e_{-\alpha_1}(-1)-\frac{1}{2}h_1(-1)e_{\theta_0}(-1)+e_{\theta_0}(-2) \\ & -e_{\beta_2}(-1)e_{\delta_2}(-1)-\frac{1}{2}\sum_{j=3}^{l}(e_{\beta_j}(-1)e_{\delta_j}(-1)-e_{\beta_j'}(-1)e_{\delta_j'}(-1)) \end{aligned}$$

is singular for $V^k(\mathfrak{g})$ with $k=-(l+2)/2$. The verifications are left to the reader. This remark will be not used in the sequel.

Proof of Theorem 3.1. Let $\mathfrak{g}$, k be as in Theorem. Then $\sigma(w_i)$ is a singular vector of $V^k(\mathfrak{g})$ for all i by Theorem 4.2. Let N be the submodule of $V^k(\mathfrak{g})$ generated by $\sigma(w_i)$ for all i, and set $\tilde{V}_k(\mathfrak{g})=V^k(\mathfrak{g})/N$. By construction, the image of N in $A(V^k(\mathfrak{g}))\cong U(\mathfrak{g})$ is $\mathcal{J}_W$. Hence,

$$A(\tilde{V}_k(\mathfrak{g}))=U(\mathfrak{g})/\mathcal{J}_W.$$

It remains to show that $\tilde{V}_k(\mathfrak{g})=V_k(\mathfrak{g})$, that is, $\tilde{V}_k(\mathfrak{g})$ is simple. (In the case that k is admissible, that is, if $\mathfrak{g}$ is of type G_2, F_4, this follows from [30]. Also, this has been proved in [38] in the case that $\mathfrak{g}$ is of type D_4.)

Suppose that $\tilde{V}_k(\mathfrak{g})$ is not simple, or equivalently, $\tilde{V}_k(\mathfrak{g})$ is reducible as a $\widehat{\mathfrak{g}}$-module. Then there is at least one non-zero weight singular vector, say, v. Let μ be the weight of v, and let M be a submodule of $\tilde{V}_k(\mathfrak{g})$ generated by v. Since $M_{top}=L_{\mathfrak{g}}(\bar{\mu})$, $L_{\mathfrak{g}}(\bar{\mu})$ is a module over $A(\tilde{V}_k(\mathfrak{g}))=U(\mathfrak{g})/\mathcal{J}_W=\mathbb{C}\times U(\mathfrak{g})/\mathcal{J}_0$. On the other hand, $L_{\mathfrak{g}}(\bar{\mu})$ is finite-dimensional since it is a submodule of $V^k(\mathfrak{g})_d$ for some d. This implies that $L_{\mathfrak{g}}(\bar{\mu})$ cannot be a $U(\mathfrak{g})/\mathcal{J}_0$-module. Therefore, $\bar{\mu}=0$. This implies that v coincides with the highest weight vector of $\tilde{V}_k(\mathfrak{g})$ up to non-zero multiplication, which is a contradiction. □

5. Proof of Theorem 1.1

Let $\mathfrak{g}$ be of type D_l, $l\geqslant 4$, E_6, E_7, or E_8.

For $n\in\mathbb{Z}_{\geqslant 0}$, set

$$k_n=\begin{cases} n-h^\vee/6-1 & \text{if } \mathfrak{g} \text{ is of type } D_4,\ E_6,\ E_7,\ E_8, \\ n-2 & \text{if } \mathfrak{g} \text{ is of type } D_l,\ l\geqslant 5. \end{cases} \tag{3}$$

Let N be the submodule of $V^k(\mathfrak{g})$ generated by $\sigma(w_i)^{n+1}$ for all i for type D_4, E_6, E_7, E_8, and by $\sigma(w_1)^{n+l-3}$ and $\sigma(w_2)^{n+1}$ for type D_l, $l\geqslant 5$, and let

$$\tilde{V}_{k_n}(\mathfrak{g}):=V^{k_n}(\mathfrak{g})/N.$$

Conjecture 1. $\tilde{V}_{k_n}(\mathfrak{g})=V_{k_n}(\mathfrak{g})$, that is, $\tilde{V}_{k_n}(\mathfrak{g})$ is simple, if $k_n<0$.

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We have proven Conjecture 1 in the case that $n=0$ in type D_4, E_6, E_7, E_8 in the proof of Theorem 3.1.

Remark 5.1. If $k_n \geqslant 0$, $\tilde{V}_{k_n}(\mathfrak{g})$ is obviously not simple as the maximal submodule of $V^{k_n}(\mathfrak{g})$ is generated by $e_\theta(-1)^{k_n+1}$.

Proposition 5.2. *For each $n \geqslant 0$, we have $X_{\tilde{V}_{k_n}(\mathfrak{g})} = \overline{\mathbb{O}_{min}}$.*

Proof. Set $k=k_n$. The exact sequence $0 \to N \to V^k(\mathfrak{g}) \to \tilde{V}_k(\mathfrak{g}) \to 0$ induces an exact sequence

$$N/\mathfrak{g}[t^{-1}]t^{-2}N \to V^k(\mathfrak{g})/\mathfrak{g}[t^{-1}]t^{-2}V^k(\mathfrak{g}) \to \tilde{V}_k(\mathfrak{g})/\mathfrak{g}[t^{-1}]t^{-2}\tilde{V}_k(\mathfrak{g}) \to 0.$$

Under the isomorphism $V^k(\mathfrak{g})/\mathfrak{g}[t^{-1}]t^{-2}V^k(\mathfrak{g}) \cong S(\mathfrak{g})$, the image of $N/\mathfrak{g}[t^{-1}]t^{-2}N$ in $V^k(\mathfrak{g})/\mathfrak{g}[t^{-1}]t^{-2}V^k(\mathfrak{g})$ is identified with the ideal J of $S(\mathfrak{g})$ generated by some powers of w_i for all i. Hence $J \subset J_W \subset \sqrt{J}$. Therefore,

$$\sqrt{J} = \sqrt{J_W} = J_0$$

by Lemma 2.1 as required. □

Proof of Theorem 1.1. For $\mathfrak{g}$ of type A_1, A_2, G_2, F_4, the number $-h^\vee/6-1$ is admissible, and the statement (1) of the theorem is a special case of [7, Theorem 5.14]. So let us assume that $\mathfrak{g}$ is of type D_l, $l \geqslant 4$, E_6, E_7, or E_8 as above. Since $V_{k_n}(\mathfrak{g})$ is a quotient of $\tilde{V}_{k_n}(\mathfrak{g})$, Proposition 5.2 implies that

$$X_{V_{k_n}(\mathfrak{g})} \subset \overline{\mathbb{O}_{min}} = \mathbb{O}_{min} \cup \{0\}.$$

Therefore, $X_{V_{k_n}(\mathfrak{g})}$ is either $\{0\}$ or $\overline{\mathbb{O}_{min}}$. The assertion follows since $X_{V_k(\mathfrak{g})} = \{0\}$ if and only if $k \in \mathbb{Z}_{\geqslant 0}$ by [7, Proposition 4.25] (see also Theorem 6.1(2) and (3)(a)). □

The following assertion was proved in [38] in the case that $\mathfrak{g}$ is of type D_4 and $k=-2$.

Corollary 5.3. *Let $\mathfrak{g}$, k be as in Theorem 1.1. Then $V_k(\mathfrak{g})$ has only finitely many simple modules in the category $\mathcal{O}$.*

Proof. By [19, Proposition 2.17(c)], [10, Proposition 3.3] there is a surjection

$$R_{V_k(\mathfrak{g})} \twoheadrightarrow \operatorname{gr} A(V_k(\mathfrak{g}))$$

of Poisson algebras,where $\operatorname{gr} A(V_k(\mathfrak{g}))$ is the associated graded algebra of $A(V_k(\mathfrak{g}))$ with respect to Zhu's filtration [42], which coincides with the one induced from the PBW filtration of $U(\mathfrak{g})$ under the identification $A(V_k(\mathfrak{g})) = U(\mathfrak{g})/\mathcal{I}_k$. Hence $\operatorname{Specm}(\operatorname{gr} A(V_k(\mathfrak{g}))) \subset X_{V_k(\mathfrak{g})} \subset \mathcal{N}$. It follows that the action of the argumentation ideal $\mathbb{C}[\mathfrak{g}^*]_+^G$ of $\mathbb{C}[\mathfrak{g}^*]$ is nilpotent on $\operatorname{gr} A(V_k(\mathfrak{g})) = \mathbb{C}[\mathfrak{g}^*]/\operatorname{gr}\mathcal{I}_k$. Let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. The PBW filtration induces a filtration on $Z(\mathfrak{g})/(Z(\mathfrak{g}) \cap \mathcal{I}_k)$. Since the associated graded algebra $\operatorname{gr}(Z(\mathfrak{g})/(Z(\mathfrak{g}) \cap \mathcal{I}_k)) = \mathbb{C}[\mathfrak{g}^*]^G/(\mathbb{C}[\mathfrak{g}^*]^G \cap \operatorname{gr}\mathcal{I}_k)$ is a subalgebra of $\mathbb{C}[\mathfrak{g}^*]/\operatorname{gr}\mathcal{I}_k$, the nilpotentcy of generators implies that $\operatorname{gr}(Z(\mathfrak{g})/(Z(\mathfrak{g}) \cap \mathcal{I}_k))$ is finite-dimensional. Hence so is $Z(\mathfrak{g})/(Z(\mathfrak{g}) \cap \mathcal{I}_k)$. We conclude that $Z(\mathfrak{g})$ acts finitely on $A(V_k(\mathfrak{g}))$ andtherefore there are only finitely many possible central characters for the simple modules of $A(V_k(\mathfrak{g}))$. □

Remark 5.4. Let $\mathfrak{g}, f$ be as in Theorem 1.1. As in [8, Theorem 9.5], one finds that $X_{V_k(\mathfrak{g})} = \mathrm{Specm}(\mathrm{gr}\, A(V_k(\mathfrak{g})))$, which gives another evidence for [8, Conjecture 1].

Conjecture 2. We have

$$X_{V_k(\mathfrak{g})} = \overline{\mathbb{O}_{min}}$$

if and only if

(1) $\mathfrak{g}$ is of type A_1, and k is a rational admissible number that is not an integer, or $k = -2$.

(2) $\mathfrak{g}$ is of type A_2, C_l $(l \geqslant 2)$, F_4, and k is admissible with denominator 2.

(3) $\mathfrak{g}$ is of type G_2, and k is admissible with denominator 3, or $k = -1$.

(4) $\mathfrak{g}$ is of type D_4, E_6, E_7, E_8 and k is an integer such that

$$-\frac{h^\vee}{6} - 1 \leqslant k \leqslant -1.$$

(5) $\mathfrak{g}$ is of type D_l with $l \geqslant 5$, and $k = -2, -1$.

One can easily verify Conjecture 2 for type A_1. Note that the 'if' part of Conjecture 2 follows from Theorem 1.1 and [7, Theorem 5.14].

6. Proof of Theorem 1.2

Let $H_{f_\theta}^{\frac{\infty}{2}+\bullet}(M)$ denote the BRST cohomology associated with the quantized Drinfeld–Sokolov reduction corresponding to f_θ [29], so that

$$\mathcal{W}^k(\mathfrak{g}, f_\theta) = H_{f_\theta}^{\frac{\infty}{2}+0}(V^k(\mathfrak{g})).$$

The correspondence $M \mapsto H_{f_\theta}^{\frac{\infty}{2}+0}(M)$ gives a functor $\mathcal{O}_k \to \mathcal{W}^k(\mathfrak{g}, f_\theta)$-Mod, where $\mathcal{O}_k$ is the category $\mathcal{O}$ of $\widehat{\mathfrak{g}}$ of level k and $\mathcal{W}^k(\mathfrak{g}, f_\theta)$-Mod is the category of $\mathcal{W}^k(\mathfrak{g}, f_\theta)$-modules.

Recall that $\mathcal{W}_k(\mathfrak{g}, f_\theta)$ is the unique simple quotient of $\mathcal{W}^k(\mathfrak{g}, f_\theta)$.

Theorem 6.1. (1) [4, Main theorem] *The functor $\mathcal{O}_k \to \mathcal{W}^k(\mathfrak{g}, f_\theta)$-Mod, $M \mapsto H_{f_\theta}^{\frac{\infty}{2}+0}(M)$, is exact.*

(2) [4, Main theorem] *We have $H_{f_\theta}^{\frac{\infty}{2}+0}(L(\lambda)) = 0$ if $\lambda(\alpha_0^\vee) \in \mathbb{Z}_{\geqslant 0}$, where $\alpha_0^\vee = K - \theta$. Otherwise $H_{f_\theta}^{\frac{\infty}{2}+0}(L(\lambda))$ is an irreducible highest weight representation of $\mathcal{W}^k(\mathfrak{g}, f_\theta)$. In particular,*

$$H_{f_\theta}^{\frac{\infty}{2}+0}(V_k(\mathfrak{g})) \cong \begin{cases} \mathcal{W}_k(\mathfrak{g}, f_\theta) & \text{if } k \notin \mathbb{Z}_{\geqslant 0}, \\ 0 & \text{if } k \in \mathbb{Z}_{\geqslant 0}. \end{cases}$$

(3) [7, Theorem 4.21] *For any quotient V of $V^k(\mathfrak{g})$ we have*

$$X_{H_{f_\theta}^{\frac{\infty}{2}+0}(V)} = X_V \cap \mathcal{S}_{min}.$$

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Hence

(a) [7, Proposition 4.22] $H_{f_\theta}^{\frac{\infty}{2}+0}(V) \neq 0$ *if and only if* $\overline{\mathbb{O}_{min}} \subset X_V$.

(b) [7, Theorem 4.23] $H_{f_\theta}^{\frac{\infty}{2}+0}(V)$ *is a lisse vertex algebra if* $X_V = \overline{\mathbb{O}_{min}}$.

Remark 6.2. By [32, Theorem 6.3], the image $H_{f_\theta}^{\frac{\infty}{2}+0}(M(\lambda))$ of the Verma module $M(\lambda)$ of $\widehat{\mathfrak{g}}$ with highest weight λ is isomorphic to a Verma module of $\mathcal{W}^k(\mathfrak{g}, f_\theta)$. Moreover, all the Verma modules of $\mathcal{W}^k(\mathfrak{g}, f_\theta)$ appear in this way. By Theorem 6.1(1), (2), $H_{f_\theta}^{\frac{\infty}{2}+0}(L(\lambda))$ is the unique simple quotient of $H_{f_\theta}^{\frac{\infty}{2}+0}(M(\lambda))$ provided $\lambda(\alpha_0^\vee) \notin \mathbb{Z}_{\geqslant 0}$. From this, one sees that all the irreducible highest weight representations of $\mathcal{W}^k(\mathfrak{g}, f_\theta)$ appear as $H_{f_\theta}^{\frac{\infty}{2}+0}(L(\lambda))$ for some λ (see [4] for the details).

Let k be non-critical, that is, $k + h^\vee \neq 0$. By [32, Theorem 6.3], one finds that $H_{f_\theta}^{\frac{\infty}{2}+0}(M(\lambda)) \cong H_{f_\theta}^{\frac{\infty}{2}+0}(M(\mu))$ if and only if $\mu = s_0 \circ \lambda$, where s_0 is the reflection corresponding to α_0. It follows that $H_{f_\theta}^{\frac{\infty}{2}+0}(L(\lambda))$ and $H_{f_\theta}^{\frac{\infty}{2}+0}(L(\mu))$ are non-zero and isomorphic if and only if $\lambda(\alpha_0^\vee), \mu(\alpha_0^\vee) \notin \mathbb{Z}_{\geqslant 0}$ and $\mu = s_0 \circ \lambda$.

Proof of Theorem 1.2. Let $k = k_n$ with $n \geqslant 0$ as in § 5. We have shown that $X_{\tilde{V}_k(\mathfrak{g})} = \overline{\mathbb{O}_{min}}$ in Proposition 5.2. Hence, the vertex algebra $H_{f_\theta}^{\frac{\infty}{2}+0}(\tilde{V}_k(\mathfrak{g}))$ is non-zero and lisse by Theorem 6.1(3). Note that both $\mathcal{W}_k(\mathfrak{g}, f_\theta)$ and $H_{f_\theta}^{\frac{\infty}{2}+0}(\tilde{V}_k(\mathfrak{g}))$ are quotients of $\mathcal{W}^k(\mathfrak{g}, f_\theta)$. Indeed, $H_{f_\theta}^{\frac{\infty}{2}+0}(\tilde{V}_k(\mathfrak{g}))$ is a quotient of $\mathcal{W}^k(\mathfrak{g}, f_\theta) = H_{f_\theta}^{\frac{\infty}{2}+0}(V^k(\mathfrak{g}))$ by Theorem 6.1(1) since $\tilde{V}_k(\mathfrak{g})$ is a quotient of $V^k(\mathfrak{g})$. Because it is the unique simple quotient of $\mathcal{W}^k(\mathfrak{g}, f_\theta)$, $\mathcal{W}_k(\mathfrak{g}, f_\theta)$ is a quotient of $H_{f_\theta}^{\frac{\infty}{2}+0}(\tilde{V}_k(\mathfrak{g}))$, which is lisse as we have just proved. Therefore, $\mathcal{W}_k(\mathfrak{g}, f_\theta)$ is lisse as well. □

Conjecture 3. Let $\mathfrak{g}$ and k be as in Theorem 1.2. Then $H_{f_\theta}^{\frac{\infty}{2}+0}(\tilde{V}_k(\mathfrak{g})) \cong \mathcal{W}_k(\mathfrak{g}, f_\theta)$, where $\tilde{V}_k(\mathfrak{g})$ is defined above.

Remark 6.3. Let $\mathfrak{g}$ and k be as in Theorem 1.2. Then $\mathcal{W}_k(\mathfrak{g}, f_\theta) \ncong H_{f_\theta}^{\frac{\infty}{2}+0}(L(\lambda))$ for any irreducible admissible representation $L(\lambda)$ of $\widehat{\mathfrak{g}}$. Indeed, if $k \leqslant -1$ (respectively if $k \geqslant -1$), $L(k\Lambda_0) = V_k(\mathfrak{g})$ (respectively $L(s_0 \circ k\Lambda_0)$) is the unique irreducible highest weight representation of $\widehat{\mathfrak{g}}$ such that $\mathcal{W}_k(\mathfrak{g}, f_\theta) \cong H_{f_\theta}^{\frac{\infty}{2}+0}(L(\lambda))$ (see Remark 6.2). But $k\Lambda_0$ (respectively $s_0 \circ k\Lambda_0$) is not an admissible weight since it is not regular dominant.

7. Classification of lisse minimal W-algebras

Theorem 7.1. (1) $\mathcal{W}_k(\mathfrak{sp}_{2l}, f_\theta)$, $l \geqslant 2$, *is lisse if and only if* k *is admissible with denominator* 2, *that is,* $k = p/2$ *and* p *is an odd number equal to or greater than* -1.

(2) $\mathcal{W}_k(\mathfrak{so}_7, f_\theta)$ *is lisse if and only if* k *is admissible with denominator* 2, *that is,* $k = p/2$ *and* p *is an odd integer equal to or greater than* -3.

(3) $\mathcal{W}_k(\mathfrak{so}_{2l+1}, f_\theta)$, $l \geqslant 4$, *is never lisse.*

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(4) $\mathcal{W}_k(\mathfrak{so}_{2l}, f_\theta)$, $l \geqslant 2$, *is lisse if and only if* k *is an integer equal to or greater than* -2.

(5) $\mathcal{W}_k(F_4, f_\theta)$ *is lisse if and only if* k *is admissible with denominator* 2, *that is*, $k = p/2$ *and* p *is an odd number equal to or greater than* -5.

(6) $\mathcal{W}_k(E_6, f_\theta)$ *is lisse if and only if* k *is an integer equal to or greater than* -3.

(7) $\mathcal{W}_k(E_7, f_\theta)$ *is lisse if and only if* k *is an integer equal to or greater than* -4.

(8) $\mathcal{W}_k(E_8, f_\theta)$ *is lisse if and only if* k *is an integer equal to or greater than* -6.

If $\mathcal{W}_k(\mathfrak{g}, f_\theta) = \mathbb{C}$, then it is obviously lisse. Hence, it is natural to ask when $\mathcal{W}_k(\mathfrak{g}, f_\theta) = \mathbb{C}$. It turns out that not every W-algebra admits one-dimensional representations.

Theorem 7.2. *Suppose* $\mathfrak{g}$ *is not of type* A_1. *The following are equivalent:*

(1) $\mathcal{W}^k(\mathfrak{g}, f_\theta)$ *admits a (non-twisted or Ramond-twisted) one-dimensional representation,*

(2) $\mathcal{W}_k(\mathfrak{g}, f_\theta) = \mathbb{C}$,

(3) (a) $\mathfrak{g}$ *belongs to the Deligne exceptional series and* $k = -h^\vee/6 - 1$, *or*

(b) $\mathfrak{g} = \mathfrak{sp}_{2l}$, $l \geqslant 2$, *and* $k = -1/2$.

Remark 7.3. If $\mathfrak{g} = \mathfrak{sl}_2$, then $f_\theta = f_{\mathrm{reg}}$ is regular, $\mathcal{W}_k(\mathfrak{g}, f_\theta) = \mathcal{W}_k(\mathfrak{sl}_2, f_{\mathrm{reg}})$ is the simple Virasoro vertex algebra provided that $k \neq -2$, and the results are well known[2]. Namely,

- $\mathcal{W}_k(\mathfrak{sl}_2, f_{\mathrm{reg}})$ is lisse if and only if either $k+2 = p/q$, with $p, q \in \mathbb{Z}_{\geqslant 0}$, $(p, q) = 1$ and $p, q \geqslant 2$, or $k + 2 = 0$ (cf. [5]),
- $\mathcal{W}_k(\mathfrak{sl}_2, f_{\mathrm{reg}}) = \mathbb{C}$ if and only if either $k+2 = 2/3$, or $k+2 = 3/2$, or $k+2 = 0$.

The rest of this section is devoted to the proof of Theorems 7.1 and 7.2.

Let $\mathfrak{g}_0$ be the center of the reductive Lie algebra $\mathfrak{g}^\natural$, so that

$$\mathfrak{g}^\natural = \bigoplus_{i \geqslant 0} \mathfrak{g}_i.$$

Define an invariant bilinear form on $\mathfrak{g}_i$, $i \geqslant 0$, by

$$(x|y)_i^\natural := \left(k + \frac{h^\vee}{2}\right)(x|y) - \frac{1}{4}(\mathrm{tr}_{\mathfrak{g}(0)}(\mathrm{ad}\, x\, \mathrm{ad}\, y)),$$

where $(\ |\ )$ is the normalized inner product of $\mathfrak{g}$ as before. Then there exists a polynomial $k_i^\natural$ of k of degree 1 such that

$$(\ |\ )_i^\natural = k_i^\natural (\ |\ )_i,$$

where $(\ |\ )_i$ is the normalized inner product of $\mathfrak{g}_i$, that is, $(\theta_i|\theta_i) = 2$.

By [32, Theorem 5.1], we have an embedding

$$\bigotimes_{i \geqslant 0} V^{k_i^\natural}(\mathfrak{g}_i) \hookrightarrow \mathcal{W}^k(\mathfrak{g}, f_\theta)$$

of vertex algebras.

[2]Note that $\mathcal{W}_{r-2}(\mathfrak{sl}_2, f_{\mathrm{reg}}) \cong \mathcal{W}_{1/r-2}(\mathfrak{sl}_2, f_{\mathrm{reg}})$ for any $r \in \mathbb{C}^*$.

	$\mathfrak{sl}_3$	$\mathfrak{sl}_{l+1}$, $l \geqslant 3$	$\mathfrak{sp}_{2l}$, $l \geqslant 2$	$\mathfrak{so}_7$	$\mathfrak{so}_8$	$\mathfrak{so}_n$, $n \geqslant 9$
$\mathfrak{g}^\natural$	$\mathfrak{g}_0$, $\mathfrak{g}_0 \cong \mathbb{C}$	$\mathfrak{g}_0 \oplus \mathfrak{g}_1$ $\mathfrak{g}_0 \cong \mathbb{C}$, $\mathfrak{g}_1 \cong \mathfrak{sl}_{l-1}$	$\mathfrak{g}_1$, $\mathfrak{g}_1 \cong \mathfrak{sp}_{2l-2}$	$\mathfrak{g}_1 \oplus \mathfrak{g}_2$, $\mathfrak{g}_1 \cong \mathfrak{g}_2 \cong \mathfrak{sl}_2$	$\bigoplus_{i=1}^{3} \mathfrak{g}_i$, $\mathfrak{g}_i \cong \mathfrak{sl}_2$,	$\mathfrak{g}_1 \oplus \mathfrak{g}_2$, $\mathfrak{g}_1 \cong \mathfrak{sl}_2$, $\mathfrak{g}_2 \cong \mathfrak{so}_{n-4}$
$k_i^\natural$	$k_0^\natural = k + \frac{3}{2}$	$k_0^\natural = k + \dfrac{l+1}{2}$, $k_1^\natural = k+1$	$k_1^\natural = k + \frac{1}{2}$	$k_1^\natural = k + \frac{3}{2}$, $k_2^\natural = 2k+4$	$k_i^\natural = k+2$, $i \in \{1, 2, 3\}$	$k_1^\natural = k + \frac{n}{2} - 2$, $k_2^\natural = k+2$

TABLE 3. $\mathfrak{g}^\natural = \bigoplus_{i \geqslant 0} \mathfrak{g}_i$ and $k_i^\natural$ for the classical types.

	G_2	F_4	E_6	E_7	E_8
$\mathfrak{g}^\natural$	$\mathfrak{sl}_2$	$\mathfrak{sp}_6$	$\mathfrak{sl}_6$	$\mathfrak{so}_{12}$	E_7
$k_1^\natural$	$3k+5$	$k + \frac{5}{2}$	$k+3$	$k+4$	$k+6$

TABLE 4. $\mathfrak{g}^\natural = \bigoplus_{i \geqslant 0} \mathfrak{g}_i$ and $k_i^\natural$ for the exceptional types.

Lemma 7.4. (1) *Suppose that $\mathcal{W}_k(\mathfrak{g}, f_\theta)$ is lisse. Then the value of $k_i^\natural$ for all $i \geqslant 1$ must be a non-negative integer.*

(2) *Suppose that $\mathcal{W}^k(\mathfrak{g}, f_\theta)$ admits a (non-twisted or Ramond-twisted) one-dimensional representation. Then the value of $k_i^\natural$ for all $i \geqslant 0$ must be zero.*

Proof. (1) By [20], if a lisse vertex algebra V contains a quotient of an affine vertex algebra as a vertex subalgebra, this quotient must be integrable. With $V = \mathcal{W}_k(\mathfrak{g}, f_\theta)$, we deduce that the simple quotient $V_{k_i^\natural}(\mathfrak{g}_i)$ must be integrable for any $i \geqslant 1$, that is, $k_i^\natural$ is a non-negative integer for any $i \geqslant 1$.

(2) If $\mathcal{W}^k(\mathfrak{g}, f_\theta)$ admits a (non-twisted or Ramond-twisted) one-dimensional representation, by restriction we obtain that $V^{k_i^\natural}(\mathfrak{g}_i)$, for $i \geqslant 0$, admits a one-dimensional representation. Hence, $k_i^\natural = 0$ for all $i \geqslant 0$. □

Lemma 7.5. *The reductive Lie algebras $\mathfrak{g}^\natural = \bigoplus_{i \geqslant 0} \mathfrak{g}_i$ and the polynomials $k_i^\natural$ are described in Tables 3 and 4.*

Proof. The verifications are easy and left to the reader. □

Proof of Theorem 7.1. The 'if' part of Theorem 7.1 has been already proven in Theorem 1.2 and [7, Theorem 5.18], and the 'only if' part follows from Lemmas 7.4 and 7.5. □

Remark 7.6. For $\mathfrak{g} = \mathfrak{sp}_{2l}$ it is possible to show the following.

$$A(V_{-1/2}(\mathfrak{g})) \cong U(\mathfrak{g})/\mathcal{J}_{W_1} \cong \mathbb{C} \times (L_\mathfrak{g}(\varpi_1)^* \otimes_\mathbb{C} L_\mathfrak{g}(\varpi_1)) \times U(\mathfrak{g})/\mathcal{J}_0,$$

where $\mathcal{J}_{W_1}$ is the ideal generated by $W_1 := L_{\mathfrak{g}}(\theta+\theta_1) \subset W$. This implies that $\mathcal{J}_0$ is generated by $\mathcal{J}_{W_1}$ and $\Omega - c_0$.

Conjecture 4. (1) $\mathcal{W}_k(\mathfrak{sl}_3, f_\theta)$ is lisse if and only if k is admissible with denominator 2, that is, $k = p/2$ and p is an odd integer equal or greater than -3.

(2) $\mathcal{W}_k(\mathfrak{sl}_n, f_\theta)$, $n \geqslant 4$, is never lisse.

(3) $\mathcal{W}_k(G_2, f_\theta)$ is lisse if and only if k is admissible with denominator 3, or an integer equal to or greater than -1.

The 'if' part of Conjecture 4 follows from [7, Theorem 5.18].

Proof of Theorem 7.2. Clearly (2) implies (1). The direction (1) $\Rightarrow$ (3) follows from Lemmas 7.4 and 7.5.

Let us show (3) implies (2).

The A_2 case follows from [6].

Assume that $\mathfrak{g}$ is of type D_l, E_6, E_7, or E_8. Note that $k = k_0$ in (3). Let N be the submodule of $V^k(\mathfrak{g})$ generated by $v_i = \sigma(w_i)$, for all i, and set $\tilde{V}_k(\mathfrak{g}) = V^k(\mathfrak{g})/N$ as in § 5. By Theorem 6.1(1) we have an exact sequence

$$0 \to H_{f_\theta}^{\frac{\infty}{2}+0}(N) \to H_{f_\theta}^{\frac{\infty}{2}+0}(V^k(\mathfrak{g})) \to H_{f_\theta}^{\frac{\infty}{2}+0}(\tilde{V}_k(\mathfrak{g})) \to 0$$

of $\mathcal{W}^k(\mathfrak{g}, f_\theta)$-modules. The image $\bar{v}_i$ of $v_i \in N$ in $H_{f_\theta}^{\frac{\infty}{2}+0}(V^k(\mathfrak{g})) = \mathcal{W}^k(\mathfrak{g}, f_\theta)$ is non-zero, since its image in $R_{\mathcal{W}_k(\mathfrak{g}, f_\theta)} = \mathbb{C}[\mathcal{S}_{min}]$ is non-zero and coincides with e_{θ_i} under the identification $\mathbb{C}[\mathcal{S}_{min}] = S(\mathfrak{g}^{f_\theta})$, where e_{θ_i} is a highest root vector of $\mathfrak{g}_i$. By weight consideration one finds that $\bar{v}_i$ coincides with $e_{\theta_i}(-1) \in V^{k_i^\natural}(\mathfrak{g}_i) \subset \mathcal{W}^k(\mathfrak{g}, f_\theta)$ up to non-zero constant multiplication.

Since $\mathcal{W}^k(\mathfrak{g}, f_\theta)_1 = \mathfrak{g}^\natural = \bigoplus_{i \geqslant 1} \mathfrak{g}_i$, the whole weight one space $\mathcal{W}^k(\mathfrak{g}, f_\theta)_1$ is included in the image of $H_{f_\theta}^{\frac{\infty}{2}+0}(N)$. Then from the commutation relations of $\mathcal{W}^k(\mathfrak{g}, f_\theta)$ described in [32, Theorem 5.1] it follows that all the generators G^v, $v \in \mathfrak{g}_{1/2}$, defined in [32], and the conformal vector are also in the image of $H_{f_\theta}^{\frac{\infty}{2}+0}(N)$. Therefore $H_{f_\theta}^{\frac{\infty}{2}+0}(\tilde{V}_k(\mathfrak{g}))$ must be trivial, and hence, so is its simple quotient $\mathcal{W}_k(\mathfrak{g}, f_\theta)$.

Assume that $\mathfrak{g}$ is of type C_l, G_2 or F_4, so that $\mathfrak{g}^\natural$ is simple and k is admissible, and hence the maximal submodule N_k of $V^k(\mathfrak{g})$ is generated by a singular vector v. By Theorem 6.1(1), (2) we have the exact sequence

$$0 \to H_{f_\theta}^{\frac{\infty}{2}+0}(N) \to \mathcal{W}^k(\mathfrak{g}, f_\theta) \to \mathcal{W}_k(\mathfrak{g}, f_\theta) \to 0.$$

Also, by [32, Theorem 6.3.1] $H_{f_\theta}^{\frac{\infty}{2}+0}(N)$ is generated by the image $\bar{v}$ of v. Since the image of $H_{f_\theta}^{\frac{\infty}{2}+0}(N)$ in $\mathcal{W}^k(\mathfrak{g}, f_\theta)$ is non-zero as $\mathcal{W}_k(\mathfrak{g}, f_\theta)$ is lisse [7], the image of $\bar{v}$ in $\mathcal{W}^k(\mathfrak{g}, f_\theta)$ is non-zero. Hence, as above, by weight consideration it follows that $\mathcal{W}^1(\mathfrak{g}, f_\theta)_1$ is included in the image of $H_{f_\theta}^{\frac{\infty}{2}+0}(N)$, which gives that $\mathcal{W}_k(\mathfrak{g}, f_\theta) = \mathbb{C}$ as required. □

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