Proceedings of the Edinburgh Mathematical Society (2014) **57**, 811–839
DOI:10.1017/S0013091513000643

GORENSTEIN ISOLATED QUOTIENT SINGULARITIES OVER $\mathbb{C}$

D. A. STEPANOV

Department of Mathematical Modelling, Bauman Moscow State Technical University, 2-ya Baumanskaya ul. 5, Moscow 105005, Russia (dstepanov@bmstu.ru)

(Received 11 January 2011)

Abstract In this paper we review the classification of isolated quotient singularities over the field of complex numbers due to Zassenhaus, Vincent and Wolf. As an application, we describe Gorenstein isolated quotient singularities over $\mathbb{C}$, generalizing a result of Kurano and Nishi.

Keywords: Gorenstein singularity; quotient singularity; groups without fixed points

2010 *Mathematics subject classification:* Primary 14B05

1. Introduction

Let $\Bbbk$ be a field, and let R be a finitely generated $\Bbbk$-algebra. Recall that R is called *Gorenstein* if it has a canonical module generated by one element (see, for example, [**5**, § 21.3]). Geometrically, the affine algebraic variety $X = \operatorname{Spec} R$ is Gorenstein if the canonical divisor K_X of X is Cartier. Now, let $\Bbbk = \mathbb{C}$ be the field of complex numbers, let $S = \mathbb{C}[x_1, \dots, x_N]$ be the polynomial ring in N variables, let G be a finite subgroup of $\mathrm{GL}(N, \mathbb{C})$, and let $R = S^G$ be the algebra of invariants. By the Hilbert basis theorem, R is finitely generated. We say that $X = \operatorname{Spec} R$ is an *isolated singularity* if the algebraic variety X is singular and its singular set has dimension 0.

This paper is devoted to considering the following problem.

Problem 1.1. Classify all Gorenstein isolated quotient singularities of the form $\mathbb{C}^N/G$, where G is a finite subgroup of $\mathrm{GL}(N, \mathbb{C})$, up to an isomorphism of algebraic varieties.

Recently, Kurano and Nishi obtained the following result.

Theorem 1.2 (Kurano and Nishi [7, Corollary 1.3]). *Let G be a finite subgroup of $\mathrm{GL}(p, \mathbb{C})$, where p is an odd prime, and assume that $\mathbb{C}^p/G$ is an isolated Gorenstein singularity. Then $\mathbb{C}^p/G$ is isomorphic to a cyclic quotient singularity, i.e. to a singularity $\mathbb{C}^p/H$, where $H < \mathrm{GL}(p, \mathbb{C})$ is a cyclic group.*

Kurano and Nishi give a direct proof of their theorem. On the other hand, there exists a complete classification of isolated quotient singularities over $\mathbb{C}$, and it is natural to try to derive results of this sort, and even a complete description of Gorenstein isolated quotient singularities over $\mathbb{C}$, from this classification.

Isolated quotient singularities (IQS) over $\mathbb{C}$ were classified mainly in works of Zassenhaus [**15**], Vincent [**11**] and Wolf [**14**]. These authors were motivated neither by singularity theory nor by algebraic geometry. In his pioneer work, Zassenhaus studied nearfields, whereas Vincent and Wolf classified compact Riemannian manifolds of constant positive curvature. The final results were obtained by Wolf and stated in [**14**] as a classification of such manifolds. The fact that the Zassenhaus–Vincent–Wolf classification also gives a classification of IQS has been known for a long time (see, for example, [**8**]), but it seems that this topic has not attracted much of the attention of algebraic geometers yet.

An element $g \in \mathrm{GL}(N, \mathbb{C})$ is called a *quasi-reflection* if g fixes pointwise a codimension 1 linear subspace of $\mathbb{C}^N$. Gorenstein IQS over $\mathbb{C}$ are characterized by the following theorem of Watanabe.

Theorem 1.3 (Watanabe [12, 13]). *Let G be a finite subgroup of* $\mathrm{GL}(N, \mathbb{C})$*, free from quasi-reflections. Then, $\mathbb{C}^N/G$ is Gorenstein if and only if G is contained in* $\mathrm{SL}(N, \mathbb{C})$*.*

It follows that to get a classification of Gorenstein IQS we have to look through those Zassenhaus–Vincent–Wolf groups giving IQS and check which of them are contained in the special linear group. We perform this analysis in the paper and present the results in Theorem 4.1.

Chapters 5, 6 and 7 of Wolf's book [**14**] are our main reference. The presentation there is so clear, detailed and self-contained that it is hard to imagine it improved. But the classification is done there from the point of view of the Riemannian geometry. We have anyway to outline a lot of notation, and this requires considerable introduction. Therefore, we decided to include in our work a review of the Zassenhaus–Vincent–Wolf classification written from the point of view of the theory of singularities.

This paper has the following structure. In § 2 we list some preliminaries, observations and general results on IQS and their classification. In particular, we explain the connection of IQS to the classification of compact Riemannian manifolds of constant positive curvature. In § 3 we present the classification of IQS over $\mathbb{C}$ as it is given in [**14**]. The results are summarized in Theorem 3.11. The only new thing we add is the computation of determinants of irreducible representations, which we use later in the description of Gorenstein singularities. We also show that this classification generalizes almost literally to other algebraically closed fields of characteristic 0, and to algebraically closed fields of characteristic p, where p does not divide the order of G (the last generalization was pointed out to us by Mazurov). In § 4 we show how the result of Kurano and Nishi can be deduced and generalized using the classification. As an example we give a description of Gorenstein IQS over $\mathbb{C}$ in dimensions $N \leqslant 7$.

We hope that the beautiful classification of Zassenhaus *et al*. will be extended to as many different fields $\Bbbk$ as possible, and will find other interesting applications in algebraic geometry.

2. Preliminaries

The results of this section are either well known or easy, so even if we state some of them without reference, there is no claim of originality.

2.1. Generalities on isolated quotient singularities and groups without fixed points

We start by formulating general problems on the classification of IQS. Let V be an algebraic variety defined over a field $\Bbbk$. Let G be a finite group acting on V by automorphisms, and let $P \in V$ be a closed non-singular fixed point of this action. Denote by $\pi\colon V \to V/G$ the canonical projection to the quotient variety, and let $Q = \pi(P)$. Assume that Q is an isolated singular point of V/G.

Problem 2.1. Classify all singularities $Q \in V/G$ of the form described above up to a formal equivalence or, when $\Bbbk = \mathbb{C}$, up to an analytic equivalence.

At a non-singular point P any algebraic variety is formally equivalent to $0 \in \Bbbk^N$, $N = \dim V$, i.e. we have an isomorphism of complete local rings $\hat{\mathcal{O}}_{P,V} \simeq \hat{\mathcal{O}}_{0,\Bbbk^N}$ (see Remark 2 after Proposition 11.24 in [**2**]). With the additional condition that the characteristic of $\Bbbk$ does not divide the order of G, the action of G can be formally linearized at P (see Lemma 2.3). In the analytic case $\Bbbk = \mathbb{C}$, V is a manifold at P and the action of G can be linearized in some local analytic coordinates at P (see, for example, [**1**, p. 35]). Thus, we see that if $\operatorname{char}\Bbbk \nmid |G|$, then Problem 2.1 is equivalent to the following.

Problem 2.2. Classify all isolated quotient singularities of the form $\Bbbk^N/G$, where G is a finite subgroup of $\mathrm{GL}(N, \Bbbk)$.

We prove the following lemma on formal linearization.

Lemma 2.3. *Let $R = \Bbbk[\![x_1, \dots, x_N]\!]$ be the local ring of formal power series in N variables over a field $\Bbbk$, and let G be a finite group acting on R by local automorphisms. Assume that the characteristic of the field $\Bbbk$ does not divide the order of G. We can then choose new local parameters $y_1, \dots, y_N$ in R such that G acts by linear substitutions in $y_1, \dots, y_N$.*

Proof. Recall that $R = \Bbbk[\![x_1, \dots, x_N]\!]$ can be identified with the inverse limit $\varprojlim R/\mathfrak{m}^n$, where $\mathfrak{m}$ is the maximal ideal $(x_1, \dots, x_N)$. Since G acts by local automorphisms, it preserves all the ideals $\mathfrak{m}, \mathfrak{m}^2, \dots, \mathfrak{m}^n, \dots,$ and, therefore, acts linearly on all quotients $\mathfrak{m}/\mathfrak{m}^n$, $n \geqslant 1$, which in turn are finite-dimensional vector spaces over $\Bbbk$. Moreover, if $\pi_{nn'}$, $n \geqslant n'$, denotes the natural truncation map $\mathfrak{m}/\mathfrak{m}^{n+1} \to \mathfrak{m}/\mathfrak{m}^{n'+1}$, the action of G is compatible with $\pi_{nn'}$. We construct $y_1, \dots, y_N$ as limits of Cauchy sequences $y_1^{(n)}, \dots, y_N^{(n)}$, where, for $i = 1, \dots, N$,

$$y_i^{(n)} \in \mathfrak{m}/\mathfrak{m}^{n+1} \quad \text{and} \quad \pi_{nn'}(y_i^{(n)}) = y_i^{(n')}.$$

Let $y_1^{(1)}, \dots, y_N^{(1)}$ be any basis of $\mathfrak{m}/\mathfrak{m}^2$. Note that G acts on $y_1^{(1)}, \dots, y_N^{(1)}$ by linear substitutions. Now suppose that $y_1^{(n)}, \dots, y_N^{(n)}$ have already been constructed. Due to our

https://doi.org/10.1017/S0013091513000643 Published online by Cambridge University Press

assumption on the characteristic of $\Bbbk$, the representation of G on $\mathfrak{m}/\mathfrak{m}^{n+2}$ is completely reducible. Thus, we have a decomposition

$$\mathfrak{m}/\mathfrak{m}^{n+2} \simeq \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2} \bigoplus V,$$

where V projects isomorphically onto $\mathfrak{m}/\mathfrak{m}^{n+1}$ under the map $\pi_{n+1,n}$. We can now uniquely lift $y_1^{(n)},\dots,y_N^{(n)}$ to some elements $y_1^{(n+1)},\dots,y_N^{(n+1)}$ of $\mathfrak{m}/\mathfrak{m}^{n+2}$. By construction, G acts on $y_1^{(n+1)},\dots,y_N^{(n+1)}$ by linear substitutions; hence, the same holds for the limits $y_1,\dots,y_N$ of the sequences $y_1^{(n)},\dots,y_N^{(n)}$. $\square$

From now on, if not stated otherwise, we work only with the case when the field $\Bbbk$ is algebraically closed and $\operatorname{char}\Bbbk \nmid |G|$. We consider Problem 2.2. Recall that an element $g \in \mathrm{GL}(N,\Bbbk)$ is called a *quasi-reflection* if g fixes pointwise a codimension 1 subspace of $\Bbbk^N$. By the Chevalley–Shephard–Todd theorem (see [**3**, Theorem 7.2.1]), a quotient $\Bbbk^N/H$, where $H < \mathrm{GL}(N,\Bbbk)$ is finite and $\operatorname{char}\Bbbk \nmid |H|$, is smooth if and only if the group H is generated by quasi-reflections. Quasi-reflections contained in a finite group $G < \mathrm{GL}(N,\Bbbk)$ generate a normal subgroup $H \lhd G$. It follows that, from the point of view of Problem 2.2, we can restrict to the groups $G < \mathrm{GK}(N,\Bbbk)$ free from quasi-reflections. We also require the singularity $\Bbbk^N/G$ to be isolated. This imposes additional strong restrictions on the group G.

Lemma 2.4 (compare with Kurano and Nishi [7, pp. 2, 3]). *Let G be a finite subgroup of $\mathrm{GL}(N,\Bbbk)$. Assume that G is free from quasi-reflections and $\operatorname{char}\Bbbk \nmid |G|$. Then, $\Bbbk^N/G$ has isolated singularities if and only if 1 is not an eigenvalue of any element of G except the unity.*

Proof. Sufficiency of the condition given in the lemma is clear, so we prove necessity. Suppose, on the contrary, that G has some non-unit element with eigenvalue 1. Let $\mathcal{U}$ be the set of linear subspaces of $\Bbbk^N$ with non-trivial stabilizers in G, and let U be a maximal element in $\mathcal{U}$ with respect to inclusion. Denote by H the stabilizer of U. If U is positive dimensional, the given representation of H is reducible, so we consider the splitting $\Bbbk^N = U \bigoplus U'$, where U' is the complementary invariant subspace of H. Let $\bar{U}$ be the image of U under the canonical projection $\pi\colon \Bbbk^N \to \Bbbk^N/G$. At a general point P of $\bar{U}$ the quotient $P \in \Bbbk^N/H$ is formally isomorphic to the direct product $(0,\bar{0}) \in U \times (U'/H)$. On the other hand, $\Bbbk^N/G$ must be non-singular at P; thus, by the Chevalley–Shephard–Todd theorem, the group H acting on U' is generated by quasi-reflections. But, then, H acting on $\Bbbk^N$, and thus G, also contain quasi-reflections, which contradicts the conditions of the lemma. $\square$

Definition 2.5. Following [**14**], we call a group G satisfying the conditions of Lemma 2.4 (i.e. 1 is not an eigenvalue of any element of G except the unity) a *group without fixed points.*

https://doi.org/10.1017/S0013091513000643 Published online by Cambridge University Press

2.2. The Clifford–Klein problem and *pq*-conditions

To introduce the context in which a solution for Problem 2.2 was first obtained, set $\Bbbk = \mathbb{C}$. It is a standard fact of the representation theory that any finite subgroup of $\mathrm{GL}(N, \mathbb{C})$ is conjugate to a subgroup of the unitary group $\mathrm{U}(N)$. On the other hand, conjugate groups G and G' obviously give isomorphic quotient singularities $\mathbb{C}^N/G$ and $\mathbb{C}^N/G'$. Thus, we may assume from the beginning that G is a subgroup without fixed points of $\mathrm{U}(N)$. If we equip $\mathbb{C}^N$ with the standard Hermitian product and the corresponding metric, we see that G acts by isometries on the unit sphere S^{2N-1} of $\mathbb{C}^N$. Moreover, G acts on S^{2N-1} without fixed points in the usual sense, which justifies Definition 2.5. Furthermore, the quotient S^{2N-1}/G with the induced metric is a compact Riemannian manifold of constant positive curvature. We immediately get the following proposition.

Proposition 2.6. *Let $P \in X$ be an isolated quotient singularity over the field $\mathbb{C}$. The link of P in X can then be equipped with a Riemannian metric of constant positive curvature.*

Classification of Riemannian manifolds of constant curvature was an important field of research in the twentieth century. The following fundamental theorem is a key result on manifolds of positive curvature.

Theorem 2.7 (Killing, Hopf; see [14, Corollary 2.4.10]). *Let M be a Riemannian manifold of dimension $n \geqslant 2$, and let $K > 0$ be a real number. Then, M is a compact connected manifold of constant curvature K if and only if M is isometric to a quotient space of the form S^n_K/Γ, where S^n_K is the standard n-sphere of radius $1/K^2$ in $\mathbb{R}^{n+1}$, and Γ is a finite subgroup of $\mathrm{O}(n+1)$ acting on S^n_K without fixed points.*

The problem of classification of such spaces S^n_K/Γ was called, by Killing, the *Clifford–Klein space form problem*. In the following we simply call it the *Clifford–Klein problem.*

It follows from the discussion above that Problem 2.2 for $\Bbbk = \mathbb{C}$ is a part of the Clifford–Klein problem. But in fact they are almost equivalent. Indeed, it is shown in [**14**, § 7.4] that for even n we have only two solutions for the Clifford–Klein problem: the sphere S^n ($\Gamma = 1$) and the real projective space $\mathbb{RP}^n$ ($\Gamma = \mathbb{Z}/2$). For odd n, all necessary subgroups Γ of $\mathrm{O}(n+1)$ (orthogonal representations) are obtained from the subgroups of $\mathrm{U}((n+1)/2)$ (unitary representations) without fixed points, by means of the standard representation theory (the Frobenius–Schur theorem; see [**14**, Theorem 4.7.3]).

Certainly, the problem of classification of finite subgroups $G < \mathrm{GL}(N, \Bbbk)$ is wider than the problem of classification of IQS themselves, because different subgroups G and H of $\mathrm{GL}(N, \Bbbk)$ can give isomorphic quotients $\Bbbk^N/G$ and $\Bbbk^N/H$. This happens, for example, if G and H are conjugate in $\mathrm{GL}(N, \Bbbk)$. At least for $\Bbbk = \mathbb{C}$, the converse statement also holds.

Lemma 2.8. *Let G and H be two finite subgroups of $\mathrm{GL}(N, \mathbb{C})$, $N \geqslant 2$. Assume that G and H act without fixed points, so that $\mathbb{C}^N/G$ and $\mathbb{C}^N/H$ have isolated singularities. The quotients $\mathbb{C}^N/G$ and $\mathbb{C}^N/H$ are then isomorphic if and only if G and H are conjugate in $\mathrm{GL}(N, \mathbb{C})$.*

Proof. Denote by g and h the canonical projections $g\colon \mathbb{C}^N \to \mathbb{C}^N/G$ and $h\colon \mathbb{C}^N \to C^N/H$, denote by 0 the origin in $\mathbb{C}^N$, and let $P = g(0)$, $Q = h(0)$. The sufficiency condition of the lemma is obvious, so we prove necessity.

Let $f\colon \mathbb{C}^N/G \to \mathbb{C}^N/H$ be an isomorphism. Since G and H act freely on $\mathbb{C}^N \setminus \{0\}$, g and h restrict to universal coverings

$$\mathbb{C}^N \setminus \{0\} \to (\mathbb{C}^N/G) \setminus \{P\} \quad \text{and} \quad \mathbb{C}^N \setminus \{0\} \to (\mathbb{C}^N/G) \setminus \{Q\}.$$

Clearly, $f(P) = Q$; thus, f lifts to an analytic isomorphism $\bar{f}\colon \mathbb{C}^N \setminus \{0\} \to \mathbb{C}^N \setminus \{0\}$. But f is also continuous at P, and it follows that $\bar{f}$ extends continuously to 0 by setting $\bar{f}(0) = 0$. This means that 0 is a removable singular point of the analytic map $\bar{f}$, and, actually, $\bar{f}$ is an analytic isomorphism in the following diagram:

$$\begin{tikzcd} \mathbb{C}^N \arrow[r, "\bar{f}"] \arrow[d, "g"'] & \mathbb{C}^N \arrow[d, "h"] \\ \mathbb{C}^N/G \arrow[r, "f"] & \mathbb{C}^N/H \end{tikzcd}$$

After fixing a reference point O in $(\mathbb{C}^N/G) \setminus \{P\}$, f induces an isomorphism

$$f_*\colon \pi_1((\mathbb{C}^N/G) \setminus \{P\}, O) \to \pi_1((\mathbb{C}^N/H) \setminus \{Q\}, f(O))$$

between fundamental groups, which in turn are isomorphic to G and H, respectively. Fixing the lifting $\bar{f}$, we also fix an isomorphism between G and H such that $\bar{f}$ becomes equivariant with respect to it. Now, observe that G and H also act naturally on the tangent space $T_0\mathbb{C}^N$ that is canonically isomorphic to $\mathbb{C}^N$. Let $A\colon T_0\mathbb{C}^N \to T_0\mathbb{C}^N$ be the differential of $\bar{f}$ at 0. Then $H = AGA^{-1}$. □

Lemma 2.9. *Let $\varphi, \psi\colon G \to \mathrm{GL}(N, \Bbbk)$ be two exact linear representations of a finite group G over a field $\Bbbk$. The images $\varphi(G)$ and $\psi(G)$ are then conjugate in $\mathrm{GL}(N, \Bbbk)$ if and only if there exists an automorphism α of the group G such that representations $\psi \circ \alpha$ and φ are equivalent.*

Proof. The proof is straightforward and left to the reader. See also [**14**, Lemma 4.7.1]. □

Definition 2.10. Let p and q be prime numbers, not necessarily distinct. We say that a finite group G satisfies the *pq-condition* if every subgroup of G of order pq is cyclic.

The following theorem is the cornerstone of the whole classification of groups without fixed points. Note that it does not impose any conditions on the characteristic of the base field $\Bbbk$.

Theorem 2.11. *Let G be a finite subgroup of $\mathrm{GL}(N, \Bbbk)$ without fixed points, where the field $\Bbbk$ is arbitrary. Then G satisfies the pq-conditions for all primes p and q.*

Proof. See [**14**, Theorem 5.3.1]. □

We can now formulate a programme that should give a classification of IQS $\Bbbk^N/G$ in the non-modular case $\operatorname{char}\Bbbk \nmid |G|$. This programme was first suggested by Vincent [**11**] for $\Bbbk = \mathbb{C}$ as a method of solution of the Clifford–Klein problem. First we classify all finite groups satisfying all the pq-conditions. This is a purely group-theoretic problem, completely settled for solvable groups by Zassenhaus, Vincent, Wolf and Suzuki. Next, for every such group G, we study its linear representations over the given field $\Bbbk$, $\operatorname{char}\Bbbk \nmid |G|$, and determine all exact irreducible representations without fixed points. Arbitrary representation without fixed points is a direct sum of irreducible ones. Finally, for every G we calculate its group of automorphisms and determine when two irreducible representations without fixed points are equivalent modulo an automorphism. The last two steps were performed by Vincent and Wolf for $\Bbbk = \mathbb{C}$. It follows from Lemma 2.8 and other results of this section that in this way we indeed get a complete classification of isolated quotient singularities over $\mathbb{C}$ up to an analytic equivalence. For other fields some further identifications may be needed, i.e. different output classes of the Vincent programme might still give isomorphic quotient singularities. However, we do not have any examples of such phenomena.

2.3. Consequences of *pq*-conditions and some representations

In the particular case $p = q$, the pq-condition is called the *p^2-condition*.

Theorem 2.12 (Wolf [14, Theorem 5.3.2]). *If G is a finite group, then the following conditions are equivalent.*

(1) *G satisfies the p^2-conditions for all primes p.*

(2) *Every abelian subgroup of G is cyclic.*

(3) *If p is an odd prime, then every Sylow p-subgroup of G is cyclic; Sylow 2-subgroups of G are either cyclic, or generalized quaternion groups.*

We recall the definition of generalized quaternion groups below (see (2.1)).

Groups G such that all their Sylow p-subgroups are cyclic constitute the simplest class of groups possessing representations without fixed points (provided that G also satisfies all the pq-conditions). All such groups are solvable (see [**14**, Lemma 5.4.3]) and even metacyclic (see [**14**, Lemma 5.4.5]). Non-solvable groups without fixed points necessarily contain Sylow 2-subgroups isomorphic to generalized quaternion groups. A complete classification of non-solvable groups satisfying all the pq-conditions was obtained by Suzuki in [**10**]. We use only the following result of Zassenhaus.

Theorem 2.13 (Wolf [14, Theorem 6.2.1]). *The binary icosahedral group I^* is the only finite perfect group possessing representations without fixed points over $\mathbb{C}$.*

The proof of this theorem is very technical and occupies about 15 pages of [**14**] (the original proof of Zassenhaus was not complete).

We now describe some groups and their complex representations, which we use in the following.

https://doi.org/10.1017/S0013091513000643 Published online by Cambridge University Press

The *generalized quaternion group* $\mathrm{Q}2^a$, $a \geqslant 3$, is a group of order 2^a with generators P, Q, and relations

$$Q^{2^{a-1}} = 1, \qquad P^2 = Q^{2^{a-2}}, \qquad PQP^{-1} = Q^{-1}. \tag{2.1}$$

We get the usual quaternion group $\mathrm{Q}8 = \{\pm 1, \pm i, \pm j, \pm k\}$ for $a = 3$.

Lemma 2.14 (Wolf [14, Lemma 5.6.2]). *Let $\mathrm{Q}2^a$, $a \geqslant 3$, be a generalized quaternion group, and let k be one of 2^{a-3} numbers $1, 3, 5, \dots, 2^{a-2}-1$. Any irreducible complex representation without fixed points of the group $\mathrm{Q}2^a$ is equivalent to one of the following two-dimensional representations α_k:*

$$\alpha_k(P) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \alpha_k(Q) = \begin{pmatrix} \mathrm{e}^{2\pi \mathrm{i}k/2^{a-1}} & 0 \\ 0 & \mathrm{e}^{-2\pi \mathrm{i}k/2^{a-1}} \end{pmatrix}.$$

The representations α_k are pairwise non-equivalent.

The *generalized binary tetrahedral group* T_v^*, $v \geqslant 1$, has generators X, P, Q, and relations

$$X^{3^v} = P^4 = 1, \qquad P^2 = Q^2,$$
$$XPX^{-1} = Q, \qquad XQX^{-1} = PQ, \qquad PQP^{-1} = Q^{-1}.$$

The usual binary tetrahedral group T^* is T_1^*. T_v^* has order $8 \cdot 3^v$.

Lemma 2.15 (Wolf [14, Lemma 7.1.3]). *The binary tetrahedral group T^* has only one irreducible complex representation without fixed points. This is the standard representation $\tau\colon\ \mathrm{T}^* \to \mathrm{SU}(2)$ obtained from the tetrahedral group $\mathrm{T} < \mathrm{SO}(3)$ and the double covering $\mathrm{SU}(2) \to \mathrm{SO}(3)$. If $v > 1$, then T_v^* has exactly $2 \cdot 3^{v-1}$ pairwise non-equivalent irreducible complex representations without fixed points τ_k, where $1 \leqslant k < 3^v$, $(k, 3) = 1$, given by*

$$\tau_k(X) = -\tfrac{1}{2}\mathrm{e}^{2\pi \mathrm{i}k/3^v} \begin{pmatrix} 1+\mathrm{i} & 1+\mathrm{i} \\ -1+\mathrm{i} & 1-\mathrm{i} \end{pmatrix},$$
$$\tau_k(P) = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \qquad \tau_k(Q) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The representation τ of T^* is also given by these formulae if we formally set $v = 0$.

Proof. The lemma is proved in [**14**]; only the precise matrices are not given there. But their computation is an easy exercise, which we omit. □

The *generalized binary octahedral group* O_v^*, $v \geqslant 1$, has generators X, P, Q, R, and relations

$$X^{3^v} = P^4 = 1, \qquad P^2 = Q^2 = R^2,$$
$$PQP^{-1} = Q^{-1}, \qquad XPX^{-1} = Q, \qquad XQX^{-1} = PQ,$$
$$RXR^{-1} = X^{-1}, \qquad RPR^{-1} = QP, \qquad RQR^{-1} = Q^{-1}.$$

The usual binary octahedral group O^* is O_1^*. O_v^* has order $16 \cdot 3^v$.

Lemma 2.16 (Wolf [14, Lemma 7.1.5]). *The group* O^* *has exactly two irreducible complex representations without fixed points, say* o_1 *and* o_2*. Images of* o_1 *and* o_2 *are conjugate subgroups of* $\mathrm{SU}(2)$*. For* o_1 *we may take the standard representation obtained from the octahedral group* $\mathrm{O} < \mathrm{SO}(3)$ *and the double covering* $\mathrm{SU}(2) \to \mathrm{SO}(3)$*. We may set* $o_1(X) = \tau(X)$*,* $o_1(P) = \tau(P)$*,* $o_1(Q) = \tau(Q)$ *(see Lemma 2.15) and*

$$o_1(R) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1+\mathrm{i} & 0 \\ 0 & 1-\mathrm{i} \end{pmatrix}.$$

If $v > 1$*, then* O_v^* *has exactly* 3^{v-1} *pairwise non-equivalent irreducible complex representations without fixed points* o_k*, where* $1 \leqslant k < 3^v$*,* $k \equiv 1 \pmod 3$*, induced by the representations* τ_k *(see Lemma 2.15) of the subgroup* $\mathrm{T}_v^* = \langle X, P, Q \rangle$*, i.e.* o_k *has dimension* 4 *and is given by*

$$o_k(X) = \begin{pmatrix} \tau_k(X) & 0 \\ 0 & \tau_k(X^{-1}) \end{pmatrix}, \qquad o_k(P) = \begin{pmatrix} \tau_k(P) & 0 \\ 0 & \tau_k(QP) \end{pmatrix},$$
$$o_k(Q) = \begin{pmatrix} \tau_k(Q) & 0 \\ 0 & \tau_k(Q^{-1}) \end{pmatrix}, \qquad o_k(R) = \begin{pmatrix} 0 & 1 \\ \tau_k(R^2) & 0 \end{pmatrix}.$$

Recall that the *binary icosahedral group* I^* is obtained from the group I of rotations of an icosahedron by the double covering $\mathrm{SU}(2) \to \mathrm{SO}(3)$. For generators of I we may take one of the rotations by angle $2\pi/5$, which we denote by V, a rotation T of order 2, whose axis has angle $\pi/3$ with the axis of V, and another rotation U of order 2 whose axis is perpendicular to the axes of V and T. We denote the corresponding generators of I^* by $\pm V$, $\pm T$ and $\pm U$.

Lemma 2.17 (Wolf [14, Lemma 7.1.7]). *The group* I^* *has only two irreducible complex representations without fixed points* ι_1 *and* ι_{-1}*. They both have dimension* 2*, and their images are conjugate subgroups of* $\mathrm{SU}(2)$*. For* ι_1 *we may take the standard representation*

$$\iota_1(\pm V) = \begin{pmatrix} \pm\varepsilon^3 & 0 \\ 0 & \pm\varepsilon^2 \end{pmatrix}, \qquad \iota_1(\pm U) = \begin{pmatrix} 0 & \mp 1 \\ \pm 1 & 0 \end{pmatrix},$$
$$\iota_1(\pm T) = \frac{1}{\sqrt{5}} \begin{pmatrix} \mp(\varepsilon - \varepsilon^4) & \pm(\varepsilon^2 - \varepsilon^3) \\ \pm(\varepsilon^2 - \varepsilon^3) & \pm(\varepsilon - \varepsilon^4) \end{pmatrix},$$

where $\varepsilon = \mathrm{e}^{2\pi\mathrm{i}/5}$*.*

Proof. The matrices of ι_1 can be found in [**6**, Chapter II, § 6]. □

3. Classification of isolated quotient singularities over $\mathbb{C}$

This is the central section of our paper. Here we state the main theorems on classification of IQS over the field $\mathbb{C}$ following [**14**, Chapters 6 and 7].

https://doi.org/10.1017/S0013091513000643 Published online by Cambridge University Press

Table 1. *Solvable groups with pq-conditions.*

type	generators	relations	conditions	order
I	A, B	$A^m = B^n = 1$, $BAB^{-1} = A^r$	$m \geqslant 1$, $n \geqslant 1$, $(n(r-1), m) = 1$, $r^n \equiv 1(m)$	mn
II	A, B, R	as in I; also, $R^2 = B^{n/2}$, $RAR^{-1} = A^l$, $RBR^{-1} = B^k$	as in I; also, $l^2 \equiv r^{k-1} \equiv 1(m)$, $n = 2^u v$, $u \geqslant 2$, $k \equiv -1(2^u)$, $k^2 \equiv 1(n)$	$2mn$
III	A, B, P, Q	as in I; also, $P^4 = 1$, $P^2 = Q^2 = (PQ)^2$, $AP = PA$, $AQ = QA$, $BPB^{-1} = Q$, $BQB^{-1} = PQ$	as in I; also, $n \equiv 1(2)$, $n \equiv 0(3)$	$8mn$
IV	A, B, P, Q, R	as in III; also, $R^2 = P^2$, $RPR^{-1} = QP$, $RQR^{-1} = Q^{-1}$, $RAR^{-1} = A^l$, $RBR^{-1} = B^k$	as in III; also, $k^2 \equiv 1(n)$, $k \equiv -1(3)$, $r^{k-1} \equiv l^2 \equiv 1(m)$	$16mn$

As was said in § 2, we first have to classify finite groups with *pq*-conditions. The classification is given in Theorem 3.1. Theorem 3.2 classifies non-solvable groups with *pq*-conditions possessing representations without fixed points over $\mathbb{C}$. Some information about other fields is given at the end of this section.

Theorem 3.1 (Wolf [14, Theorem 6.1.11]). *Let G be a finite solvable group satisfying all the pq-conditions. Then, G is isomorphic to one of the groups listed in Table 1. Moreover, G satisfies an additional condition: if d is of order r in the multiplicative group of residues modulo m coprime with m, then every prime divisor of d divides n/d (r, m and n are defined in Table 1).*

Theorem 3.2 (Wolf [14, Theorem 6.3.1]). *Let G be a non-solvable finite group. If G admits a representation without fixed points over $\mathbb{C}$, then G belongs to one of the following two types.*

Type V. *$G = K \times \mathrm{I}^*$, where K is a solvable group of type I (see Table 1), the order of K is coprime with* 30, *and I^* is the binary icosahedral group.*

Type VI. *G is generated by a normal subgroup G_1 of index* 2 *and an element S, where $G_1 = K \times \mathrm{I}^*$ is of type V and S satisfies the following conditions. If we identify I^* with* *the group* $\mathrm{SL}(2,5)$ *of* 2×2 *matrices of determinant* 1 *over the field* $\mathbb{Z}/5$*, then* $S^2=-I\in\mathrm{SL}(2,5)$*, and, for any* $L\in\mathrm{SL}(2,5)$*,* $SLS^{-1}=\theta(L)$*, where* θ *is an automorphism of* I^* *given by*

$$\theta(L)=\begin{pmatrix}0 & -1\\ 2 & 0\end{pmatrix} L \begin{pmatrix}0 & -1\\ 2 & 0\end{pmatrix}^{-1}.$$

Moreover, any such L *belongs to the normalizer of the subgroup* K*. If* A *and* B *are the generators of the group* K*, then* $SBS^{-1}=B^k$*,* $SAS^{-1}=A^l$*, where the integers* k *and* l *satisfy* $l^2\equiv 1(m)$*,* $k^2\equiv 1(n)$*,* $r^{k-1}\equiv 1(m)$ *(see Table 1).*

Conversely, every group of type V or VI is non-solvable and admits representations without fixed points over $\mathbb{C}$*.*

Remark 3.3. All groups listed in Theorems 3.1 and 3.2, with the exception of groups of type I, contain a generalized quaternion group $\mathrm{Q}2^a$ as a subgroup. This follows from Theorem 2.12 and [**14**, Theorem 5.4.1], the latter stating that if all Sylow subgroups of a finite group G are cyclic, then G is a group of type I.

Remark 3.4. The Kleinian subgroups of $\mathrm{SL}(2,\mathbb{C})$ are certainly among the groups of types I–VI. The cyclic group $\mathbb{Z}/n$ belongs to type I ($m=1$). The binary dihedral group D_b^* belongs to type I for odd b ($m=b$, $n=4$, $r=-1$), and to type II for even b ($m=1$, $n=2b$, $k=-1$). The binary tetrahedral group T^* belongs to type III ($m=1$, $n=3$). The binary octahedral group O^* belongs to type IV ($m=1$, $n=3$, $k=-1$), and the binary icosahedral group belongs to type V ($K=\{1\}$).

Remark 3.5. It would be useful to write the action of the automorphism θ from Theorem 3.2 in terms of matrices $\iota_1(\pm V)$, $\iota_1(\pm U)$ and $\iota_1(\pm T)$ of the standard representation ι_1 of the group I^*. We identify the image of this representation with I^*. Note that the matrices $-V$ and $-T$ alone generate I^*. We can fix an isomorphism with $\mathrm{SL}(2,5)$ by setting, for example,

$$-V\leftrightarrow\begin{pmatrix}-1 & -1\\ 0 & -1\end{pmatrix},\qquad -T\leftrightarrow\begin{pmatrix}2 & 0\\ -1 & 3\end{pmatrix}.$$

Then, θ is given by

$$\theta(-V)=\frac{1}{5}\begin{pmatrix}1-\varepsilon+2\varepsilon^2-2\varepsilon^4 & -2+2\varepsilon+\varepsilon^2-\varepsilon^4\\ 2+\varepsilon-\varepsilon^3-2\varepsilon^4 & 1-2\varepsilon+2\varepsilon^3-\varepsilon^4\end{pmatrix},\qquad \theta(-T)=\begin{pmatrix}0 & -\varepsilon\\ \varepsilon^4 & 0\end{pmatrix},$$

where $\varepsilon=\mathrm{e}^{2\pi\mathrm{i}/5}$.

Next, we need a classification of irreducible complex representations without fixed points of groups from Theorems 3.1 and 3.2. The reference for this classification is [**14**, § 7.2]. We give a list of such representations for all types I–VI. The set of all irreducible complex representations without fixed points of a given group G is denoted by $\mathfrak{F}_{\mathbb{C}}(G)$; φ here denotes the Euler function. We also use the notation introduced in Theorems 3.1, 3.2 and Table 1.

The list of irreducible complex representations without fixed points

Type I. Let G be a group of type I. Recall that d is of order r in the group of residues modulo m coprime with m. The set $\mathfrak{F}_{\mathbb{C}}(G)$ then consists of the following $\varphi(mn)/d^2$ representations $\pi_{k,l}$ of dimension d:

$$\pi_{k,l}(A) = \begin{pmatrix} \mathrm{e}^{2\pi \mathrm{i}k/m} & 0 & \cdots & 0 \\ 0 & \mathrm{e}^{2\pi \mathrm{i}kr/m} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{e}^{2\pi \mathrm{i}kr^{d-1}/m} \end{pmatrix},$$

$$\pi_{k,l}(B) = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \mathrm{e}^{2\pi \mathrm{i}l/n'} & 0 & \cdots & 0 \end{pmatrix}.$$

Here, $(k,m) = (l,n) = 1$, $n' = n/d$.

Type II. Let G be a group of type II. It then contains a subgroup $\langle A, B\rangle$ of type I. All representations without fixed points of the group G are induced from the representations of the subgroup $\langle A, B\rangle$. Namely, $\mathfrak{F}_{\mathbb{C}}(G)$ consists of the following $\varphi(2mn)/(2d)^2$ representations $\alpha_{k',l'}$ of dimension $2d$:

$$\alpha_{k',l'}(A) = \begin{pmatrix} \pi_{k',l'}(A) & 0 \\ 0 & \pi_{k',l'}(A^l) \end{pmatrix}, \qquad \alpha_{k',l'}(B) = \begin{pmatrix} \pi_{k',l'}(B) & 0 \\ 0 & \pi_{k',l'}(B^k) \end{pmatrix},$$

$$\alpha_{k',l'}(R) = \begin{pmatrix} 0 & I \\ \pi_{k',l'}(B^{n/2}) & 0 \end{pmatrix}.$$

Here, $(k', m) = (l', n) = 1$, k and l are defined under type II in Table 1, and I is the unit $d \times d$ matrix.

Type III. Let G be a group of type III. It then contains a subgroup $\langle A, B\rangle$, which is a group of type I of odd order. We have to distinguish three cases.

Case 1. $9 \nmid n$, in particular, $3 \nmid d$. In this case

$$G = \langle A, B^3\rangle \times \langle B^{n''}, P, Q\rangle,$$

where $n'' = n/3$, $\langle B^{n''}, P, Q\rangle$ is the binary tetrahedral group T^*, and $\langle A, B^3\rangle$ is a group of type I with the same value of d as the group $\langle A, B\rangle$. Then, $\mathfrak{F}_{\mathbb{C}}(G)$ consists of the following $\varphi(mn)/2d^2$ representations $\nu_{k,l}$ of dimension $2d$:

$$\nu_{k,l} = \pi_{k,l} \otimes \tau,$$

where $\pi_{k,l} \in \mathfrak{F}_{\mathbb{C}}(\langle A, B^3\rangle)$, and τ is the only irreducible representation without fixed points of the group T^* (see Lemma 2.15).

https://doi.org/10.1017/S0013091513000643 Published online by Cambridge University Press

Case 2. $9 \mid n$, $3 \nmid d$. Let $n = 3^v n''$, where $(3, n'') = 1$ and $v > 1$. Then

$$G = \langle A, B^{3^v} \rangle \times \langle B^{n''}, P, Q \rangle = \langle A, B^{3^v} \rangle \times \mathrm{T}^*_v,$$

where $\langle A, B^{3^v} \rangle$ is a group of type I with the same value of d as $\langle A, B \rangle$, and T^*_v is a generalized tetrahedral group. Then, $\mathfrak{F}_{\mathbb{C}}(G)$ consists of the following $\varphi(mn)/d^2$ representations $\nu_{k,l,j}$ of dimension $2d$:

$$\nu_{k,l,j} = \pi_{k,l} \otimes \tau_j,$$

where $\pi_{k,l} \in \mathfrak{F}_{\mathbb{C}}(\langle A, B^{3^v} \rangle)$, and τ_j are defined in Lemma 2.15.

Case 3. $9 \mid n$ and $3 \mid d$. Then, G contains a normal subgroup of index d,

$$\langle A, B^d, P, Q \rangle = \langle A \rangle \times \langle B^d \rangle \times \langle P, Q \rangle,$$

where $\langle P, Q \rangle$ is the quaternion group Q8. Irreducible representations without fixed points of G are induced from irreducible representations without fixed points of the subgroup $\langle A, B^d, P, Q \rangle$. The set $\mathfrak{F}_{\mathbb{C}}(G)$ consists of the following $\varphi(mn)/d^2$ representations $\mu_{k,l}$ of dimension $2d$:

$$\mu_{k,l}(A) = \begin{pmatrix} \mathrm{e}^{2\pi \mathrm{i} k/m} I_2 & 0 & \cdots & 0 \\ 0 & \mathrm{e}^{2\pi \mathrm{i} kr/m} I_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{e}^{2\pi \mathrm{i} kr^{d-1}/m} I_2 \end{pmatrix},$$

$$\mu_{k,l}(B) = \begin{pmatrix} 0 & I_{2d-2} \\ \mathrm{e}^{2\pi \mathrm{i} l/n'} I_2 & 0 \end{pmatrix},$$

$$\mu_{k,l}(P) = \begin{pmatrix} \alpha(P) & 0 & \cdots & 0 \\ 0 & \alpha(Q) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha(B^{d-1} P B^{1-d}) \end{pmatrix},$$

$$\mu_{k,l}(Q) = \begin{pmatrix} \alpha(Q) & 0 & \cdots & 0 \\ 0 & \alpha(PQ) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha(B^{d-1} Q B^{1-d}) \end{pmatrix}.$$

Here, $(k, m) = (l, n) = 1$, $n' = n/d$, I_2 and I_{2d-2} are the unit matrices of dimensions 2×2 and $(2d-2) \times (2d-2)$, respectively, and α is the representation of the quaternion group Q8 defined in Lemma 2.14.

Type IV. Let G be a group of type IV. Note that elements A, B, P, Q of G generate a subgroup of type III. Here, we again have to consider several cases.

https://doi.org/10.1017/S0013091513000643 Published online by Cambridge University Press

Case 1. $9 \nmid n$, in particular, $3 \nmid d$.

Subcase 1 (a). Assume that there exists an element $\pi \in \mathfrak{F}(\langle A, B^3\rangle)$ ($\langle A, B^3\rangle$ is a group of type I), which is equivalent to the representation $\pi' \colon g \mapsto \pi(RgR^{-1})$ of the group $\langle A, B^3\rangle$. Then, G is a direct product

$$G = \langle A, B^3\rangle \times \langle R, B^{n/3}, P, Q\rangle = \langle A, B^3\rangle \times \mathrm{O}^*,$$

where O^* is the binary octahedral group. Then, $\mathfrak{F}_{\mathbb{C}}(G)$ consists of the following $\varphi(mn)/d^2$ representations $\psi_{k,l,j}$ of dimension $2d$:

$$\psi_{k,l,j} = \pi_{k,l} \otimes o_j,$$

where $\pi_{k,l} \in \mathfrak{F}_{\mathbb{C}}(\langle A, B^3\rangle)$, and $o_j = o_1$ or o_{-1} is a representation of the binary octahedral group O^* defined in Lemma 2.16.

Subcase 1 (b). If the assumption of Subcase 1 (a) is not satisfied, the group G is not the direct product of subgroups $\langle A, B^3\rangle$ and O^*. Then, $\mathfrak{F}_{\mathbb{C}}(G)$ consists of $\varphi(mn)/4d^2$ representations $\gamma_{k',l'}$ of dimension $4d$ induced by representations $\nu_{k',l'}$ of the subgroup $\langle A, B, P, Q\rangle$. We have that

$$\gamma_{k',l'}(A) = \begin{pmatrix} \nu_{k',l'}(A) & 0 \\ 0 & \nu_{k',l'}(A^l) \end{pmatrix},$$
$$\gamma_{k',l'}(B) = \begin{pmatrix} \nu_{k',l'}(B) & 0 \\ 0 & \nu_{k',l'}(B^k) \end{pmatrix},$$
$$\gamma_{k',l'}(P) = \begin{pmatrix} \nu_{k',l'}(P) & 0 \\ 0 & \nu_{k',l'}(QP) \end{pmatrix},$$
$$\gamma_{k',l'}(Q) = \begin{pmatrix} \nu_{k',l'}(Q) & 0 \\ 0 & \nu_{k',l'}(Q^{-1}) \end{pmatrix},$$
$$\gamma_{k',l'}(R) = \begin{pmatrix} 0 & I_{2d} \\ \nu_{k',l'}(P^2) & 0 \end{pmatrix}.$$

Here, $(k', m) = (l', n) = 1$, k and l are defined under type IV in Table 1, and I_{2d} is the unit matrix of dimension $2d \times 2d$.

Case 2. $9 \mid n$, but $3 \nmid d$. Let $n = 3^v n''$, $(3, n'') = 1$. The subgroup $\langle R, B^{n''}, P, Q\rangle$ of G is then isomorphic to the generalized binary octahedral group O^*_v.

Subcase 2 (a). Assume that there exists an element $\pi \in \mathfrak{F}(\langle A, B^{3^v}\rangle)$ ($\langle A, B^{3^v}\rangle$ is a group of type I), which is equivalent to the representation $\pi' \colon g \mapsto \pi(RgR^{-1})$ of the group $\langle A, B^{3^v}\rangle$. Then, G is a direct product,

$$G = \langle A, B^{3^v}\rangle \times \langle R, B^{n''}, P, Q\rangle = \langle A, B^{3^v}\rangle \times \mathrm{O}^*_v,$$

https://doi.org/10.1017/S0013091513000643 Published online by Cambridge University Press

and the set $\mathfrak{F}_{\mathbb{C}}(G)$ consists of the following $\varphi(mn)/2d^2$ representations $\xi_{k,l,j}$ of dimension $4d$:

$$\xi_{k,l,j} = \pi_{k,l} \otimes o_j,$$

where $\pi_{k,l} \in \mathfrak{F}_{\mathbb{C}}(\langle A, B^{3^v}\rangle)$, and o_j is a representation of the generalized binary octahedral group O_v^* defined in Lemma 2.16.

Subcase 2 (b). If the assumption of Subcase 2 (a) is not satisfied, then the group G is not the direct product of subgroups $\langle A, B^{3^v}\rangle$ and O_v^*. Then, $\mathfrak{F}_{\mathbb{C}}(G)$ consists of $\varphi(mn)/2d^2$ representations $\gamma_{k',l',j}$ of dimension $4d$ induced by representations $\nu_{k',l',j}$ of the subgroup $\langle A, B, P, Q\rangle$. We have that

$$\gamma_{k',l',j}(A) = \begin{pmatrix} \nu_{k',l',j}(A) & 0 \\ 0 & \nu_{k',l',j}(A^l) \end{pmatrix},$$

$$\gamma_{k',l',j}(B) = \begin{pmatrix} \nu_{k',l',j}(B) & 0 \\ 0 & \nu_{k',l',j}(B^k) \end{pmatrix},$$

$$\gamma_{k',l',j}(P) = \begin{pmatrix} \nu_{k',l',j}(P) & 0 \\ 0 & \nu_{k',l',j}(QP) \end{pmatrix},$$

$$\gamma_{k',l',j}(Q) = \begin{pmatrix} \nu_{k',l',j}(Q) & 0 \\ 0 & \nu_{k',l',j}(Q^{-1}) \end{pmatrix},$$

$$\gamma_{k',l',j}(R) = \begin{pmatrix} 0 & I_{2d} \\ \nu_{k',l',j}(P^2) & 0 \end{pmatrix}.$$

Here, $(k', m) = (l', n) = 1$, and the index j numerates representations of the group O_v^* (see Lemma 2.16), k and l are defined under type IV in Table 1, and I_{2d} is the unit matrix of dimension $2d \times 2d$.

Case 3. $3 \mid d$, in particular, $9 \mid n$. The set $\mathfrak{F}_{\mathbb{C}}(G)$ consists of $\varphi(mn)/2d^2$ representations $\eta_{k',l'}$ of dimension $4d$ induced by representations $\mu_{k',l'}$ of the subgroup $\langle A, B, P, Q\rangle$. We have that

$$\eta_{k',l'}(A) = \begin{pmatrix} \mu_{k',l'}(A) & o \\ 0 & \mu_{k',l'}(A^l) \end{pmatrix},$$

$$\eta_{k',l'}(B) = \begin{pmatrix} \mu_{k',l'}(B) & 0 \\ 0 & \mu_{k',l'}(B^k) \end{pmatrix},$$

$$\eta_{k',l'}(P) = \begin{pmatrix} \mu_{k',l'}(P) & 0 \\ 0 & \mu_{k',l'}(QP) \end{pmatrix},$$

$$\eta_{k',l'}(Q) = \begin{pmatrix} \mu_{k',l'}(Q) & 0 \\ 0 & \mu_{k',l'}(Q^{-1}) \end{pmatrix},$$

$$\eta_{k',l'}(R) = \begin{pmatrix} 0 & I_{2d} \\ \mu_{k',l'}(P^2) & 0 \end{pmatrix}.$$

https://doi.org/10.1017/S0013091513000643 Published online by Cambridge University Press

Here, $(k',m) = (l',n) = 1$, k and l are defined under type IV in Table 1, and I_{2d} is the unit matrix of dimension $2d \times 2d$.

Type V. Let G be a group of type V. Then, G has the form $G = K \times \mathrm{I}^*$ and the set $\mathfrak{F}_{\mathbb{C}}(G)$ consists of $2\varphi(mn)/d^2$ representations $\iota_{k,l,j}$ of dimension $2d$:

$$\iota_{k,l,j} = \pi_{k,l} \otimes \iota_j,$$

where $\pi_{k,l}$ are described for type I in the list of irreducible complex representations without fixed points in this section, and $\iota_j = \iota_1$ or ι_{-1} are described in Lemma 2.17.

Type VI. Let G be a group of type VI. The set $\mathfrak{F}_{\mathbb{C}}(G)$ then consists of $\varphi(mn)/d^2$ representations $\varkappa_{k',l',j}$ of dimension $4d$ induced by representations $\iota_{k',l',j}$ of the subgroup $K \times \mathrm{I}^*$. We have, in particular, that

$$\varkappa_{k',l',j}(A) = \begin{pmatrix} \iota_{k',l',j}(A) & 0 \\ 0 & \iota_{k',l',j}(A^l) \end{pmatrix},$$

$$\varkappa_{k',l',j}(B) = \begin{pmatrix} \iota_{k',l',j}(B) & 0 \\ 0 & \iota_{k',l',j}(B^k) \end{pmatrix},$$

$$\varkappa_{k',l',j}(S) = \begin{pmatrix} 0 & I_{2d} \\ -I_{2d} & 0 \end{pmatrix},$$

where k and l are defined in Theorem 3.2, and I_{2d} is the unit matrix of dimension $2d \times 2d$. Matrices $\varkappa_{k',l',j}(\pm V)$, $\varkappa_{k',l',j}(\pm T)$, $\varkappa_{k',l',j}(\pm U)$ of the generators of I^* can be obtained using Lemma 2.17 and Theorem 3.2.

Theorem 3.6. *Let G be a finite group possessing a complex representation without fixed points. Then, G is one of the groups of types I–VI and all irreducible complex representations of G without fixed points are given in Table 2. The columns of the table have the following meaning. The first column gives the type of the group G, the second gives additional conditions on the group G, the third gives irreducible representations of G without fixed points, the fourth gives the dimension of representations, the fifth gives the determinant of the matrix of representation corresponding to the generator B (sometimes some power of B) of the group G, and the sixth gives the conditions when the image of a representation is contained in $\mathrm{SL}(d,\mathbb{C})$ ($\mathrm{SL}(2d,\mathbb{C})$, $\mathrm{SL}(4d,\mathbb{C})$). All the other generators of the group G have determinant 1. In the table we use the notation introduced in Theorems 3.1, 3.2, and the list of irreducible representations without fixed points in this section. D_n^* denotes the binary dihedral group of order $4n$.*

Proof. Everything with the exception of the computation of determinants is done in [**14**, § 7.2]. So, we concentrate on columns 5 and 6 of Table 2.

Type I. From the matrices given in the corresponding entry of the list of irreducible representations, without fixed points we easily find that

$$|\pi_{k,l}(A)| = \exp\left(\frac{2\pi\mathrm{i}k}{m}(1 + \cdots + r^{d-1})\right) = 1,$$

Table 2. *Irreducible representations without fixed points.*

type	case	representations	dim	$\det(B)$	$G < \mathrm{SL}$
I		$\pi_{k,l}$	d	$(-1)^{d-1}\mathrm{e}^{2\pi\mathrm{i}l/n'}$	$n = 2^{s+1}$, $d = 2^s$, $s \geqslant 1$, or $d = 1$ and $G = \{1\}$
II		$\alpha_{k',l'}$	$2d$	$\mathrm{e}^{2\pi\mathrm{i}l'(k+1)/n'}$	$d = 1$ and $G = \mathrm{D}_n^*$, or $d = 2$ and $k = -1$
III	$9 \nmid n$	$\nu_{k,l}$	$2d$	$\lvert\nu_{k,l}(B^3)\rvert = \mathrm{e}^{12\pi\mathrm{i}l/n'}$	$d = 1$ and $G = \mathrm{T}^*$
III	$9 \mid n$, $3 \nmid d$	$\nu_{k,l,j}$	$2d$	$\lvert\nu_{k,l,j}(B^{n''})\rvert = \mathrm{e}^{4\pi\mathrm{i}jd/3^v}$, $\lvert\nu_{k,l,j}(B^{3^v})\rvert = \mathrm{e}^{4\pi\mathrm{i}l3^v/n'}$	never
III	$3 \mid d$	$\mu_{k,l}$	$2d$	$\mathrm{e}^{4\pi\mathrm{i}l/n'}$	never
IV	$9 \nmid n$, $G = \langle A, B^3\rangle \times \mathrm{O}^*$	$\psi_{k,l,j}$	$2d$	$\lvert\psi_{k,l,j}(B^3)\rvert = \mathrm{e}^{4\pi\mathrm{i}l/(n'/3)}$	$d = 1$ and $G = \mathrm{O}^*$
IV	$9 \nmid n$, $G \neq \langle A, B^3\rangle \times \mathrm{O}^*$	$\gamma_{k',l'}$	$4d$	$\lvert\gamma_{k',l'}(B^3)\rvert = \mathrm{e}^{4\pi\mathrm{i}l'(k+1)/(n'/3)}$	$d = 1,\ A = 1,\ k = -1$
IV	$9 \mid n$, $3 \nmid d$, $G = \langle A, B^{3^v}\rangle \times \mathrm{O}_v^*$	$\xi_{k,l,j}$	$4d$	$\lvert\xi_{k,l,j}(B^{3^v})\rvert = \mathrm{e}^{2\pi\mathrm{i}l/(n''/d)}$	$d = 1$ and $G = \mathrm{O}_v^*$
IV	$9 \mid n$, $3 \nmid d$, $G \neq \langle A, B^{3^v}\rangle \times \mathrm{O}_v^*$	$\gamma_{k',l',j}$	$4d$	$\lvert\gamma_{k',l',j}(B^{3^v})\rvert = \mathrm{e}^{2\pi\mathrm{i}l'(k+1)/(n''/d)}$	$d = 1,\ A = 1$
IV	$3 \mid d$	$\eta_{k',l'}$	$4d$	$\mathrm{e}^{4\pi\mathrm{i}l'(k+1)/n'}$	never
V		$\iota_{k,l,j}$	$2d$	$\mathrm{e}^{4\pi\mathrm{i}l/n'}$	$d = 1$ and $G = \mathrm{I}^*$
VI		$\varkappa_{k',l',j}$	$4d$	$\mathrm{e}^{4\pi\mathrm{i}l'(k+1)/n'}$	$d = 1,\ A = 1,\ k = -1$

because $m \mid (r^d - 1)$ but $(r-1, m) = 1$ (see Table 1). Furthermore,

$$|\pi_{k,l}(B)| = (-1)^{d-1} \mathrm{e}^{2\pi \mathrm{i} l/n'}.$$

Since $(l, n') = 1$, $\mathrm{e}^{2\pi \mathrm{i} l/n'}$ is a primitive root of unity of degree n'. It follows that the last determinant can be equal to 1 either when $d = n' = n = m = 1$, that is, when the group G is trivial, or when $n' = 2$. Since any prime divisor of d divides n', we have $d = 2^s$, $n = n'd = 2^{s+1}$, $s \geqslant 1$. The case $s = 0$, i.e. $d = 1$, again leads to the trivial group G. We have proven the theorem for groups of type I.

Type II. From the matrices of the representation $\alpha_{k',l'}$ and under the conditions of type I in Table 2, which we just proved, we easily find that

$$\begin{aligned} |\alpha_{k',l'}(A)| &= |\pi_{k',l'}(A)|\,|\pi_{k',l'}(A^l)| = 1, \\ |\alpha_{k',l'}(R)| &= (-1)^d |\pi_{k',l'}(B^{n/2})| = (-1)^d \mathrm{e}^{\pi \mathrm{i} l' d} = 1 \end{aligned}$$

and

$$|\alpha_{k',l'}(B)| = |\pi_{k',l'}(B)|\,|\pi_{k',l'}(B^k)| = \mathrm{e}^{2\pi \mathrm{i}(k+1)l'/n'},$$

where one has to use that n and $k+1$ are divisible by 4. If the determinant of B is equal to 1, then $n' \mid k+1$, and, thus, any prime divisor of d also divides $k+1$. On the other hand, d divides $k-1$ (see type II in Table 1). It follows that $d = 1$ or $d = 2$. In the first case we get $m = 1$ and $A = 1$; thus,

$$G = \langle B, R \mid R^2 = B^{n/2},\ RBR^{-1} = B^{-1},\ B^n = 1 \rangle$$

is the binary dihedral group and $\alpha_{k',l'}$ is its standard representation. In the second case we get $n' = 2^{u-1}v$. On the other hand, $(k+1)/n'$ must be integer, but k is defined modulo n. Thus, we may take $k = -1$ and the group reduces to

$$\begin{aligned} G = \langle A, B, R \mid A^m = B^n = 1,\ R^2 = B^{n/2}, \\ BAB^{-1} = A^r,\ RAR^{-1} = A^l,\ RBR^{-1} = B^{-1} \rangle. \end{aligned}$$

Type III. Recall that if S and T are matrices of size $m \times m$ and $n \times n$, respectively, then

$$|S \otimes T| = |A|^n |B|^m.$$

Case 1. $9 \nmid n$. From the definition of the representation $\nu_{k,l}$ we get that

$$\begin{aligned} |\nu_{k,l}(A)| &= |\pi_{k,l} \otimes \tau(A)| = 1, \\ |\nu_{k,l}(B^3)| &= |\pi_{k,l}(B^3)|^2 = \mathrm{e}^{12\pi \mathrm{i} l/n'}. \end{aligned}$$

Matrices of $B^{n/3}$, P and Q all have determinant 1 since they generate the binary tetrahedral group. Recall that now n' is odd; thus, $|\nu_{k,l}(B^3)| = 1$ only if $n' = 1$ or $n' = 3$. But the first case is impossible, because it would imply that $n = 1$, but n is divisible by 3. In the second case we have $d = 1$ or $d = 3$. Again, the second case is impossible, because it would imply that $9 \mid n$. We conclude that $d = 1$, $G = \mathrm{T}^*$ and $\nu_{k,l} = \tau$.

Case 2. $9 \mid n$, but $3 \nmid d$. In this case the group G is the direct product of the subgroups $\langle A, B^{3^v}\rangle$ and $\langle B^{n''}, P, Q\rangle \simeq \mathrm{T}_v^*$, where $n = 3^v n''$, $v \geqslant 2$, and $B^{n''}$ corresponds to the generator X of the group Tv*. It is easy to check that $|\tau_j(X)| = \mathrm{e}^{4\pi \mathrm{i} j/3^v}$ (see Lemma 2.15), which is never equal to 1. It follows that $|\nu_{k,l,j}(B^{n''})| = \mathrm{e}^{4\pi \mathrm{i} jd/3^v}$ and the image of the group G is never contained in the group $\mathrm{SL}(2d, \mathbb{C})$. It is also a straightforward verification that $|\nu_{k,l,j}(A)| = |\nu_{k,l,j}(P)| = |\nu_{k,l,j}(Q)| = 1$ and $|\nu_{k,l,j}(B^{3^v})| = \mathrm{e}^{4\pi \mathrm{i} l 3^v/n'}$.

Case 3. $3 \mid n$. From the definition of the representation $\mu_{k,l}$, we find that determinants of the matrices corresponding to A, P and Q are all equal to 1, whereas $|\mu_{k,l}(B)| = \mathrm{e}^{4\pi \mathrm{i} l/n'}$. Note that n' is divisible by 3, $(n', l) = 1$; thus, the last determinant is never equal to 1 and the image of $\mu_{k,l}$ is never contained in $\mathrm{SL}(2d, \mathbb{C})$.

Type IV.

Subcase 1 (a). $9 \nmid n$, G is the direct product $\langle A, B^3\rangle \times \mathrm{O}^*$, where $\mathrm{O}^* = \langle B^{n/3}, P, Q, R\rangle$. From the definition of the representation $\psi_{k,l,j}$, we find that

$$|\psi_{k,l,j}(A)| = |\psi_{k,l,j}(B^{n/3})| = |\psi_{k,l,j}(P)| = |\psi_{k,l,j}(Q)| = |\psi_{k,l,j}(R)| = 1,$$
$$|\psi_{k,l,j}(B^3)| = |\pi_{k,l} \otimes o_j(B^3)| = \mathrm{e}^{4\pi \mathrm{i} l/(n'/3)}.$$

Note that n' is an odd number, so the only possibility for the last determinant to be equal to 1 is that $d = 1$, $n = n' = 3$; thus, $G = \mathrm{O}^*$.

Subcase 1 (b). Conditions are as in Subcase 1 (a), but G is not the direct product. From the definition of the representation $\gamma_{k',l'}$, it follows easily that determinants of matrices corresponding to A, $B^{n/3}$, P, Q, R are all equal to 1. For B^3 we have that

$$|\gamma_{k',l'}(B^3)| = |\nu_{k',l'}(B^3)|\,|\nu_{k',l'}(B^{3k})| = \mathrm{e}^{4\pi \mathrm{i} l'(k+1)/(n'/3)}.$$

By an argument analogous to that of Type II we prove that either $d = 2$ or $d = 1$. But here n is odd; thus, $d = 1$. This implies that $A = 1$. Furthermore, $n/3$ divides $k+1$, and at the same time 3 divides $k+1$. Since k is determined modulo n, we may take $k = -1$.

Subcase 2 (a). $9 \mid n$, $3 \nmid d$, and G is the direct product $\langle A, B^{3^v}\rangle \times \mathrm{O}_v^*$, where O_v^* is generated by $B^{n''}$, P, Q and R, $n = 3^v n''$, $(3, n'') = 1$. First, we check that $|o_j(X)| = 1$, where X is a generator of the generalized binary octahedral group. This follows easily from Lemma 2.16. It then follows from the definition of the representation $\xi_{k,l}$ that the matrices corresponding to A, $B^{n''}$, P, Q, R all have determinant 1. Furthermore,

$$|\xi_{k,l}(B^{3^v})| = \mathrm{e}^{8\pi \mathrm{i} l/(n''/d)}.$$

This determinant equals 1 only if $n'' = d$. But any prime divisor of d divides n''/d; hence, $n'' = d = 1$, and $G = \mathrm{O}_v^*$.

Subcase 2 (b). The conditions are as in Subcase 2 (a), but G is not the direct product. From the definition of the representation $\gamma_{k',l',j}$, we easily compute that all determinants are equal to 1, with the exception of

$$|\gamma_{k',l',j}(B^{3^v})| = \mathrm{e}^{2\pi \mathrm{i} l'(k+1)/(n''/d)}.$$

https://doi.org/10.1017/S0013091513000643 Published online by Cambridge University Press

If this determinant equals 1 and p is a prime divisor of d, we again see that p must divide both $k-1$ and $k+1$; thus, $p=2$. But in this case n is odd, so the only possibility is that $d=1$ and $A=1$.

Case 3. $3 \mid d$. From the definition of the representation $\eta_{k',l'}$, we directly find that all the determinants involved equal 1, with the exception of

$$|\eta_{k',l'}(B)| = \mathrm{e}^{4\pi \mathrm{i} l'(k+1)/n'}.$$

This determinant can be equal to 1 only if $d=1$, but this would contradict $3 \mid d$. Therefore, $\eta_{k',l'}(G)$ is never contained in $\mathrm{SL}(4d,\mathbb{C})$ in this case.

Type V. Now G is a direct product $K \times \mathrm{I}^*$, where K is a group of type I and the order of G is coprime with 30. It follows from the definition of the representation $\iota_{k,l,j}$ and Lemma 2.17 that matrices corresponding to generators of G all have determinant 1, with the exception of $|\iota_{k,l,j}(B)|$, which equals $\mathrm{e}^{4\pi \mathrm{i} l/n'}$. The number n' is odd; thus, this determinant can be equal to 1 only if $n'=1$. This implies that $d=n=1$ and K is trivial. Hence, $G=\mathrm{I}^*$ is the binary icosahedral group and $\iota_{k,l,j}$ is one of its representations described in Lemma 2.17.

Type VI. Again, an easy computation, which we omit, shows that only the determinant $|\varkappa_{k',l',j}(B)|$ is in question. From the definition of $\varkappa_{k',l',j}$, we find that

$$|\varkappa_{k',l',j}(B)| = \mathrm{e}^{4\pi \mathrm{i} l'(k+1)}.$$

The same type of argument as for type II shows that this can be equal to 1 only if $d=1$, $A=1$ and $k=-1$. $\square$

Remark 3.7. Note that if one of the irreducible representations without fixed points of a group G is contained in the special linear group, then the others are too. Thus, this seems to be a property of the group G rather than of a particular representation.

Remark 3.8. For a given group G from Theorems 3.1 and 3.2, all its irreducible representations have the same dimension d, $2d$ or $4d$. Thus, any representation without fixed points of G has dimension a multiple of d, $2d$, or $4d$, respectively.

Remark 3.9. Note also that all representations from Theorem 3.6 are imprimitive and induced from primitive representations of dimension 1 or 2.

Finally, we have to determine the automorphisms of groups of types I–VI and the action of the automorphisms on the irreducible representations of these groups by the rule $\varphi \to \varphi \circ \alpha$, where φ is a representation and α is an automorphism. This will give the conditions when the images of two representations are conjugate in GL.

Theorem 3.10 (Wolf [14, Theorem 7.3.21]). *Let G be a finite group possessing a representation without fixed points, i.e. one of the groups listed in Theorems 3.1 and 3.2. The action of the automorphisms on the irreducible representations of G is then described in Table 3. We use here the notation introduced in Table 1, Theorem 3.2 and the list of irreducible complex representations without fixed points.*

https://doi.org/10.1017/S0013091513000643 Published online by Cambridge University Press

Table 3. *Action of automorphisms on representations.*

type	case	action	conditions
I		$A_{a,b}\colon \pi_{k,l} \mapsto \pi_{ak,bl}$	$(a,m)=1$, $(b,n)=1$, $b \equiv 1(d)$
II		$A_{a,b}\colon \alpha_{k',l'} \mapsto \alpha_{ak',bl'}$	$(a,m)=1$, $(b,n)=1$, $b \equiv 1(d)$
III	$9 \nmid n$	$A_{a,b}\colon \nu_{k,l} \mapsto \nu_{ak,bl}$	$(a,m)=1$, $(b,n/3)=1$, $b \equiv 1(d)$
III	$9 \mid n$, $3 \nmid d$	$A_{a,b,c}\colon \nu_{k,l,j} \mapsto \nu_{ak,bl,cj}$	$(a,m)=1$, $(b,n/3^v)=1$, $(c,3)=1$, $b \equiv 1(d)$
III	$3 \mid d$	$A_{a,b}\colon \mu_{k,l} \mapsto \mu_{ak,bl}$	$(a,m)=1$, $(b,n)=1$, $b \equiv 1(d)$
IV	$9 \nmid n$, $G = \langle A, B^3 \rangle \times \mathrm{O}^*$	$A_{a,b,c}\colon \psi_{k,l,j} \mapsto \psi_{ak,bl,cj}$	$(a,m)=1$, $(b,n/3)=1$, $b \equiv 1(d)$, $c=\pm 1$
IV	$9 \nmid n$, $G \neq \langle A, B^3 \rangle \times \mathrm{O}^*$	$A_{a,b}\colon \gamma_{k',l'} \mapsto \gamma_{ak',bl'}$	$(a,m)=1$, $(b,n/3)=1$, $b \equiv 1(d)$
IV	$9 \mid n$, $3 \nmid d$, $G = \langle A, B^{3^v} \rangle \times \mathrm{O}_v^*$	$A_{a,b,c}\colon \xi_{k,l,j} \mapsto \xi_{ak,bl,cj}$	$(a,m)=1$, $(b,n/3^v)=1$, $(c,3)=1$, $b \equiv 1(d)$
IV	$9 \mid n$, $3 \nmid d$, $G \neq \langle A, B^{3^v} \rangle \times \mathrm{O}_v^*$	$A_{a,b,c}\colon \gamma_{k',l',j} \mapsto \gamma_{ak',bl',cj}$	$(a,m)=1$, $(b,n/3^v)=1$, $(c,3)=1$, $b \equiv 1(d)$
IV	$3 \mid d$	$A_{a,b}\colon \eta_{k',l'} \mapsto \eta_{ak',bl'}$	$(a,m)=1$, $(b,n)=1$, $b \equiv 1(d)$
V		$A_{a,b,c}\colon \iota_{k,l,j} \mapsto \iota_{ak,bl,cj}$	$(a,m)=1$, $(b,n)=1$, $b \equiv 1(d)$, $c=\pm 1$
VI		$A_{a,b,c}\colon \varkappa_{k',l',j} \mapsto \varkappa_{ak',bl',cj}$	$(a,m)=1$, $(b,n)=1$, $b \equiv 1(d)$, $c=\pm 1$

https://doi.org/10.1017/S0013091513000643 Published online by Cambridge University Press

We can formulate the main theorem on classification of IQS over the field $\mathbb{C}$.

Theorem 3.11. *Let $Q \in X$ be an isolated quotient singularity of a variety X defined over the field $\mathbb{C}$, as it is described in § 2. Then, locally analytically at Q, the variety X is isomorphic to the quotient $\mathbb{C}^N/\varphi(G)$, where $N = \dim X$, G is one of the groups described in Theorems 3.1 and 3.2, and φ is a direct sum of irreducible representations of G described in Theorem 3.6.*

Conversely, any group from Theorems 3.1 and 3.2 acting on $\mathbb{C}^N$ via a direct sum φ of irreducible representations described in Theorem 3.6 gives an isolated singularity $O \in \mathbb{C}^N/\varphi(G)$. If φ and ψ are two such representations, then the singularities $\mathbb{C}^N/\varphi(G)$ and $\mathbb{C}^N/\psi(G)$ are isomorphic if and only if φ and ψ can be transformed into each other by an action of an automorphism of the group G as described in Theorem 3.10.

Remark 3.12. The conditions for the singularities $\mathbb{C}^N/\varphi(G)$ and $\mathbb{C}^N/\psi(G)$ to be isomorphic can be stated more explicitly. An interested reader can consult [**14**, §§ 7.3 and 7.4].

This classification generalizes almost literally to other algebraically closed fields $\Bbbk$ of characteristic 0 or p, where p does not divide the order of G. First, we recall some terminology. Let $\varphi\colon G \to \mathrm{GL}(V_\Bbbk)$ be a representation of a group G on a finite-dimensional $\Bbbk$-vector space $V_\Bbbk$. We say that φ *is defined over a subfield* $L \subseteq \Bbbk$ if φ is obtained from a representation $\varphi'\colon G \to \mathrm{GL}(V_L)$ over L by extension of scalars: $V_\Bbbk = V_L \otimes \Bbbk$. Now, in view of Lemma 2.3 and the Chevalley–Shephard–Todd theorem, the main problem is to describe IQS of the form $\Bbbk^N/G$, where G is a finite linear group without quasi-reflections. Recall that if φ is a complex representation of G and p is a prime number, then there is a standard procedure of reduction of φ modulo p that gives a modular representation of G over a field of characteristic p (see [**4**, Chapter XII]). Assume that p does not divide the order of G. Choose an integer q such that $(p, q) = 1$ (if $p > 0$) and any irreducible complex representation of G is defined over the field $\mathbb{Q}(\sqrt[q]{1})$. Such a choice is possible by Brower's theorem (see [**9**, 12.2]). Finally, fix an isomorphism between the group μ_q of complex roots of unity of degree q and the group of roots of unity of degree q contained in the field $\Bbbk$.

Theorem 3.13. *Let $\Bbbk$ be an algebraically closed field of characteristic* 0 *or p. Let G be a finite group acting without quasi-reflections on the space $\Bbbk^N$, and assume that $p \nmid |G|$. The quotient $\Bbbk^N/G$ is then an isolated singularity if and only if G is one of the groups described in Theorems 3.1 and 3.2 and it acts via a direct sum of irreducible representations without fixed points of G described in Theorem 3.6, where, in matrices of representations, we have to replace the complex roots of unity with the corresponding roots of unity contained in the field $\Bbbk$.*

Proof. First, suppose that $\Bbbk$ is an algebraically closed field of characteristic 0. We may then assume that it contains the field $\mathbb{Q}(\sqrt[q]{1})$, and, by Brower's theorem, that the given representation of G is defined over $\mathbb{Q}(\sqrt[q]{1})$. It follows that everything that we need for the classification of groups without fixed points and their linear representations, and that we have proven over $\mathbb{C}$, also holds over $\Bbbk$.

Now suppose that $\operatorname{char} \mathbb{k} = p > 0$. Since $p \nmid |G|$, it follows from the theory of modular representations that we can establish a one-to-one correspondence between complex irreducible and irreducible $\mathbb{k}$-representations of G (see [**4**, Chapter XII]). Moreover, it is easy to see that a $\mathbb{k}$-representation has no fixed points if and only if its lift to $\mathbb{C}$ has no fixed points. This implies the theorem. $\square$

4. Gorenstein isolated quotient singularities

In this section we work over the field $\mathbb{C}$ of complex numbers. We first deduce Theorem 1.2 of Kurano and Nishi from the classification of IQS over the field $\mathbb{C}$.

Proof of Theorem 1.2. Let p be an odd prime, and let G be a finite group acting on $\mathbb{C}^p$ via a representation $\varphi\colon G \to \mathrm{GL}(p, \mathbb{C})$. We may assume that G acts without quasi-reflections. If $\mathbb{C}^p/\varphi(G)$ is a Gorenstein isolated singularity, by Theorem 3.11 G is one of the groups of types I–VI, φ is a direct sum of irreducible representations of dimension d, $2d$ or $4d$, from Theorem 3.6, and $\varphi(G) \subset \mathrm{SL}(p, \mathbb{C})$ by Theorem 1.3. Since p is odd, G has type I. Thus, either $d = 1$ and G is cyclic, or $d = p$. But it follows from Theorem 3.6, type I, that the last case is impossible. $\square$

If we want to classify all Gorenstein IQS over the field $\mathbb{C}$, we have only to refine the classification of Theorem 3.11 by taking into account the determinants of representations.

Theorem 4.1. *Let $X = \mathbb{C}^N/G'$ be a Gorenstein isolated quotient singularity, where G' is a finite subgroup of $\mathbb{C}^N$. The variety X is then isomorphic to the quotient $\mathbb{C}^N/\varphi(G)$, where G is one of the groups described in Theorems 3.1 and 3.2, and φ is a direct sum $\varphi = \varphi_1 \oplus \cdots \oplus \varphi_s$, $s \geqslant 1$, of irreducible representations described in Theorem 3.6 and satisfying the additional condition*

$$\det(\varphi_1(B)) \cdot \det(\varphi_2(B)) \cdots \det(\varphi_s(B)) = 1,$$

where the values $\det(\varphi_i(B))$, $1 \leqslant i \leqslant s$, are given in the fifth column of Table 2. More explicitly, this condition is stated in Table 4, where we use the same notation as in Table 2. The representations φ_i have one and the same dimension d, $2d$ or $4d$, $d \mid N$ ($2d \mid N$, $4d \mid N$).

Proof. The theorem is a direct consequence of Theorems 1.3, 3.6, 3.11, and conditions on the parameters n, d, etc. For instance, assume that G is a group of type I, and φ is a direct sum of s d-dimensional irreducible representations π_{k_j,l_j} described in the list of irreducible representations without fixed points in § 3, $1 \leqslant j \leqslant s$, $s = N/d$. From Table 2 we get the value $|\pi_{k_j,l_j}(B)| = (-1)^{d-1} \mathrm{e}^{2\pi \mathrm{i} l_j/n'}$, $n' = n/d$. It follows that

$$|\varphi(B)| = \prod_{j=1}^{s} (-1)^{d-1} \mathrm{e}^{2\pi \mathrm{i} l_j/n'} = (-1)^{s(d-1)} \exp\left(\frac{2\pi \mathrm{i}}{n'} \sum_{j=1}^{s} l_j\right).$$

This number is equal to 1 if and only if $(1/n')\sum l_j - (sn'(d-1))/2$ is integer, which is equivalent to the condition under type I in Table 4.

Table 4. *When the image of a representation is contained in* $\mathrm{SL}(N, \mathbb{C})$.

type	case	representation	condition
I		$\varphi_i = \pi_{k_i,l_i}$	$2\sum_{i=1}^{s} l_i \equiv sn'(d-1) \pmod{2n'}$
II		$\varphi_i = \alpha_{k'_i,l'_i}$	$(k+1)\sum_{i=1}^{s} l'_i \equiv 0 \pmod{n'}$
III	$9 \nmid n$	$\varphi_i = \nu_{k_i,l_i}$	$\sum_{i=1}^{s} l_i \equiv 0 \pmod{n'/3}$
III	$9 \mid n$, $3 \nmid d$	$\varphi_i = \nu_{k_i,l_i,j_i}$	$\sum_{i=1}^{s} j_i \equiv 0 \pmod{3^v}$, $\sum_{i=1}^{s} l_i \equiv 0 \pmod{n'/3^v}$
III	$3 \mid d$	$\varphi_i = \mu_{k_i,l_i}$	$\sum_{i=1}^{s} l_i \equiv 0 \pmod{n'}$
IV	$9 \nmid n$, $G = \langle A, B^3\rangle \times \mathrm{O}^*$	$\varphi_i = \psi_{k_i,l_i,j_i}$	$\sum_{i=1}^{s} l_i \equiv 0 \pmod{n'/3}$
IV	$9 \nmid n$, $G \neq \langle A, B^3\rangle \times \mathrm{O}^*$	$\varphi_i = \gamma_{k'_i,l'_i}$	$(k+1)\sum_{i=1}^{s} l'_i \equiv 0 \pmod{n'/3}$
IV	$9 \mid n$, $3 \nmid d$, $G = \langle A, B^{3^v}\rangle \times \mathrm{O}^*_v$	$\varphi_i = \xi_{k_i,l_i,j_i}$	$\sum_{i=1}^{s} l_i \equiv 0 \pmod{n''/d}$
IV	$9 \mid n$, $3 \nmid d$, $G \neq \langle A, B^{3^v}\rangle \times \mathrm{O}^*_v$	$\varphi_i = \gamma_{k'_i,l'_i,j}$	$(k+1)\sum_{i=1}^{s} l'_i \equiv 0 \pmod{n''/d}$
IV	$3 \mid d$	$\varphi_i = \eta_{k'_i,l'_i}$	$(k+1)\sum_{i=1}^{s} l'_i \equiv 0 \pmod{n'}$
V		$\varphi_i = \iota_{k_i,l_i,j_i}$	$\sum_{i=1}^{s} l_i \equiv 0 \pmod{n'}$
VI		$\varphi_i = \varkappa_{k'_i,l'_i,j}$	$(k+1)\sum_{i=1}^{s} l'_i \equiv 0 \pmod{n'}$

We also consider the condition under type III ($9 \mid n, 3 \nmid d$) in Table 4. It follows from Table 2 that $\varphi(B)$ has determinant 1 if and only if

$$6\sum_{i=1}^{s} l_i \equiv 0 \pmod{n'}.$$

But recall that n is odd here, divisible by 3, but not divisible by 9, and d is not divisible by 3. Thus, n' is odd and divisible by 3. Hence, we can rewrite the condition in the form given in Table 4. The rest of Table 4 can be checked in the same way. $\square$

Remark 4.2. By Lemma 2.8, two Gorenstein IQS $\mathbb{C}^N/\varphi(G)$ and $\mathbb{C}^N/\psi(G)$ are isomorphic if and only if $\varphi(G)$ and $\psi(G)$ are conjugate subgroups of $\mathrm{SL}(N,\mathbb{C})$. Explicit conditions can be obtained from Theorem 3.10. Also note that, since the numbers n'' and 3^v are coprime, two conditions of the case $9|n$, $3 \nmid d$ under type III Table 4 can be reduced to one.

To illustrate Theorem 4.1, we describe complex Gorenstein IQS in dimensions N, with $2 \leqslant N \leqslant 7$.

$N = 2$. Here, we have to classify up to conjugacy finite subgroups of $\mathrm{SL}(2,\mathbb{C})$. This case is classical, and it is well known that all such groups are either cyclic (they belong to type I in Wolf's terminology), binary dihedral (type I and II), binary tetrahedral (type III), binary octahedral (type IV) or binary icosahedral (type V). The corresponding quotients $\mathbb{C}^2/G$ are the Kleinian singularities A_n, D_n, E_6, E_7, E_8.

$N = 3$. By Theorem 1.2 of Kurano and Nishi, all Gorenstein IQS of dimension 3 are cyclic, i.e. they are isomorphic to quotients $\mathbb{C}^3/(\mathbb{Z}/n)$, $n \geqslant 2$, where the cyclic group is generated by the matrix

$$\begin{pmatrix} \mathrm{e}^{2\pi \mathrm{i} l_1/n} & 0 & 0 \\ 0 & \mathrm{e}^{2\pi \mathrm{i} l_2/n} & 0 \\ 0 & 0 & \mathrm{e}^{2\pi \mathrm{i} l_3/n} \end{pmatrix},$$

and $(l_1,n) = (l_2,n) = (l_3,n) = 1$, $l_1 + l_2 + l_3 \equiv 0 \pmod n$ (in fact, it follows from this that n must be odd).

$N = 4$. Let $\mathbb{C}^4/\varphi(G)$ be a four-dimensional Gorenstein isolated quotient singularity, as described in Theorem 4.1. The representation φ is a direct sum of irreducible representations of dimension q, $q = 1, 2$, or 4. The case $q = 1$ corresponds to cyclic IQS analogous to case $N = 3$. Now let $q = 2$. If G is a group of type I, then $\varphi = \pi_{k_1,l_1} \oplus \pi_{k_2,l_2}$; the corresponding condition of Table 4 takes the form $l_1 + l_2 \equiv 0 \pmod{n/2}$, and n must be divisible by 4. The corresponding singularities can be described in the following way. Consider all ordered collections $(m, n, r, k_1, l_1, k_2, l_2)$ of seven positive integers satisfying the conditions $4 \mid n$, $r^2 \equiv 1 \pmod n$, $((r-1)n, m) = 1$, where k_i, l_i are defined modulo and coprime to m and n, respectively, $i = 1, 2$, and $l_1 + l_2 \equiv 0 \pmod{n/2}$. Consider a group $G = G(m,n,r)$ of type I defined in Table 1, and its irreducible two-dimensional representations π_{k_i,l_i}, $i = 1, 2$, defined in the list of irreducible representations without fixed points in § 3 (simply the list in what follows). For any such collection, we then have a Gorenstein isolated quotient singularity

$$X_I(m,n,r,k_1,l_1,k_2,l_2) = \mathbb{C}^4/\pi_{k_1,l_1}(G) \oplus \pi_{k_2,l_2}(G).$$

Let G be a group of type II. Then $\varphi = \alpha_{k_1',l_1'} \oplus \alpha_{k_2',l_2'}$. To get a two-dimensional irreducible representation of G, we have to take $d = 1$; thus, we also have $A = 1$ (see Table 1). The corresponding condition of Table 4 takes the form $(k+1)(l_1' + l_2') \equiv 0 \pmod n$. The corresponding singularities can be described in the following way. Consider all ordered collections (n, k, l'_1, l'_2) of four positive integers such that $n = 2^u v$, $u \geqslant 2$, $k \equiv -1(2^u)$, $k^2 \equiv 1(n)$, $(l_1, n) = (l_2, n) = 1$ are defined modulo n, $(k+1)(l'_1 + l'_2) \equiv 0(n)$. Consider a group $G = G(m = 1, n, r = 1, l = 1, k)$ of type II defined in Table 1, and its irreducible representations $\alpha_{k'_i=1,l'_i}$, $i = 1, 2$, defined in the list. For any such collection, we then have a Gorenstein isolated quotient singularity

$$X_{II}(n, k, l'_1, l'_2) = \mathbb{C}^4/\alpha_{1,l'_1}(G) \oplus \alpha_{1,l'_2}(G).$$

Let G be a group of type III. From the condition $q = 2$, we again conclude that $d = 1$ and $A = 1$. We have three possibilities for the group G. If $9 \nmid n$, then $\varphi = \nu_{k_1,l_1} \oplus \nu_{k_2,l_2}$. The corresponding condition of Table 4 takes the form $l_1 + l_2 \equiv 0(n/3)$. The corresponding singularities can be described in the following way. Consider all triples (n, l_1, l_2) of positive integers such that n is odd, $3 \mid n$ but $9 \nmid n$, l_1 and l_2 are defined modulo and coprime to n, and $l_1 + l_2 \equiv 0(n/3)$. Consider a group $G = G(m = 1, n, r = 1)$ of type III defined in Table 1, and its irreducible representations $\nu_{k_i=1,l_i}$, $i = 1, 2$, defined in the list. For any such triple, we then have a Gorenstein isolated quotient singularity

$$X^{(1)}_{III}(n, l_1, l_2) = \mathbb{C}^4/\nu_{1,l_1}(G) \oplus \nu_{1,l_2}(G).$$

If $9 \mid n$, but $3 \nmid d$, then $\varphi = \nu_{k_1,l_1,j_1} \oplus \nu_{k_2,l_2,j_2}$. Here, we get the following description of the corresponding singularities. Consider all ordered collections (n, l_1, l_2, j_1, j_2) of five positive integers satisfying the conditions $n = 3^v n''$, $v \geqslant 2$, where $(l_i, n) = 1$ are defined modulo n, $(j_i, 3) = 1$ are defined modulo 3^v, $i = 1, 2$, $l_1 + l_2 \equiv 0(n/3^v)$, $j_1 + j_2 \equiv 0(3^v)$. Consider a group $G = G(m = 1, n, r = 1)$ of type III defined in Table 1, and its irreducible representations $\nu_{k_i=1,l_i,j_i}$, $i = 1, 2$, defined in the list. For any such collection, we then have a Gorenstein isolated quotient singularity

$$X^{(2)}_{III}(n, l_1, l_2, j_1, j_2) = \mathbb{C}^4/\nu_{1,l_1,j_1}(G) \oplus \nu_{1,l_2,j_2}(G).$$

The third possibility $3 \mid d$ obviously does not appear here.

Assume that G is a group of type IV. Two-dimensional irreducible representations exist only in the case $G \simeq \mathrm{O}^* \times \langle A, B^3 \rangle$. Again, there must be $d = 1$ and $A = 1$. We deduce the following description of the corresponding singularities. Consider all ordered collections $(n, k, l_1, l_2, j_1, j_2)$ of six positive integers such that n is odd, $3 \mid n$ but $9 \nmid n$, k is defined modulo n, $k \equiv -1(3)$, $k^2 \equiv 1(n)$, l_i are defined modulo n, $(l_i, n) = 1$, $j_i = \pm 1$, $i = 1, 2$, and $l_1 + l_2 \equiv 0(n/3)$. Consider a group $G = G(m = 1, n, r = 1, l = 1, k)$ of type IV defined in Table 1, and its irreducible representations $\psi_{k_i=1,l_i,j_i}$, $i = 1, 2$, defined in the list. For any such collection, we then have a Gorenstein isolated quotient singularity

$$X_{IV}(n, k, l_1, l_2, j_1, j_2) = \mathbb{C}^4/\psi_{1,l_1,j_1}(G) \oplus \psi_{1,l_2,j_2}(G).$$

Assume that G is a group of type V. We have two-dimensional irreducible representations only in the case $d = 1$, $A = 1$. The corresponding singularities can be described in the following way. Consider all ordered collections (n, l_1, l_2, j_1, j_2) of five positive integers such that $(n, 30) = 1$, $(l_i, n) = 1$ are defined modulo n, $j_i = \pm 1$, $i = 1, 2$, and

$l_1 + l_2 \equiv 0(n)$. Consider a group $G = G(m = 1, n, r = 1)$ of type V defined in Theorem 3.2, and its irreducible representations $\iota_{k_i=1,l_i,j_i}$, $i = 1, 2$, defined in the list. For any such collection, we then have a Gorenstein isolated quotient singularity

$$X_V(n, l_1, l_2, j_1, j_2) = \mathbb{C}^4/\iota_{1,l_1,j_1}(G) \oplus \iota_{1,l_2,j_2}(G).$$

There are no two-dimensional irreducible representations of groups of type VI.

Now let $q = 4$, i.e. the group G acts on $\mathbb{C}^4$ via an irreducible representation φ. Irreducible representations of groups of types I–VI with determinants 1 are described in Theorem 3.6 and Table 2. We obtain the following singularities. Let G be a group of type I. Consider all triples (m, r, k) of positive integers such that m is odd, $r^4 \equiv 1(m)$, $(r - 1, m) = 1$, and k is defined modulo m, $(k, m) = 1$. Consider a group $G = G(m, n = 8, r)$ of type I defined in Table 1, and its irreducible representation $\pi_{k,l=2}$ defined in the list. For any such triple, we then have a Gorenstein isolated quotient singularity

$$X_I(m, r, k) = \mathbb{C}^4/\pi_{k,2}(G).$$

Let G be a group of type II. Consider all ordered collections (m, n, r, l, k', l') of six positive integers such that (m, n, r, l) satisfy the conditions of type II in Table 1, with $d = 2$, $k = -1$, where k' is defined modulo m, l' is defined modulo n, $(k', m) = (l', n) = 1$. Consider a group $G = G(m, n, r, l, k = -1)$ of type II defined in Table 1, and its irreducible representation $\alpha_{k',l'}$ defined in the list. For any such collection, we then have a Gorenstein isolated quotient singularity

$$X_{II}(m, n, r, l, k', l') = \mathbb{C}^4/\alpha_{k',l'}(G).$$

There are no irreducible representations of dimension 4 with determinant 1 of groups of type III.

Let G be a group of type IV. We get the following singularities. Consider a pair (n, l') of positive integers such that n is odd, $3 \mid n$ but $9 \nmid n$, $(l', n) = 1$ is defined modulo n. Consider a group $G = G(m = 1, n, r = 1, l = 1, k = -1)$, $G \neq \langle A, B^3\rangle \times \mathrm{O}^*$, of type IV defined in Table 1 and in the list, and its irreducible representation $\gamma_{k'=1,l'}$ defined in the list. For any such pair, we then have a Gorenstein isolated quotient singularity

$$X_{IV}^{(1)}(n, l') = \mathbb{C}^4/\gamma_{1,l'}(G).$$

For any $v > 1$ and k, $1 \leqslant k < 3^v$, $k \equiv 1(3)$, we have a Gorenstein isolated quotient singularity

$$X_{IV}^{(2)}(v, k) = \mathbb{C}^4/o_k(\mathrm{O}_v^*),$$

where O_v^* is the generalized octahedral group and o_k is its irreducible representation defined in Lemma 2.16.

Consider all ordered collections (n, k, l', j) of four positive integers such that the pair (n, k) satisfies the conditions under type IV in Table 1, with $m = 1$, $r = 1$, $l = 1$, $n = 3^v n''$, $v \geqslant 2$, where l' is defined modulo n, $(l', n) = 1$, $1 \leqslant j < 3^v$, $j \equiv 1(3)$. Consider a group $G = G(m = 1, n, r = 1, l = 1, k)$, $G \neq \langle A, B^{3^v}\rangle \times \mathrm{O}_v^*$, of type IV defined in

https://doi.org/10.1017/S0013091513000643 Published online by Cambridge University Press

Table 1 and in the list, and its irreducible representation $\gamma_{k'=1,l',j}$ defined in the list. For any such collection, we then have a Gorenstein isolated quotient singularity

$$X_{IV}^{(3)}(n,k,l',j) = \mathbb{C}^4/\gamma_{1,l',j}(G).$$

There are no irreducible representations of dimension 4 with determinant 1 of groups of type V.

Let G be a group of type VI. Consider all triples (n,l',j) satisfying the conditions $(n,30) = 1$, where l' is defined modulo n, $(l',n) = 1$, $j = \pm 1$. Consider a group $G = G(m = 1, n, r = 1, l = 1, k = -1)$ of type VI defined in Theorem 3.2, and its irreducible representation $\varkappa_{k'=1,l',j}$ defined in the list. For any such triple, we then have a Gorenstein isolated quotient singularity

$$X_{VI}(n,l',j) = \mathbb{C}^4/\varkappa_{1,l',j}(G).$$

$N = 5$. By Theorem 1.2, in this case we have only cyclic quotient singularities $\mathbb{C}^5/(\mathbb{Z}/n)$, where the group is generated by a diagonal matrix with $\mathrm{e}^{2\pi\mathrm{i}l_j/n}$, $(l_j,n) = 1$, $j = 1,\dots,5$, on the diagonal, $l_1 + \cdots + l_5 \equiv 0 \pmod n$.

$N = 6$. Again we have to consider all divisors q of 6. If $q = 1$, we get cyclic quotient singularities. Let $q = 2$. In this case we get singularities of the form $\mathbb{C}^6/\varphi(G)$, where G is a group of one of the types I–VI, and φ is a direct sum of three two-dimensional irreducible representations of G without fixed points, satisfying the Gorenstein condition of Table 4. For example, let G be a group of type I. Consider all ordered collections $(m,n,r,k_1,l_1,k_2,l_2,k_3,l_3)$ of nine positive integers, where (m,n,r) satisfy the conditions of type I in Table 1, $r^2 \equiv 1(m)$, k_i are defined modulo m, $(k_i,m) = 1$, l_i are defined modulo n, $(l_i,n) = 1$, and $2(l_1 + l_2 + l_3) \equiv 3(n/2) \pmod n$. It is not difficult to see that the last condition is equivalent to the following: $n/4$ is odd and divides $l_1 + l_2 + l_3$. Consider a group $G = G(m,n,r)$ of type I defined in Table 1, and its irreducible representations π_{k_i,l_i}, $i = 1,2,3$, defined in the list. For any such collection, we then have a Gorenstein isolated singularity

$$X_I(m,n,r,k_1,l_1,k_2,l_2,k_3,l_3) = \mathbb{C}^6/\pi_{k_1,l_1}(G) \oplus \pi_{k_2,l_2}(G) \oplus \pi_{k_3,l_3}(G).$$

We leave a more precise description of the singularities corresponding to types II–V to an interested reader. Note that groups of type VI do not give any singularities in this case.

Assume that $q = 3$. In this case all singularities are produced by groups of type I, since only they have irreducible representations without fixed points of odd dimension. Consider all ordered collections (m,n,r,k_1,l_1,k_2,l_2) of seven positive integers, where (m,n,r) satisfy conditions of type I in table 1, $r^3 \equiv 1(m)$, k_i are defined modulo m, $(k_i,m) = 1$, l_i are defined modulo n, $(l_i,n) = 1$, $i = 1,2$, and $l_1 + l_2 \equiv 0(n/3)$. Consider a group $G = G(m,n,r)$ of type I defined in Table 1, and its irreducible representations π_{k_i,l_i}, $i = 1,2$, defined in the list. For any such collection, we then have a Gorenstein isolated quotient singularity

$$X_I(m,n,r,k_1,l_1,k_2,l_2) = \mathbb{C}^6/\pi_{k_1,l_1}(G) \oplus \pi_{k_2,l_2}(G).$$

There are no irreducible representations without fixed points of dimension 6 of groups of types I–VI.

$N = 7$. By Theorem 1.2, there exist only cyclic Gorenstein IQS in dimension 7.

Acknowledgements. The research was supported by a Russian Grant for Leading Scientific Schools (Grant 5139.2012.1) and by the RFBR (Grant 11-01-00336-a). This paper originates from the talk given by the author at the workshop 'Quotient Singularities' organized by Ivan Cheltsov in June 2010 at the University of Edinburgh. The author thanks the organizer and all the participants of the workshop for the warm and stimulating atmosphere.

References

1. D. N. Akhiezer, *Lie group actions in complex analysis*, Aspects of Mathematics, Volume 27 (Vieweg, Braunschweig, 1995).
2. M. F. Atiyah and I. G. MacDonald, *Introduction to commutative algebra* (Addison-Wesley, 1969).
3. D. J. Benson, *Polynomial invariants of finite groups*, London Mathematical Society Lecture Note Series, Volume 190 (Cambridge University Press, 1993).
4. C. Curtis and J. Reiner, *Representation theory of finite groups and associative algebras*, American Mathematical Society Chelsea Publishing Series, Volume 356 (American Mathematical Society, Providence, RI, 1962).
5. D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Graduate Texts in Mathematics, Volume 150 (Springer, 1995).
6. F. Klein, *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen von fünften Grade* (Leipzig, 1884).
7. K. Kurano and S. Nishi, Gorenstein isolated quotient singularities of odd prime dimension are cyclic, *Commun. Alg.* **40**(8) (2012), 3010–3020.
8. V. L. Popov, When are the stabilizers of all nonzero semisimple points finite?, in *Operator algebras, unitary representations, enveloping algebras and invariant theory*, Progress in Mathematics, Volume 92, pp. 541–559 (Birkhäuser, 1990).
9. J.-P. Serre, *Représentations linéaires des groupes finis* (Hermann, Paris, 1967).
10. M. Suzuki, On finite groups with cyclic Sylow subgroups for all odd primes, *Am. J. Math.* **77** (1955), 657–691.
11. G. Vincent, Les groupes linéares finis sans points fixes, *Comment. Math. Helv.* **20** (1947), 117–171.
12. K.-I. Watanabe, Certain invariant subrings are Gorenstein, I, *Osaka J. Math.* **11**(1) (1974), 1–8.
13. K.-I. Watanabe, Certain invariant subrings are Gorenstein, II, *Osaka J. Math.* **11**(2) (1974), 379–388.
14. J. A. Wolf, *Spaces of constant curvature*, 6th edn (American Mathematical Society, Providence, RI, 2011).
15. H. Zassenhaus, Über endliche Fastkörper, *Abh. Math. Sem. Univ. Hamburg* **11** (1935), 187–220.