Adv. Appl. Prob. **48,** 274–297 (2016)
*doi:*10.1017/apr.2015.17

JOINT DISTRIBUTION OF A SPECTRALLY NEGATIVE LÉVY PROCESS AND ITS OCCUPATION TIME, WITH STEP OPTION PRICING IN VIEW

HÉLÈNE GUÉRIN,* *Université de Rennes 1*
JEAN-FRANÇOIS RENAUD,** *Université du Québec à Montréal (UQAM)*

Abstract

We study the distribution $\mathbb{E}_x[\exp(-q\int_0^t \mathbf{1}_{(a,b)}(X_s)\,\mathrm{d}s);\ X_t \in \mathrm{d}y]$, where $-\infty \leq a < b < \infty$, and where $q, t > 0$ and $x \in \mathbb{R}$ for a spectrally negative Lévy process X. More precisely, we identify the Laplace transform with respect to t of this measure in terms of the scale functions of the underlying process. Our results are then used to price step options and the particular case of an exponential spectrally negative Lévy jump-diffusion model is discussed.

Keywords: Occupation time; spectrally negative Lévy process; fluctuation theory; scale function; step option

2010 Mathematics Subject Classification: Primary 60G51
Secondary 91G20

1. Introduction

One of Paul Lévy's arcsine laws gives the distribution of the occupation time of the positive/negative half-line for a standard Brownian motion. More precisely, if $\{B_t, t \geq 0\}$ is a standard Brownian motion then, for $s \leq t$,

$$\mathbb{P}\bigg(\int_0^t \mathbf{1}_{(-\infty,0)}(B_u)\mathrm{d}u \leq s\bigg) = \frac{2}{\pi}\arcsin\bigg(\sqrt{\frac{s}{t}}\bigg).$$

This result was then extended to a Brownian motion with drift by Akahori [1] and Takács [16].

In the last few years several papers have looked at the distribution of functionals involving occupation times of a spectrally negative Lévy process (SNLP), in each case over an infinite time horizon. First, in [10], the Laplace transform of the occupation time of semi-infinite intervals for a SNLP was derived. More precisely, the Laplace transform of

$$\int_0^\infty \mathbf{1}_{(-\infty,0)}(X_s)\,\mathrm{d}s \quad \text{and} \quad \int_0^{\tau_{-b}^-} \mathbf{1}_{(-\infty,0)}(X_s)\,\mathrm{d}s,$$

where $X = \{X_t, t \geq 0\}$ is a SNLP and $\tau_{-b}^- = \inf\{t > 0\colon X_t < -b\}$ with $b > 0$, were expressed in terms of the Laplace exponent and the scale functions of the underlying process X. Then, in [14], those results were significantly extended, first by considering more general quantities,

Received 13 June 2014; revision received 22 December 2014.
* Postal address: IRMAR, Université de Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France.
Email address: helene.guerin@univ-rennes1.fr
** Postal address: Département de Mathématiques, Université du Québec à Montréal (UQAM), 201 av. Président-Kennedy, Montréal, QC H2X 3Y7, Canada. Email address: renaud.jf@uqam.ca

i.e.

$$\Big(\tau_0^-, \int_0^{\tau_0^-} \mathbf{1}_{(a,b)}(X_s)\,\mathrm{d}s\Big) \quad \text{and} \quad \Big(\tau_c^+, \int_0^{\tau_c^+} \mathbf{1}_{(a,b)}(X_s)\,\mathrm{d}s\Big),$$

where $\tau_0^- = \inf\{t > 0\colon X_t < 0\}$, $\tau_c^+ = \inf\{t > 0\colon X_t > c\}$, and $0 \le a \le b \le c$, and by obtaining considerably simpler expressions for the joint Laplace transforms. Note that the authors of [8] and [15] have also looked at the abovementioned quantities involving occupation times, but for a so-called refracted Lévy process, while similar quantities for diffusion processes were studied in [9] and [12].

In this paper we are interested in the joint distribution of a SNLP and its occupation time when both are sampled at a fixed time. This is closer in spirit to Lévy's arcsine law and also much more useful for financial applications, especially for the pricing of occupation time options.

1.1. Occupation time options

Let the risk-neutral price of an asset $S = \{S_t, t \ge 0\}$ be of the form

$$S_t = S_0 \mathrm{e}^{X_t},$$

where $X = \{X_t, t \ge 0\}$ is the log-return process. For example, in the Black–Scholes–Merton model, X is a Brownian motion with drift. The time spent by S in an interval I, or, equivalently, the time spent by X in an interval I', from time 0 to time T, is given by

$$\int_0^T \mathbf{1}_{\{S_t \in I\}}\,\mathrm{d}t = \int_0^T \mathbf{1}_{\{X_t \in I'\}}\,\mathrm{d}t.$$

Introduced by Linetsky [13], (barrier) step options are exotic options linked to occupation times of the underlying asset price process. They are generalized barrier options: instead of being activated (or canceled) when the underlying asset price crosses a barrier, which is a problem from a risk management point of view, the payoff of occupation-time options will depend on the time spent above/below this barrier. Therefore, the change of value occurs more gradually. For instance, a (down-and-out call) step option admits the following payoff:

$$\mathrm{e}^{-\rho\int_0^T \mathbf{1}_{\{S_t \le L\}}\,\mathrm{d}t}(S_T - K)_+ = \mathrm{e}^{-\rho\int_0^T \mathbf{1}_{\{X_t \le \ln(L/S_0)\}}\,\mathrm{d}t}(S_0\mathrm{e}^{X_T} - K)_+,$$

where $\rho > 0$ is called the *knock-out rate*. Therefore, its price can be written as

$$\begin{aligned} C(T) &:= \mathrm{e}^{-rT}\mathbb{E}[\mathrm{e}^{-\rho\int_0^T \mathbf{1}_{\{S_t \le L\}}\,\mathrm{d}t}(S_T - K)_+] \\ &= \mathrm{e}^{-rT}\int_{\ln(K/S_0)}^{\infty}(S_0\mathrm{e}^y - K)\mathbb{E}[\mathrm{e}^{-\rho\int_0^T \mathbf{1}_{\{X_s \le \ln(L/S_0)\}}\,\mathrm{d}s}; X_T \in \mathrm{d}y], \end{aligned}$$

where r is the risk-free interest rate, and its Laplace transform, with respect to the time of maturity T, can be written as

$$\begin{aligned} &\int_0^\infty \mathrm{e}^{-pT}C(T)\,\mathrm{d}T \\ &\quad= \int_0^\infty \mathrm{e}^{-(p+r)T}\int_{\ln(K/S_0)}^{\infty}(S_0\mathrm{e}^y - K)\mathbb{E}[\mathrm{e}^{-\rho\int_0^T \mathbf{1}_{\{X_s \le \ln(L/S_0)\}}\,\mathrm{d}s}; X_T \in \mathrm{d}y]\,\mathrm{d}T \\ &\quad= \int_{\ln(K/S_0)}^{\infty}(S_0\mathrm{e}^y - K)\int_0^\infty \mathrm{e}^{-(p+r)T}\mathbb{E}[\mathrm{e}^{-\rho\int_0^T \mathbf{1}_{\{X_s \le \ln(L/S_0)\}}\,\mathrm{d}s}; X_T \in \mathrm{d}y]\,\mathrm{d}T. \end{aligned}$$

Thus, pricing step options boils down to identyfing this distribution:

$$\mathbb{E}[\mathrm{e}^{-\rho \int_0^T \mathbf{1}_{(-\infty,b)}(X_s)\,\mathrm{d}s}; X_T \in \mathrm{d}y]$$

for a given value $b \in \mathbb{R}$. Other occupation time options can also be priced using this distribution. For references on occupation-time option pricing, see, e.g. [4], [5].

The rest of the paper is organized as follows. In Section 2 we give the necessary background on SNLPs and their scale functions. In Section 3 the main results are presented, while their proofs are left until Appendix C. Finally, in Section 4 we consider the pricing step options question and a specific example of a jump-diffusion process with hyperexponential jumps.

2. SNLPs

On the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, let $X = \{X_t, t \geq 0\}$ be a SNLP, that is a process with stationary and independent increments and no positive jumps. Hereby, we exclude the case of X having monotone paths. As the Lévy process X has no positive jumps, its Laplace transform exists: for $\lambda, t \geq 0$,

$$\mathbb{E}[\mathrm{e}^{\lambda X_t}] = \mathrm{e}^{t\psi(\lambda)},$$

where

$$\psi(\lambda) = \gamma\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_0^\infty (\mathrm{e}^{-\lambda z} - 1 + \lambda z\,\mathbf{1}_{(0,1]}(z))\Pi(\mathrm{d}z)$$

for $\gamma \in \mathbb{R}$ and $\sigma \geq 0$, and where Π is a σ-finite measure on $(0, \infty)$ such that

$$\int_0^\infty (1 \wedge z^2)\Pi(\mathrm{d}z) < \infty.$$

This measure is called the Lévy measure of X, while (γ, σ, Π) is refered to as the Lévy triplet of X. Note that for convenience we define the Lévy measure in such a way that it is a measure on the positive half-line instead of the negative half-line. Furthermore, note that $\mathbb{E}[X_1] = \psi'(0+)$.

There exists a function $\Phi\colon [0, \infty) \to [0, \infty)$ defined by $\Phi(q) = \sup\{\lambda \geq 0 \mid \psi(\lambda) = q\}$ (the right-inverse of ψ) such that

$$\psi(\Phi(q)) = q, \qquad q \geq 0.$$

It follows that $\Phi(q) = 0$ if and only if $q = 0$ and $\psi'(0+) \geq 0$.

The process X has jumps of bounded variation (BV) if $\int_0^1 z\Pi(\mathrm{d}z) < \infty$. In that case, we denote by $\mathtt{c} := \gamma + \int_0^1 z\Pi(\mathrm{d}z) > 0$ the so-called drift of X which can now be written as

$$X_t = \mathtt{c}t - S_t + \sigma B_t,$$

where $S = \{S_t, t \geq 0\}$ is a driftless subordinator (for example, a gamma process or a compound Poisson process with positive jumps).

For more details on SNLPs, we refer the reader to [7].

2.1. Scale functions and fluctuation identities

The law of X such that $X_0 = x$ is denoted by $\mathbb{P}_x$ and the corresponding expectation by $\mathbb{E}_x$. We write $\mathbb{P}$ and $\mathbb{E}$ when $x = 0$. Finally, for a random variable Z and an event A, $\mathbb{E}[Z; A] := \mathbb{E}[Z\,\mathbf{1}_A]$.

https://doi.org/10.1017/apr.2015.17 Published online by Cambridge University Press

We now recall the definition of the q-scale function $W^{(q)}$. For $q \geq 0$, the q-scale function of the process X is defined as the continuous function on $[0, \infty)$ with Laplace transform

$$\int_0^\infty \mathrm{e}^{-\lambda y} W^{(q)}(y)\,\mathrm{d}y = \frac{1}{\psi(\lambda) - q} \quad \text{for } \lambda > \Phi(q). \tag{1}$$

This function is unique, positive, and strictly increasing for $x \geq 0$ and is further continuous for $q \geq 0$. We extend $W^{(q)}$ to the whole real line by setting $W^{(q)}(x) = 0$ for $x < 0$. We write $W = W^{(0)}$ when $q = 0$. The initial value of $W^{(q)}$ is known to be

$$W^{(q)}(0) = \begin{cases} \dfrac{1}{\mathrm{c}} & \text{when } \sigma = 0 \text{ and } \int_0^1 z\Pi(\mathrm{d}z) < \infty, \\ 0 & \text{otherwise,} \end{cases}$$

where we used the following definition: $W^{(q)}(0) = \lim_{x\downarrow 0} W^{(q)}(x)$. We also have, when $\psi'(0+) > 0$,

$$\lim_{x\to\infty} W(x) = \frac{1}{\psi'(0+)}.$$

Finally, we recall the following useful identity (see [14, Equation (6)]): for $p, q \geq 0$ and $x \in \mathbb{R}$,

$$(q-p)\int_0^x W^{(p)}(x-y)W^{(q)}(y)\,\mathrm{d}y = W^{(q)}(x) - W^{(p)}(x). \tag{2}$$

We will also frequently use the function

$$Z^{(q)}(x) = 1 + q\int_0^x W^{(q)}(y)\,\mathrm{d}y, \qquad x \in \mathbb{R}. \tag{3}$$

Now, for any $a, b \in \mathbb{R}$, define the stopping times

$$\tau_a^- = \inf\{t > 0\colon X_t < a\} \quad \text{and} \quad \tau_b^+ = \inf\{t > 0\colon X_t > b\},$$

with the convention $\inf \varnothing = \infty$. It is well known that, if $a \leq x \leq c$ then the solution to the two-sided exit problem for X is given by

$$\mathbb{E}_x[\mathrm{e}^{-q\tau_c^+}; \tau_c^+ < \tau_a^-] = \frac{W^{(q)}(x-a)}{W^{(q)}(c-a)},$$
$$\mathbb{E}_x[\mathrm{e}^{-q\tau_a^-}; \tau_a^- < \tau_c^+] = Z^{(q)}(x-a) - \frac{Z^{(q)}(c-a)}{W^{(q)}(c-a)}W^{(q)}(x-a).$$

It is well known (see, e.g. [7]) that under the change of measure given by

$$\left.\frac{\mathrm{d}\mathbb{P}_x^c}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \exp\{c(X_t - x) - \psi(c)t\}$$

for c such that $\mathbb{E}_x[\mathrm{e}^{cX_1}] < \infty$, X is a SNLP with Laplace exponent

$$\psi_c(\theta) = \psi(\theta + c) - \psi(c) \quad \text{for } \theta \geq -c. \tag{4}$$

Its right-inverse function is then given by

$$\Phi_c(q) = \Phi(q + \psi(c)) - c.$$

We write $W_c^{(q)}$ and $Z_c^{(q)}$ for the corresponding scale functions. Note that [7, Lemma 8.4] states that, for $x \geq 0$,

$$W^{(q)}(x) = \mathrm{e}^{cx} W_c^{(q-\psi(c))}(x).$$

A direct application then gives us, for $p, q \geq 0$,

$$W^{(p+q)}(x) = \mathrm{e}^{\Phi(p)x} W_{\Phi(p)}^{(q)}(x). \tag{5}$$

Note also that in this case, $\psi'_{\Phi(p)}(0+) = \psi'(\Phi(p)) > 0$ and $\Phi_{\Phi(p)}(0) = 0$ if $p > 0$.

For examples and numerical techniques related to the computation of scale functions, we refer the reader to [6].

3. Main results

We are interested in the following distribution: for fixed $q, t > 0$, and for all $x \in \mathbb{R}$,

$$\mathbb{E}_x[\mathrm{e}^{-q\int_0^t \mathbf{1}_{(0,a)}(X_s)\mathrm{d}s}; X_t \in \mathrm{d}y]. \tag{6}$$

We will consider the Laplace transform, with respect to t, of the expectation in (6): for all $x \in \mathbb{R}$, set, for $p > 0$,

$$v(x, \mathrm{d}y) := \int_0^\infty \mathrm{e}^{-pt}\mathbb{E}_x[\mathrm{e}^{-q\int_0^t \mathbf{1}_{(0,a)}(X_s)\,\mathrm{d}s}; X_t \in \mathrm{d}y]\,\mathrm{d}t.$$

We note that it can be written as

$$v(x, \mathrm{d}y) = \frac{1}{p}\mathbb{E}_x[\mathrm{e}^{-q\int_0^{\boldsymbol{e}_p} \mathbf{1}_{(0,a)}(X_s)\,\mathrm{d}s}; X_{\boldsymbol{e}_p} \in \mathrm{d}y], \tag{7}$$

where $\boldsymbol{e}_p$ is an exponentially distributed random variable (independent of X) with mean $1/p$.

Remark 1. In what follows, for the sake of simplicity, we will omit indicator functions of the form $\mathbf{1}_{\tau<\infty}$ in our expectations, where τ is a first-passage stopping time when there is no confusion. For example, we will write $\mathbb{E}_x[\mathrm{e}^{-q\tau_a^-}]$ instead of $\mathbb{E}_x[\mathrm{e}^{-q\tau_a^-}; \tau_a^- < \infty]$.

As in [14], for the sake of compactness of the next result, we introduce the following functions. First, for $p, p+q \geq 0$ and $x \in \mathbb{R}$, set

$$\begin{aligned}\mathcal{W}_a^{(p,q)}(x) &= W^{(p+q)}(x) - q\int_0^a W^{(p+q)}(x-z)W^{(p)}(z)\,\mathrm{d}z \\ &= W^{(p)}(x) + q\int_a^x W^{(p+q)}(x-z)W^{(p)}(z)\,\mathrm{d}z,\end{aligned} \tag{8}$$

the second equality following from (2). We note that $\mathcal{W}_a^{(p,q)}(x) = W^{(p)}(x)$ for $x \leq a$.

Secondly, for $p \geq 0$, $q \in \mathbb{R}$ with $p+q \geq 0$ and $x \in \mathbb{R}$, set

$$\mathcal{H}^{(p,q)}(x) = \mathrm{e}^{\Phi(p)x}\biggl[1 + q\int_0^x \mathrm{e}^{-\Phi(p)z}W^{(p+q)}(z)\,\mathrm{d}z\biggr].$$

From (5), we easily obtain that

$$\mathcal{H}^{(p,q)}(x) = \mathrm{e}^{\Phi(p)x} Z_{\Phi(p)}^{(q)}(x). \tag{9}$$

https://doi.org/10.1017/apr.2015.17 Published online by Cambridge University Press

Finally, note that the Laplace transform of $\mathcal{H}^{(p,q)}$ on $[0,\infty)$ is explicitly given by

$$\int_0^\infty \mathrm{e}^{-\lambda x}\mathcal{H}^{(p,q)}(x)\,\mathrm{d}x = \frac{1}{\lambda-\Phi(p)}\bigg(1+\frac{q}{\psi(\lambda)-p-q}\bigg) \quad \text{for } \lambda > \Phi(p+q).$$

Here is our main result.

Theorem 1. *Fix $a>0$, $q\geq 0$, and $x\in\mathbb{R}$. For $p>0$, $y\in\mathbb{R}$,*

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-pt}\mathbb{E}_x[\mathrm{e}^{-q\int_0^t \mathbf{1}_{(0,a)}(X_s)\,\mathrm{d}s}; X_t\in\mathrm{d}y]\,\mathrm{d}t &= \mathrm{e}^{-\Phi(p)a}\bigg(\frac{\mathcal{H}^{(p,q)}(x)-q\int_a^x W^{(p)}(x-z)\mathcal{H}^{(p,q)}(z)\,\mathrm{d}z}{\psi'(\Phi(p))+q\int_0^a \mathrm{e}^{-\Phi(p)z}\mathcal{H}^{(p,q)}(z)\,\mathrm{d}z}\bigg)\\ &\quad\times\bigg\{\mathcal{H}^{(p,q)}(a-y)-q\int_0^{-y}\mathcal{H}^{(p,q)}(a-y-z)W^{(p)}(z)\,\mathrm{d}z\bigg\}\,\mathrm{d}y\\ &\quad-\bigg\{\mathcal{W}_{x-a}^{(p,q)}(x-y)-q\int_0^{-y}\mathcal{W}_{x-a}^{(p,q)}(x-y-z)W^{(p)}(z)\,\mathrm{d}z\bigg\}\,\mathrm{d}y.\end{aligned}$$

The proof of this theorem can be found in Appendix C.

We now extend the result of Theorem 1 to any finite and then semi-infinite interval, i.e. we study the Laplace transform of

$$\mathbb{E}_x[\mathrm{e}^{-q\int_0^t \mathbf{1}_{(a,b)}(X_s)\,\mathrm{d}s}; X_t\in\mathrm{d}y] \quad\text{and}\quad \mathbb{E}_x[\mathrm{e}^{-q\int_0^t \mathbf{1}_{(-\infty,b)}(X_s)\,\mathrm{d}s}; X_t\in\mathrm{d}y],$$

where $t>0$.

We could reapply the same methodology (in a quite shorter form) as in the proof of Theorem 1 for proving the second part of the next result, but we will instead take limits in our main result.

Corollary 1. *Fix $a,b\in\mathbb{R}$ with $a<b$. For all $x\in\mathbb{R}$ and $q\geq 0$, we have, for $p>0$,*

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-pt}\mathbb{E}_x[\mathrm{e}^{-q\int_0^t \mathbf{1}_{(a,b)}(X_s)\,\mathrm{d}s}; X_t\in\mathrm{d}y]\,\mathrm{d}t &= \mathrm{e}^{-\Phi(p)(b-a)}\bigg(\frac{\mathcal{H}^{(p,q)}(x-a)-q\int_b^x W^{(p)}(x-z)\mathcal{H}^{(p,q)}(z-a)\,\mathrm{d}z}{\psi'(\Phi(p))+q\int_0^{b-a} \mathrm{e}^{-\Phi(p)z}\mathcal{H}^{(p,q)}(z)\,\mathrm{d}z}\bigg)\\ &\quad\times\bigg\{\mathcal{H}^{(p,q)}(b-y)-q\int_0^{-y+a}\mathcal{H}^{(p,q)}(b-y-z)W^{(p)}(z)\,\mathrm{d}z\bigg\}\,\mathrm{d}y\\ &\quad-\bigg\{\mathcal{W}_{x-b}^{(p,q)}(x-y)-q\int_0^{-y+a}\mathcal{W}_{x-b}^{(p,q)}(x-y-z)W^{(p)}(z)\,\mathrm{d}z\bigg\}\,\mathrm{d}y,\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-pt}\mathbb{E}_x[\mathrm{e}^{-q\int_0^t \mathbf{1}_{(-\infty,b)}(X_s)\,\mathrm{d}s}; X_t\in\mathrm{d}y]\,\mathrm{d}t &= \bigg\{\bigg(\frac{\Phi(p+q)-\Phi(p)}{q}\bigg)\mathcal{H}^{(p+q,-q)}(x-b)\mathcal{H}^{(p,q)}(b-y)-\mathcal{W}_{x-b}^{(p,q)}(x-y)\bigg\}\,\mathrm{d}y,\end{aligned} \tag{10}$$

where we understand $(\Phi(p+q)-\Phi(p))/q$ in the limiting sense for $q=0$ and, hence, we replace it by $1/\psi'(\Phi(p))$.

https://doi.org/10.1017/apr.2015.17 Published online by Cambridge University Press

Proof. From the spatial homogeneity of the process X, we have

$$\mathbb{E}_x[\mathrm{e}^{-q\int_0^T \mathbf{1}_{(a,b)}(X_s)\,\mathrm{d}s}; X_T \in \mathrm{d}y] = \mathbb{E}_{x-a}[\mathrm{e}^{-q\int_0^T \mathbf{1}_{(0,b-a)}(X_s)\,\mathrm{d}s}; a + X_T \in \mathrm{d}y],$$

so the first result is just an easy consequence of Theorem 1.

For the second result, we will take the limit of the first result when a goes to $-\infty$. First note that, from (5) and (9),

$$\begin{aligned}\frac{\mathcal{H}^{(p,q)}(x+c)}{W^{(p+q)}(c)} &= \mathrm{e}^{\Phi(p)x}\frac{Z^{(q)}_{\Phi(p)}(x+c)}{W^{(q)}_{\Phi(p)}(x+c)}\left(\frac{W^{(q)}_{\Phi(p)}(x+c)}{W^{(q)}_{\Phi(p)}(c)}\right)\\ &\longrightarrow \mathrm{e}^{\Phi(p)x}\frac{q}{\Phi_{\Phi(p)}(q)}\mathrm{e}^{\Phi_{\Phi(p)}(q)x} \quad \text{as } c\to\infty\\ &= \mathrm{e}^{\Phi(p+q)x}\frac{q}{\Phi(p+q)-\Phi(p)},\end{aligned} \tag{11}$$

where, to compute the limit, we used the fact that $\lim_{c\to\infty} Z^{(q)}(c)/W^{(q)}(c) = q/\Phi(q)$ and, to obtain the last expression, we used the results from the beginning of Section 3. As a consequence, using Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned}&\lim_{a\to-\infty}\frac{\mathcal{H}^{(p,q)}(x-a) - q\int_b^x W^{(p)}(x-z)\mathcal{H}^{(p,q)}(z-a)\,\mathrm{d}z}{W^{(p+q)}(-a)}\\ &\quad= \frac{q}{\Phi(p+q)-\Phi(p)}\left(\mathrm{e}^{\Phi(p+q)x} - q\int_b^x W^{(p)}(x-z)\mathrm{e}^{\Phi(p+q)z}\,\mathrm{d}z\right)\\ &\quad= \frac{q}{\Phi(p+q)-\Phi(p)}\mathrm{e}^{\Phi(p+q)b}\mathcal{H}^{(p+q,-q)}(x-b).\end{aligned} \tag{12}$$

On the other hand, using again Lebesgue's dominated convergence theorem with the limit in (5) and (11), we can write

$$\begin{aligned}&\lim_{a\to-\infty}\frac{\mathrm{e}^{\Phi(p)(b-a)}}{W^{(p+q)}(-a)}\left(\psi'(\Phi(p)) + q\int_0^{b-a}\mathrm{e}^{-\Phi(p)z}\mathcal{H}^{(p,q)}(z)\,\mathrm{d}z\right)\\ &\quad= \frac{q^2}{(\Phi(p+q)-\Phi(p))^2}\mathrm{e}^{\Phi(p+q)b},\end{aligned} \tag{13}$$

since we also have

$$\mathrm{e}^{\Phi(p)(b-a)}\frac{\psi'(\Phi(p))}{W^{(p+q)}(-a)} = \mathrm{e}^{\Phi(p)b}\frac{\psi'(\Phi(p))}{W^{(q)}_{\Phi(p)}(-a)} \to 0 \quad \text{as } a\to-\infty.$$

Finally, combining (12) and (13), we have

$$\begin{aligned}&\lim_{a\to-\infty}\mathrm{e}^{-\Phi(p)(b-a)}\frac{\mathcal{H}^{(p,q)}(x-a) - q\int_b^x W^{(p)}(x-z)\mathcal{H}^{(p,q)}(z-a)\,\mathrm{d}z}{\psi'(\Phi(p)) + q\int_0^{b-a}\mathrm{e}^{-\Phi(p)z}\mathcal{H}^{(p,q)}(z)\,\mathrm{d}z}\\ &\quad= \frac{(\Phi(p+q)-\Phi(p))}{q}\mathcal{H}^{(p+q,-q)}(x-b).\end{aligned}$$

Using the fact that $W^{(p)}(x) = 0$ for $x < 0$, the proof is complete. □

https://doi.org/10.1017/apr.2015.17 Published online by Cambridge University Press

Remark 2. (i) We note that $\mathcal{W}_0^{(p,q)}(x) = W^{(p+q)}(x)$. Thus, for $x = b$, the expression in (10) can be written as

$$\int_0^\infty \mathrm{e}^{-pt}\mathbb{E}_b[\mathrm{e}^{-q\int_0^t \mathbf{1}_{(-\infty,b)}(X_s)\mathrm{d}s}; X_t \in \mathrm{d}y]\,\mathrm{d}t$$
$$= \bigg\{\bigg(\frac{\Phi(p+q)-\Phi(p)}{q}\bigg)\mathcal{H}^{(p,q)}(b-y) - W^{(p+q)}(b-y)\bigg\}\,\mathrm{d}y.$$

(ii) Note that when $q = 0$, the result in (10) can be written as

$$\int_0^\infty \mathrm{e}^{-pt}\mathbb{P}_x(X_t \in \mathrm{d}y)\,\mathrm{d}t = \bigg\{\frac{\mathrm{e}^{\Phi(p)(x-y)}}{\psi'(\Phi(p))} - W^{(p)}(x-y)\bigg\}\,\mathrm{d}y,$$

which agrees with [7, Corollary 8.9] for the density of the p-potential measure of X without killing.

4. Pricing step options

4.1. General case

Following the notations introduced in Section 1.1, we consider the risk-neutral price process $S_t = S_0\mathrm{e}^{X_t}$, where $X = \{X_t, t \geq 0\}$ is a SNLP. Recall that the price of a (down-and-out call) step option with knock-out rate ρ and risk-free interest rate r is given by

$$C(T) := \mathrm{e}^{-rT}\mathbb{E}[\mathrm{e}^{-\rho\int_0^T \mathbf{1}_{\{S_t\le L\}}\mathrm{d}t}(S_T-K)_+].$$

In what follows, without loss of generality, we take $r = 0$. The Laplace transform of the price is then given by

$$\int_0^\infty \mathrm{e}^{-pT}C(T)\,\mathrm{d}T$$
$$= \int_{\ln(K/S_0)}^\infty (S_0\mathrm{e}^y - K)\int_0^\infty \mathrm{e}^{-pT}\mathbb{E}[\mathrm{e}^{-\rho\int_0^T \mathbf{1}_{(-\infty,\ln(L/S_0))}(X_s)\,\mathrm{d}s}; X_T \in \mathrm{d}y]\,\mathrm{d}T,$$

where, by Corollary 1 or more precisely (10), we have

$$\int_0^\infty \mathrm{e}^{-pT}\mathbb{E}[\mathrm{e}^{-\rho\int_0^T \mathbf{1}_{(-\infty,\ln(L/S_0))}(X_s)\,\mathrm{d}s}; X_T \in \mathrm{d}y]\,\mathrm{d}T$$
$$= \bigg\{\bigg(\frac{\Phi(p+\rho)-\Phi(p)}{\rho}\bigg)\mathcal{H}^{(p+\rho,-\rho)}\bigg(\ln\bigg(\frac{S_0}{L}\bigg)\bigg)\mathcal{H}^{(p,\rho)}\bigg(\ln\bigg(\frac{L}{S_0}\bigg)-y\bigg)$$
$$- \mathcal{W}^{(p,\rho)}_{\ln(S_0/L)}(-y)\bigg\}\,\mathrm{d}y.$$

From a financial point of view, as we are interested in a down-and-out step option, it is natural to look at the case when $S_0 > L$ and $K > L$.

Corollary 2. *In an exponential spectrally negative Lévy model, the Laplace transform of the price $C(T)$ of a (down-and-out call) step option with $S_0, K > L$ is given by, for $p > 0$ such* *that* $\Phi(p) > 1$,

$$\int_0^\infty e^{-pT} C(T)\,dT = \frac{\Phi(p+\rho)-\Phi(p)}{\rho\Phi(p)(\Phi(p)-1)} \mathcal{H}^{(p+\rho,-\rho)}\Big(\ln\Big(\frac{S_0}{L}\Big)\Big) K \Big(\frac{L}{K}\Big)^{\Phi(p)} - \int_0^{\ln(S_0/K)} (S_0 e^{-y} - K) W^{(p)}(y)\,dy,$$

with

$$\mathcal{H}^{(p+\rho,-\rho)}\Big(\ln\Big(\frac{S_0}{L}\Big)\Big) = \Big(\frac{S_0}{L}\Big)^{\Phi(p+\rho)} \Big(1 - \rho\int_0^{\ln(S_0/L)} e^{-\Phi(p+\rho)y} W^{(p)}(y)\,dy\Big).$$

Proof. We first note that, under the assumptions on S_0, K, and L, on the interval $[\ln(K/S_0), +\infty)$, we have $y > \ln(L/S_0)$ and, thus,

$$\mathcal{H}^{(p,\rho)}\Big(\ln\Big(\frac{L}{S_0}\Big) - y\Big) = \Big(\frac{L}{S_0}\Big)^{\Phi(p)} e^{-\Phi(p)y} \quad \text{and} \quad \mathcal{W}^{(p,\rho)}_{\ln(S_0/L)}(-y) = W^{(p)}(-y).$$

Then, when $K > L$, we have, for $p > 0$ such that $\Phi(p) > 1$,

$$\begin{aligned}\int_0^\infty e^{-pT} C(T)\,dT &= \Big(\frac{\Phi(p+\rho)-\Phi(p)}{\rho}\Big) \mathcal{H}^{(p+\rho,-\rho)}\Big(\ln\Big(\frac{S_0}{L}\Big)\Big)\Big(\frac{L}{S_0}\Big)^{\Phi(p)} \\ &\quad \times \int_{\ln(K/S_0)}^\infty (S_0 e^y - K) e^{-\Phi(p)y}\,dy - \int_{\ln(K/S_0)}^\infty (S_0 e^y - K) W^{(p)}(-y)\,dy \\ &= \frac{\Phi(p+\rho)-\Phi(p)}{\rho\Phi(p)(\Phi(p)-1)} \mathcal{H}^{(p+\rho,-\rho)}\Big(\ln\Big(\frac{S_0}{L}\Big)\Big) K \Big(\frac{L}{K}\Big)^{\Phi(p)} \\ &\quad - \int_{\ln(K/S_0)}^0 (S_0 e^y - K) W^{(p)}(-y)\,dy,\end{aligned}$$

which concludes the proof. □

Of course, using Corollary 1, we could easily derive similar expressions for down-and-out put step options, as well as for up-and-out call/put step options.

4.2. Particular case of a Lévy jump-diffusion process with hyperexponential jumps

We now present the case of a Lévy jump-diffusion process where the jump distribution is a mixture of exponential distributions, as in [14]. In other words, let

$$X_t = ct + \sigma B_t - \sum_{i=1}^{N_t} \xi_i,$$

where $\sigma \geq 0$, $c \in \mathbb{R}$, $B = \{B_t, t \geq 0\}$ is a Brownian motion, $N = \{N_t, t \geq 0\}$ is a Poisson process with intensity $\eta > 0$, and $\{\xi_1, \xi_2, \ldots\}$ are independent and identically distributed random variables with common probability density function given by

$$f_\xi(y) = \Big(\sum_{i=1}^n a_i \alpha_i e^{-\alpha_i y}\Big) \mathbf{1}_{\{y>0\}},$$

https://doi.org/10.1017/apr.2015.17 Published online by Cambridge University Press

where n is a positive integer, $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n$, and $\sum_{i=1}^n a_i = 1$, where $a_i > 0$ for all $i = 1, \ldots, n$. All of the aforementioned objects are mutually independent.

The Laplace exponent of X is then clearly given by

$$\psi(\lambda) = \mathrm{c}\lambda + \frac{1}{2}\sigma^2\lambda^2 + \eta\left(\sum_{i=1}^n \frac{a_i\alpha_i}{\lambda+\alpha_i} - 1\right) \quad \text{for } \lambda > -\alpha_1.$$

In this case

$$\mathbb{E}[X_1] = \psi'(0+) = \mathrm{c} - \eta\sum_{i=1}^n \frac{a_i}{\alpha_i}.$$

We note that $\lambda \mapsto (\psi(\lambda) - q)^{-1}$ is a rational fraction of the form $P(\lambda)/Q(\lambda)$, where P and Q are polynomial functions given by

$$P(\lambda) = \prod_{i=1}^n (\lambda + \alpha_i) \quad \text{and} \quad Q(\lambda) = (\psi(\lambda) - q)\, P(\lambda).$$

Note that if $\sigma > 0$ then Q is a polynomial of degree $n+2$ with $\sigma^2/2$ as the coefficient of the leading term, and if $\sigma = 0$ then Q is a polynomial of degree $n+1$ with c as the coefficient of the leading term. In all cases, P is a polynomial of degree n. To ease the presentation, set $N = (n+1) + \mathbf{1}_{\{\sigma>0\}}$. For $q \geq 0$, the function $\lambda \mapsto (\psi(\lambda) - q)^{-1}$ has N simple poles such that $\theta_N^{(q)} < \theta_{N-1}^{(q)} < \cdots < \theta_2^{(q)} \leq 0 \leq \theta_1^{(q)}$ with $\theta_1^{(q)} = \Phi(q)$. Studying the functions $\lambda \mapsto \psi(\lambda) - q$ and $\lambda \mapsto Q(\lambda)$ extended on $\mathbb{R} \setminus \{-\alpha_n, \ldots, -\alpha_1\}$, we deduce that, for $\sigma > 0$,

$$\theta_{n+2}^{(q)} < -\alpha_n < \theta_{n+1}^{(q)} < -\alpha_{n-1} < \theta_n^{(q)} \cdots < -\alpha_1 < \theta_2^{(q)} \leq 0 \leq \theta_1^{(q)},$$

and, for $\sigma = 0$,

$$-\alpha_n < \theta_{n+1}^{(q)} < -\alpha_{n-1} < \theta_n^{(q)} \cdots < -\alpha_1 < \theta_2^{(q)} \leq 0 \leq \theta_1^{(q)},$$

where, in both cases, we have one of the following three cases:

- $\theta_2^{(q)} < \theta_1^{(q)} = \Phi(q) = 0$ if and only if $q = 0$ and $\psi'(0) > 0$;
- $\theta_2^{(q)} = \theta_1^{(q)} = \Phi(q) = 0$ if and only if $q = 0$ and $\psi'(0) = 0$;
- $\theta_2^{(q)} = 0 < \theta_1^{(q)} = \Phi(q)$ if and only if $q = 0$ and $\psi'(0) < 0$.

For more details see, e.g. [3], where a similar analysis is undertaken for a closely related jump-diffusion process. For simplicity, in what follows, we assume that either $q > 0$ or $q = 0$ and $\psi'(0) \neq 0$, in which cases $\theta_2^{(q)} \neq \theta_1^{(q)}$. Consequently, using the classical decomposition of rational fractions and noting that

$$\frac{P(\theta_i^{(q)})}{Q'(\theta_i^{(q)})} = \frac{1}{\psi'(\theta_i^{(q)})},$$

we can write

$$\frac{1}{\psi(\lambda) - q} = \sum_{i=1}^N \frac{1}{\psi'(\theta_i^{(q)})} \frac{1}{(\lambda - \theta_i^{(q)})}. \tag{14}$$

In conclusion, by Laplace inversion, we have, for $x \geq 0$,

$$W^{(q)}(x) = \sum_{i=1}^{N} \frac{\mathrm{e}^{\theta_i^{(q)} x}}{\psi'(\theta_i^{(q)})}, \tag{15}$$

$$Z^{(q)}(x) = \begin{cases} q \displaystyle\sum_{i=1}^{N} \frac{\mathrm{e}^{\theta_i^{(q)} x}}{\theta_i^{(q)} \psi'(\theta_i^{(q)})} & \text{if } q > 0, \\ 1 & \text{if } q = 0, \end{cases}$$

since taking $\lambda = 0$ in (14) gives $q \sum_{i=1}^{N} (\theta_i^{(q)} \psi'(\theta_i^{(q)}))^{-1} = 1$.

Remark 3. We recover well-known expressions of scale functions in two specific cases; see, e.g. [6]. When $\sigma > 0$ and $n = 0$, X is a Brownian motion with drift and, for $x \geq 0$,

$$W^{(q)}(x) = \frac{\mathrm{e}^{\theta_1^{(q)} x}}{\sqrt{\Delta_q}} - \frac{\mathrm{e}^{\theta_2^{(q)} x}}{\sqrt{\Delta_q}}$$

with $\theta_1^{(q)} = (1/\sigma^2)(\sqrt{\Delta_q} - \mathrm{c})$, $\theta_2^{(q)} = (-1/\sigma^2)(\sqrt{\Delta_q} + \mathrm{c})$, and $\Delta_q = \mathrm{c}^2 + 2\sigma^2 q$.

When $\sigma = 0$ and $n = 1$, X is a compound Poisson process with drift and exponential jumps and, for $x \geq 0$,

$$W^{(q)}(x) = \frac{\alpha + \theta_1^{(q)}}{\sqrt{\Delta_q}} \mathrm{e}^{\theta_1^{(q)} x} - \frac{\alpha + \theta_2^{(q)}}{\sqrt{\Delta_q}} \mathrm{e}^{\theta_2^{(q)} x}$$

with $\theta_1^{(q)} = (1/2\mathrm{c})(q + \eta - \mathrm{c}\alpha + \sqrt{\Delta_q})$, $\theta_2^{(q)} = (1/2\mathrm{c})(q + \eta - \mathrm{c}\alpha - \sqrt{\Delta_q})$, and $\Delta_q = (q + \eta - \mathrm{c}\alpha)^2 + 4\mathrm{c}\alpha q$.

Now, let us explicitly compute the Laplace transform of the price $C(T)$, given by Corollary 2, for this particular SNLP.

We first give an expression of $\mathcal{H}^{(p,q)}$. Let us recall, from (9), that $\mathcal{H}^{(p,q)}(x) = \mathrm{e}^{\Phi(p)x} Z_{\Phi(p)}^{(q)}(x)$ with $\Phi(p) = \theta_1^{(p)}$. Denoting by $\theta_{\Phi(p),i}^{(q)}$ the poles of $(\psi_{\Phi(p)}(\lambda) - q)^{-1}$ and using (4), we have $\theta_{\Phi(p),i}^{(q)} = \theta_i^{(p+q)} - \Phi(p)$ and $\psi'_{\Phi(p)}(\theta_{\Phi(p),i}^{(q)}) = \psi'(\theta_i^{(p+q)})$. Consequently, for $q > 0$ and $x \geq 0$,

$$\mathcal{H}^{(p,q)}(x) = q \sum_{i=1}^{N} \frac{\mathrm{e}^{\theta_i^{(p+q)} x}}{(\theta_i^{(p+q)} - \theta_1^{(p)}) \psi'(\theta_i^{(p+q)})}, \tag{16}$$

and, for $x < 0$, $\mathcal{H}^{(p,q)}(x) = \exp(\theta_1^{(p)} x)$.

We also compute, for $a < x$,

$$\mathcal{W}_a^{(p,q)}(x) = q \sum_{i,j=1}^{N} \frac{\exp(\theta_i^{(p+q)} x + (\theta_j^{(p)} - \theta_i^{(p+q)}) a)}{(\theta_i^{(p+q)} - \theta_j^{(p)}) \psi'(\theta_i^{(p+q)}) \psi'(\theta_j^{(p)})}.$$

Note that $\mathcal{W}_a^{(p,q)}$ had already been computed in [14]. However, our expression here is slightly simpler because we made one more simplification.

Corollary 3. *In a Lévy jump-diffusion model with hyperexponential jumps, the Laplace transform of the price $C(T)$ of a (down-and-out call) step option with $S_0, K > L$ is given by,*

https://doi.org/10.1017/apr.2015.17 Published online by Cambridge University Press

for $p > 0$ such that $\Phi(p) > 1$,

$$\int_0^\infty \mathrm{e}^{-pT} C(T)\,\mathrm{d}T = K \sum_{i=2}^{N} \frac{1}{\psi'(\theta_i^{(p)})} \Bigg[\frac{\Phi(p+\rho) - \Phi(p)}{(\Phi(p+\rho) - \theta_i^{(p)})\Phi(p)(\Phi(p)-1)} \bigg(\frac{L}{K}\bigg)^{\Phi(p)} \bigg(\frac{S_0}{L}\bigg)^{\theta_i^{(p)}} - \frac{1}{\theta_i^{(p)}(\theta_i^{(p)}-1)} \bigg(\frac{S_0}{K}\bigg)^{\theta_i^{(p)}} \Bigg],$$

when $S_0 > K > L$, and

$$\int_0^\infty \mathrm{e}^{-pT} C(T)\,\mathrm{d}T = K \frac{\Phi(p+\rho) - \Phi(p)}{\Phi(p)(\Phi(p)-1)} \sum_{i=1}^{N} \frac{1}{(\Phi(p+\rho) - \theta_i^{(p)})\psi'(\theta_i^{(p)})} \bigg(\frac{L}{K}\bigg)^{\Phi(p)} \bigg(\frac{S_0}{L}\bigg)^{\theta_i^{(p)}},$$

when $K \geq S_0 > L$.

Proof. From (16), we deduce that, for $S_0 > L$,

$$\mathcal{H}^{(p+\rho,-\rho)}\bigg(\ln\bigg(\frac{S_0}{L}\bigg)\bigg) = \rho \sum_{i=1}^{N} \frac{1}{(\theta_1^{(p+\rho)} - \theta_i^{(p)})\psi'(\theta_i^{(p)})} \bigg(\frac{S_0}{L}\bigg)^{\theta_i^{(p)}}.$$

Using (15), we also deduce that, when $S_0 > K$,

$$\int_0^{\ln(S_0/K)} (S_0\mathrm{e}^{-y} - K) W^{(p)}(y)\,\mathrm{d}y = K \sum_{i=1}^{N} \frac{1}{\theta_i^{(p)}(\theta_i^{(p)}-1)\psi'(\theta_i^{(p)})} \bigg(\frac{S_0}{K}\bigg)^{\theta_i^{(p)}},$$

otherwise, when $K \geq S_0$ the integral is equal to 0. Then, from Corollary 2, for $S_0 > K$,

$$\begin{aligned}
\int_0^\infty \mathrm{e}^{-pT} C(T)\,\mathrm{d}T &= \frac{\theta_1^{(p+\rho)} - \theta_1^{(p)}}{\rho\theta_1^{(p)}(\theta_1^{(p)}-1)} \mathcal{H}^{(p+\rho,-\rho)}\bigg(\ln\bigg(\frac{S_0}{L}\bigg)\bigg) K \bigg(\frac{L}{K}\bigg)^{\theta_1^{(p)}} \\
&\quad - \int_0^{\ln(S_0/K)} (S_0\mathrm{e}^{-y} - K) W^{(p)}(y)\,\mathrm{d}y \\
&= K \frac{\theta_1^{(p+\rho)} - \theta_1^{(p)}}{\theta_1^{(p)}(\theta_1^{(p)}-1)} \sum_{i=1}^{N} \frac{1}{(\theta_1^{(p+\rho)} - \theta_i^{(p)})\psi'(\theta_i^{(p)})} \bigg(\frac{S_0}{L}\bigg)^{\theta_i^{(p)}} \bigg(\frac{L}{K}\bigg)^{\theta_1^{(p)}} \\
&\quad - K \sum_{i=1}^{N} \frac{1}{\theta_i^{(p)}(\theta_i^{(p)}-1)\psi'(\theta_i^{(p)})} \bigg(\frac{S_0}{K}\bigg)^{\theta_i^{(p)}} \\
&= \sum_{i=2}^{N} \frac{1}{\psi'(\theta_i^{(p)})} S_0^{\theta_i^{(p)}} \Bigg[\frac{\theta_1^{(p+\rho)} - \theta_1^{(p)}}{(\theta_1^{(p+\rho)} - \theta_i^{(p)})\theta_1^{(p)}(\theta_1^{(p)}-1)} L^{(\theta_1^{(p)} - \theta_i^{(p)})} K^{1-\theta_1^{(p)}} \\
&\qquad - \frac{1}{\theta_i^{(p)}(\theta_i^{(p)}-1)} K^{1-\theta_i^{(p)}} \Bigg].
\end{aligned}$$

The result follows since $\theta_1^{(s)} = \Phi(s)$ for all $s \geq 0$. □

https://doi.org/10.1017/apr.2015.17 Published online by Cambridge University Press

Appendix A. A probabilistic decomposition

In our proof of Theorem 1, we will use a probabilistic decomposition of $v(x, \mathrm{d}y)$ (see Lemma 1 below). In order to do so, we first recall known results about resolvents; see, e.g. [6, Theorem 2.7].

First, for $0 \le x \le a$,

$$U^{(p)}(x,\, \mathrm{d}y; a) = \int_0^{+\infty} \mathrm{e}^{-pt} \mathbb{P}_x(X_t \in \mathrm{d}y; t < \tau_0^- \wedge \tau_a^+)\, \mathrm{d}t$$

is the p-potential measure of X killed on exiting $[0, a]$ (see also [7, Theorem 8.7]) and it has a density supported on $[0, a]$ given by

$$U^{(p)}(x,\, \mathrm{d}y; a) = \left\{ \frac{W^{(p)}(x) W^{(p)}(a-y)}{W^{(p)}(a)} - W^{(p)}(x-y) \right\} \mathrm{d}y. \tag{17}$$

Also, for $x \le a$,

$$U_+^{(p)}(x,\, \mathrm{d}y; a) = \int_0^{+\infty} \mathrm{e}^{-pt} \mathbb{P}_x(X_t \in \mathrm{d}y; t < \tau_a^+)\, \mathrm{d}t$$

is the p-potential measure of X killed on exiting $(-\infty, a]$ and it has a density supported on $(-\infty, a]$ given by

$$U_+^{(p)}(x,\, \mathrm{d}y; a) = \{\mathrm{e}^{\Phi(p)(x-a)} W^{(p)}(a-y) - W^{(p)}(x-y)\}\, \mathrm{d}y. \tag{18}$$

Finally, for $x \ge a$,

$$U_-^{(p)}(x,\, \mathrm{d}y; a) = \int_0^{+\infty} \mathrm{e}^{-pt} \mathbb{P}_x(X_t \in \mathrm{d}y; t < \tau_a^-)\, \mathrm{d}t$$

is the p-potential measure of X killed on exiting $[a, +\infty)$ and it has a density supported on $[a, +\infty)$ given by

$$U_-^{(p)}(x,\, \mathrm{d}y; a) = \{\mathrm{e}^{-\Phi(p)(y-a)} W^{(p)}(x-a) - W^{(p)}(x-y)\}\, \mathrm{d}y. \tag{19}$$

Recall that if we assume X has paths of BV, then, for any $b \in \mathbb{R}$, we have $X_{\tau_b^-} < b$ almost surely (a.s.). In fact, this means that X does not creep downward, which is true as soon as the Brownian part is 0 ($\sigma = 0$).

Lemma 1. *Assume that $a > 0$. For all $x \in \mathbb{R}$, $v(x, \mathrm{d}y)$ satisfies*

$$\begin{aligned} v(x,\, \mathrm{d}y) = {} & U_-^{(p)}(x,\, \mathrm{d}y; a)\, \mathbf{1}_{y>a} + \mathbb{E}_x[\mathrm{e}^{-p\tau_a^-} U^{(p+q)}(X_{\tau_a^-},\, \mathrm{d}y; a)]\, \mathbf{1}_{y\in[0,a]} \\ & + \frac{v(a,\, \mathrm{d}y)}{W^{(p+q)}(a)} \mathbb{E}_x[\mathrm{e}^{-p\tau_a^-} W^{(p+q)}(X_{\tau_a^-})] \\ & + v(0,\, \mathrm{d}y) \mathbb{E}_x[\mathrm{e}^{-p\tau_a^-} \mathbb{E}_{X_{\tau_a^-}}[\mathrm{e}^{-(p+q)\tau_0^- + \Phi(p) X_{\tau_0^-}}; \tau_0^- < \tau_a^+]] \\ & + \mathbb{E}_x[\mathrm{e}^{-p\tau_a^-} \mathbb{E}_{X_{\tau_a^-}}[\mathrm{e}^{-(p+q)\tau_0^-} U_+^{(p)}(X_{\tau_0^-},\, \mathrm{d}y; 0); \tau_0^- < \tau_a^+]]\, \mathbf{1}_{y<0}\, . \end{aligned}$$

Proof. For simplicity, we assume that X has paths of BV. The proof is almost identical when X has paths of unbounded variation (UBV), so the details are left to the reader.

We use representation (7) of $v(\cdot,\ \mathrm{d}y)$. For $x < 0$, we note that

$$\begin{aligned} v(x,\ \mathrm{d}y) &= \frac{1}{p}\mathbb{P}_x(X_{\boldsymbol{e}_p} \in \mathrm{d}y; \boldsymbol{e}_p < \tau_0^+) + v(0,\ \mathrm{d}y)\mathbb{P}_x(\tau_0^+ < \boldsymbol{e}_p) \\ &= U_+^{(p)}(x,\ \mathrm{d}y; 0)\,\mathbf{1}_{y<0} + \mathrm{e}^{\Phi(p)x} v(0,\ \mathrm{d}y). \end{aligned} \tag{20}$$

Let us now consider $0 \le x < a$. We have

$$\begin{aligned} v(x, \mathrm{d}y) &= \frac{1}{p}\mathbb{E}_x[\mathrm{e}^{-q\boldsymbol{e}_p}; X_{\boldsymbol{e}_p} \in \mathrm{d}y; \boldsymbol{e}_p < \tau_0^- \wedge \tau_a^+] + v(a,\ \mathrm{d}y)\mathbb{E}_x[\mathrm{e}^{-q\tau_a^+}; \tau_a^+ < \tau_0^- \wedge \boldsymbol{e}_p] \\ &\quad + \mathbb{E}_x[\mathrm{e}^{-q\tau_0^-} v(X_{\tau_0^-},\ \mathrm{d}y); \tau_0^- < \boldsymbol{e}_p \wedge \tau_a^+] \\ &= U^{(p+q)}(x,\ \mathrm{d}y; a)\,\mathbf{1}_{y\in[0,a]} + v(a,\ \mathrm{d}y)\mathbb{E}_x[\mathrm{e}^{-(p+q)\tau_a^+}; \tau_a^+ < \tau_0^-] \\ &\quad + \mathbb{E}_x[\mathrm{e}^{-(p+q)\tau_0^-} v(X_{\tau_0^-},\ \mathrm{d}y); \tau_0^- < \tau_a^+] \\ &= U^{(p+q)}(x,\ \mathrm{d}y; a)\,\mathbf{1}_{y\in[0,a]} + v(a,\ \mathrm{d}y)\frac{W^{(p+q)}(x)}{W^{(p+q)}(a)} \\ &\quad + \mathbb{E}_x[\mathrm{e}^{-(p+q)\tau_0^-} v(X_{\tau_0^-},\ \mathrm{d}y); \tau_0^- < \tau_a^+]. \end{aligned}$$

Since X is of BV, $X_{\tau_0^-} < 0$ a.s. Consequently, using (20), we note that for any $x < a$,

$$\begin{aligned} v(x,\ \mathrm{d}y) &= U^{(p+q)}(x,\ \mathrm{d}y; a)\,\mathbf{1}_{y\in[0,a]} + v(a, \mathrm{d}y)\frac{W^{(p+q)}(x)}{W^{(p+q)}(a)} \\ &\quad + v(0,\ \mathrm{d}y)\mathbb{E}_x[\mathrm{e}^{-(p+q)\tau_0^- + \Phi(p)X_{\tau_0^-}}; \tau_0^- < \tau_a^+] \\ &\quad + \mathbb{E}_x[\mathrm{e}^{-(p+q)\tau_0^-} U_+^{(p)}(X_{\tau_0^-},\ \mathrm{d}y; 0); \tau_0^- < \tau_a^+]\,\mathbf{1}_{y<0}\,. \end{aligned} \tag{21}$$

We now study the last case. Let $x \ge a$,

$$\begin{aligned} v(x,\ \mathrm{d}y) &= \frac{1}{p}\mathbb{P}_x(X_{\boldsymbol{e}_p} \in \mathrm{d}y; \boldsymbol{e}_p < \tau_a^-) + \mathbb{E}_x[v(X_{\tau_a^-},\ \mathrm{d}y); \tau_a^- < \boldsymbol{e}_p] \\ &= U_-^{(p)}(x,\ \mathrm{d}y; a)\,\mathbf{1}_{y>a} + \mathbb{E}_x[\mathrm{e}^{-p\tau_a^-} v(X_{\tau_a^-},\ \mathrm{d}y)]. \end{aligned}$$

Since $X_{\tau_a^-} < a$ a.s., we deduce from (21) that, for $x \ge a$,

$$\begin{aligned} v(x,\ \mathrm{d}y) &= U_-^{(p)}(x,\ \mathrm{d}y; a)\,\mathbf{1}_{y>a} + \mathbb{E}_x[\mathrm{e}^{-p\tau_a^-} U^{(p+q)}(X_{\tau_a^-},\ \mathrm{d}y; a)]\,\mathbf{1}_{y\in[0,a]} \\ &\quad + \frac{v(a,\ \mathrm{d}y)}{W^{(p+q)}(a)}\mathbb{E}_x[\mathrm{e}^{-p\tau_a^-} W^{(p+q)}(X_{\tau_a^-})] \\ &\quad + v(0,\ \mathrm{d}y)\mathbb{E}_x[\mathrm{e}^{-p\tau_a^-}\mathbb{E}_{X_{\tau_a^-}}[\mathrm{e}^{-(p+q)\tau_0^- + \Phi(p)X_{\tau_0^-}}; \tau_0^- < \tau_a^+]] \\ &\quad + \mathbb{E}_x[\mathrm{e}^{-p\tau_a^-}\mathbb{E}_{X_{\tau_a^-}}[\mathrm{e}^{-(p+q)\tau_0^-} U_+^{(p)}(X_{\tau_0^-},\ \mathrm{d}y; 0); \tau_0^- < \tau_a^+]]\,\mathbf{1}_{y<0}\,. \end{aligned}$$

To conclude the proof, we just note that this last expression is satisfied for any $x \in \mathbb{R}$. □

https://doi.org/10.1017/apr.2015.17 Published online by Cambridge University Press

Appendix B. Technical results

It is easy to show that the Laplace transform of $x \mapsto \int_0^x W^{(p)}(x-y)\mathcal{H}^{(p,q)}(y)\,\mathrm{d}y$ is given by

$$\lambda \mapsto \bigg(\frac{1}{\lambda - \Phi(p)}\bigg)\bigg(\frac{1}{\psi(\lambda) - (p+q)}\bigg).$$

Therefore,

$$\int_0^x W^{(p)}(x-z)\mathcal{H}^{(p,q)}(z)\,\mathrm{d}z = \int_0^x \mathrm{e}^{\Phi(p)(x-z)} W^{(p+q)}(z)\,\mathrm{d}z,$$

from which we deduce that, in the spirit of (8) and [14, Equation (6)],

$$\mathcal{H}^{(p,q)}(x) - q\int_a^x W^{(p)}(x-y)\mathcal{H}^{(p,q)}(y)\,\mathrm{d}y = \mathrm{e}^{\Phi(p)x} + q\int_0^a W^{(p)}(x-y)\mathcal{H}^{(p,q)}(y)\,\mathrm{d}y.$$

Note that, we can use the last relation to rewrite our main result in Theorem 1 and Lemma 5.

The next result is well known (see, e.g. [7] and [11]), but we rewrite it in terms of $\mathcal{H}^{(p,q)}$ for future reference.

Lemma 2. *For $x \in \mathbb{R}$, $a \le b$, $p, q \ge 0$, we have*

$$\begin{aligned}
&\mathbb{E}_x[\mathrm{e}^{-(p+q)\tau_a^- + \Phi(p)X_{\tau_a^-}}; \tau_a^- < \tau_b^+] \\
&\quad = \mathrm{e}^{\Phi(p)a}\mathcal{H}^{(p,q)}(x-a) - \frac{\mathcal{H}^{(p,q)}(b-a)}{W^{(p+q)}(b-a)}\mathrm{e}^{\Phi(p)a}W^{(p+q)}(x-a).
\end{aligned}$$

Moreover, for $x, a \in \mathbb{R}$, $p \ge 0$, and $q > 0$, we have

$$\begin{aligned}
&\mathbb{E}_x[\mathrm{e}^{-(p+q)\tau_a^- + \Phi(p)X_{\tau_a^-}}; \tau_a^- < \infty] \\
&\quad = \mathrm{e}^{\Phi(p)a}\mathcal{H}^{(p,q)}(x-a) - \frac{q}{\Phi(p+q) - \Phi(p)}\mathrm{e}^{\Phi(p)a}W^{(p+q)}(x-a)
\end{aligned}$$

and, when $q \to 0$, we obtain

$$\mathbb{E}_x[\mathrm{e}^{-p\tau_a^- + \Phi(p)X_{\tau_a^-}}; \tau_a^- < \infty] = \mathrm{e}^{\Phi(p)x} - \psi'(\Phi(p))\mathrm{e}^{\Phi(p)a}W^{(p)}(x-a).$$

The next lemma is an immediate consequence of [14, Lemma 2.2].

Lemma 3. *Let $p, q \ge 0$. For $a \le b$ and $x, y \in \mathbb{R}$, we have*

$$\begin{aligned}
&\mathbb{E}_x[\mathrm{e}^{-p\tau_a^-}W^{(q)}(X_{\tau_a^-} - y); \tau_a^- < \tau_b^+] \\
&\quad = W^{(p)}(x-y) + (q-p)\int_0^{a-y} W^{(p)}(x-y-z)W^{(q)}(z)\,\mathrm{d}z \\
&\qquad - \frac{W^{(p)}(x-a)}{W^{(p)}(b-a)}\bigg(W^{(p)}(b-y) + (q-p)\int_0^{a-y} W^{(p)}(b-y-z)W^{(q)}(z)\,\mathrm{d}z\bigg),
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}_x[\mathrm{e}^{-p\tau_a^-}W^{(q)}(X_{\tau_a^-} - y); \tau_a^- < +\infty] \\
&\quad = W^{(p)}(x-y) + (q-p)\int_0^{a-y} W^{(p)}(x-y-z)W^{(q)}(z)\,\mathrm{d}z \\
&\qquad - W^{(p)}(x-a)\mathcal{H}^{(p,q-p)}(a-y).
\end{aligned}$$

https://doi.org/10.1017/apr.2015.17 Published online by Cambridge University Press

Proof. First, let us note that, by the spatial homogeneity of X, we have

$$\mathbb{E}_x[\mathrm{e}^{-p\tau_a^-}W^{(q)}(X_{\tau_a^-}-y);\ \tau_a^- < \tau_b^+] = \mathbb{E}_{x-y}[\mathrm{e}^{-p\tau_{a-y}^-}W^{(q)}(X_{\tau_{a-y}^-});\ \tau_{a-y}^- < \tau_{b-y}^+],$$

$$\mathbb{E}_x[\mathrm{e}^{-p\tau_a^-}W^{(q)}(X_{\tau_a^-}-y);\ \tau_a^- < +\infty] = \mathbb{E}_{x-y}[\mathrm{e}^{-p\tau_{a-y}^-}W^{(q)}(X_{\tau_{a-y}^-});\ \tau_{a-y}^- < +\infty].$$

So, it suffices to prove the lemma for the $y=0$ case.

Let $y=0$. We know from [14, Lemmas 2.1 and 2.2] that

$$\begin{aligned}\mathbb{E}_x&[\mathrm{e}^{-p\tau_a^-}W^{(q)}(X_{\tau_a^-});\ \tau_a^- < \tau_b^+]\\ &= W^{(q)}(x)-(q-p)\int_a^x W^{(p)}(x-z)W^{(q)}(z)\,\mathrm{d}z\\ &\quad -\frac{W^{(p)}(x-a)}{W^{(p)}(b-a)}\bigg(W^{(q)}(b)-(q-p)\int_a^b W^{(p)}(b-z)W^{(q)}(z)\,\mathrm{d}z\bigg).\end{aligned}$$

Using (2), the expression can be written as

$$\begin{aligned}\mathbb{E}_x&[\mathrm{e}^{-p\tau_a^-}W^{(q)}(X_{\tau_a^-});\ \tau_a^- < \tau_b^+]\\ &= W^{(p)}(x)+(q-p)\int_0^a W^{(p)}(x-z)W^{(q)}(z)\,\mathrm{d}z\\ &\quad -\frac{W^{(p)}(x-a)}{W^{(p)}(b-a)}\bigg(W^{(p)}(b)+(q-p)\int_0^a W^{(p)}(b-z)W^{(q)}(z)\,\mathrm{d}z\bigg).\end{aligned}$$

Since

$$\mathbb{E}_x[\mathrm{e}^{-p\tau_a^-}W^{(q)}(X_{\tau_a^-});\ \tau_a^- < \infty] = \lim_{b\to+\infty}\mathbb{E}_x[\mathrm{e}^{-p\tau_a^-}W^{(q)}(X_{\tau_a^-});\ \tau_a^- < \tau_b^+],$$

$$\frac{W^{(p)}(b)}{W^{(p)}(b-a)} = \mathrm{e}^{\Phi(p)a}\frac{W_{\Phi(p)}(b)}{W_{\Phi(p)}(b-a)},$$

$$\frac{W^{(p)}(b-z)}{W^{(p)}(b-a)} = \mathrm{e}^{\Phi(p)(a-z)}\frac{W_{\Phi(p)}(b-z)}{W_{\Phi(p)}(b-a)},$$

by Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}\mathbb{E}_x&[\mathrm{e}^{-p\tau_a^-}W^{(q)}(X_{\tau_a^-});\ \tau_a^- < +\infty]\\ &= W^{(p)}(x)+(q-p)\int_0^a W^{(p)}(x-z)W^{(q)}(z)\,\mathrm{d}z\\ &\quad -\mathrm{e}^{\Phi(p)(a)}W^{(p)}(x-a)\bigg(1+(q-p)\int_0^a \mathrm{e}^{-\Phi(p)z}W^{(q)}(z)\,\mathrm{d}z\bigg).\end{aligned}$$ □

Using the same tools as in the previous lemma, we also have the following result for $Z^{(q)}$.

Lemma 4. *Let $p,q\ge 0$. For $a\le b$ and $x,y\in\mathbb{R}$, we have*

$$\begin{aligned}\mathbb{E}_x&[\mathrm{e}^{-p\tau_a^-}Z^{(q)}(X_{\tau_a^-}-y);\ \tau_a^- < \tau_b^+]\\ &= Z^{(q)}(x-y)+(q-p)\int_0^{a-y} W^{(p)}(x-y-z)Z^{(q)}(z)\,\mathrm{d}z\\ &\quad -\frac{W^{(p)}(x-a)}{W^{(p)}(b-a)}\bigg(Z^{(q)}(b-y)+(q-p)\int_0^{a-y} W^{(p)}(b-y-z)Z^{(q)}(z)\,\mathrm{d}z\bigg).\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}_x[\mathrm{e}^{-p\tau_a^-}Z^{(q)}(X_{\tau_a^-}-y);\ \tau_a^-<+\infty]\\&\quad=Z^{(p)}(x-y)+(q-p)\int_0^{a-y}W^{(p)}(x-y-z)Z^{(q)}(z)\,\mathrm{d}z\\&\qquad-W^{(p)}(x-a)\mathrm{e}^{\Phi(p)(a-y)}\bigg[\frac{p}{\Phi(p)}+(q-p)\int_0^{a-y}\mathrm{e}^{\Phi(p)z}Z^{(q)}(z)\,\mathrm{d}z\bigg],\end{aligned}$$

where $\lim_{p\to 0} p/\Phi(p)\to\psi'(0)\vee 0$ *in the* $p=0$ *case.*

Lemma 5. *For all* $a, x\in\mathbb{R}$, $p\geq 0$, *and* $p+q\geq 0$,

$$\begin{aligned}&\mathbb{E}_x[\mathrm{e}^{-p\tau_a^-}\mathcal{H}^{(p,q)}(X_{\tau_a^-});\ \tau_a^-<\infty]\\&\quad=\mathcal{H}^{(p,q)}(x)-q\int_a^x W^{(p)}(x-z)\mathcal{H}^{(p,q)}(z)\,\mathrm{d}z\\&\qquad-W^{(p)}(x-a)\mathrm{e}^{\Phi(p)a}\bigg(\psi'(\Phi(p))+q\int_0^a\mathrm{e}^{-\Phi(p)z}\mathcal{H}^{(p,q)}(z)\,\mathrm{d}z\bigg).\end{aligned}$$

Proof. First, using (9), we note that

$$\begin{aligned}\mathbb{E}_x[\mathrm{e}^{-p\tau_a^-}\mathcal{H}^{(p,q)}(X_{\tau_a^-});\ \tau_a^-<\infty]&=\mathbb{E}_x[\mathrm{e}^{-p\tau_a^-+\Phi(p)X_{\tau_a^-}}Z^{(q)}_{\Phi(p)}(X_{\tau_a^-});\ \tau_a^-<\infty]\\&=\mathrm{e}^{\Phi(p)x}\mathbb{E}_x^{\Phi(p)}[Z^{(q)}_{\Phi(p)}(X_{\tau_a^-});\ \tau_a^-<\infty]\\&=\mathrm{e}^{\Phi(p)x}\mathbb{E}_x[Z^{(q)}_{\Phi(p)}(Y_{\nu_a^-});\ \nu_a^-<\infty],\end{aligned}$$

where Y is the SNLP obtained from X by the change of measure with coefficient $\Phi(p)$ and $\nu_a^-=\inf\{t>0\colon Y_t<a\}$. Therefore, we can apply Lemma 4 and write

$$\begin{aligned}&\mathbb{E}_x[\mathrm{e}^{-p\tau_a^-}\mathcal{H}^{(p,q)}(X_{\tau_a^-});\ \tau_a^-<\infty]\\&\quad=Z_{\Phi(p)}(x)+q\int_0^a W_{\Phi(p)}(x-y)Z^{(q)}_{\Phi(p)}(y)\,\mathrm{d}y\\&\qquad-W_{\Phi(p)}(x-a)\mathrm{e}^{\Phi_{\Phi(p)}(0)a}\bigg[\psi'_{\Phi(p)}(0)+q\int_0^a\mathrm{e}^{\Phi_{\Phi(p)}(0)y}Z^{(q)}_{\Phi(p)}(y)\,\mathrm{d}y\bigg]\end{aligned}$$

since $\psi'_{\Phi(p)}(0)=\psi'(\Phi(p))\geq 0$ and $\Phi_{\Phi(p)}(0)=0$. The result follows from the discussion at the end of Section 2.1. □

Appendix C. Proofs of the main results

Proof of Theorem 1 when X is of BV. First, we assume that X has paths of BV. In order to simplify the manipulations, we introduce the following quantities:

$$\begin{aligned}A(x)&:=\mathbb{E}_x[\mathrm{e}^{-p\tau_a^-}\mathbb{E}_{X_{\tau_a^-}}[\mathrm{e}^{-(p+q)\tau_0^-+\Phi(p)X_{\tau_0^-}};\ \tau_0^-<\tau_a^+]],\\B(x;a)&:=\mathbb{E}_x[\mathrm{e}^{-p\tau_a^-}W^{(p+q)}(X_{\tau_a^-})],\qquad C(x,\mathrm{d}y):=\mathbb{E}_x[\mathrm{e}^{-p\tau_a^-}U^{(p+q)}(X_{\tau_a^-},\mathrm{d}y;a)],\\D(x,\mathrm{d}y)&:=\mathbb{E}_x[\mathrm{e}^{-p\tau_a^-}\mathbb{E}_{X_{\tau_a^-}}[\mathrm{e}^{-(p+q)\tau_0^-}U_+^{(p)}(X_{\tau_0^-},\mathrm{d}y;0);\ \tau_0^-<\tau_a^+]],\end{aligned}$$

https://doi.org/10.1017/apr.2015.17 Published online by Cambridge University Press

where, as mentioned previously, all these expectations are taken over the set $\{\tau_a^- < \infty\}$. At this point, let us note that these quantities can be made more explicit (in terms of scale functions) using results from Appendix B; this will be done later on.

Then, using Lemma 1, we can write, for all $x \in \mathbb{R}$,

$$v(x,\,\mathrm{d}y) = A(x)v(0,\,\mathrm{d}y) + \frac{B(x;a)}{W^{(p+q)}(a)}v(a,\,\mathrm{d}y) \\ + U_-^{(p)}(x,\,\mathrm{d}y;a)\,\mathbf{1}_{y>a} + C(x,\,\mathrm{d}y)\,\mathbf{1}_{y\in[0,a]} + D(x,\,\mathrm{d}y)\,\mathbf{1}_{y<0}\,. \tag{22}$$

Therefore, the quantities $v(0,\,\mathrm{d}y)$ and $v(a,\,\mathrm{d}y)$ satisfy the following 2×2 linear system:

$$(1-A(0))v(0,\,\mathrm{d}y) - \frac{W^{(p+q)}(0)}{W^{(p+q)}(a)}v(a,\,\mathrm{d}y) = C(0,\,\mathrm{d}y)\,\mathbf{1}_{y\in[0,a]} + D(0,\,\mathrm{d}y)\,\mathbf{1}_{y<0},$$

$$-A(a)v(0,\,\mathrm{d}y) + \left(1 - \frac{B(a;a)}{W^{(p+q)}(a)}\right)v(a,\,\mathrm{d}y) \\ = U_-^{(p)}(a,\,\mathrm{d}y;a)\,\mathbf{1}_{y>a} + C(a,\,\mathrm{d}y)\,\mathbf{1}_{y\in[0,a]} + D(a,\,\mathrm{d}y)\,\mathbf{1}_{y<0}\,.$$

Our aim is to exhibit the values of $v(0,\,\mathrm{d}y)$ and $v(a,\,\mathrm{d}y)$ from this linear system using results from Appendix B in order to obtain an explicit expression of $v(x,\,\mathrm{d}y)$.

From Lemma 2, we note that

$$A(x) = \mathbb{E}_x[\mathrm{e}^{-p\tau_a^-}\mathcal{H}^{(p,q)}(X_{\tau_a^-})] - \frac{\mathcal{H}^{(p,q)}(a)}{W^{(p+q)}(a)}B(x;a), \tag{23}$$

and from Lemma 3, we have

$$B(x;a) = W^{(p)}(x) + q\int_0^a W^{(p)}(x-z)W^{(p+q)}(z)\,\mathrm{d}z - W^{(p)}(x-a)\mathcal{H}^{(p,q)}(a). \tag{24}$$

We deduce that

$$A(0) = 1 - \frac{W^{(p+q)}(0)}{W^{(p+q)}(a)}\mathcal{H}^{(p,q)}(a),$$

and, using (2), that

$$B(a;a) = W^{(p+q)}(a) - W^{(p)}(0)\mathcal{H}^{(p,q)}(a). \tag{25}$$

Then, since $W^{(p+q)}(0) > 0$, the linear system can be written as

$$\mathcal{H}^{(p,q)}(a)v(0,\,\mathrm{d}y) - v(a,\,\mathrm{d}y) = \frac{W^{(p+q)}(a)}{W^{(p+q)}(0)}[C(0,\,\mathrm{d}y)\,\mathbf{1}_{y\in[0,a]} + D(0,\,\mathrm{d}y)\,\mathbf{1}_{y<0}],$$

$$-A(a)v(0,\,\mathrm{d}y) + \frac{W^{(p)}(0)}{W^{(p+q)}(a)}\mathcal{H}^{(p,q)}(a)v(a,\,\mathrm{d}y) \\ = U_-^{(p)}(a,\,\mathrm{d}y;a)\,\mathbf{1}_{y>a} + C(a,\,\mathrm{d}y)\,\mathbf{1}_{y\in[0,a]} + D(a,\,\mathrm{d}y)\,\mathbf{1}_{y<0}\,.$$

The determinant of the matrix related to this linear system is equal to

$$\Delta = \frac{W^{(p)}(0)}{W^{(p+q)}(a)}\left(\mathcal{H}^{(p,q)}(a)\right)^2 - A(a)$$

https://doi.org/10.1017/apr.2015.17 Published online by Cambridge University Press

with, from (23), (25), and Lemma 5,

$$A(a) = -W^{(p)}(0)\mathrm{e}^{\Phi(p)a}\biggl(\psi'(\Phi(p)) + q\int_0^a \mathrm{e}^{-\Phi(p)z}\mathcal{H}^{(p,q)}(z)\,\mathrm{d}z\biggr) + \frac{W^{(p)}(0)}{W^{(p+q)}(a)}(\mathcal{H}^{(p,q)}(a))^2.$$

We deduce that

$$\Delta = W^{(p)}(0)\mathrm{e}^{\Phi(p)a}\biggl(\psi'(\Phi(p)) + q\int_0^a \mathrm{e}^{-\Phi(p)z}\mathcal{H}^{(p,q)}(z)\,\mathrm{d}z\biggr). \tag{26}$$

Since ψ is increasing on $[\Phi(0), +\infty)$ and $W^{(p)}(0) = 1/c > 0$ when X has paths of BV, the determinant Δ is not equal to 0 and there is a unique solution to the linear system satisfied by $v(0, \mathrm{d}y)$ and $v(a, \mathrm{d}y)$.

We first note, using (23) and the first equation of the linear system, that (22) can be written as

$$\begin{aligned} v(x, \mathrm{d}y) &= \mathbb{E}_x[\mathrm{e}^{-p\tau_a^-}\mathcal{H}^{(p,q)}(X_{\tau_a^-})]v(0, \mathrm{d}y) + U_-^{(p)}(x, \mathrm{d}y; a)\,\mathbf{1}_{y>a} \\ &\quad + \biggl[C(x, \mathrm{d}y) - \frac{B(x; a)}{W^{(p+q)}(0)}C(0, \mathrm{d}y)\biggr]\mathbf{1}_{y\in[0,a]} \\ &\quad + \biggl[D(x, \mathrm{d}y) - \frac{B(x; a)}{W^{(p+q)}(0)}D(0, \mathrm{d}y)\biggr]\mathbf{1}_{y<0} \end{aligned} \tag{27}$$

with, by Lemma 5,

$$\mathbb{E}_x[\mathrm{e}^{-p\tau_a^-}\mathcal{H}^{(p,q)}(X_{\tau_a^-})] = \mathcal{H}^{(p,q)}(x) - q\int_a^x W^{(p)}(x-z)\mathcal{H}^{(p,q)}(z)\,\mathrm{d}z - \frac{W^{(p)}(x-a)}{W^{(p)}(0)}\Delta. \tag{28}$$

Case 1. We first consider $y > a$. The linear system satisfied by $v(0, \mathrm{d}y)$ and $v(a, \mathrm{d}y)$ is then

$$\begin{aligned} \mathcal{H}^{(p,q)}(a)v(0, \mathrm{d}y) - v(a, \mathrm{d}y) &= 0, \\ -A(a)v(0, \mathrm{d}y) + \frac{W^{(p)}(0)}{W^{(p+q)}(a)}\mathcal{H}^{(p,q)}(a)v(a, \mathrm{d}y) &= U_-^{(p)}(a, \mathrm{d}y; a). \end{aligned}$$

We deduce that $v(a, \mathrm{d}y) = \mathcal{H}^{(p,q)}(a)v(0, \mathrm{d}y)$ and $v(0, \mathrm{d}y) = \Delta^{-1}U_-^{(p)}(a, \mathrm{d}y; a)$. Consequently, from the expression of $v(x, \mathrm{d}y)$ given by (27), we finally have, for $y > a$,

$$v(x, \mathrm{d}y) = \Delta^{-1}U_-^{(p)}(a, \mathrm{d}y; a)\mathbb{E}_x[\mathrm{e}^{-p\tau_a^-}\mathcal{H}^{(p,q)}(X_{\tau_a^-})] + U_-^{(p)}(x, \mathrm{d}y; a)$$

with $U_-^{(p)}(\cdot, \mathrm{d}y; a)$ given by (19) and $\mathbb{E}_x[\mathrm{e}^{-p\tau_a^-}\mathcal{H}^{(p,q)}(X_{\tau_a^-})]$ given by (28). We finally have, for $y > a$,

$$\begin{aligned} v(x, \mathrm{d}y) &= -W^{(p)}(x-y)\,\mathrm{d}y \\ &\quad + \Delta^{-1}W^{(p)}(0)\mathrm{e}^{\Phi(p)(a-y)}\biggl(\mathcal{H}^{(p,q)}(x) - q\int_a^x W^{(p)}(x-z)\mathcal{H}^{(p,q)}(z)\,\mathrm{d}z\biggr)\mathrm{d}y. \end{aligned}$$

https://doi.org/10.1017/apr.2015.17 Published online by Cambridge University Press

Case 2. We now consider $y \in [0, a]$. The linear system satisfied by $v(0,\, \mathrm{d}y)$ and $v(a,\, \mathrm{d}y)$ becomes

$$\mathcal{H}^{(p,q)}(a)v(0,\, \mathrm{d}y) - v(a,\, \mathrm{d}y) = \frac{W^{(p+q)}(a)}{W^{(p+q)}(0)}C(0,\, \mathrm{d}y),$$
$$-A(a)v(0,\, \mathrm{d}y) + \frac{W^{(p)}(0)}{W^{(p+q)}(a)}\mathcal{H}^{(p,q)}(a)v(a,\, \mathrm{d}y) = C(a,\, \mathrm{d}y),$$

and its unique solution is given by

$$v(0,\, \mathrm{d}y) = \Delta^{-1}(\mathcal{H}^{(p,q)}(a)C(0,\, \mathrm{d}y) + C(a,\, \mathrm{d}y)),$$
$$v(a,\, \mathrm{d}y) = \Delta^{-1}\bigg(A(a)\frac{W^{(p+q)}(a)}{W^{(p+q)}(0)}C(0,\, \mathrm{d}y) + \mathcal{H}^{(p,q)}(a)C(a,\, \mathrm{d}y)\bigg).$$

Let us now study $C(x,\, \mathrm{d}y)$. From (17), we obtain

$$C(x,\, \mathrm{d}y) = \bigg\{\frac{W^{(p+q)}(a-y)}{W^{(p+q)}(a)}\mathbb{E}_x[\mathrm{e}^{-p\tau_a^-}W^{(p+q)}(X_{\tau_a^-})] - \mathbb{E}_x[\mathrm{e}^{-p\tau_a^-}W^{(p+q)}(X_{\tau_a^-} - y)]\bigg\}\,\mathrm{d}y,$$

from which we deduce that

$$C(0,\, \mathrm{d}y) = \frac{W^{(p+q)}(0)W^{(p+q)}(a-y)}{W^{(p+q)}(a)}\,\mathrm{d}y.$$

Introducing the notation $B(\cdot;\cdot)$ and using (25), we obtain

$$C(x,\, \mathrm{d}y) = \bigg\{\frac{W^{(p+q)}(a-y)}{W^{(p+q)}(a)}B(x; a) - B(x-y; a-y)\bigg\}\,\mathrm{d}y,$$
$$C(a,\, \mathrm{d}y) = W^{(p)}(0)\bigg(\mathcal{H}^{(p,q)}(a-y) - \frac{W^{(p+q)}(a-y)}{W^{(p+q)}(a)}\mathcal{H}^{(p,q)}(a)\bigg)\,\mathrm{d}y.$$

We deduce that

$$v(x,\, \mathrm{d}y) = v(0,\, \mathrm{d}y)\mathbb{E}_x[\mathrm{e}^{-p\tau_a^-}\mathcal{H}^{(p,q)}(X_{\tau_a^-})] - B(x-y; a-y)\,\mathrm{d}y$$

with $v(0,\, \mathrm{d}y) = \Delta^{-1}W^{(p)}(0)\mathcal{H}^{(p,q)}(a-y)\,\mathrm{d}y$. So using (24) and (28), we obtain, for $y \in [0, a]$,

$$v(x,\, \mathrm{d}y) = \bigg\{\Delta^{-1}W^{(p)}(0)\mathcal{H}^{(p,q)}(a-y)\bigg(\mathcal{H}^{(p,q)}(x) - q\int_a^x W^{(p)}(x-z)\mathcal{H}^{(p,q)}(z)\,\mathrm{d}z\bigg)$$
$$- W^{(p)}(x-y) - q\int_0^{a-y} W^{(p)}(x-y-z)W^{(p+q)}(z)\,\mathrm{d}z\bigg\}\,\mathrm{d}y.$$

Case 3. We finally consider the case where $y < 0$. Now the linear system becomes

$$\mathcal{H}^{(p,q)}(a)v(0,\, \mathrm{d}y) - v(a,\, \mathrm{d}y) = \frac{W^{(p+q)}(a)}{W^{(p+q)}(0)}D(0,\, \mathrm{d}y),$$
$$-A(a)v(0,\, \mathrm{d}y) + \frac{W^{(p)}(0)}{W^{(p+q)}(a)}\mathcal{H}^{(p,q)}(a)v(a,\, \mathrm{d}y) = D(a,\, \mathrm{d}y),$$

https://doi.org/10.1017/apr.2015.17 Published online by Cambridge University Press

and its unique solution is

$$v(0,\,\mathrm{d}y) = \Delta^{-1}(\mathcal{H}^{(p,q)}(a)D(0,\,\mathrm{d}y) + D(a,\,\mathrm{d}y))$$
$$v(a,\,\mathrm{d}y) = \Delta^{-1}\bigg(A(a)\frac{W^{(p+q)}(a)}{W^{(p+q)}(0)}D(0,\,\mathrm{d}y) + \mathcal{H}^{(p,q)}(a)D(a,\,\mathrm{d}y)\bigg).$$

Let us now study $D(x,\,\mathrm{d}y)$.

From (18), for $y < 0$, we have $U_+^{(p)}(x,\,\mathrm{d}y;0) = \{\mathrm{e}^{\Phi(p)x}W^{(p)}(-y) - W^{(p)}(x-y)\}\,\mathrm{d}y$ for $x \le a$ and then

$$\begin{aligned} D(x,\,\mathrm{d}y) &= \mathbb{E}_x[\mathrm{e}^{-p\tau_a^-}\mathbb{E}_{X_{\tau_a^-}}[\mathrm{e}^{-(p+q)\tau_0^-}U_+^{(p)}(X_{\tau_0^-},\,\mathrm{d}y;0);\,\tau_0^- < \tau_a^+]], \\ &= \{W^{(p)}(-y)A(x) - \mathbb{E}_x[\mathrm{e}^{-p\tau_a^-}\mathbb{E}_{X_{\tau_a^-}}[\mathrm{e}^{-(p+q)\tau_0^-}W^{(p)}(X_{\tau_0^-} - y);\,\tau_0^- < \tau_a^+]]\}\,\mathrm{d}y. \end{aligned}$$

Using Lemma 3 and $B(x-y;a-y) = \mathbb{E}_x[\mathrm{e}^{-p\tau_a^-}W^{(p+q)}(X_{\tau_a^-} - y)]$, with an explicit form given by (24), we obtain

$$\begin{aligned} &\mathbb{E}_x[\mathrm{e}^{-p\tau_a^-}\mathbb{E}_{X_{\tau_a^-}}[\mathrm{e}^{-(p+q)\tau_0^-}W^{(p)}(X_{\tau_0^-} - y);\,\tau_0^- < \tau_a^+]] \\ &\quad= B(x-y;a-y) - q\int_0^{-y} B(x-y-z;a-y-z)W^{(p)}(z)\,\mathrm{d}z \\ &\qquad - \frac{B(x;a)}{W^{(p+q)}(a)}\bigg(W^{(p+q)}(a-y) - q\int_0^{-y} W^{(p+q)}(a-y-z)W^{(p)}(z)\,\mathrm{d}z\bigg). \end{aligned}$$

From (23), we deduce that

$$\begin{aligned} D(x,\,\mathrm{d}y) = \bigg\{ & W^{(p)}(-y)\mathbb{E}_x[\mathrm{e}^{-p\tau_a^-}\mathcal{H}^{(p,q)}(X_{\tau_a^-})] - B(x-y;a-y) \\ & + q\int_0^{-y} B(x-y-z;a-y-z)W^{(p)}(z)\,\mathrm{d}z \\ & + \frac{B(x;a)}{W^{(p+q)}(a)}\bigg(W^{(p+q)}(a-y) - q\int_0^{-y} W^{(p+q)}(a-y-z)W^{(p)}(z)\,\mathrm{d}z \\ & \qquad - W^{(p)}(-y)\mathcal{H}^{(p,q)}(a)\bigg)\bigg\}\,\mathrm{d}y. \end{aligned}$$

We easily compute $B(h;a+h) = W^{(p+q)}(h)$ for $h \ge 0$ and then, using (2), we have

$$\begin{aligned} D(0,\,\mathrm{d}y) = \frac{W^{(p+q)}(0)}{W^{(p+q)}(a)}\bigg\{ & W^{(p+q)}(a-y) - q\int_0^{-y} W^{(p+q)}(a-y-z)W^{(p)}(z)\,\mathrm{d}z \\ & - W^{(p)}(-y)\mathcal{H}^{(p,q)}(a)\bigg\}\,\mathrm{d}y. \end{aligned}$$

Consequently, we have

$$\begin{aligned} & D(x,\,\mathrm{d}y) - \frac{B(x;a)}{W^{(p+q)}(0)}D(0,\,\mathrm{d}y) \\ &\quad = \bigg\{W^{(p)}(-y)\mathbb{E}_x[\mathrm{e}^{-p\tau_a^-}\mathcal{H}^{(p,q)}(X_{\tau_a^-})] - B(x-y;a-y) \\ &\qquad + q\int_0^{-y} B(x-y-z;a-y-z)W^{(p)}(z)\,\mathrm{d}z\bigg\}\,\mathrm{d}y \end{aligned}$$

https://doi.org/10.1017/apr.2015.17 Published online by Cambridge University Press

and then, from (27) satisfied by $v(x,\, \mathrm{d}y)$, we deduce that

$$v(x,\, \mathrm{d}y) = (v(0,\, \mathrm{d}y) + pW^{(p)}(-y)\,\mathrm{d}y)\mathbb{E}_x[\mathrm{e}^{-p\tau_a^-}\mathcal{H}^{(p,q)}(X_{\tau_a^-})] - \bigg\{B(x-y; a-y) - q\int_0^{-y} B(x-y-z; a-y-z)W^{(p)}(z)\,\mathrm{d}z\bigg\}\,\mathrm{d}y.$$

Taking $x = a$ in (28), we note that $\mathbb{E}_a[\mathrm{e}^{-p\tau_a^-}\mathcal{H}^{(p,q)}(X_{\tau_a^-})] = \mathcal{H}^{(p,q)}(a) - \Delta$. Then, using the expression of the solution of the linear system, the expression of $D(\cdot,\, \mathrm{d}y)$, and (25), we have

$$\begin{aligned} v(0,\, \mathrm{d}y) &+ W^{(p)}(-y)\,\mathrm{d}y \\ &= \Delta^{-1}(\mathcal{H}^{(p,q)}(a)D(0,\, \mathrm{d}y) + D(a,\, \mathrm{d}y) + \Delta W^{(p)}(-y)\,\mathrm{d}y) \\ &= \Delta^{-1}W^{(p)}(0)\bigg\{\mathcal{H}^{(p,q)}(a-y) - q\int_0^{-y}\mathcal{H}^{(p,q)}(a-y-z)W^{(p)}(z)\,\mathrm{d}z\bigg\}\,\mathrm{d}y. \end{aligned}$$

Introducing this value in the last expression of $v(x,\, \mathrm{d}y)$ and using (28), we have

$$\begin{aligned} v(x,\, \mathrm{d}y) \\ = \bigg\{&-(W^{(p)}(x-a)\mathcal{H}^{(p,q)}(a-y) + B(x-y; a-y)) \\ &+ q\int_0^{-y}(W^{(p)}(x-a)\mathcal{H}^{(p,q)}(a-y-z) + B(x-y-z; a-y-z))W^{(p)}(z)\,\mathrm{d}z \\ &+ \Delta^{-1}W^{(p)}(0)\bigg(\mathcal{H}^{(p,q)}(a-y) - q\int_0^{-y}\mathcal{H}^{(p,q)}(a-y-z)W^{(p)}(z)\,\mathrm{d}z\bigg) \\ &\times\bigg(\mathcal{H}^{(p,q)}(x) - q\int_a^x W^{(p)}(x-z)\mathcal{H}^{(p,q)}(z)\,\mathrm{d}z\bigg)\bigg\}\,\mathrm{d}y. \end{aligned}$$

From (24), we have

$$W^{(p)}(x-a)\mathcal{H}^{(p,q)}(a) + B(x; a) = W^{(p)}(x) + q\int_0^a W^{(p)}(x-u)W^{(p+q)}(u)\,\mathrm{d}u,$$

then we finally deduce that, for $y < 0$,

$$\begin{aligned} v(x,\, \mathrm{d}y) = \bigg\{&-W^{(p)}(x-y) - q\int_0^{a-y} W^{(p)}(x-y-u)W^{(p+q)}(u)\,\mathrm{d}u \\ &+ q\int_0^{-y}\bigg(W^{(p)}(x-y-z) \\ &\qquad + q\int_0^{a-y-z} W^{(p)}(x-y-z-u)W^{(p+q)}(u)\,\mathrm{d}u\bigg)W^{(p)}(z)\,\mathrm{d}z \\ &+ \Delta^{-1}W^{(p)}(0)\bigg(\mathcal{H}^{(p,q)}(a-y) - q\int_0^{-y} W^{(p)}(z)\mathcal{H}^{(p,q)}(a-y-z)\,\mathrm{d}z\bigg) \\ &\times\bigg(\mathcal{H}^{(p,q)}(x) - q\int_a^x W^{(p)}(x-z)\mathcal{H}^{(p,q)}(z)\,\mathrm{d}z\bigg)\bigg\}\,\mathrm{d}y. \end{aligned}$$

Introducing the value of Δ given by (26) and noting that the function $\mathcal{W}^{(p,q)}$ satisfies the relations

$$\mathcal{W}_{x-a}^{(p,q)}(x-y) = \mathcal{W}_{a-y}^{(p+q,-q)}(x-y),$$

and, if $y > a$ then it is also equal to $W^{(p)}(x-y)$, Theorem 1 is proved for any $y \in \mathbb{R}$ when X is a process of BV. □

https://doi.org/10.1017/apr.2015.17 Published online by Cambridge University Press

Proof of Theorem 1 when X is of UBV. We follow the argument of [14] to extend the result to SNLPs with paths of UBV.

The proof uses an approximation argument for which we need to introduce a sequence $(X^n)_{n\geq 1}$ of SNLPs of BV. To this end, suppose that X is a SNLP having paths of UBV, with Lévy triplet (γ, σ, Π). Set, for each $n \geq 1$, the SNLP $X^n = \{X_t^n, t \geq 0\}$ with Lévy triplet $(\gamma, 0, \Pi^n)$, where

$$\Pi^n(\mathrm{d}\theta) := \mathbf{1}_{\{\theta\geq 1/n\}}\,\Pi(\mathrm{d}\theta) + \sigma^2 n^2 \delta_{1/n}(\mathrm{d}\theta)$$

with $\delta_{1/n}(\mathrm{d}\theta)$ standing for the Dirac point mass at $1/n$. Note that X^n has paths of BV with drift $\mathrm{c}^n := \gamma + \int_{1/n}^1 \theta\,\Pi(\mathrm{d}\theta) + \sigma^2 n^2$, which means that c^n may be negative for small n. Note that X^n is a *true* SNLP when n is large enough n. By Bertoin [2, p. 210] , X^n converges a.s. to X uniformly on compact time intervals. We denote by $W_n^{(p)}$ the $p-$scale function corresponding to the SNLP X^n. We also introduce ψ_n the Laplace exponent and Φ_n its right-inverse of X^n, with the convention $\inf\varnothing = \infty$.

From the result for BV processes, we have, for $y \geq 0$,

$$\begin{aligned}
&\int_0^\infty \mathrm{e}^{-pt}\mathbb{E}_x[\mathrm{e}^{-q\int_0^t \mathbf{1}_{(0,a)}(X_s^n)\,\mathrm{d}s}; X_t^n \in \mathrm{d}y]\,\mathrm{d}t \\
&\quad = \mathrm{e}^{-\Phi_n(p)a}\biggl(\frac{\mathcal{H}_n^{(p,q)}(x) - q\int_a^x W_n^{(p)}(x-z)\mathcal{H}_n^{(p,q)}(z)\,\mathrm{d}z}{\psi_n'(\Phi_n(p)) + q\int_0^a \mathrm{e}^{-\Phi_n(p)z}\mathcal{H}_n^{(p,q)}(z)\,\mathrm{d}z}\biggr) \\
&\qquad \times \biggl\{\mathcal{H}_n^{(p,q)}(a-y) - q\int_0^{-y}\mathcal{H}_n^{(p,q)}(a-y-z)W_n^{(p)}(z)\,\mathrm{d}z\biggr\}\,\mathrm{d}y \\
&\qquad - \biggl\{\mathcal{W}_{n,x-a}^{(p,q)}(x-y) - q\int_0^{-y}\mathcal{W}_{n,x-a}^{(p,q)}(x-y-z)W_n^{(p)}(z)\,\mathrm{d}z\biggr\}\,\mathrm{d}y
\end{aligned} \tag{29}$$

with

$$\mathcal{W}_{n,a}^{(p,q)}(x) := W_n^{(p+q)}(x) - q\int_0^a W_n^{(p+q)}(x-z)W_n^{(p)}(z)\,\mathrm{d}z$$

and

$$\mathcal{H}_n^{(p,q)}(x) := \mathrm{e}^{\Phi_n(p)x}\biggl[1 + q\int_0^x \mathrm{e}^{-\Phi_n(p)z}W_n^{(p+q)}(z)\,\mathrm{d}z\biggr].$$

Our aim is to let $n \to \infty$ on both sides of (29).

By Bertoin [2, p. 210] , X^n converges a.s. to X uniformly on compact time intervals, i.e. for all $t > 0$, $\lim_{n\to\infty}\sup_{s\in[0,t]}|X_s^n - X_s| = 0$ $\mathbb{P}_x$-a.s. Let f be a real-valued continuous and bounded function. Therefore, $\mathrm{e}^{-q\int_0^t \mathbf{1}_{(0,a)}(X_s^n)\,\mathrm{d}s} f(X_t^n)$ converges to $\mathrm{e}^{-q\int_0^t \mathbf{1}_{(0,a)}(X_s)\,\mathrm{d}s} f(X_t)$ a.s. Since $\mathrm{e}^{-q\int_0^t \mathbf{1}_{(0,a)}(X_s^n)\,\mathrm{d}s} f(X_t^n)$ is bounded, it is easy to show that the left-hand side of (29) converges to the desired expression.

Let us now study the convergence of the right-hand side of (29) when $n \to \infty$. The Laplace exponent ψ_n of X^n converges to the Laplace exponent ψ of X, which implies that Φ_n and ψ_n' converge to Φ and ψ', respectively. It also means via (1) that the Laplace transform of $W_n^{(p)}$ converges to the Laplace transform of $W^{(p)}$, and thanks to the continuity theorem of Laplace transforms, $W_n^{(p)}(x) \to W^{(p)}(x)$ for all $x \in \mathbb{R}$ and $p \geq 0$. Finally, since all the functions involved are continuous and since we consider compact sets, using the dominated convergence theorem on the definition of $\mathcal{W}_{n,a}^{(p,q)}$ and $\mathcal{H}_n^{(p,q)}$, we deduce the convergence of $\mathcal{W}_{n,a}^{(p,q)}$ and $\mathcal{H}_n^{(p,q)}$ to $\mathcal{W}_a^{(p,q)}$ and $\mathcal{H}^{(p,q)}$, respectively, when $n \to \infty$. □

https://doi.org/10.1017/apr.2015.17 Published online by Cambridge University Press

Acknowledgements

We would like to thank Xiaowen Zhou for his help and comments on an earlier version of this paper. We would also like to thank the CNRS, its UMI 3457 and the Centre de recherches mathématiques for providing the research infrastructure.

Funding in support of this work was provided by the Natural Sciences and Engineering Research Council of Canada and the Institut de Finance Mathématique de Montréal (IFM2).

References

[1] Akahori, J. (1995). Some formulae for a new type of path-dependent option. *Ann. Appl. Prob.* **5,** 383–388.
[2] Bertoin, J. (1996). *Lévy Processes.* Cambridge University Press.
[3] Cai, N. (2009). On first passage times of a hyper-exponential jump diffusion process. *Operat. Res. Lett.* **37,** 127–134.
[4] Cai, N., Chen, N. and Wan, X. (2010). Occupation times of jump-diffusion processes with double exponential jumps and the pricing of options. *Math. Operat. Res.* **35,** 412–437.
[5] Hugonnier, J.-N. (1999). The Feynman–Kac formula and pricing occupation time derivatives. *Internat. J. Theoret. Appl. Finance* **2,** 153–178.
[6] Kuznetsov, A., Kyprianou, A. E. and Rivero, V. (2012). The theory of scale functions for spectrally negative Lévy processes. In *Lévy Matters* (Lecture Notes Math. **2061**), Springer, Heidelberg, pp. 97–186.
[7] Kyprianou, A. E. (2006). *Introductory Lectures on Fluctuations of Lévy Processes with Applications.* Springer, Berlin.
[8] Kyprianou, A. E., Pardo, J. C. and Pérez, J. L. (2014). Occupation times of refracted Lévy processes. *J. Theoret. Prob.* **27,** 1292–1315.
[9] Lachal, A. (2013). Sojourn time in an union of intervals for diffusions. *Methodology Comput. Appl. Prob.* **15,** 743–771.
[10] Landriault, D., Renaud, J.-F. and Zhou, X. (2011). Occupation times of spectrally negative Lévy processes with applications. *Stoch. Process. Appl.* **121,** 2629–2641.
[11] Landriault, D., Renaud, J.-F. and Zhou, X. (2014). An insurance risk model with Parisian implementation delays. *Methodology Comput. Appl. Prob.* **16,** 583–607.
[12] Li, B. and Zhou, X. (2013). The joint Laplace transforms for diffusion occupation times. *Adv. Appl. Prob.* **45,** 1049–1067.
[13] Linetsky, V. (1999). Step options. *Math. Finance* **9,** 55–96.
[14] Loeffen, R. L., Renaud, J.-F. and Zhou, X. (2014). Occupation times of intervals until first passage times for spectrally negative Lévy processes. *Stoch. Process. Appl.* **124,** 1408–1435.
[15] Renaud, J.-F. (2014). On the time spent in the red by a refracted Lévy risk process. *J. Appl. Prob.* **51,** 1171–1188.
[16] Takács, L. (1996). On a generalization of the arc-sine law. *Ann. Appl. Prob.* **6,** 1035–1040.