

PREDICATIVE COLLAPSING PRINCIPLES

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Abstract. We show that arithmetical transfinite recursion is equivalent to a suitable formalization of the following: For every ordinal α there exists an ordinal β such that $1 + \beta \cdot (\beta + \alpha)$ (ordinal arithmetic) admits an almost order preserving collapse into β . Arithmetical comprehension is equivalent to a statement of the same form, with $\beta \cdot \alpha$ at the place of $\beta \cdot (\beta + \alpha)$. We will also characterize the principles that any set is contained in a countable coded ω -model of arithmetical transfinite recursion and arithmetical comprehension, respectively.

§1. Introduction. Well-ordering principles (of type one) are statements which assert that “ $T(X)$ is well-founded for any well-order X ,” for some transformation T of linear orders. We will consider such statements from the viewpoint of reverse mathematics (see [20] for a comprehensive introduction). In this setting X ranges over ordered subsets of \mathbb{N} . The fact that T is a transformation of linear orders can usually be proved in \mathbf{RCA}_0 , so that the entire strength of the well-ordering principle lies in the preservation of well-foundedness.

The literature contains many results that characterize important Π_2^1 -statements in terms of well-ordering principles. In order to explain our approach we focus on the following equivalence (but further results will be covered below):

THEOREM 1.1 (H. Friedman [unpublished]; M. Rathjen and A. Weiermann [18]).
The following are equivalent over \mathbf{RCA}_0 :

- (i) *arithmetical transfinite recursion (i. e. the principal axiom of \mathbf{ATR}_0),*
- (ii) *the statement that $\varphi(1 + X)0$ is well-founded for any well-order X .*

The transformation in (ii) is related to the Veblen function, which iterates derivatives of normal functions into the transfinite (cf. [14, 19]). In the context of reverse mathematics, the relevant values of this function can be represented by relativized ordinal notation systems $\varphi(1 + X)0$ (see [18, Definition 2.2] for details; our summand 1 corresponds to the minimal element 0_Q in the cited definition).

The present article shows that complicated well-ordering principles can, in a certain sense, be reduced to much simpler ones. In particular we will reduce the well-ordering principle $X \mapsto \varphi(1 + X)0$ to the family of order transformations

$$Y \mapsto T_X^\varphi(Y) := 1 + (Y + X) \times Y,$$

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indexed by the order X (unless stated otherwise, “order” will always mean “linear order”). Here 1 denotes the order with a single element. Also recall that the sum of two orders $X = (X, <_X)$ and $Y = (Y, <_Y)$ has underlying set

$$X + Y = \{\langle 0, x \rangle \mid x \in X\} \cup \{\langle 1, y \rangle \mid y \in Y\}.$$

For $x <_X x'$ and $y <_Y y'$ we have $\langle 0, x \rangle <_{X+Y} \langle 0, x' \rangle$ resp. $\langle 1, y \rangle <_{X+Y} \langle 1, y' \rangle$, and $\langle 0, x \rangle <_{X+Y} \langle 1, y \rangle$ holds for any $x \in X$ and $y \in Y$. The product is given by

$$X \times Y = \{\langle x, y \rangle \mid x \in X \text{ and } y \in Y\},$$

where $\langle x, y \rangle <_{X \times Y} \langle x', y' \rangle$ holds if we have $x <_X x'$, or $x = x'$ and $y <_Y y'$. Clearly the definition of sum and product is much simpler than the construction of $\varphi(1 + X)0$ in [18, Definition 2.2]. The fact that sums and products of well orders are themselves well ordered can be proved in **RCA**₀, in contrast to Theorem 1.1.

So how can $X \mapsto \varphi(1 + X)0$ be reduced to the transformations T_X^φ ? The idea is to consider fixed points of a certain type. Let us first observe that $T_X^\varphi(Y) \cong Y$ cannot hold for any well orders X and Y : If the latter have order types α resp. β , then $T_X^\varphi(Y)$ has order type $1 + \beta \cdot (\beta + \alpha) > \beta$. The best we can hope for is an “almost” order preserving function

$$\vartheta : T_X^\varphi(Y) \rightarrow Y.$$

To make this precise we need some terminology: A transformation $Y \mapsto T(Y)$ of linear orders is called inclusive if $T(Y_0)$ is a suborder of $T(Y)$ whenever Y_0 is a suborder of Y . This property allows us to introduce the following notion:

DEFINITION 1.2. Let $Y \mapsto T(Y)$ be an inclusive transformation of orders. Given any order Y , we define the support $\text{supp}_Y^T(\sigma) \subseteq Y$ of an element $\sigma \in T(Y)$ by

$$\text{supp}_Y^T(\sigma) = \bigcap \{Y_0 \subseteq Y \mid \sigma \in T(Y_0)\}.$$

For the above transformations T_X^φ , the supports have a concrete description: The element of the summand 1 has empty support. The support of an element $\langle \langle 0, y \rangle, y' \rangle$ resp. $\langle \langle 1, x \rangle, y' \rangle$ in the other summand is equal to $\{y, y'\}$ resp. $\{y'\}$. We can now say what we mean by an “almost” order preserving function:

DEFINITION 1.3. Consider an inclusive transformation $Y \mapsto T(Y)$ of linear orders. A function $\vartheta : T(Y) \rightarrow Y$ is called a Bachmann–Howard collapse if the following holds for all $\sigma, \tau \in T(Y)$:

- (i) $\sigma <_{T(Y)} \tau$ implies $\vartheta(\sigma) <_Y \vartheta(\tau)$, under the side condition that $y <_Y \vartheta(\tau)$ holds for all $y \in \text{supp}_Y^T(\sigma)$,
- (ii) we have $y <_Y \vartheta(\sigma)$ for all $y \in \text{supp}_Y^T(\sigma)$.

An order Y that admits such a function is called a Bachmann–Howard fixed point of T . If Y can be embedded into any other Bachmann–Howard fixed point of T , then it is called a minimal Bachmann–Howard fixed point.

In Remark 2.4 we will discuss a stronger notion of minimality, which may be more appealing from a categorical standpoint. We can now state our characterization of the transformation $X \mapsto \varphi(1 + X)0$, which will be proved in Section 3.

THEOREM 1.4 (RCA₀). *The order $\varphi(1 + X)0$ is a minimal Bachmann–Howard fixed point of the transformation $Y \mapsto 1 + (Y + X) \times Y$, for any linear order X .*

Due to minimality, a descending sequence in $\varphi(1 + X)0$ propagates to any Bachmann–Howard fixed point of T_X^φ . Hence $\varphi(1 + X)0$ is well-founded if, and only if, the transformation T_X^φ has a well-founded Bachmann–Howard fixed point. Together with Theorem 1.1 we obtain the following:

COROLLARY 1.5. *The following principles are equivalent over **RCA₀**:*

- (i) *arithmetical transfinite recursion,*
- (ii) *for every well-order X the transformation $Y \mapsto 1 + (Y + X) \times Y$ has a well-founded Bachmann–Howard fixed point.*

As mentioned above, the literature contains several results that have the same form as Theorem 1.1. In each line of the following table, the principle in the left column is equivalent to the assertion that the transformation in the middle column preserves well-foundedness (so the third line is Theorem 1.1). Precise definitions and proofs can be found in the references that are given in the right column.

Arithmetical comprehension	$X \mapsto \omega^X$	[12, 13]
The ω -jump of every set exists.	$X \mapsto \varepsilon_X$	[3, 14]
Arithmetical transfinite recursion	$X \mapsto \varphi(1 + X)0$	[18]
Every set lies in an ω -model of ATR .	$X \mapsto \Gamma_X$	[15]

Note that the existence of ω -jumps is equivalent to the statement that every set lies in a (countable coded) ω -model of **ACA**, over the base theory **ACA₀** (see [3, Lemma 3.4]; ω -models are explained in [20, Section 7.2]). We will characterize all transformations from the previous table in terms of collapsing principles. In each line of the next table, the order in the left column is a minimal Bachmann–Howard fixed point of the transformation in the middle column, for any linear order X . The right column refers to the corresponding theorem of the present article.

ω^{ω^X}	$Y \mapsto T_X^\omega(Y) := 1 + (1 + X) \times Y$	Theorem 2.2
ε_X	$Y \mapsto T_X^\varepsilon(Y) := 1 + Y^2 + X$	Theorem 2.6
$\varphi(1 + X)0$	$Y \mapsto T_X^\varphi(Y) := 1 + (Y + X) \times Y$	Theorem 1.4
Γ_X	$Y \mapsto T_X^\Gamma(Y) := 1 + 2 \times Y^2 + X$	Theorem 3.5

As in Corollary 1.5, we obtain new characterizations of the (broadly) predicative principles from above: arithmetical comprehension (Corollary 2.3), the existence of ω -jumps (Corollary 2.7), and the existence of ω -models of **ATR** (Corollary 3.6).

In the rest of this introduction we explain the wider context of our results: Let us first recall that J.-Y. Girard [11] has singled out a class of particularly uniform well-ordering principles, which are known as dilators. More precisely, a dilator is an endofunctor on the category of well-orders that preserves direct limits and pullbacks. In the inclusive case, these requirements correspond to the following properties of the supports from Definition 1.2 (cf. [5, Remark 2.2.2]):

- each support $\text{supp}_Y^T(\sigma) \subseteq Y$ is finite,
- we have $\sigma \in T(\text{supp}_Y^T(\sigma))$ for any $\sigma \in T(Y)$.

Girard has shown that dilators are determined by their restrictions to the category of natural numbers (up to natural equivalence). This crucial property makes it possible to represent dilators in second order arithmetic. For the purpose of the present article we do not need this general representation, since our families of order transformations come with an explicit parametrization.

The notion of Bachmann–Howard fixed point has been introduced in [5, 6], for arbitrary (i. e. not necessarily inclusive) dilators. In the cited articles it was shown that Π_1^1 -comprehension is equivalent to the statement that every dilator has a well-founded Bachmann–Howard fixed point. Furthermore, a minimal Bachmann–Howard fixed point of a given dilator can already be constructed in \mathbf{RCA}_0 , as shown in [7, 8]. Due to its minimality, that fixed point must be well-founded, but \mathbf{RCA}_0 cannot prove this fact. Applied to Corollary 1.5, this confirms that the strength of statement (ii) does not lie in the existence of a Bachmann–Howard fixed point as such, but rather in its well-foundedness.

The name “Bachmann–Howard fixed point” refers to the fact that our definitions are inspired by the Bachmann–Howard ordinal, in particular by the notation system from [17, Section 1]. It is well known that values of the Veblen function also arise in the construction of the Bachmann–Howard ordinal (see e. g. [4]). For this reason a result such as Theorem 1.4 may not be entirely unexpected. Nevertheless, it seems that the connection on the level of predicative well-ordering principles has not been made before. The literature does contain an impredicative well-ordering principle that is related to the Bachmann–Howard ordinal: As shown by M. Rathjen and P. Valencia Vizcaíno [16], the statement that every set lies in an ω -model of bar induction is equivalent to the principle that a relativized notation system ϑ_X is well-founded for any well-order X . In contrast to our approach, the notation system ϑ_X incorporates the collapsing function into the term structure.

In the present article we are concerned with “almost” order preserving collapsing functions of transformations that do not have well-founded fixed points in the usual sense. A class of transformations that correspond to normal functions has been singled out by P. Aczel [1, 2]: these transformations have well-founded fixed points of arbitrarily large order type. In [9, 10] it was shown that an appropriate formalization of the statement that “every normal function has a derivative (resp. at least one fixed point)” is equivalent to Π_1^1 -induction along arbitrary well-orders (resp. along \mathbb{N}). These induction principles are considerably weaker than the principle of Π_1^1 -comprehension, which is equivalent to the existence of well-founded Bachmann–Howard fixed points. The present article appears to show that the great strength of Bachmann–Howard fixed points translates into particularly simple characterizations of weaker principles.

§2. Collapsing and ordinal exponentiation. In the present section we show how the orders ω^X and ε_X can be constructed as Bachmann–Howard fixed points. As mentioned in the introduction, this yields characterizations of arithmetical comprehension and the principle that the ω -jump of every set exists.

Let us recall some definitions: Given an order $X = (X, \leq_X)$, the underlying set

$$\omega^X = \{ \langle x_0, \dots, x_{n-1} \rangle \mid x_{n-1} \leq_X \dots \leq_X x_0 \}$$

of the order $\omega^X = (\omega^X, \leq_{\omega^X})$ consists of the finite decreasing sequences with entries from X . The relation \leq_{ω^X} is defined as the lexicographic order on this set (cf. [13, Definition 2.2]). Intuitively, the elements of ω^X correspond to Cantor normal forms. To convey this intuition we will write $\omega^{x_0} + \dots + \omega^{x_{n-1}}$ rather than $\langle x_0, \dots, x_{n-1} \rangle$, and in particular 0 rather than $\langle \rangle \in \omega^X$. If X is a well-order of type α , then ω^X has order type ω^α , in the usual sense of ordinal arithmetic.

Addition on ω^X can be defined in terms of Cantor normal forms: We agree that 0 is neutral and that we have

$$(\omega^{x_0} + \dots + \omega^{x_n}) + (\omega^{y_0} + \dots + \omega^{y_m}) = \omega^{x_0} + \dots + \omega^{x_i} + \omega^{y_0} + \dots + \omega^{y_m},$$

where i is maximal with $y_0 \leq_X x_i$ (note $i = -1$ if $x_0 <_X y_0$). It is well known that basic properties of ordinal addition can be proved in \mathbf{RCA}_0 (cf. e. g. [19, 21]).

In order to define multiplication we must consider ω^{ω^X} rather than ω^X (note that an ordinal of the form ω^α does not need to be multiplicatively principal). The general definition of multiplication in terms of Cantor normal forms is somewhat cumbersome, since ordinal arithmetic does not validate right distributivity. Luckily, we will only need to multiply terms of a particular form: Given elements $\alpha \in \omega^X$ and $\beta_{n-1} \leq_{\omega^X} \dots \leq_{\omega^X} \beta_0$, we can set

$$\omega^\alpha \cdot (\omega^{\beta_0} + \dots + \omega^{\beta_{n-1}}) = \omega^{\alpha+\beta_0} + \dots + \omega^{\alpha+\beta_{n-1}},$$

where the exponents are added in ω^X . Since ω^X contains a minimal element $0 = \langle \rangle$, the order ω^{ω^X} contains a minimal non-zero element $1 = \omega^0$. This allows us to distinguish between successor and limit elements. Again, basic properties of these notions can be proved in \mathbf{RCA}_0 . To avoid iterated superscripts we will abbreviate $\omega_2(X) := \omega^{\omega^X}$, as well as $\omega_2(x) := \omega^{\omega^x} \in \omega_2(X)$ for $x \in X$.

On an intuitive level one would like to prove certain statements by induction along the order $\leq_{\omega_2(X)}$, but this induction principle is not available in our setting. Instead we argue by induction over the length of terms. For this purpose we define functions $l_X^\omega : \omega^X \rightarrow \mathbb{N}$ and $L_X^\omega : \omega_2(X) \rightarrow \mathbb{N}$ by setting

$$\begin{aligned} l_X^\omega(\omega^{x_0} + \dots + \omega^{x_{n-1}}) &= n, \\ L_X^\omega(\omega^{\alpha_0} + \dots + \omega^{\alpha_{n-1}}) &= l_X^\omega(\alpha_0) + \dots + l_X^\omega(\alpha_{n-1}) + n. \end{aligned}$$

The following observation will be crucial for our analysis of the order $\omega_2(X)$.

LEMMA 2.1 (\mathbf{RCA}_0). *Let X be a linear order. Any limit element of $\omega_2(X)$ can be uniquely written as $\omega_2(x) \cdot \eta$ with $0 <_{\omega_2(X)} \eta <_{\omega_2(X)} \omega_2(x) \cdot \eta$. Furthermore we have $L_X^\omega(\eta) < L_X^\omega(\omega_2(x) \cdot \eta)$ for any such decomposition.*

PROOF. To establish existence we consider an arbitrary limit element

$$\omega^{\beta_0} + \dots + \omega^{\beta_n} \in \omega_2(X).$$

Since we are concerned with a limit, the last exponent is different from $0 \in \omega^X$. Hence there are elements $x \in X$ and $\gamma_n \in \omega^X$ with

$$\beta_n = \omega^x + \gamma_n > \gamma_n.$$

Let us also record $l_X^\omega(\gamma_n) < l_X^\omega(\beta_n)$. Left subtraction is readily defined on the level of Cantor normal forms. In view of $\omega^x \leq_{\omega^x} \beta_n \leq_{\omega^x} \dots \leq_{\omega^x} \beta_0$ we can thus write

$$\beta_i = \omega^x + \gamma_i$$

for all $i < n$. One readily checks $\gamma_i \leq_{\omega^x} \beta_i$ and $l_X^\omega(\gamma_i) \leq l_X^\omega(\beta_i)$ (note that the inequalities may not be strict for $i < n$). Due to the monotonicity of addition we must also have $\gamma_n \leq_{\omega^x} \dots \leq_{\omega^x} \gamma_0$. We can thus define

$$\eta := \omega^{\gamma_0} + \dots + \omega^{\gamma_n} \in \omega_2(X).$$

By construction we have

$$\omega_2(x) \cdot \eta = \omega^{\omega^x + \gamma_0} + \dots + \omega^{\omega^x + \gamma_n} = \omega^{\beta_0} + \dots + \omega^{\beta_n}.$$

The above inequalities between β_i and γ_i imply

$$0 <_{\omega_2(X)} \eta <_{\omega_2(X)} \omega^{\beta_0} + \dots + \omega^{\beta_n} = \omega_2(x) \cdot \eta.$$

In view of $l_X^\omega(\gamma_n) < l_X^\omega(\beta_n)$ we also get

$$L_X^\omega(\eta) = l_X^\omega(\gamma_0) + \dots + l_X^\omega(\gamma_n) + n + 1 < l_X^\omega(\beta_0) + \dots + l_X^\omega(\beta_n) + n + 1 = L_X^\omega(\omega_2(x) \cdot \eta).$$

It remains to establish uniqueness. Due to the monotonicity of multiplication it suffices to show that

$$\eta <_{\omega_2(X)} \omega_2(x) \cdot \eta = \omega_2(y) \cdot \xi >_{\omega_2(X)} \xi$$

implies $x = y$. Aiming at a contradiction, let us assume that we have $x <_X y$. Then we get $\omega^x + \omega^y = \omega^y$ and hence $\omega_2(x) \cdot \omega_2(y) = \omega_2(y)$. We can deduce

$$\omega_2(x) \cdot \eta = \omega_2(y) \cdot \xi = \omega_2(x) \cdot \omega_2(y) \cdot \xi = \omega_2(x) \cdot \omega_2(x) \cdot \eta,$$

which is incompatible with the assumption $\eta <_{\omega_2(X)} \omega_2(x) \cdot \eta$. ⊥

Our goal is to characterize $\omega_2(X)$ as a minimal Bachmann–Howard fixed point of the order transformation

$$Y \mapsto T_X^\omega(Y) = 1 + (1 + X) \times Y.$$

Let us write \perp for the unique element of 1 . The elements of $T_X^\omega(Y)$ will be written as \perp , $\langle \perp, y \rangle$ and $\langle 1 + x, y \rangle$, rather than $\langle 0, \perp \rangle$, $\langle 1, \langle \langle 0, \perp \rangle, y \rangle \rangle$ and $\langle 1, \langle \langle 1, x \rangle, y \rangle \rangle$, respectively. Sometimes we also use x to denote an arbitrary element of $1 + X$. The supports from Definition 1.2 take the forms

$$\text{supp}_Y^\omega(\perp) = \emptyset \quad \text{and} \quad \text{supp}_Y^\omega(\langle x, y \rangle) = \{y\}.$$

In view of Definition 1.3, this means that a function $\vartheta : T_X^\omega(Y) \rightarrow Y$ is a Bachmann–Howard collapse if, and only if, the following conditions are satisfied:

- (i) we have $\vartheta(\perp) <_Y \vartheta(\langle x, y \rangle)$ for any $\langle x, y \rangle \in (1 + X) \times Y$,
- (i') $\langle x, y \rangle <_{(1+X) \times Y} \langle x', y' \rangle$ implies $\vartheta(\langle x, y \rangle) <_Y \vartheta(\langle x', y' \rangle)$, under the side condition that we have $y <_Y \vartheta(\langle x', y' \rangle)$,
- (ii) we have $y <_Y \vartheta(\langle x, y \rangle)$ for any $x \in 1 + X$ and $y \in Y$.

We can now establish the promised characterization, improving [8, Proposition 3.3]:

THEOREM 2.2 (RCA₀). *The order ω^{ω^x} is a minimal Bachmann–Howard fixed point of the transformation T_X^ω , for any order X .*

PROOF. In order to show that $\omega_2(X) = \omega^{\omega^x}$ is a Bachmann–Howard fixed point of T_X^ω we must define a collapsing function

$$\vartheta : 1 + (1 + X) \times \omega_2(X) \rightarrow \omega_2(X).$$

Using the successor operation and multiplication in $\omega_2(X)$, we set

$$\begin{aligned} \vartheta(\perp) &:= 0, \\ \vartheta(\langle \perp, \eta \rangle) &:= \eta + 1, \\ \vartheta(\langle 1 + x, \eta \rangle) &:= \omega_2(x) \cdot (\eta + 1). \end{aligned}$$

The above Condition (i) is immediate. Condition (ii) is satisfied in view of

$$\eta <_{\omega_2(X)} \eta + 1 \leq_{\omega_2(X)} \omega_2(x) \cdot (\eta + 1).$$

To verify Condition (i') one needs to distinguish several cases. In the first interesting case we are concerned with an inequality

$$\langle \perp, \eta \rangle <_{(1+X) \times Y} \langle 1 + x, \eta' \rangle.$$

Due to the side condition in (i') we may assume

$$\eta <_{\omega_2(X)} \vartheta(\langle 1 + x, \eta' \rangle) = \omega_2(x) \cdot (\eta' + 1).$$

The element on the right side is a limit (note that the last exponent in its Cantor normal form is equal to $\omega^x \neq 0$). Hence we obtain

$$\vartheta(\langle \perp, \eta \rangle) = \eta + 1 <_{\omega_2(X)} \vartheta(\langle 1 + x, \eta' \rangle),$$

as required. Let us also consider the case of an inequality

$$\langle 1 + x, \eta \rangle <_{(1+X) \times Y} \langle 1 + x', \eta' \rangle$$

with $x <_X x'$. Yet again, the side condition yields $\eta + 1 <_{\omega_2(X)} \omega_2(x') \cdot (\eta' + 1)$. Also observe that $x <_X x'$ implies $\omega_2(x) \cdot \omega_2(x') = \omega_2(x')$, as in the proof of Lemma 2.1. Using the monotonicity of multiplication we can deduce

$$\begin{aligned} \vartheta(\langle 1 + x, \eta \rangle) &= \omega_2(x) \cdot (\eta + 1) <_{\omega_2(X)} \omega_2(x) \cdot \omega_2(x') \cdot (\eta' + 1) = \\ &= \omega_2(x') \cdot (\eta' + 1) = \vartheta(\langle 1 + x', \eta' \rangle). \end{aligned}$$

So far we have shown that $\omega_2(X)$ is a Bachmann–Howard fixed point of T_X^ω . To establish minimality we consider an arbitrary order Y that admits a Bachmann–Howard collapse

$$\vartheta : 1 + (1 + X) \times Y \rightarrow Y.$$

We need to construct an embedding $f : \omega_2(X) \rightarrow Y$. In view of Lemma 2.1 we can define f by recursion over the length of terms, by setting

$$\begin{aligned} f(0) &:= \vartheta(\perp), \\ f(\eta + 1) &:= \vartheta(\langle \perp, f(\eta) \rangle), \\ f(\omega_2(x) \cdot \eta) &:= \vartheta(\langle 1 + x, f(\eta) \rangle), \quad \text{where } 0 <_{\omega_2(X)} \eta <_{\omega_2(X)} \omega_2(x) \cdot \eta. \end{aligned}$$

To show that $\xi <_{\omega_2(X)} \xi'$ implies $f(\xi) <_Y f(\xi')$ we argue by induction on the combined length $L_X^\omega(\xi) + L_X^\omega(\xi')$ of ξ and ξ' (note that this amounts to an induction

over a Π_1^0 -statement, which is available in \mathbf{RCA}_0). In the first interesting case we consider an inequality

$$\xi = \eta + 1 <_{\omega_2(X)} \omega_2(x) \cdot \eta' = \xi'.$$

In view of $\perp <_{1+X} 1+x$ we clearly have $\langle \perp, f(\eta) \rangle <_{(1+X) \times Y} \langle 1+x, f(\eta') \rangle$. Invoking the induction hypothesis, we also see that $\eta <_{\omega_2(X)} \xi'$ implies

$$f(\eta) <_Y f(\xi') = \vartheta(\langle 1+x, f(\eta') \rangle).$$

This is the side condition required in clause (i') above. We can thus conclude

$$f(\xi) = \vartheta(\langle \perp, f(\eta) \rangle) <_Y \vartheta(\langle 1+x, f(\eta') \rangle) = f(\xi').$$

Let us now consider an inequality of the form

$$\xi = \omega_2(x) \cdot \eta <_{\omega_2(X)} \eta' + 1 = \xi'.$$

Using the induction hypothesis and clause (ii) above we get

$$f(\xi) \leq_Y f(\eta') <_Y \vartheta(\langle \perp, f(\eta') \rangle) = f(\xi'),$$

as required. To conclude the proof we consider an inequality of the form

$$\xi = \omega_2(x) \cdot \eta <_{\omega_2(X)} \omega_2(x') \cdot \eta' = \xi'.$$

We need to distinguish three cases: First assume that we have $x <_X x'$. Then we immediately get $\langle 1+x, f(\eta) \rangle <_{(1+X) \times Y} \langle 1+x', f(\eta') \rangle$. In view of Lemma 2.1 we have $\eta <_{\omega_2(X)} \xi <_{\omega_2(X)} \xi'$ and $L_X^\omega(\eta) < L_X^\omega(\xi)$. Hence the induction hypothesis yields

$$f(\eta) <_Y f(\xi') = \vartheta(\langle 1+x', f(\eta') \rangle).$$

Hence the side condition from clause (i') is satisfied, and we obtain

$$f(\xi) = \vartheta(\langle 1+x, f(\eta) \rangle) <_Y \vartheta(\langle 1+x', f(\eta') \rangle) = f(\xi').$$

Now assume $x = x'$. In view of $\xi <_{\omega_2(X)} \xi'$ we must have $\eta <_{\omega_2(X)} \eta'$. Then the induction hypothesis yields $\langle 1+x, f(\eta) \rangle <_{(1+X) \times Y} \langle 1+x', f(\eta') \rangle$, and we can conclude as in the previous case. Finally assume $x >_X x'$. In this case we observe that

$$\omega_2(x') \cdot \xi = \xi <_{\omega_2(X)} \xi' = \omega_2(x') \cdot \eta'$$

implies $\xi <_{\omega_2(X)} \eta'$. Using the induction hypothesis and clause (ii) we obtain

$$f(\xi) <_Y f(\eta') <_Y \vartheta(\langle 1+x', f(\eta') \rangle) = f(\xi'),$$

just as needed. ⊣

The statement that ω^X is well-founded for every well-order X is equivalent to arithmetical comprehension, as shown by J.-Y. Girard [12, Section 5.4] (cf. also the computability-theoretic proof by J. Hirst [13]). We can deduce the following:

COROLLARY 2.3. *The following are equivalent over \mathbf{RCA}_0 :*

- (i) *arithmetical comprehension (which is the principal axiom of \mathbf{ACA}_0),*
- (ii) *for every well-order X the transformation $Y \mapsto 1 + X \times Y$ has a well-founded Bachmann–Howard fixed point.*

PROOF. To deduce (ii) from (i) we consider an arbitrary well-order X . In view of Girard’s result, we can invoke (i) to infer that ω^X and $\omega_2(X)$ are well-founded. Theorem 2.2 yields a Bachmann–Howard collapse

$$\vartheta : 1 + (1 + X) \times \omega_2(X) \rightarrow \omega_2(X).$$

The restriction of ϑ to $1 + X \times \omega_2(X)$ witnesses that $\omega_2(X)$ is a Bachmann–Howard fixed point of $Y \mapsto 1 + X \times Y$, as one readily verifies. To show that (ii) implies (i) we again invoke Girard’s result. Hence we must establish that ω^X is well-founded for an arbitrary well-order X . Since $1 + X$ is still well-founded, we can use (ii) to get a well-founded Bachmann–Howard fixed point Y of the transformation

$$Y \mapsto 1 + (1 + X) \times Y = T_X^\omega(Y).$$

From Theorem 2.2 we know that $\omega_2(X)$ can be embedded into Y . Hence $\omega_2(X)$ must be well-founded as well. In view of the embedding

$$\omega^X \ni \alpha \mapsto \omega^\alpha \in \omega_2(X)$$

we can infer that ω^X is well-founded, as required. ◻

To conclude the first half of the present section we discuss a possible improvement of Theorem 2.2:

REMARK 2.4. The Bachmann–Howard collapse $\vartheta : T_X^\omega(\omega_2(X)) \rightarrow \omega_2(X)$ that we have constructed in the proof of Theorem 2.2 does not look quite optimal: For an element $\eta \in \omega_2(X)$ with $\eta <_{\omega_2(X)} \omega_2(x) \cdot \eta$ it might have been more natural to define $\vartheta(\langle 1 + x, \eta \rangle)$ as $\omega_2(x) \cdot \eta$ rather than $\omega_2(x) \cdot (\eta + 1)$. To make this intuition precise we can observe the following: In the second half of the proof of Theorem 2.2 we have constructed an embedding $f : \omega_2(X) \rightarrow Y$ into an arbitrary Bachmann–Howard fixed point Y of the transformation T_X^ω . If we construct this embedding with respect to the given Bachmann–Howard collapse for $Y = \omega_2(X)$, then we get

$$f(\omega_2(x) \cdot \eta) = \vartheta(\langle 1 + x, f(\eta) \rangle) = \omega_2(x) \cdot (f(\eta) + 1),$$

which means that f cannot be the identity on $\omega_2(Y)$. In order to understand this phenomenon in general we recall that the notion of Bachmann–Howard fixed point was defined for dilators, i. e. for particularly uniform endofunctors on the category of linear orders. Functoriality allows us to define the following notion: Given Bachmann–Howard fixed points X and Y with fixed collapsing functions $\vartheta_X : T(X) \rightarrow X$ and $\vartheta_Y : T(Y) \rightarrow Y$, we say that $f : X \rightarrow Y$ is a morphism of Bachmann–Howard fixed points if we have

$$f \circ \vartheta_X = \vartheta_Y \circ T(f).$$

Following the usual categorical terminology, an initial Bachmann–Howard fixed point consists of an order X and a Bachmann–Howard collapse $\vartheta : T(X) \rightarrow X$ that admit a unique morphism into any Bachmann–Howard fixed point of the same dilator. The proofs of [7, Theorem 3.4] and [8, Theorem 4.5] reveal that every dilator has an initial Bachmann–Howard fixed point, which is necessarily unique up to isomorphism. Note that any initial fixed point is minimal in the sense of Definition 1.3. The notion of initial fixed point is certainly more satisfactory from a theoretical perspective. On the other hand, minimal fixed points are entirely

sufficient to deduce Corollary 2.3 and similar results. We can also observe that the order type of a minimal fixed point is necessarily unique in the well-founded case. For these reasons we have decided to avoid the additional technicalities that would be necessary to determine initial fixed points, rather than just minimal ones.

In the second half of this section we are concerned with the orders $\varepsilon_X = (\varepsilon_X, <_{\varepsilon_X})$ that have been mentioned in the introduction. In contrast to the case of ω^X , the set ε_X and the relation $<_{\varepsilon_X}$ have to be defined simultaneously. The underlying set consists of the terms that are generated by the following clauses:

- The set ε_X contains a symbol 0, and a symbol ε_x for each element $x \in X$.
- If $\alpha \in \varepsilon_X$ is not of the form ε_x , then we have a term $\omega^\alpha \in \varepsilon_X$.
- Given $n > 1$ elements $\alpha_n \leq_{\varepsilon_X} \dots \leq_{\varepsilon_X} \alpha_1$ of ε_X , we get $\omega^{\alpha_1} + \dots + \omega^{\alpha_n} \in \varepsilon_X$.

The order $<_{\varepsilon_X}$ reflects the intuition that any term of the form ε_x represents an ε -number, i. e. an ordinal α that satisfies $\omega^\alpha = \alpha$. We refer to [8, Definition 3.4] for full details of the somewhat lengthy definition.

On the set ε_X one can define counterparts of addition, multiplication and exponentiation to the base ω , taking into account that ε -numbers are closed under these operations (cf. [19]). In particular we have an operation

$$\varepsilon_X \ni \alpha \mapsto \omega_2(\alpha) := \omega^{\omega^\alpha} \in \varepsilon_X,$$

which plays a similar (though somewhat less important) role as in the analysis of the order $\omega_2(X)$. To define a length function $L_X^\varepsilon : \varepsilon_X \rightarrow \mathbb{N}$ we recursively set

$$\begin{aligned} L_X^\varepsilon(0) &:= L_X^\varepsilon(\varepsilon_x) := 0, \\ L_X^\varepsilon(\omega^{\alpha_1} + \dots + \omega^{\alpha_n}) &:= L_X^\varepsilon(\alpha_1) + \dots + L_X^\varepsilon(\alpha_n) + n. \end{aligned}$$

We say that an element of ε_X is decomposable if it is neither equal to 0 nor of the form ε_x . This terminology is justified in view of the following (cf. Lemma 2.1).

LEMMA 2.5 (RCA₀). *Any decomposable element of ε_X can be uniquely written as $\omega^\alpha + \beta$ with $\alpha, \beta <_{\varepsilon_X} \omega^\alpha + \beta$. Furthermore we have $L_X^\varepsilon(\alpha), L_X^\varepsilon(\beta) < L_X^\varepsilon(\omega^\alpha + \beta)$ for any such decomposition.*

PROOF. We first establish existence: For a decomposable $\omega^{\alpha_1} + \dots + \omega^{\alpha_n} \in \varepsilon_X$, we set $\alpha := \alpha_1$ and

$$\beta := \begin{cases} 0 & \text{if } n = 1, \\ \varepsilon_x & \text{if } n = 2 \text{ and } \alpha_2 = \varepsilon_x, \\ \omega^{\alpha_2} + \dots + \omega^{\alpha_n} & \text{otherwise.} \end{cases}$$

By construction (and by the definition of addition and exponentiation on ε_X) we have $\omega^\alpha + \beta = \omega^{\alpha_1} + \dots + \omega^{\alpha_n}$. A straightforward induction on the term α_1 yields

$$\alpha_1 <_{\varepsilon_X} \omega^{\alpha_1} + \dots + \omega^{\alpha_n},$$

which amounts to $\alpha <_{\varepsilon_X} \omega^\alpha + \beta$. In all cases it is straightforward to verify that we have $\beta <_{\varepsilon_X} \omega^\alpha + \beta$ as well as $L_X^\varepsilon(\alpha), L_X^\varepsilon(\beta) < L_X^\varepsilon(\omega^\alpha + \beta)$. Due to the monotonicity of addition, uniqueness reduces to the claim that

$$\beta <_{\varepsilon_X} \omega^\alpha + \beta = \omega^\gamma + \delta >_{\varepsilon_X} \delta$$

implies $\alpha = \gamma$. Aiming at a contradiction, we assume $\alpha <_{\varepsilon_X} \gamma$. The latter yields

$$\omega^\alpha + \beta = \omega^\gamma + \delta = \omega^\alpha + \omega^\gamma + \delta = \omega^\alpha + \omega^\alpha + \beta,$$

which is incompatible with $\beta <_{\varepsilon_X} \omega^\alpha + \beta$. ⊥

We now want to characterize ε_X as a minimal Bachmann–Howard fixed point of the order transformation

$$Y \mapsto T_X^\varepsilon(Y) = 1 + Y^2 + X.$$

Elements of the summands 1 , Y^2 and X will be written as \perp , $\langle y_0, y_1 \rangle$ and x , respectively. The supports from Definition 1.2 amount to

$$\text{supp}_Y^\varepsilon(\perp) = \text{supp}_Y^\varepsilon(x) = \emptyset \quad \text{and} \quad \text{supp}_Y^\varepsilon(\langle y_0, y_1 \rangle) = \{y_0, y_1\}.$$

Together with Definition 1.3, this means that a function $\vartheta : T_X^\varepsilon(Y) \rightarrow Y$ is a Bachmann–Howard collapse if, and only if, the following conditions are satisfied:

- (i) $\vartheta(\perp) <_Y \vartheta(\langle y_0, y_1 \rangle)$ holds for all $y_0, y_1 \in Y$, and $\vartheta(\perp) <_Y \vartheta(x) <_Y \vartheta(x')$ holds for all $x, x' \in X$ with $x <_X x'$,
- (i') $\langle y_0, y_1 \rangle <_{Y^2} \langle y'_0, y'_1 \rangle$ implies $\vartheta(\langle y_0, y_1 \rangle) <_Y \vartheta(\langle y'_0, y'_1 \rangle)$, under the side condition that we have $y_0, y_1 <_Y \vartheta(\langle y'_0, y'_1 \rangle)$,
- (i'') $\vartheta(\langle y_0, y_1 \rangle) <_Y \vartheta(x)$ holds for $y_0, y_1 \in Y$ and $x \in X$ with $y_0, y_1 <_Y \vartheta(x)$,
- (ii) we have $y_0, y_1 <_Y \vartheta(\langle y_0, y_1 \rangle)$ for all $y_0, y_1 \in Y$.

We can now establish the desired characterization:

THEOREM 2.6 (RCA₀). *The order ε_X is a minimal Bachmann–Howard fixed point of the transformation T_X^ε , for any order X .*

PROOF. To witness that ε_X is a Bachmann–Howard fixed point of T_X^ε we need a collapsing function

$$\vartheta : 1 + \varepsilon_X \times \varepsilon_X + X \rightarrow \varepsilon_X.$$

Relying on the ordinal arithmetic that is available in ε_X , we set

$$\begin{aligned} \vartheta(\perp) &:= 0, \\ \vartheta(\langle \alpha, \beta \rangle) &:= \omega_2(\alpha + 1) \cdot (\beta + 1), \\ \vartheta(x) &:= \varepsilon_x. \end{aligned}$$

It is straightforward to see that the above Conditions (i) and (ii) are satisfied (note that Condition (ii) could fail if we were to replace $\omega_2(\alpha + 1)$ by $\omega_2(\alpha)$, as the proof of Theorem 2.2 might suggest). Condition (i') is verified as in the proof of Theorem 2.2. To establish Condition (i'') we consider arbitrary $\alpha, \beta \in \varepsilon_X$ and $x \in X$ with

$$\alpha, \beta <_{\varepsilon_X} \vartheta(x) = \varepsilon_x.$$

Considering the order on ε_X (cf. [8, Definition 3.4]), it is straightforward to see that the element ε_x behaves like an ε -number. Hence we obtain

$$\vartheta(\langle \alpha, \beta \rangle) = \omega_2(\alpha + 1) \cdot (\beta + 1) <_{\varepsilon_X} \varepsilon_x = \vartheta(x).$$

This completes the proof that ε_X is a Bachmann–Howard fixed point of T_X^ε . Let us now consider an arbitrary Bachmann–Howard collapse

$$\vartheta : 1 + Y^2 + X \rightarrow Y.$$

We need to construct an embedding $f : \varepsilon_X \rightarrow Y$. In view of Lemma 2.5 we can recursively define

$$\begin{aligned} f(0) &:= \vartheta(\perp), \\ f(\omega^\alpha + \beta) &:= \vartheta(\langle f(\alpha), f(\beta) \rangle), \quad \text{where } \alpha, \beta < \omega^\alpha + \beta, \\ f(\varepsilon_x) &:= \vartheta(x). \end{aligned}$$

By induction on $L_X^\varepsilon(\eta) + L_X^\varepsilon(\xi)$ we can show that $\eta <_{\varepsilon_X} \xi$ implies $f(\eta) <_Y f(\xi)$. The first interesting case concerns an inequality

$$\eta = \varepsilon_x <_{\varepsilon_X} \omega^\alpha + \beta = \xi.$$

Since ε_x behaves like an ε -number we must have $\varepsilon_x \leq_{\varepsilon_X} \alpha$. Using the induction hypothesis and clause (ii) above we get

$$f(\eta) \leq_Y f(\alpha) <_Y \vartheta(\langle f(\alpha), f(\beta) \rangle) = f(\xi).$$

Let us now consider an inequality

$$\eta = \omega^\alpha + \beta <_{\varepsilon_X} \varepsilon_x = \xi.$$

By Lemma 2.5 we get $\alpha, \beta <_{\varepsilon_X} \xi$, so that the induction hypothesis yields

$$f(\alpha), f(\beta) <_Y f(\xi) = \vartheta(x).$$

Invoking clause (i'') we can infer

$$f(\eta) = \vartheta(\langle f(\alpha), f(\beta) \rangle) <_{\varepsilon_X} \vartheta(x) = f(\xi).$$

Finally, we consider an inequality

$$\eta = \omega^\alpha + \beta <_{\varepsilon_X} \omega^\gamma + \delta = \xi.$$

Considering the proof of Lemma 2.5, it is straightforward to see that we must have $\alpha \leq_{\varepsilon_X} \gamma$. If we have $\alpha = \gamma$, then we get $\beta <_{\varepsilon_X} \delta$. In any case we can use the induction hypothesis to infer

$$\langle f(\alpha), f(\beta) \rangle <_{Y^2} \langle f(\gamma), f(\delta) \rangle.$$

In view of $\alpha, \beta <_{\varepsilon_X} \eta <_{\varepsilon_X} \xi$ the induction hypothesis also yields

$$f(\alpha), f(\beta) <_Y f(\xi) = \vartheta(\langle f(\gamma), f(\delta) \rangle).$$

By Condition (i') we now obtain

$$f(\eta) = \vartheta(\langle f(\alpha), f(\beta) \rangle) <_Y \vartheta(\langle f(\gamma), f(\delta) \rangle) = f(\xi),$$

as required. ⊣

The statement that ε_X is well-founded for any well-order X is equivalent to the assertion that the ω -jump of any set exists, as shown by A. Marcone and A. Montalbán [14] (see also the proof-theoretic argument due to B. Afshari and M. Rathjen [3]). Together with Theorem 2.6 we obtain the following:

COROLLARY 2.7. *The following are equivalent over \mathbf{RCA}_0 :*

- (i) *the ω -jump of every set exists (which is the principal axiom of \mathbf{ACA}_0^+),*
- (ii) *for every well-order X the transformation $Y \mapsto 1 + Y^2 + X$ has a well-founded Bachmann–Howard fixed point.*

§3. Collapsing and the Veblen hierarchy. In this section we show how the orders $\varphi(1 + X)0$ and Γ_X can be constructed as Bachmann–Howard fixed points. This will yield characterizations of arithmetical transfinite recursion and of the principle that every set lies in an ω -model of **ATR**.

Let us begin by recalling the Veblen hierarchy: A function f from ordinals to ordinals is called a normal function if it is strictly increasing and continuous at limit stages. Equivalently, f is the unique increasing enumeration of a closed and unbounded (club) class of ordinals. If f is a normal function, then the class

$$\{\alpha \mid f(\alpha) = \alpha\}$$

of its fixed points is itself closed and unbounded. The normal function that enumerates these fixed points is called the derivative of f and is denoted by f' . The Veblen hierarchy is a family of normal functions φ_α , indexed by the ordinals. The first function in this hierarchy is usually given as

$$\varphi_0(\beta) = \omega^\beta.$$

Since the intersection of set-many clubs is itself a club, the function at stage $\alpha > 0$ can be recursively defined by

$$\varphi_\alpha := \text{“the increasing enumeration of } \bigcap_{\gamma < \alpha} \{\beta \mid \varphi_\gamma(\beta) = \beta\}\text{”}.$$

In particular we have $\varphi_{\alpha+1} = \varphi_\alpha'$ at successor stages. Since the values of φ_α are fixed points of all previous functions in the hierarchy, we obtain

$$\varphi_\gamma \circ \varphi_\alpha = \varphi_\alpha \quad \text{whenever } \gamma < \alpha.$$

It is straightforward to deduce that we have

$$\varphi_\alpha(\beta) < \varphi_\gamma(\delta) \iff \begin{cases} \text{either } \alpha < \gamma \text{ and } \beta < \varphi_\gamma(\delta), \\ \text{or } \alpha = \gamma \text{ and } \beta < \delta, \\ \text{or } \alpha > \gamma \text{ and } \varphi_\alpha(\beta) < \delta. \end{cases} \quad (\star)$$

Also note that the values of φ_α are additively closed; for $\alpha > 0$ they are ε -numbers.

Relativized notation systems $\varphi(1 + X)0$ for values of the Veblen function have been described in [18, Definition 2.2] (note that our summand 1 corresponds to the minimal element 0_Q that was required in the cited reference). As in the case of ε_X , the underlying set of $\varphi(1 + X)0$ needs to be defined simultaneously with the order relation. The set $\varphi(1 + X)0$ and the auxiliary function $h : \varphi(1 + X)0 \rightarrow 1 + X$ are recursively defined by the following clauses (recall that \perp denotes the unique element of 1, which coincides with the minimal element of the order $1 + X$):

- We have an element $0 \in \varphi(1 + X)0$ with $h(0) = \perp$.
- Given elements $x \in 1 + X$ and $\alpha \in \varphi(1 + X)0$ with $h(\alpha) \leq_{1+X} x$, we get a term $\varphi_x\alpha \in \varphi(1 + X)0$ with $h(\varphi_x\alpha) = x$.
- Given $n > 1$ elements $\varphi_{x_n}\alpha_n \leq_{\varphi(1+X)0} \dots \leq_{\varphi(1+X)0} \varphi_{x_1}\alpha_1$ of the indicated form, we get $\alpha := \varphi_{x_1}\alpha_1 + \dots + \varphi_{x_n}\alpha_n \in \varphi(1 + X)0$ with $h(\alpha) = \perp$.

The order on $\varphi(1 + X)0$ (which we will usually denote by $<$ rather than $<_{\varphi(1+X)0}$) reflects equivalence (\star) , as well as the intuition that elements of the form $\varphi_x\alpha$ are additively closed. Full details can be found in [18, Section 2]. Note that we write $\varphi_x\alpha$ (without parentheses) for terms in $\varphi(1 + X)0$ but $\varphi_\gamma(\alpha)$ (with parentheses) for values of the Veblen function on actual ordinals (an exception is made when

parentheses in a term are needed to avoid ambiguity). In the sequel, we also write (\star) for the “term version” of this equivalence in $\varphi(1 + X)0$.

Similarly to the previous section, we define a length function $L_X^\varphi : \varphi(1 + X)0 \rightarrow \mathbb{N}$ by the recursive clauses

$$L_X^\varphi(0) := 0,$$

$$L_X^\varphi(\varphi_{x_1}\alpha_1 + \cdots + \varphi_{x_n}\alpha_n) := L_X^\varphi(\alpha_1) + \cdots + L_X^\varphi(\alpha_n) + n,$$

where the second clause includes the case $n = 1$. We will need the following fact:

LEMMA 3.1 (RCA₀). *We have $\alpha < \varphi_x \alpha$ for any element $\varphi_x \alpha \in \varphi(1 + X)0$.*

Before we prove the lemma, let us explain how it can be reconciled with the intuition that we should have $\varphi_y \beta = \varphi_x(\varphi_y \beta)$ in case $x <_{1+X} y$. The point is that $\varphi(1 + X)0$ does not even allow to form the “superfluous” term $\varphi_x(\varphi_y \beta)$, which violates the condition $h(\varphi_y \beta) \leq_{1+X} x$.

PROOF. We show the following stronger claim by induction on $L_X^\varphi(\alpha) + L_X^\varphi(\gamma)$:

“if α is a proper subterm of $\gamma \in \varphi(1 + X)0$, then we have $\alpha < \gamma$.”

Let us consider the most interesting case, in which we have $\alpha = \varphi_x \beta$ and $\gamma = \varphi_y \delta$. In view of equivalence (\star) we need to distinguish three cases: First assume $x <_X y$. By induction hypothesis we get $\beta < \varphi_y \delta$, which does indeed imply $\varphi_x \beta < \varphi_y \delta$. Now assume that we have $x = y$. Since β is a proper subterm of δ , the induction hypothesis yields $\beta < \delta$. Once again (\star) yields the claim. Finally, assume that we have $y <_X x$. In view of

$$h(\delta) \leq_X y <_X x = h(\varphi_x \beta)$$

we see that $\varphi_x \beta$ cannot be equal to δ ; hence it must be a proper subterm. Then the induction hypothesis yields $\varphi_x \beta < \delta$, as needed to conclude by (\star) . \dashv

Above we have used x to denote an arbitrary element of $1 + X$. If we want to distinguish the elements of the two summands, then we write them as \perp and $1 + x$. On $\varphi(1 + X)0$ one readily defines an operation of addition with the usual properties. Exponentiation to the base ω can be given by

$$\omega^\alpha = \begin{cases} \alpha & \text{if } \alpha = \varphi_x \beta \text{ with } \perp <_{1+X} x, \\ \varphi_\perp \alpha & \text{otherwise.} \end{cases}$$

This allows to develop a notion of Cantor normal form, which supports the usual definition of multiplication. Let us observe that values of the form $\varphi_{1+x}(\beta)$ do indeed behave like ε -numbers: In view of $\perp <_{1+X} 1 + x$ equivalence (\star) reveals that $\alpha < \varphi_{1+x} \beta$ implies $\omega^\alpha < \varphi_{1+x} \beta$. An element of $\varphi(1 + X)0$ will be called decomposable if it is not equal to 0 and not of the form $\varphi_{1+x} \alpha$ (hence $\varphi_\perp \alpha$ is considered as decomposable). Let us state an appropriate version of Lemma 2.5:

LEMMA 3.2 (RCA₀). *Any decomposable element of $\varphi(1 + X)0$ can be uniquely written as $\omega^\alpha + \beta$ with $\alpha, \beta < \omega^\alpha + \beta$. Furthermore $L_X^\varphi(\alpha), L_X^\varphi(\beta) < L_X^\varphi(\omega^\alpha + \beta)$ holds for any such decomposition.*

PROOF. Given a decomposable element $\varphi_{x_1}\alpha_1 + \cdots + \varphi_{x_n}\alpha_n \in \varphi(1 + X)0$, possibly with $n = 1$ and $x_1 = \perp$, we set

$$\alpha := \begin{cases} \alpha_1 & \text{if } x_1 = \perp, \\ \varphi_{x_1} \alpha_1 & \text{otherwise,} \end{cases}$$

as well as $\beta := \varphi_{x_2} \alpha_2 + \dots + \varphi_{x_n} \alpha_n$ (in particular we have $\beta = 0$ in case $n = 1$). By construction we have $\omega^\alpha = \varphi_{x_1} \alpha_1$ and hence $\omega^\alpha + \beta = \varphi_{x_1} \alpha_1 + \dots + \varphi_{x_n} \alpha_n$. It is straightforward to see that we have $\alpha, \beta < \omega^\alpha + \beta$, except when we have $x_1 = \perp$. In that case the claim reduces to $\alpha_1 < \varphi_\perp \alpha_1$, which requires Lemma 3.1. The condition $L_X^\varphi(\alpha), L_X^\varphi(\beta) < L_X^\varphi(\omega^\alpha + \beta)$ is readily verified. Uniqueness follows from basic properties of addition and exponentiation, as in Lemma 2.5. \dashv

Our goal is to characterize $\varphi(1 + X)0$ as a minimal Bachmann–Howard fixed point of the transformation

$$Y \mapsto T_X^\varphi(Y) = 1 + (Y + X) \times Y \cong 1 + Y^2 + X \times Y.$$

Elements of $T_X^\varphi(Y)$ will be written as \perp , $\langle y_0, y_1 \rangle$ and $\langle x, y \rangle$, with $y_0, y_1, y \in Y$ and $x \in X$. The supports from Definition 1.2 can be given as

$$\text{supp}_Y^\varphi(\perp) = \emptyset, \quad \text{supp}_Y^\varphi(\langle y_0, y_1 \rangle) = \{y_0, y_1\}, \quad \text{supp}_Y^\varphi(\langle x, y \rangle) = \{y\}.$$

Hence a function $\vartheta : T_X^\varphi(Y) \rightarrow Y$ is a Bachmann–Howard collapse if, and only if, the following conditions are satisfied:

- (i) we have $\vartheta(\perp) <_Y \vartheta(\langle y_0, y_1 \rangle)$ for arbitrary elements $y_0, y_1 \in Y$, as well as $\vartheta(\perp) <_Y \vartheta(\langle x, y \rangle)$ for arbitrary $y \in Y$ and $x \in X$,
- (i') $\langle y_0, y_1 \rangle <_{Y^2} \langle y'_0, y'_1 \rangle$ implies $\vartheta(\langle y_0, y_1 \rangle) <_Y \vartheta(\langle y'_0, y'_1 \rangle)$, under the side condition that we have $y_0, y_1 <_Y \vartheta(\langle y'_0, y'_1 \rangle)$,
- (i'') if we have $y_0, y_1 <_Y \vartheta(\langle x, y \rangle)$, then we have $\vartheta(\langle y_0, y_1 \rangle) <_Y \vartheta(\langle x, y \rangle)$,
- (i''') $\langle x, y \rangle <_{X \times Y} \langle x', y' \rangle$ implies $\vartheta(\langle x, y \rangle) <_Y \vartheta(\langle x', y' \rangle)$, under the side condition that we have $y <_Y \vartheta(\langle x', y' \rangle)$,
- (ii) we have $y_0, y_1 <_Y \vartheta(\langle y_0, y_1 \rangle)$ for arbitrary $y_0, y_1 \in Y$,
- (ii') we have $y <_Y \vartheta(\langle x, y \rangle)$ for arbitrary $y \in Y$ and $x \in X$.

We can now prove the theorem that was stated in the introduction:

PROOF OF THEOREM 1.4. In the first half of the proof we show that $\varphi(1 + X)0$ is a Bachmann–Howard fixed point of the transformation T_X^φ . For this purpose we must specify a Bachmann–Howard collapse

$$\vartheta : 1 + \varphi(1 + X)0 \times \varphi(1 + X)0 + X \times \varphi(1 + X)0 \rightarrow \varphi(1 + X)0.$$

Above we have discussed basic ordinal arithmetic on $\varphi(1 + X)0$. As in the previous section we abbreviate $\omega_2(\alpha) := \omega^{\omega^\alpha}$, as well as $1 := \omega^0$. We can now set

$$\begin{aligned} \vartheta(\perp) &:= 0, \\ \vartheta(\langle \alpha, \beta \rangle) &:= \omega_2(\alpha + 1) \cdot (\beta + 1), \\ \vartheta(\langle x, y \rangle) &:= \varphi_{1+x}(y + 1). \end{aligned}$$

Concerning the third clause, we observe that $\varphi_{1+x}(y + 1) \in \varphi(1 + X)0$ holds because of $h(y + 1) = \perp$ (note that $y + 1$ cannot be of the form $\varphi_y \delta$ with $y \neq \perp$). Terms of the form $\varphi_\perp \delta$ are used implicitly, via the definition of exponentiation. We need to verify the conditions stated above: Conditions (i) and (ii) are immediate, and Condition (ii') follows from Lemma 3.1. To verify Condition (i') one argues just as

in the proof of Theorem 2.2. For Condition (i'') it suffices to recall that $\varphi_{1+x}(\gamma + 1)$ behaves like an ε -number (cf. also the proof of Theorem 2.6). In order to establish Condition (i''') we consider an inequality

$$\langle x, \gamma \rangle <_{X \times \varphi(1+X)0} \langle y, \delta \rangle.$$

If we have $x = y$ and $\gamma < \delta$, then

$$\vartheta(\langle x, \gamma \rangle) = \varphi_{1+x}(\gamma + 1) < \varphi_{1+y}(\delta + 1) = \vartheta(\langle y, \delta \rangle)$$

follows from equivalence (\star). It remains to consider the case where we have $x <_X y$. Due to the side condition in (i''') we may assume

$$\gamma < \vartheta(\langle y, \delta \rangle) = \varphi_{1+y}(\delta + 1),$$

which can be strengthened to $\gamma + 1 < \varphi_{1+y}(\delta + 1)$. As we also have $1 + x <_{1+X} 1 + y$, we can again infer $\varphi_{1+x}(\gamma + 1) < \varphi_{1+y}(\delta + 1)$ by equivalence (\star). In the rest of this proof we show that the Bachmann–Howard fixed point $\varphi(1 + X)0$ is minimal. For this purpose we consider an arbitrary order Y with a Bachmann–Howard collapse

$$\vartheta : 1 + Y^2 + X \times Y \rightarrow Y.$$

We need to construct an order embedding $f : \varphi(1 + X)0 \rightarrow Y$. In view of Lemma 3.2 we can recursively define

$$\begin{aligned} f(0) &:= \vartheta(\perp), \\ f(\omega^\alpha + \beta) &:= \vartheta(\langle f(\alpha), f(\beta) \rangle), \quad \text{where } \alpha, \beta < \omega^\alpha + \beta, \\ f(\varphi_{1+x}\gamma) &:= \vartheta(\langle x, f(\gamma) \rangle). \end{aligned}$$

Note that elements of the form $\varphi_\perp \alpha = \omega^\alpha + 0$ are covered by the second clause. To show that $\eta < \xi$ implies $f(\eta) <_Y f(\xi)$ we argue by induction on $L_X^\varphi(\eta) + L_X^\varphi(\xi)$. In most cases one argues just as in the proof of Theorem 2.6. The only case that is essentially new concerns an inequality

$$\eta = \varphi_{1+x}\gamma < \varphi_{1+z}\delta = \xi.$$

In view of (\star) we first assume that this inequality holds because we have $x <_X z$ and $\gamma < \xi$. Then we immediately get

$$\langle x, f(\gamma) \rangle <_{X \times Y} \langle z, f(\delta) \rangle.$$

Due to the induction hypothesis we also obtain

$$f(\gamma) <_Y f(\xi) = \vartheta(\langle z, f(\delta) \rangle).$$

This is the side condition required by (i''') above. We can thus infer

$$f(\eta) = \vartheta(\langle x, f(\gamma) \rangle) <_Y \vartheta(\langle z, f(\delta) \rangle) = f(\xi).$$

Now assume $x = z$ and $\gamma < \delta$. The induction hypothesis yields $f(\gamma) <_Y f(\delta)$, so that we obtain $\langle x, f(\gamma) \rangle <_{X \times Y} \langle z, f(\delta) \rangle$ once again. Using Lemma 3.1 we also get $\gamma < \eta < \xi$, which allows us to conclude as in the previous case. Finally, assume that we have $z <_X x$ and $\eta < \delta$. Using the induction hypothesis and Condition (ii') we obtain

$$f(\eta) <_Y f(\delta) <_Y \vartheta(\langle z, f(\delta) \rangle) = f(\xi),$$

just as required. –

The statement that $\varphi(1 + X)0$ is well founded for any well-order X is equivalent to the principle of arithmetical transfinite recursion, as shown by H. Friedman (the first published proof seems to appear in [18], where a draft by Friedman, Montalbán and Weiermann is cited as the original reference). Now that we have proved Theorem 1.4, we immediately obtain Corollary 1.5 from the introduction.

In the rest of this section we show how the orders Γ_X can be characterized in terms of minimal Bachmann–Howard fixed points. To motivate the definition of these orders we observe that the function $\alpha \mapsto \varphi_\alpha(0)$ is normal. We write $\gamma \mapsto \Gamma_\gamma$ for the derivative of this function. Its range is the club class

$$\{\Gamma_\gamma \mid \gamma \text{ an ordinal}\} = \{\alpha \mid \varphi_\alpha(0) = \alpha\} = \{\alpha > 0 \mid \varphi_\beta(\gamma) < \alpha \text{ for all } \beta, \gamma < \alpha\} \quad (\dagger)$$

of Γ -numbers. Note that any Γ -number is an ε -number.

A relativized notation system Γ_X for all ordinals below the X -th Γ -number has been described in [15, Section 2]. As for the notation system $\varphi(1 + X)0$, the underlying set Γ_X and the order relation are defined by a simultaneous recursion. In the present case we must also specify a function $h : \Gamma_X \rightarrow \Gamma_X$ and a set $\mathbf{H} \subseteq \Gamma_X$ (here \mathbf{H} refers to the German “Hauptzahlen” for (additively) principal ordinals):

- We have an element $0 \in \Gamma_X \setminus \mathbf{H}$ with $h(0) = 0$.
- For each $x \in X$ we have an element $\Gamma_x \in \mathbf{H} \subseteq \Gamma_X$ with $h(\Gamma_x) = \Gamma_x$.
- Given elements $\alpha, \beta \in \Gamma_X$, we get a term $\varphi_\alpha\beta \in \mathbf{H} \subseteq \Gamma_X$ with $h(\varphi_\alpha\beta) = \alpha$, provided that the following conditions are satisfied:
 - we have $h(\beta) \leq_{\Gamma_X} \alpha$,
 - if α is of the form Γ_x , then we have $\beta \neq 0$.
- Given $n > 1$ elements $\alpha_n \leq_{\Gamma_X} \dots \leq_{\Gamma_X} \alpha_1$ from $\mathbf{H} \subseteq \Gamma_X$, we get a term $\alpha := \alpha_1 + \dots + \alpha_n \in \Gamma_X \setminus \mathbf{H}$ with $h(\alpha) = 0$.

The order on Γ_X is determined by (\star) and (\dagger) , where the latter are to be read as statements about terms from Γ_X rather than actual ordinals. Full details of the somewhat lengthy definition can be found in [15, Section 2].

Addition, multiplication and exponentiation on Γ_X can be defined as in the case of $\varphi(1 + X)0$. We point out that $\varphi_0\alpha \in \Gamma_X$ assumes the role of $\varphi_\perp\alpha \in \varphi(1 + X)0$. Elements of the form Γ_x or $\varphi_\alpha\beta$ with $\alpha \neq 0$ behave like ε -numbers. To define a length function $L_X^\Gamma : \Gamma_X \rightarrow \mathbb{N}$ we set

$$\begin{aligned} L_X^\Gamma(0) &:= L_X^\Gamma(\Gamma_x) := 0, \\ L_X^\Gamma(\varphi_\alpha\beta) &:= L_X^\Gamma(\alpha) + L_X^\Gamma(\beta) + 1, \\ L_X^\Gamma(\alpha_1 + \dots + \alpha_n) &:= L_X^\Gamma(\alpha_1) + \dots + L_X^\Gamma(\alpha_n) + n. \end{aligned}$$

It will be convenient to use a somewhat different decomposition than before:

LEMMA 3.3 (RCA₀). *Any non-zero element of $\Gamma_X \setminus \mathbf{H}$ can be uniquely written as $\alpha + \beta$ with $\alpha, \beta <_{\Gamma_X} \alpha + \beta$ and $\alpha \in \mathbf{H}$. Furthermore $L_X^\Gamma(\alpha), L_X^\Gamma(\beta) < L_X^\Gamma(\alpha + \beta)$ holds for any such decomposition.*

PROOF. Given an element $\alpha_1 + \dots + \alpha_n$ we set $\alpha := \alpha_1$ and $\beta := \alpha_2 + \dots + \alpha_n$ (in particular $\beta = \alpha_2$ in case $n = 2$). It is straightforward to see that this satisfies the desired properties. To establish uniqueness it suffices to observe that $\gamma + \alpha = \alpha$ holds for any $\gamma <_{\Gamma_X} \alpha \in \mathbf{H}$ (cf. the proof of Lemma 2.5). ⊥

We will also need the following variant of Lemma 3.1 (cf. the explanation after the statement of that result):

LEMMA 3.4 (RCA₀). *We have $\alpha, \beta <_{\Gamma_X} \varphi_\alpha \beta$ for any element $\varphi_\alpha \beta \in \Gamma_X$.*

PROOF. Yet again, the following stronger claim can be established by induction on the joint complexity $L_X^\Gamma(\eta) + L_X^\Gamma(\xi)$ of η and ξ :

“if η is a proper subterm of $\xi \in \Gamma_X$, then we have $\eta <_{\Gamma_X} \xi$.”

Let us consider the case of $\eta = \varphi_\alpha \beta$ and $\xi = \varphi_\gamma \delta$. In contrast to Lemma 3.1, we must now distinguish two possibilities: If η is a subterm of δ , then one argues just as before. Now assume that η is a subterm of γ . Then the induction hypothesis yields both $\alpha <_{\Gamma_X} \gamma$ and $\beta <_{\Gamma_X} \varphi_\gamma \delta$. We can conclude $\varphi_\alpha \beta <_{\Gamma_X} \varphi_\gamma \delta$ by (\star) . \dashv

Our aim is to characterize Γ_X in terms of the order transformation

$$Y \mapsto T_X^\Gamma(Y) = 1 + 2 \times Y^2 + X.$$

Elements of $T_X^\Gamma(Y)$ will be written as $\perp, \langle i, y_0, y_1 \rangle$ and x , with $i \in \{0, 1\}, y_0, y_1 \in Y$ and $x \in X$. The supports from Definition 1.2 amount to

$$\text{supp}_Y^\Gamma(\perp) = \text{supp}_Y^\Gamma(x) = \emptyset \quad \text{and} \quad \text{supp}_Y^\Gamma(\langle i, y_0, y_1 \rangle) = \{y_0, y_1\}.$$

In view of Definition 1.3 this means that a function $\vartheta : T_X^\Gamma(Y) \rightarrow Y$ is a Bachmann–Howard collapse if, and only if, the following conditions are satisfied:

- (i) we have $\vartheta(\perp) <_Y \vartheta(\langle i, y_0, y_1 \rangle)$ for arbitrary $i \leq 1$ and $y_0, y_1 \in Y$, as well as $\vartheta(\perp) <_Y \vartheta(x) <_Y \vartheta(x')$ for any $x, x' \in X$ with $x <_X x'$,
- (i') $\langle y_0, y_1 \rangle <_{Y^2} \langle y'_0, y'_1 \rangle$ implies $\vartheta(\langle i, y_0, y_1 \rangle) <_Y \vartheta(\langle i, y'_0, y'_1 \rangle)$ for each $i \leq 1$, under the side condition that we have $y_0, y_1 <_Y \vartheta(\langle i, y'_0, y'_1 \rangle)$,
- (i'') $y_0, y_1 <_Y \vartheta(\langle 1, y'_0, y'_1 \rangle)$ implies $\vartheta(\langle 0, y_0, y_1 \rangle) <_Y \vartheta(\langle 1, y'_0, y'_1 \rangle)$,
- (i''') $y_0, y_1 <_Y \vartheta(x)$ implies $\vartheta(\langle i, y_0, y_1 \rangle) <_Y \vartheta(x)$ for each $i \leq 1$,
- (ii) we have $y_0, y_1 <_Y \vartheta(\langle i, y_0, y_1 \rangle)$ for each $i \leq 1$.

We can now establish the promised characterization:

THEOREM 3.5 (RCA₀). *The order Γ_X is a minimal Bachmann–Howard fixed point of the transformation T_X^Γ , for each order X .*

PROOF. Let us first construct a Bachmann–Howard collapse

$$\vartheta : 1 + 2 \times \Gamma_X \times \Gamma_X + X \rightarrow \Gamma_X.$$

As before we abbreviate $\omega_2(\alpha) := \omega^{\omega^\alpha}$. We then set

$$\begin{aligned} \vartheta(\perp) &:= 0, \\ \vartheta(\langle 0, \alpha, \beta \rangle) &:= \omega_2(\alpha + 1) \cdot (\beta + 1), \\ \vartheta(\langle 1, \alpha, \beta \rangle) &:= \varphi_{1+\alpha}(\beta + 1), \\ \vartheta(x) &:= \Gamma_x. \end{aligned}$$

To see that we have $\varphi_{1+\alpha}(\beta + 1) \in \Gamma_X$ it suffices to observe that we have $h(\beta + 1) = 0$ (as $\beta + 1$ does not lie in $\mathbf{H} \setminus \{\varphi_0 0\}$) and $\beta + 1 \neq 0$ (which is only relevant if $\alpha = \Gamma_x$). We need to show that the above conditions are satisfied. Condition (i) is immediate. To establish Condition (i') one argues just as in the proofs of Theorems 2.2 and 1.4. Conditions (i'') and (i''') hold because $\varphi_{1+\alpha}(\beta + 1)$ behaves like an ε -number (due to $1 + \alpha \neq 0$) while Γ_x behaves like a Γ -number (cf. equation (\dagger)). Using Lemma 3.4, one readily checks that Condition (ii) is satisfied. To show that Γ_X is minimal we consider an arbitrary order Y with a Bachmann–Howard collapse

$$\vartheta : 1 + 2 \times Y^2 + X \rightarrow Y.$$

Relying on Lemma 3.3, we define $f : \Gamma_X \rightarrow Y$ by the recursive clauses

$$\begin{aligned} f(0) &:= \vartheta(\perp), \\ f(\alpha + \beta) &:= \vartheta(\langle 0, f(\alpha), f(\beta) \rangle), \quad \text{where } \alpha, \beta < \alpha + \beta \text{ and } \alpha \in \mathbf{H}, \\ f(\varphi_\gamma \delta) &:= \vartheta(\langle 1, f(\gamma), f(\delta) \rangle), \\ f(\Gamma_x) &:= \vartheta(x). \end{aligned}$$

In order to show that $\eta <_{\Gamma_X} \xi$ implies $f(\eta) <_Y f(\xi)$ one argues by induction on the number $L_X^\Gamma(\eta) + L_X^\Gamma(\xi)$. In the following we discuss the cases that are not already covered by the proofs of Theorems 2.6 and 1.4 (where Lemma 3.4 assumes the role of Lemma 3.1). Let us first consider an inequality

$$\eta = \varphi_\alpha \beta <_{\Gamma_X} \Gamma_x = \xi.$$

By Lemma 3.4 we get $\alpha, \beta <_{\Gamma_X} \eta <_{\Gamma_X} \xi$, so that the induction hypothesis yields

$$f(\alpha), f(\beta) <_Y f(\xi) = \vartheta(x).$$

Using Condition (i''') from above we can infer

$$f(\eta) = \vartheta(\langle 1, f(\alpha), f(\beta) \rangle) <_Y \vartheta(x) = f(\xi).$$

The case of an inequality $\alpha + \beta <_{\Gamma_X} \Gamma_x$ is treated similarly. Let us now establish the induction step for an inequality

$$\eta = \Gamma_x <_{\Gamma_X} \varphi_\alpha \beta = \xi.$$

In view of equation (†) we must have $\Gamma_x \leq_{\Gamma_X} \alpha$ or $\Gamma_x \leq_{\Gamma_X} \beta$. In either case we can invoke the induction hypothesis and Condition (ii) to get

$$f(\eta) \leq_Y \max_Y \{f(\alpha), f(\beta)\} <_Y \vartheta(\langle 1, f(\alpha), f(\beta) \rangle) = f(\xi).$$

A similar argument covers the case of an inequality $\Gamma_x <_{\Gamma_X} \alpha + \beta$ (where we must have $\Gamma_x \leq_{\Gamma_X} \alpha$). Finally we consider an inequality

$$\eta = \Gamma_x <_{\Gamma_X} \Gamma_z = \xi.$$

Since $x \mapsto \Gamma_x$ represents a normal function we have $x <_X z$. By Condition (i) we get $f(\eta) = \vartheta(x) <_Y \vartheta(z) = f(\xi)$, just as required. \dashv

The statement that Γ_X is well-founded for any well-order X is equivalent to the assertion that every set lies in a countable coded ω -model of arithmetical transfinite recursion, as shown by Rathjen [15]. We can conclude with the following:

COROLLARY 3.6. *The following are equivalent over \mathbf{RCA}_0 :*

- (i) *every set is contained in a countable coded ω -model of \mathbf{ATR} ,*
- (ii) *for every well-order X the transformation $Y \mapsto 1 + 2 \times Y^2 + X$ has a well-founded Bachmann–Howard fixed point.*

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