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ON THE DISCRETE-TIME $G/GI/\infty$ QUEUE*

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The discrete-time $G/GI/\infty$ queue model is explored. Jobs arrive to an infinite-server queuing system following an *arbitrary* input process X; job sizes are general independent and identically distributed random variables. The system's output process Y (of job departures) and queue process N (tracking the number of jobs present in the system) are analyzed. Various statistics of the stochastic maps $X \mapsto Y$ and $X \mapsto N$ are explicitly obtained, including means, variances, autocovariances, cross-covariances, and multidimensional probability generating functions. In the case of stationary inputs, we further compute the spectral densities of the stochastic maps, characterize the fixed points (in the L^2 sense) of the input–output map, precisely determine when the output and queue processes display either short-ranged or long-ranged temporal dependencies, and prove a decomposition result regarding the intrinsic L^2 structure of general stationary $G/GI/\infty$ systems.

1. INTRODUCTION

Infinite-server queuing systems have attracted considerable interest in the scientific literature, the quintessential example being the $M/GI/\infty$ model. In this fundamental model, jobs arrive, in a Markovian fashion, to a service system with an infinite service workforce (i.e., an infinite number of servers). Each job, upon arrival, is attended by a server, admits service, and then leaves the system. The sizes of incoming jobs are independent and identically distributed (IID) random variables, drawn from a general distribution on the positive half-line.

* Dedicated to Professor Uri Yechiali—a memorable mentor, an inspirational colleague, and a true friend—on the occasion of his retirement.

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The $M/GI/\infty$ model can be considered either in discrete-time or continuous-time settings. In the discrete-time setting, job arrivals follow an IID sequence of Poisson-distributed random variables, and job sizes are integer-valued. In the continuous-time setting, job arrivals follow a Poisson process and job sizes are positive valued.

The origins of the continuous-time $M/GI/\infty$ model come from physics; specifically, from the analysis of data recorded by Geiger–Müller counters. Particles "hit" a Geiger–Müller counter following a standard Poisson process. A particle, after hitting the counter, blocks it for a random duration of time. The blocking durations "carried in" by the hitting particles are IID random variables. The counter records particles hitting it only when it is unblocked. These types of Geiger–Müller systems were referred to as "Type II counters" [22]. The first studies of such counters, conducted in the 1930s and 1940s, considered deterministic blockings [8,12,18]. Subsequent studies, conducted in the 1950s, considered random blockings [10,16,23].

The pioneering works mentioned earlier focused on the analysis of the sequence of "recording epochs," rather than on the analysis of the system's "queue process"—the fluctuating number of particles present in the system. The latter was investigated in [5,6], in the context of a mathematical model for either a textile yarn, an immigration-death process, or an infinite-server queuing system.

One particularly important feature of the $M/GI/\infty$ model is that its queue process can display both short-ranged and long-ranged temporal dependencies [4]: The queue process is short-range dependent (SRD) if the job sizes are of finite variance and is long-range dependent (LRD) if the job sizes are of infinite variance. This important feature, as well as the analytical tractability of the $M/GI/\infty$ model, led researchers to use the $M/GI/\infty$ queue process as an input-process model in the modeling of networks with LRD inflows [9,13–15]. Moreover, this modeling approach was supported by the development of efficient algorithms for the simulation of $M/GI/\infty$ queue processes displaying LRD [20,21].

Recently, generalizations of the "classic" $M/GI/\infty$ model were proposed and investigated. In [2], a discrete-time model considering stationary ergodic inflows and phase-type distributed job sizes was studied, using stochastic recursive equations. In [7], a continuous-time model considering Poisson point process inflows (with infinite arrival rates) was studied, focusing on issues such as heavy-tailed stationary distributions, LRD, and "reverse engineering".

This research is devoted to the exploration of the discrete-time $G/GI/\infty$ model; namely we consider the case of *arbitrary inflows* and general IID service times and analyze the system's departure process and queue process. Informally, if X denotes the input process of job arrivals, Y denotes the output process of job departures, and N denotes the system's queue process (counting the number of job present in the system), then the following stochastic maps are investigated:

$$X \mapsto Y \quad \text{and} \quad X \mapsto N.$$

Closed-form formulas for various statistics of these stochastic maps are derived explicitly, including means, variances, autocovariances, cross-covariances, multidimensional probability generating functions, and, in the case of stationary inputs, spectral densities. Moreover, in the case of stationary inputs, we characterize the fixed points (in the L^2 sense) of the stochastic map $X \mapsto Y$, precisely determine when SRD/LRD is displayed by the output and queue processes, and prove an L^2 decomposition result regarding the intrinsic L^2 structure of general stationary $G/GI/\infty$ systems. An application of this research to the analysis of "source-medium-sink flows" appeared recently [25].

The sequel is organized as follows: The discrete-time $G/GI/\infty$ model is formally defined in Section 2; analysis of systems with general input processes is conducted in Section 3; L^2 analysis of systems with general stationary input processes is conducted in Section 4; and the case of stationary Cox-type input processes is treated in Section 5.

Notation: Throughout the article, $\mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$ denotes the integers, $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$ denotes the nonnegative integers, and, $\mathbb{N} := \{1, 2, \ldots\}$ denotes the positive integers. Also, $\mathbf{I}\{\mathcal{S}\}$ denotes the indicator function of the set $\mathcal{S}$ and $\delta(t) := \mathbf{I}\{t = 0\}$.

2. MODEL SETTING

Time is considered discrete $t \in \mathbb{Z}$, and three $\mathbb{Z}_+$-valued random processes are defined:

1. **The input process** $X = (X(t))_{t \in \mathbb{Z}}$, where $X(t)$ denotes the number of jobs arriving to the system at time t.
2. **The output process** $Y = (Y(t))_{t \in \mathbb{Z}}$, where $Y(t)$ denotes the number of jobs departing the system at time t.
3. **The queue process** $N = (N(t))_{t \in \mathbb{Z}}$, where $N(t)$ denotes the number of jobs present in the system at time t.

The service time required by the kth job arriving at time t is denoted $\xi(t; k)$, $k = 1, 2, \ldots, X(t)$ (provided that $X(t) \geq 1$). The service times are $\mathbb{N}$-valued random variables.

The output process and queue process admit the following "integral representations":

$$Y(t) = \sum_{s \leq t} \sum_{k=1}^{X(s)} \mathbf{I}\{\xi(s; k) = t - s\} \tag{1}$$

and

$$N(t) = \sum_{s \leq t} \sum_{k=1}^{X(s)} \mathbf{I}\{\xi(s; k) > t - s\} \tag{2}$$

(using the convention that $X(s) = 0$ implies that the sum $\sum_{k=1}^{X(s)}$ is empty).

Equations (1) and (2) define the stochastic maps $X \mapsto Y$ and $X \mapsto N$, which transform the input process X into respectively the output process Y and the queue process N. The inherent randomness of these maps stems from the nonlinear "interaction"

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between the input process X and the jobs' service times $\xi(t;k)$. Our aim, in the sequel, is to explore the statistical properties of these stochastic maps, under the following model assumptions:

1. **Input**: The input process X is an arbitrary random sequence.
2. **Service**: The service times $\xi(t;k)$ are IID random variables, independent of the input process X.

Henceforth, let ξ denote the duration of a generic service time, and set

$$\begin{aligned} g(n) &:= \mathbf{P}(\xi = n), \qquad & \overline{g}(n) &:= \mathbf{P}(\xi \neq n), \\ G(x) &:= \mathbf{P}(\xi \leq x), \qquad & \overline{G}(x) &:= \mathbf{P}(\xi > x) \end{aligned} \tag{3}$$

($n \in \mathbb{N}$ and $x \geq 0$).

3. ANALYSIS

In this section we study the statistical properties of the stochastic maps $X \mapsto Y$ and $X \mapsto N$, under the general model assumptions postulated earlier. The proofs of the results stated below are given in Appendix A.

- **Means**. The means of the output process Y and the queue process N are given, respectively, by

$$\mathbf{E}[Y(t)] = \sum_{s \leq t} g(t-s)\mathbf{E}[X(s)] \tag{4}$$

  and

$$\mathbf{E}[N(t)] = \sum_{s \leq t} \overline{G}(t-s)\, \mathbf{E}[X(s)]. \tag{5}$$

- **Variances**. The variances of the output process Y and the queue process N are given, respectively, by

$$\begin{aligned} \mathrm{Var}[Y(t)] = &\sum_{s \leq t} g(t-s)\overline{g}(t-s)\mathbf{E}[X(s)] \\ &+ \sum_{s_1 \leq t} \sum_{s_2 \leq t} g(t-s_1)g(t-s_2)\, \mathrm{Cov}[X(s_1), X(s_2)] \end{aligned} \tag{6}$$

  and

$$\begin{aligned} \mathrm{Var}[N(t)] = &\sum_{s \leq t} G(t-s)\overline{G}(t-s)\mathbf{E}[X(s)] \\ &+ \sum_{s_1 \leq t} \sum_{s_2 \leq t} \overline{G}(t-s_1)\overline{G}(t-s_2)\, \mathrm{Cov}[X(s_1), X(s_2)]. \end{aligned} \tag{7}$$

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- **Autocovariances**. The autocovariance functions of the output process Y and the queue process N are given, respectively, by $(t_1 < t_2)$

$$\mathrm{Cov}[Y(t_1), Y(t_2)] = -\sum_{s\le t_1} g(t_1 - s)g(t_2 - s)\mathbf{E}[X(s)] + \sum_{s_1\le t_1}\sum_{s_2\le t_2} g(t_1-s_1)g(t_2 - s_2)\,\mathrm{Cov}[X(s_1), X(s_2)] \tag{8}$$

and

$$\mathrm{Cov}[N(t_1), N(t_2)] = \sum_{s\le t_1} G(t_1 - s)\overline{G}(t_2 - s)\mathbf{E}[X(s)] + \sum_{s_1\le t_1}\sum_{s_2\le t_2} \overline{G}(t_1-s_1)\overline{G}(t_2 - s_2)\,\mathrm{Cov}[X(s_1), X(s_2)]\,. \tag{9}$$

- **Cross-covariances**. The cross-covariance between the output process Y and the queue process N is given by

$$\mathrm{Cov}[Y(t_1), N(t_2)] = -\sum_{s\le t_1} g(t_1 - s)\overline{G}(t_2 - s)\mathbf{E}[X(s)] + \sum_{s_1\le t_1}\sum_{s_2\le t_2} g(t_1-s_1)\overline{G}(t_2 - s_2)\,\mathrm{Cov}[X(s_1), X(s_2)] \tag{10}$$

for $t_1 \le t_2$ and by

$$\mathrm{Cov}[N(t_1), Y(t_2)] = \sum_{s\le t_1} G(t_1 - s)g(t_2 - s)\mathbf{E}[X(s)] + \sum_{s_1\le t_1}\sum_{s_2\le t_2} \overline{G}(t_1-s_1)g(t_2 - s_2)\,\mathrm{Cov}[X(s_1), X(s_2)]. \tag{11}$$

for $t_1 < t_2$.

- **Probability Generating Functions**. The PGFs of the output process Y and the queue process N are given, respectively, by $(|\theta| \le 1)$

$$\mathbf{E}\big[\theta^{Y(t)}\big] = \mathbf{E}\left[\prod_{n=1}^{\infty}\Big(\overline{g}(n) + \theta g(n)\Big)^{X(t-n)}\right] \tag{12}$$

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and

$$\mathbf{E}\left[\theta^{N(t)}\right]=\mathbf{E}\left[\prod_{n=0}^{\infty}\left(G(n)+\theta\overline{G}(n)\right)^{X(t-n)}\right]. \tag{13}$$

These results are in fact one-dimensional "projections" of the following multidimensional results:

- **Multidimensional PGFs**. Let $\{\theta(t)\}_{t\in\mathbb{Z}}$ be an arbitrary sequence of complex numbers taking values in the unit disk ($|\theta(t)|\leq 1$). The multidimensional PGFs of the output process Y and the queue process N are given, respectively, by

$$\mathbf{E}\left[\prod_{t\in\mathbb{Z}}\theta(t)^{Y(t)}\right]=\mathbf{E}\left[\prod_{t\in\mathbb{Z}}\alpha(t)^{X(t)}\right], \tag{14}$$

where $\alpha(t)=\mathbf{E}[\theta(t+\xi)]$, and

$$\mathbf{E}\left[\prod_{t\in\mathbb{Z}}\theta(t)^{N(t)}\right]=\mathbf{E}\left[\prod_{t\in\mathbb{Z}}\beta(t)^{X(t)}\right], \tag{15}$$

where $\beta(t)=\mathbf{E}[\theta(t)\theta(t+1)\cdots\theta(t+\xi-1)]$.

4. STATIONARY INPUTS: L^2 ANALYSIS

In this section we consider the case where the input process X is a *stationary* random sequence (in the wide sense) and conduct an L^2 analysis of the stochastic maps $X\mapsto Y$ and $X\mapsto N$. Henceforth, by "stationary" we mean "stationary in the wide sense."

4.1. Preliminaries

Given a stationary random sequence $Z=(Z(t))_{t\in\mathbb{Z}}$, we denote by μ_Z its mean, by $R_Z(t)$ its autocovariance function ($t\in\mathbb{Z}$), and by $S_Z(\omega)$ its spectral density ($-\pi\leq\omega<\pi$):

$$S_Z(\omega)=\sum_{t\in\mathbb{Z}}R_Z(t)\,\exp\{it\omega\} \tag{16}$$

($-\pi\leq\omega<\pi$). The autocovariance function is retrievable from the spectral density via

$$R_Z(t)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\{-it\omega\}S_Z(\omega)\,d\omega \tag{17}$$

($t\in\mathbb{Z}$).

Two stationary random sequences are said to be equal in L^2 if their means and autocovariance functions (spectral densities) coincide. For a detailed account of the L^2 theory of stationary random sequences, the readers are referred to [19, Chap. VI].

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A stationary random sequence $Z = (Z(t))_{t\in\mathbb{Z}}$ is said to be *short-range dependent* (SRD) if its autocovariance function is summable—in which case,

$$S_Z(0) = \sum_{t\in\mathbb{Z}} R_Z(t). \tag{18}$$

Conversely, Z is said to be *long-range dependent* (LRD) if its autocovariance function is nonsummable, in which case its spectral density diverges at the origin. A "finer resolution" of the LRD case is provided by the following result [4] ($0 < \nu < 1$).

The autocovariance function $R_Z(t)$ is regularly varying[1] at infinity with exponent $-\nu$ if and only if the spectral density $S_Z(\omega)$ is regularly varying at the origin with exponent $-(1-\nu)$, in which case,

$$tR_Z(t) \underset{t\to\infty}{\sim} c_\nu S_Z\left(\frac{1}{t}\right), \tag{19}$$

where $c_\nu = \Gamma(\nu)\cos(\pi\nu/2)/\pi$.

For a comprehensive exposition of both the theory and applications of LRD, the readers are referred to [17,24].

4.2. The Base Case

Let us begin with a special case henceforth referred to as the '*base case*': the input X being an uncorrelated stationary random sequence whose mean and variance are equal; namely, we consider stationary inputs satisfying

$$R_X(t) = \mu_X\delta(t) \quad \Leftrightarrow \quad S_X(\omega) = \mu_X \tag{20}$$

($t \in \mathbb{Z}$; $-\pi \le \omega < \pi$).

Note that the $M/GI/\infty$ model—in which the input process X is a sequence of Poisson-distributed IID random variables—resides within the realm of the base case.

The results regarding the base case are as follows:

- **The output process**. The output process Y is stationary with mean

$$\mu_Y = \mu_X \tag{21}$$

and correlation structure

$$R_Y(t) = \mu_X\delta(t) \quad \Leftrightarrow \quad S_Y(\omega) = \mu_X \tag{22}$$

($t \in \mathbb{Z}$; $-\pi \le \omega < \pi$).

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- **The queue process**. The queue process N is stationary with mean

$$\mu_N = \mu_X \mathbf{E}[\xi], \tag{23}$$

autocovariance

$$R_N(t) = \mu_X \sum_{k=|t|}^{\infty} \overline{G}(k) = \mu_X \, \mathbf{E}\big[(\xi - |t|)_+\big] \tag{24}$$

($t \in \mathbb{Z}$), and spectral density

$$S_N(\omega) = \mu_X \frac{1 - \mathbf{E}[\cos(\omega\xi)]}{1 - \cos(\omega)} \tag{25}$$

($-\pi \leq \omega < \pi$).

- **SRD/LRD**. The queue process N is SRD if and only if the service times are of finite variance, in which case

$$S_N(0) = \mu_X \mathbf{E}\big[\xi^2\big]. \tag{26}$$

In the LRD case, the autocovariance $R_N(t)$ is regularly varying at infinity with exponent $-\nu$ if and only if the service times' probability tail $\overline{G}(x)$ is regularly varying at infinity with exponent $-(1+\nu)$, in which case,

$$\nu \cdot R_N(t) \underset{t\to\infty}{\sim} \mu_X t \overline{G}(t) \tag{27}$$

(the exponent ν being in the range $0 < \nu < 1$).

The proofs of (21)–(25) are given in Appendix B. Equation (26) follows straightforwardly from (25). Equation (27) follows from (24) due to the "monotone density theorem" of the theory of Regular Variation (see [1, Sect. 1.7.3]).

Equations (21) and (22) imply that the output process Y equals, in L^2, the input process X. In fact, this property *characterizes* the base case.

- **Characterization**. The base case characterizes the *fixed points* of the stochastic map $X \mapsto Y$ (in the L^2 sense); namely the output process Y equals, in L^2, the input process X if and only if (20) is satisfied.

The proof of this assertion is given in Appendix B.

4.3. The General Case

Having analyzed the base case in the previous subsection, we now turn to analyze the general case, in which the input X is a general stationary process.

The results regarding the general case are as follows:

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- **The output process**. The output process Y *decomposes* into the sum of two *uncorrelated* stationary processes:

$$Y = Y_{\text{Base}} + Y_{\text{Fluc}}. \tag{28}$$

Y_{Base} is the base case output process. Y_{Fluc} is a zero-mean stationary process with autocovariance

$$R_{Y_{\text{Fluc}}}(t) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} g(k)g(l)\left(R_X - \mu_X\delta\right)(k-l+|t|) \tag{29}$$

($t \in \mathbb{Z}$) and spectral density

$$S_{Y_{\text{Fluc}}}(\omega) = \left|\mathbf{E}\left[\exp\{i\omega\xi\}\right]\right|^2 \left(S_X(\omega) - \mu_X\right) \tag{30}$$

($-\pi \leq \omega < \pi$).

- **The queue process**. The queue process N *decomposes* into the sum of two *uncorrelated* stationary processes:

$$N = N_{\text{Base}} + N_{\text{Fluc}}. \tag{31}$$

N_{Base} is the base case queue process. N_{Fluc} is a zero-mean stationary process with autocovariance

$$R_{N_{\text{Fluc}}}(t) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \overline{G}(k)\overline{G}(l)\left(R_X - \mu_X\delta\right)(k-l+|t|) \tag{32}$$

($t \in \mathbb{Z}$) and spectral density

$$S_{N_{\text{Fluc}}}(\omega) = \left|\frac{1-\mathbf{E}\left[\exp\{i\omega\xi\}\right]}{1-\exp\{i\omega\}}\right|^2 \left(S_X(\omega) - \mu_X\right) \tag{33}$$

($-\pi \leq \omega < \pi$).

- **SRD/LRD**. The input process X and the fluctuation processes Y_{Fluc} and N_{Fluc} are SRD/LRD *jointly*. If SRD, then

$$S_X(0) - \mu_X = S_{Y_{\text{Fluc}}}(0) = \frac{1}{\mathbf{E}[\xi]^2} S_{N_{\text{Fluc}}}(0). \tag{34}$$

Conversely, if LRD, then

$$S_X(\omega) \underset{\omega\to 0}{\sim} S_{Y_{\text{Fluc}}}(\omega) \underset{\omega\to 0}{\sim} \frac{1}{\mathbf{E}[\xi]^2} S_{N_{\text{Fluc}}}(\omega). \tag{35}$$

In the LRD case, if one of the autocovariances is regularly varying at infinity with exponent $-\nu$, then so are the others and

$$R_X(t) \underset{t\to\infty}{\sim} R_{Y_{\text{Fluc}}}(t) \underset{t\to\infty}{\sim} \frac{1}{\mathbf{E}[\xi]^2} R_{N_{\text{Fluc}}}(t) \tag{36}$$

(the exponent ν being in the range $0 < \nu < 1$).

The proofs of (28)–(33) are given in Appendix B. Equations (34) and (35) follow straightforwardly from (30) and (33). Equation (36) follows from (35) due to the general result of (19).

Equations (28) and (31) imply that the output process Y and the queue process N *decompose* into two *uncorrelated* processes: (i) a *base process*, which is independent of the correlation structure of the input process X and represents a corresponding $M/GI/\infty$ system, and (ii) a *fluctuating process*, which is contingent on the correlation structure of the input process X and represents the deviation of the $G/GI/\infty$ system under consideration from the corresponding $M/GI/\infty$ system.

Equations (30) and (33) imply that the spectral densities of the output process Y and the queue process N *factorize* into two terms: (i) a term contingent on the distribution of the service times and (ii) a term contingent on the L^2 properties of the input process X.

The output process Y is LRD if and only if the input process X is such. Long-range dependence of the queue process N, on the other hand, stems from either infinite variance of the service times or LRD of the input process X (the "or" being inclusive).

5. STATIONARY COX INPUTS

In this last section we explore the stochastic maps $X \mapsto Y$ and $X \mapsto N$ in the case where the inputs are stationary *Cox-type* processes [3].

5.1. Analysis

Let $\Lambda = (\Lambda(t))_{t\in\mathbb{Z}}$ be a stationary and nonnegative-valued process representing random "underlying Poissonian rates." Given the rate process Λ, the input X is assumed to be a sequence of independent and Poisson distributed random variables, where $X(t)$ is Poisson distributed with mean $\Lambda(t)$ ($t \in \mathbb{Z}$). For further details regarding Cox processes, the readers are referred to [11, Chap. 6].

The L^2 characteristics of the Cox input process X are induced by the L^2 characteristics of the underlying rate process Λ:

- **The input process**. The input process X is stationary with mean

$$\mu_X = \mu_\Lambda, \tag{37}$$

autocovariance

$$R_X(t) = \mu_\Lambda \delta(t) + R_\Lambda(t) \tag{38}$$

($t \in \mathbb{Z}$), and spectral density

$$S_X(\omega) = \mu_\Lambda + S_\Lambda(\omega) \tag{39}$$

($-\pi \le \omega < \pi$).

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The proof of (37)–(39) is straightforward (using conditioning and the definition of the Cox process X). Equations (37)–(39), substituted into the results of Section 4, yield the full L^2 structure of $G/GI/\infty$ systems "fed" by Cox inputs. Moreover, we have the following:

- **Multidimensional PGFs**. Let $\{\theta(t)\}_{t\in\mathbb{Z}}$ be an arbitrary sequence of complex numbers taking values in the unit disk ($|\theta(t)| \le 1$). The multidimensional PGFs of the output process Y and the queue process N are given, respectively, by

$$\mathbf{E}\left[\prod_{t\in\mathbb{Z}} \theta(t)^{Y(t)}\right] = \mathbf{E}\left[\exp\left\{-\sum_{t\in\mathbb{Z}} a(t)\Lambda(t)\right\}\right], \tag{40}$$

where $a(t) = 1 - \mathbf{E}[\theta(t+\xi)]$, and

$$\mathbf{E}\left[\prod_{t\in\mathbb{Z}} \theta(t)^{N(t)}\right] = \mathbf{E}\left[\exp\left\{-\sum_{t\in\mathbb{Z}} b(t)\Lambda(t)\right\}\right], \tag{41}$$

where $b(t) = 1 - \mathbf{E}[\theta(t)\theta(t+1)\cdots\theta(t+\xi-1)]$.

Equations (40) and (41), whose proofs are given in Appendix C, yield the multidimensional PGFs of the output process Y and the queue process N in terms of the multidimensional *Laplace transform* of the underlying rate process Λ.

5.2. An Example

As an example of stationary Cox inputs, consider the case in which the rate process Λ is a moving-average of a "white noise":

$$\Lambda(t) = \sum_{s\in\mathbb{Z}} w(t-s)\eta(s), \tag{42}$$

where (i) $\{w(n)\}_{n\in\mathbb{Z}}$ is a sequence of nonnegative weights summing up to unity and (ii) $(\eta(t))_{t\in\mathbb{Z}}$ is a nonnegative "white noise"—a sequence of nonnegative IID random variables [$\eta(t)$ being the "innovation" at time t].

Let η denote a generic "noise variable", and let W denote a $\mathbb{Z}$-valued random variable governed by the probability distribution $\mathbf{P}(W = n) = w(n)$. Then [19]

$$\mu_\Lambda = \mathbf{E}[\eta] \quad \text{and} \quad S_\Lambda(\omega) = \mathrm{var}[\eta]\left|\mathbf{E}\left[\exp\{i\omega W\}\right]\right|^2. \tag{43}$$

Moreover, setting $\Phi(\lambda) := \mathbf{E}\left[\exp\{-\lambda\eta\}\right]$ ($\lambda \ge 0$) to denote the Laplace transform of the noise variable η, the multidimensional PGF of the output process Y and

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the queue process N are given respectively by

$$\mathbf{E}\left[\prod_{t\in\mathbb{Z}}\theta(t)^{Y(t)}\right]=\prod_{t\in\mathbb{Z}}\Phi\left(1-\mathbf{E}\left[\theta(t+W+\xi)\right]\right) \tag{44}$$

and

$$\mathbf{E}\left[\prod_{t\in\mathbb{Z}}\theta(t)^{N(t)}\right]=\prod_{t\in\mathbb{Z}}\Phi\left(1-\mathbf{E}\left[\theta(t+W)\cdots\theta(t+W+\xi-1)\right]\right). \tag{45}$$

The proofs of (44) and (45) are given in Appendix C.

6. CONCLUSIONS

The discrete-time $G/GI/\infty$ queuing model—"fed" by *arbitrary input processes* X and with general IID service times—was studied. Focusing attention on the system's output process Y (of job departures) and queue process N (tracking the number of jobs present in the system), the stochastic maps $X \mapsto Y$ and $X \mapsto N$ were investigated. Closed-form formulas for various statistics of these stochastic maps were explicitly obtained, including means, variances, autocovariances, cross-covariances, and multidimensional probability generating functions.

The general results obtained were applied to the case of *stationary input processes*, facilitating a thorough L^2 analysis of stationary $G/GI/\infty$ systems: The spectral densities of the stochastic maps $X \mapsto Y$ and $X \mapsto N$ were computed, the fixed points (in L^2) of the map $X \mapsto Y$ were characterized, and, the display of either short-ranged or long-ranged temporal dependencies, by the output process Y and the queue process N, were precisely determined. Moreover, a general L^2 decomposition result, governing the intrinsic L^2 structure of stationary $G/GI/\infty$ systems, was obtained.

Notes

1. Regarding the notion of *regular variation* [1]: A real function ϕ is said to be regularly varying at the limit point l if the limit $\lim_{x\to l}\phi(\theta x)/\phi(x)$ exists for all positive constants θ. Theory shows that if the function ϕ is regularly varying, then $\lim_{x\to l}\phi(\theta x)/\phi(x)=\theta^{\varepsilon}$, where the exponent ε is a real parameter called the *exponent* of regular variation. Regularly varying functions are generalizations of power laws and play a key role in various fields of Probability Theory (see [1, Chap. 8]).

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APPENDIX A

This Appendix contains the proofs of the results stated in Section 3.

A.1. MEANS AND VARIANCES

Using the "integral representation" of the output process Y (Eq. (1)), the model assumptions, and the notation introduced in Eq. (3), we obtain the following:

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1. The *conditional* mean of $Y(t)$, *given* the input process X, is

$$\begin{aligned}\mathbf{E}[Y(t)|X] = & \ \mathbf{E}\left[\sum_{s\le t}\sum_{k=1}^{X(s)} \mathbf{I}\left\{\xi(s;k)=t-s\right\}|X\right] \\ &= \sum_{s\le t}\sum_{k=1}^{X(s)} \mathbf{E}[\mathbf{I}\left\{\xi(s;k)=t-s\right\}] \\ &= \sum_{s\le t}\sum_{k=1}^{X(s)} \mathbf{P}(\xi(s;k)=t-s) \\ &= \sum_{s\le t} g(t-s)X(s). \end{aligned} \quad \textbf{(A.1)}$$

2. The *conditional* variance of $Y(t)$, *given* the input process X, is

$$\begin{aligned}\mathrm{Var}[Y(t)|X] = \mathrm{Var}&\left[\sum_{s\le t}\sum_{k=1}^{X(s)} \mathbf{I}\{\xi(s;k)=t-s\}|X\right] \\ &= \sum_{s\le t}\sum_{k=1}^{X(s)} \mathrm{Var}[\mathbf{I}\left\{\xi(s;k)=t-s\right\}] \\ &= \sum_{s\le t}\sum_{k=1}^{X(s)} \mathbf{P}\left(\xi(s;k)=t-s\right)\mathbf{P}\left(\xi(s;k)\neq t-s\right) \\ &= \sum_{s\le t} g(t-s)\left(1-g(t-s)\right)X(s) \\ &= \sum_{s\le t} g(t-s)\overline{g}(t-s)X(s) \end{aligned} \quad \textbf{(A.2)}$$

(the transition from the first line to the second line of (A.2) is due to the fact that the service times $\xi(s;k)$ are IID random variables).

The conditional mean and variance of (A.1) and (A.2), in turn, imply that

$$\begin{aligned}\mathbf{E}[Y(t)] &= \mathbf{E}[\mathbf{E}\left[Y(t)|X\right]] \\ &= \mathbf{E}\left[\sum_{s\le t} g(t-s)X(s)\right] \\ &= \sum_{s\le t} g(t-s)\mathbf{E}[X(s)] \end{aligned} \quad \textbf{(A.3)}$$

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and

$$\begin{aligned}\mathrm{Var}[Y(t)] &= \mathrm{E}[\mathrm{Var}[Y(t)|X]] + \mathrm{Var}[\mathbf{E}[Y(t)|X]] \\ &= \mathbf{E}\left[\sum_{s\le t} g(t-s)\overline{g}(t-s)X(s)\right] + \mathrm{Var}\left[\sum_{s\le t} g(t-s)X(s)\right] \\ &= \sum_{s\le t} g(t-s)\overline{g}(t-s)\mathbf{E}[X(s)] \\ &\quad + \sum_{s_1\le t}\sum_{s_2\le t} g(t-s_1)g(t-s_2)\,\mathrm{Cov}[X(s_1),X(s_2)] \qquad \textbf{(A.4)}\end{aligned}$$

(since the input random variables $X(s)$ are dependent, the variance of their linear combination equals the double linear combination of their covariances).

The computation of the mean $\mathbf{E}[N(t)]$ and the variance $\mathrm{Var}[N(t)]$ follows analogously.

A.2. AUTOCOVARIANCES

Using the "integral representation" of the output process Y (Eq. (1)), the model assumptions, and the notation introduced in Eq. (3), we obtain that ($t_1 < t_2$) the *conditional* covariance of the random variables $Y(t_1)$ and $Y(t_2)$, *given* the input process X, is

$$\begin{aligned}&\mathrm{Cov}[Y(t_1), Y(t_2)|X] \\ &= \mathrm{Cov}\left[\sum_{s_1\le t_1}\sum_{i=1}^{X(s_1)} \mathbf{I}\{\xi(s_1;i)=t_1-s_1\}, \sum_{s_2\le t_2}\sum_{j=1}^{X(s_2)} \mathbf{I}\{\xi(s_2;j)=t_2-s_2\}\,|X\right] \\ &= \sum_{s_1\le t_1}\sum_{i=1}^{X(s_1)}\sum_{s_2\le t_2}\sum_{j=1}^{X(s_2)} \mathrm{Cov}\big[\mathbf{I}\{\xi(s_1;i)=t_1-s_1\}, \mathbf{I}\{\xi(s_2;j)=t_2-s_2\}\big] \\ &= \sum_{s\le t_1}\sum_{k=1}^{X(s)} \mathrm{Cov}[\mathbf{I}\ \{\xi(s;k)=t_1-s\}, \mathbf{I}\{\xi(s;k)=t_2-s\}] \\ &= -\sum_{s\le t_1}\sum_{k=1}^{X(s)} \mathbf{P}(\xi(s;k)=t_1-s)\,\mathbf{P}(\xi(s;k)=t_2-s) \\ &= -\sum_{s\le t_1}\sum_{k=1}^{X(s)} g(t_1-s)\,g(t_2-s) \\ &= -\sum_{s\le t_1} g(t_1-s)\,g(t_2-s)\,X(s) \qquad \textbf{(A.5)}\end{aligned}$$

(the transition from line three to line four of (A.5) is due to the fact that the service times are IID random variables; hence, the only nonzero covariances are those corresponding to $(s_1;i) = (s_2;j)$. The transition from line four to line five of (A.5) is due to the fact that the intersection of the events $\{\xi(s;k) = t_1 - s\}$ and $\{\xi(s;k) = t_2 - s\}$ is empty, hence, yielding only the negative term of the corresponding covariance).

The conditional covariance of (A.5), in turn, implies that

$$\mathbf{E}[\mathrm{Cov}[Y(t_1), Y(t_2)|X]] = -\sum_{s\le t_1} g(t_1 - s)\, g(t_2 - s)\, \mathbf{E}[X(s)]. \tag{A.6}$$

On the other hand, using the conditional mean $\mathbf{E}[Y(t) \mid X]$ of (A.1), we have

$$\begin{aligned}
&\mathrm{Cov}[\mathbf{E}[Y(t_1)|X], \mathbf{E}[Y(t_2)|X]] \\
&\quad = \mathrm{Cov}\left[\sum_{s_1\le t_1} g(t_1 - s_1)X(s_1), \sum_{s_2\le t_2} g(t_2 - s_2)X(s_2)\right] \\
&\quad = \sum_{s_1\le t_1}\sum_{s_2\le t_2} g(t_1 - s_1)g(t_2 - s_2)\,\mathrm{Cov}[X(s_1), X(s_2)].
\end{aligned} \tag{A.7}$$

Combining (A.6) and (A.7) together, we conclude that

$$\begin{aligned}
&\mathrm{Cov}[Y(t_1), Y(t_2)] \\
&\quad = \mathbf{E}[\mathrm{Cov}[Y(t_1), Y(t_2)|X]] + \mathrm{Cov}[\mathbf{E}[Y(t_1)|X], \mathbf{E}[Y(t_2)|X]] \\
&\quad = -\sum_{s\le t_1} g(t_1 - s)g(t_2 - s)\mathbf{E}[X(s)] \\
&\qquad + \sum_{s_1\le t_1}\sum_{s_2\le t_2} g(t_1 - s_1)g(t_2 - s_2)\,\mathrm{Cov}\,[X(s_1), X(s_2)].
\end{aligned} \tag{A.8}$$

The computation of the covariance $\mathrm{Cov}[N(t_1), N(t_2)]$ follows analogously.

A.3. CROSS-COVARIANCES

Using the "integral representation" of the output process Y (Eq. (1)) and of the queue process N (Eq. (2)), the model assumptions, and the notation introduced in Eq. (3), we obtain that ($t_1 \le t_2$) the *conditional* covariance of the random variables $Y(t_1)$ and

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$N(t_2)$, *given* the input process X, is

$$\begin{aligned}
&\mathrm{Cov}[Y(t_1), N(t_2)|X] \\
&= \mathrm{Cov}\left[\sum_{s_1 \le t_1} \sum_{i=1}^{X(s_1)} \mathbf{I}\{\xi(s_1;i) = t_1 - s_1\}, \sum_{s_2 \le t_2} \sum_{j=1}^{X(s_2)} \mathbf{I}\{\xi(s_2;j) > t_2 - s_2\} \,|X\right] \\
&= \sum_{s_1 \le t_1} \sum_{i=1}^{X(s_1)} \sum_{s_2 \le t_2} \sum_{j=1}^{X(s_2)} \mathrm{Cov}\big[\mathbf{I}\{\xi(s_1;i) = t_1 - s_1\}, \mathbf{I}\{\xi(s_2;j) > t_2 - s_2\}\big] \\
&= \sum_{s \le t_1} \sum_{k=1}^{X(s)} \mathrm{Cov}[\mathbf{I}\ \{\xi(s;k) = t_1 - s\}, \mathbf{I}\{\xi(s;k) > t_2 - s\}] \\
&= -\sum_{s \le t_1} \sum_{k=1}^{X(s)} \mathbf{P}\,(\xi(s;k) = t_1 - s)\,\mathbf{P}\,(\xi(s;k) > t_2 - s) \\
&= -\sum_{s \le t_1} \sum_{k=1}^{X(s)} g\,(t_1 - s)\,\overline{G}\,(t_2 - s) \\
&= -\sum_{s \le t_1} g\,(t_1 - s)\,\overline{G}\,(t_2 - s)\,X(s) \qquad \textbf{(A.9)}
\end{aligned}$$

(the transition from line three to line four of (A.9) is due to the fact that the service times are IID random variables; hence, the only nonzero covariances are those corresponding to $(s_1; i) = (s_2; j)$. The transition from line four to line five of (A.9) is due to the fact that the intersection of the events $\{\xi(s;k) = t_1 - s\}$ and $\{\xi(s;k) > t_2 - s\}$ is empty, hence yielding only the negative term of the corresponding covariance).

The conditional covariance of (A.9), in turn, implies that

$$\mathbf{E}[\mathrm{Cov}[Y(t_1), N(t_2)|X]] = -\sum_{s \le t_1} g\,(t_1 - s)\,\overline{G}\,(t_2 - s)\,\mathbf{E}[X(s)]. \qquad \textbf{(A.10)}$$

On the other hand, using the conditional mean $\mathbf{E}[Y(t) \,|\, X]$ of (A.1) and the analogous conditional mean $\mathbf{E}[N(t) \,|\, X]$ (in which $\overline{G}(t - s)$ replaces $g\,(t - s)$), we have

$$\begin{aligned}
&\mathrm{Cov}[\mathbf{E}[Y(t_1)|X], \mathbf{E}[N(t_2)|X]] \\
&= \mathrm{Cov}\left[\sum_{s_1 \le t_1} g(t_1 - s_1)X(s_1), \sum_{s_2 \le t_2} \overline{G}(t_2 - s_2)X(s_2)\right] \\
&= \sum_{s_1 \le t_1} \sum_{s_2 \le t_2} g(t_1 - s_1)\overline{G}(t_2 - s_2)\,\mathrm{Cov}[X(s_1), X(s_2)]. \qquad \textbf{(A.11)}
\end{aligned}$$

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Combining (A.10) and (A.11) together, we conclude that

$$\begin{aligned}
&\mathrm{Cov}[Y(t_1), N(t_2)] \\
&\quad = \mathbf{E}[\mathrm{Cov}[Y(t_1), N(t_2)|X]] + \mathrm{Cov}[\mathbf{E}[Y(t_1)|X], \mathbf{E}[N(t_2)|X]] \\
&\quad = -\sum_{s\le t_1} g(t_1 - s)\overline{G}(t_2 - s)\mathbf{E}[X(s)] \\
&\qquad + \sum_{s_1\le t_1}\sum_{s_2\le t_2} g(t_1 - s_1)\overline{G}(t_2 - s_2)\,\mathrm{Cov}[X(s_1), X(s_2)]. \qquad \textbf{(A.12)}
\end{aligned}$$

The computation of the cross-covariance $\mathrm{Cov}[N(t_1), Y(t_2)]$ $(t_1 < t_2)$ follows analogously.

A.4. PROBABILITY GENERATING FUNCTIONS

A.4.1. Multidimensional PGFs

Let $\{\theta(t)\}_{t\in\mathbb{Z}}$ be an arbitrary sequence of complex numbers taking values in the unit disk ($|\theta(t)| \le 1$).

Set

$$\alpha(s) := \mathbf{E}\left[\prod_{n=0}^{\infty}\theta(s+n)^{\mathbf{I}\{\xi=n\}}\right] = \mathbf{E}[\theta(s+\xi)] \qquad \textbf{(A.13)}$$

$(s \in \mathbb{Z})$.

Using the "integral representation" of the output process Y (Eq. (1)), the model assumptions, the notation introduced in Eq. (3), and (A.13), we obtain that the *conditional* multidimensional PGF of the output process Y, *given* the input process X, is

$$\begin{aligned}
&\mathbf{E}\left[\prod_{t\in\mathbb{Z}}\theta(t)^{Y(t)}|X\right] \\
&\quad = \mathbf{E}\left[\prod_{t\in\mathbb{Z}}\left\{\prod_{s\le t}\prod_{k=1}^{X(s)}\theta(t)^{\mathbf{I}\{\xi(s;k)=t-s\}}\right\}|X\right] \\
&\quad = \prod_{s\in\mathbb{Z}}\prod_{k=1}^{X(s)}\mathbf{E}\left[\prod_{t\ge s}\theta(t)^{\mathbf{I}\{\xi(s;k)=t-s\}}\right] \\
&\quad = \prod_{s\in\mathbb{Z}}\prod_{k=1}^{X(s)}\mathbf{E}\left[\prod_{n=0}^{\infty}\theta(s+n)^{\mathbf{I}\{\xi(s;k)=n\}}\right]
\end{aligned}$$

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$$
\begin{aligned}
&= \prod_{s\in\mathbb{Z}} \prod_{k=1}^{X(s)} \alpha(s) \\
&= \prod_{s\in\mathbb{Z}} \alpha(s)^{X(s)}. \qquad \textbf{(A.14)}
\end{aligned}
$$

The conditional multidimensional PGF of (A.14), in turn, yields

$$
\begin{aligned}
\mathbf{E}\left[\prod_{t\in\mathbb{Z}} \theta(t)^{Y(t)}\right] &= \mathbf{E}\left[\mathbf{E}\left[\prod_{t\in\mathbb{Z}} \theta(t)^{Y(t)} | X\right]\right] \\
&= \mathbf{E}\left[\prod_{s\in\mathbb{Z}} \alpha(s)^{X(s)}\right]. \qquad \textbf{(A.15)}
\end{aligned}
$$

The computation of the multidimensional PGF $\mathbf{E}\left[\prod_{t\in\mathbb{Z}} \theta(t)^{N(t)}\right]$ follows analogously by setting

$$
\beta(s) := \mathbf{E}\left[\prod_{n=0}^{\infty} \theta(s+n)^{\mathbf{I}\{\xi>n\}}\right] = \mathbf{E}[\theta(s)\theta(s+1)\cdots\theta(s+\xi-1)]. \qquad \textbf{(A.16)}
$$

A.4.2. One-Dimensional PGFs

We compute the PGF $\mathbf{E}[z^{Y(t)}]$ ($|z| \le 1$) of the random variable $Y(t)$ using the multidimensional PGF result of (A.15).

Set

$$
\theta(\tau) = 1 + (z-1)\delta(\tau - t) \qquad \textbf{(A.17)}
$$

($\tau \in \mathbb{Z}$) and note that

$$
\begin{aligned}
\alpha(s) &= \mathbf{E}[\theta(s+\xi)] \\
&= \mathbf{E}[1 + (z-1)\delta(s+\xi-t)] \\
&= 1 + (z-1)\mathbf{P}(\xi = t-s) \\
&= \overline{g}(t-s) + zg(t-s) \qquad \textbf{(A.18)}
\end{aligned}
$$

($s \in \mathbb{Z}$). Further, note that

$$
\begin{aligned}
\prod_{s\in\mathbb{Z}} \alpha(s)^{X(s)} &= \prod_{s\in\mathbb{Z}} \left(\overline{g}(t-s) + zg(t-s)\right)^{X(s)} \\
&= \prod_{n=1}^{\infty} \left(\overline{g}(n) + zg(n)\right)^{X(t-n)}. \qquad \textbf{(A.19)}
\end{aligned}
$$

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Equation (A.19), substituted into (A.15), yields the desired one-dimensional PGF

$$\mathbf{E}\left[z^{Y(\tau)}\right] = \mathbf{E}\left[\prod_{t\in\mathbb{Z}} \theta(t)^{Y(t)}\right] = \mathbf{E}\left[\prod_{s\in\mathbb{Z}} \alpha(s)^{X(s)}\right] = \mathbf{E}\left[\prod_{n=1}^{\infty} \left(\overline{g}(n) + zg(n)\right)^{X(t-n)}\right]. \quad \textbf{(A.20)}$$

The computation of the PGF $\mathbf{E}[z^{N(t)}]$ ($|z| \leq 1$) of the random variable $N(t)$ follows analogously.

APPENDIX B

This appendix contains the proofs of the results stated in Section 4.

B.1. MEANS AND AUTOCOVARIANCES

B.1.1. Means

Using Eqs (4) and (5) for the means of output process Y and the queue process N yields

$$\mathbf{E}[Y(t)] = \sum_{s\leq t} g(t-s)\mathbf{E}[X(s)] = \sum_{s\leq t} g(t-s)\mu_X = \mu_X \sum_{n=0}^{\infty} g(n) = \mu_X \quad \textbf{(B.1)}$$

and

$$\mathbf{E}[N(t)] = \sum_{s\leq t} \overline{G}(t-s)\ \mathbf{E}[X(s)] = \sum_{s\leq t} \overline{G}(t-s)\mu_X = \mu_X \sum_{n=0}^{\infty} \overline{G}(n) = \mu_X \mathbf{E}[\xi]. \quad \textbf{(B.2)}$$

B.1.2. Autocovariance: The Output Process Y

Setting $t_2 - t_1 = t \geq 1$ and using Eq. (8) for the autocovariance of the output process Y yields

$$\begin{aligned}
&\mathrm{Cov}[Y(t_1), Y(t_2)] \\
&\quad = -\sum_{s \leq t_1} g(t_1 - s) g(t_2 - s) \mu_X \\
&\qquad + \sum_{s_1 \leq t_1} \sum_{s_2 \leq t_2} g(t_1 - s_1) g(t_2 - s_2) R_X(s_2 - s_1) \\
&\quad = -\mu_X \sum_{k=1}^{\infty} g(k) g(k+t) + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} g(k) g(l) R_X(k - l + t).
\end{aligned} \tag{B.3}$$

On the other hand, if $t_2 = t_1$ then Eq. (6) for the variance of the output process Y yields

$$\begin{aligned}
&\mathrm{Cov}[Y(t_1), Y(t_2)] \\
&\quad = \mathrm{Var}[Y(t_1)] = \sum_{s \leq t_1} g(t_1 - s)\,(1 - g(t_1 - s))\,\mu_X \\
&\qquad + \sum_{s_1 \leq t_1} \sum_{s_2 \leq t_1} g(t_1 - s_1) g(t_1 - s_2) R_X(s_2 - s_1) \\
&\quad = \mu_X \sum_{k=1}^{\infty} g(k)\,(1 - g(k)) + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} g(k) g(l) R_X(k - l) \\
&\quad = \mu_X - \mu_X \sum_{k=1}^{\infty} g(k)^2 + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} g(k) g(l) R_X(k - l).
\end{aligned} \tag{B.4}$$

B.1.3. Conclusion: The Output Process Y

Combining (B.1), (B.3), and (B.4) together, we conclude that the output process Y is stationary with mean $\mu_Y = \mu_X$ and autocovariance function

$$\begin{aligned}
R_Y(t) = \mu_X \delta(t) - \mu_X \sum_{k=1}^{\infty} g(k) g(k + |t|) \\
+ \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} g(k) g(l) R_X(k - l + |t|)
\end{aligned}$$

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$$= \mu_X \delta(t) + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} g(k) g(l) \left(R_X - \mu_X \delta \right) (k - l + |t|) \qquad \textbf{(B.5)}$$

$(t \in \mathbb{Z})$.

Setting $R_X(t) = \mu_X \delta(t)$ in (B.4) yields Eqs (21) and (22) and, consequently, Eqs (28) and (29).

B.1.4. Autocovariance: The Queue Process N

The computation of the autocovariance function $R_N(t)$ of the queue process N follows analogously to the computation of $R_Y(t)$, yielding

$$R_N(t) = \mu_X \sum_{k=0}^{\infty} G(k) \overline{G}(k+t) + \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \overline{G}(k) \overline{G}(l) R_X(k - l + t) \qquad \textbf{(B.6)}$$

$(t \geq 0)$. However,

$$\sum_{k=0}^{\infty} G(k) \overline{G}(k+t) = \sum_{k=0}^{\infty} \left(1 - \overline{G}(k)\right) \overline{G}(k+t)$$

$$= \sum_{k=t}^{\infty} \overline{G}(k) - \sum_{k=0}^{\infty} \overline{G}(k) \overline{G}(k+t), \qquad \textbf{(B.7)}$$

Also, note that

$$\sum_{k=t}^{\infty} \overline{G}(k) = \sum_{k=0}^{\infty} \mathbf{I}\{k \geq t\} \mathbf{E}[\mathbf{I}\{\xi > k\}]$$

$$= \mathbf{E}\left[\sum_{k=0}^{\infty} \mathbf{I}\{\xi > k \geq t\}\right]$$

$$= \mathbf{E}\left[(\xi - t)_+\right]. \qquad \textbf{(B.8)}$$

B.1.5. Conclusion: The Queue Process N

Combining (B.2), (B.6), and (B.7) together, we conclude that the queue process N is stationary with mean $\mu_N = \mu_X \mathbf{E}[\xi]$ and autocovariance function

$$R_N(t) = \mu_X \sum_{k=|t|}^{\infty} \overline{G}(k) - \mu_X \sum_{k=0}^{\infty} \overline{G}(k) \overline{G}(k + |t|)$$

$$+ \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \overline{G}(k) \overline{G}(l) R_X(k - l + |t|)$$

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$$= \mu_X \sum_{k=|t|}^{\infty} \overline{G}(k) + \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \overline{G}(k)\overline{G}(l) \left(R_X - \mu_X \delta\right)(k - l + |t|) \qquad \textbf{(B.9)}$$

$(t \in \mathbb{Z})$.

Setting $R_X(t) = \mu_X \delta(t)$ in (B.9) yields Eqs. (23) and (24) and, consequently, Eqs. (31) and (32).

B.2. SPECTRAL DENSITIES

B.2.1. Spectral Density: The Base Case

We compute the spectral density $S(\omega)$ of the autocovariance function

$$R(t) = \sum_{k=|t|}^{\infty} \overline{G}(k). \qquad \textbf{(B.10)}$$

We begin with the following calculation:

$$\begin{aligned}
S_+(\omega) &:= \sum_{n=0}^{\infty} R(n) \exp\{i\omega n\} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=n}^{\infty} \overline{G}(k) \right) \exp\{i\omega n\} \\
&= \sum_{k=0}^{\infty} \left(\sum_{n=0}^{k} \exp\{i\omega n\} \right) \overline{G}(k) \\
&= \sum_{k=0}^{\infty} \frac{1 - \exp\{i\omega(k+1)\}}{1 - \exp\{i\omega\}} \overline{G}(k) \\
&= \sum_{k=0}^{\infty} \frac{1 - \exp\{i\omega(k+1)\}}{1 - \exp\{i\omega\}} \left(\sum_{n=k+1}^{\infty} g(n) \right) \\
&= \frac{1}{1 - \exp\{i\omega\}} \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n-1} (1 - \exp\{i\omega(k+1)\}) \right) g(n)
\end{aligned}$$

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$$= \frac{1}{1-\exp\{i\omega\}} \sum_{n=1}^{\infty} \left(n - \exp\{i\omega\} \frac{1-\exp\{i\omega n\}}{1-\exp\{i\omega\}} \right) g(n)$$

$$= \frac{1}{1-\exp\{i\omega\}} \mathbf{E}\left[\xi - \exp\{i\omega\} \frac{1-\exp\{i\omega\xi\}}{1-\exp\{i\omega\}} \right]$$

$$= \frac{1}{1-\exp\{i\omega\}} \mathbf{E}[\xi] - \frac{\exp\{i\omega\}}{(1-\exp\{i\omega\})^2} \left(1-\mathbf{E}\left[\exp\{i\omega\xi\}\right]\right). \tag{B.11}$$

Some elementary algebra further implies that

$$\operatorname{Re} S_+(\omega) = \frac{1}{2} \left(\mathbf{E}[\xi] + \frac{1-\mathbf{E}[\cos(\omega\xi)]}{1-\cos(\omega)} \right). \tag{B.12}$$

Now, using (B.12), while noting that $R(0) = \mathbf{E}[\xi]$, we conclude that

$$S(\omega) = \sum_{t\in\mathbb{Z}} R(t) \exp\{it\omega\}$$

$$= \sum_{n=0}^{\infty} R(n) \exp\{-i\omega n\} + \sum_{n=0}^{\infty} R(n) \exp\{i\omega n\} - R(0) = \overline{S_+(\omega)} + S_+(\omega) - \mathbf{E}[\xi]$$

$$= 2\operatorname{Re} S_+(\omega) - \mathbf{E}[\xi] = \frac{1-\mathbf{E}[\cos(\omega\xi)]}{1-\cos(\omega)}. \tag{B.13}$$

Equation (B.13), in turn, implies the transition from Eq. (24) to Eq. (25).

B.2.2. A General Spectral-Density Computation

Consider an autocovariance function of the form

$$R_Z(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} h(k)h(l) \left(R_X - \mu_X\delta\right)(k-l+|t|) \tag{B.14}$$

($t \in \mathbb{Z}$), where $\{h(n)\}_{n=0}^{\infty}$ is an arbitrary nonnegative sequence. Set

$$H(\omega) = \sum_{n=0}^{\infty} h(n) \exp\{i\omega n\} \tag{B.15}$$

($-\pi \le \omega < \pi$).

Using Eq. (17) and the fact that the spectral density corresponding to a delta-function autocovariance is unity [i.e., $\delta(t) = (1/2\pi) \int_{-\pi}^{\pi} \exp\{-it\omega\}\, d\omega$], we have

$$R_Z(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} h(k)h(l) \left(R_X - \mu_X\delta\right)(k-l+|t|)$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} h(k)h(l) \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\{-i(k-l+|t|)\omega\} \left(S_X(\omega) - \mu_X\right) d\omega$$

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$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\{-i|t|\omega\} \left(\sum_{k=0}^{\infty} h(k) \exp\{-ik\omega\} \right)$$
$$\times \left(\sum_{l=0}^{\infty} h(l) \exp\{il\omega\} \right) (S_X(\omega) - \mu_X)\, d\omega$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\{-i|t|\omega\} H(-\omega) H(\omega)\, (S_X(\omega) - \mu_X)\, d\omega$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\{-i|t|\omega\}\, |H(\omega)|^2\, (S_X(\omega) - \mu_X)\, d\omega$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\{-it\omega\}\, |H(\omega)|^2\, (S_X(\omega) - \mu_X)\, d\omega \tag{B.16}$$

(the last transition is due to the fact that the function $|H(\omega)|^2\,(S_X(\omega) - \mu_X)$ is symmetric). Equation (B.16) implies that the spectral density $S_Z(\omega)$ corresponding to the autocovariance function $R_Z(t)$ is given by

$$S_Z(\omega) = |H(\omega)|^2\, (S_X(\omega) - \mu_X) \tag{B.17}$$

$(-\pi \le \omega < \pi)$.

B.2.3. Spectral Density: The Process Y_{Fluc}

In the case of the process Y_{Fluc}, we have $h(n) = g(n)$ $(n \in \mathbb{N};\ h(0) = 0)$ (due to Eq. (29)) and hence $H(\omega) = \mathbf{E}\big[\exp\{i\omega\xi\}\big]$. Equation (B.17) thus implies that

$$S_{Y_{\text{Fluc}}}(\omega) = \left|\mathbf{E}\big[\exp\{i\omega\xi\}\big]\right|^2 \left(S_X(\omega) - \mu_X\right) \tag{B.18}$$

$(-\pi \le \omega < \pi)$.

B.2.4. Spectral Density: The Process N_{Fluc}

In the case of the process N_{Fluc}, we have $h(n) = \overline{G}(n)$ $(n \in \mathbb{Z}_+)$ (due to Eq. (32)) and hence

$$H(\omega) = \sum_{n=0}^{\infty} \overline{G}(n) \exp\{i\omega n\}$$
$$= \sum_{n=0}^{\infty} \left(\sum_{k=n+1}^{\infty} g(k) \right) \exp\{i\omega n\}$$
$$= \sum_{k=1}^{\infty} \left(\sum_{n=0}^{k-1} \exp\{i\omega n\} \right) g(k)$$

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$$= \sum_{k=1}^{\infty} \frac{1-\exp\{i\omega k\}}{1-\exp\{i\omega\}} g(k)$$
$$= \mathbf{E}\left[\frac{1-\exp\{i\omega\xi\}}{1-\exp\{i\omega\}}\right]$$
$$= \frac{1-\mathbf{E}\big[\exp\{i\omega\xi\}\big]}{1-\exp\{i\omega\}}. \quad \textbf{(B.19)}$$

Equation (B.17) thus implies that

$$S_{Y_{\text{Fluc}}}(\omega) = \left|\frac{1-\mathbf{E}\big[\exp\{i\omega\xi\}\big]}{1-\exp\{i\omega\}}\right|^2 \cdot \Big(S_X(\omega)-\mu_X\Big) \quad \textbf{(B.20)}$$

$(-\pi \le \omega < \pi)$.

B.2.5. Characterization of the Base Case

Combining (B.5), (B.18), and the fact that the spectral density corresponding to a delta-function autocovariance is unity (i.e., $\delta(t) = (1/2\pi)\int_{-\pi}^{\pi} \exp\{-it\omega\} d\omega$), we obtain that the spectral density of the output process Y is given by

$$S_Y(\omega) = \mu_X + \left|\mathbf{E}\big[\exp\{i\omega\xi\}\big]\right|^2 (S_X(\omega)-\mu_X) \quad \textbf{(B.21)}$$

$(-\pi \le \omega < \pi)$.

Since the mean μ_Y of the output process Y equals the mean μ_X of the output process X, these processes are equal (in L^2) if and only if their spectral densities coincide. Substituting $S_Y(\omega) = S_X(\omega)$ into (B.21) yields $S_X(\omega) = \mu_X$, which, in turn, is equivalent to $R_X(t) = \mu_X \delta(t)$ $(-\pi \le \omega < \pi; t \in \mathbb{Z})$.

APPENDIX C

This Appendix contains the proofs of the results stated in Section 5.

C.1. MULTIDIMENSIONAL PGFS

Let $\{\theta(t)\}_{t\in\mathbb{Z}}$ be an arbitrary sequence of complex numbers taking values in the unit disk $(|\theta(t)| \le 1)$.

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Note that the conditional-Poisson structure of the Cox input X implies that

$$\mathbf{E}\left[\prod_{t\in\mathbb{Z}}\zeta(t)^{X(t)}|\Lambda\right] = \prod_{t\in\mathbb{Z}}\mathbf{E}\big[\zeta(t)^{X(t)}|\Lambda(t)\big]$$
$$= \prod_{t\in\mathbb{Z}}\exp\{-(1-\zeta(t))\Lambda(t)\}$$
$$= \exp\left\{-\sum_{t\in\mathbb{Z}}(1-\zeta(t))\Lambda(t)\right\}, \qquad \textbf{(C.1)}$$

where $\{\zeta(t)\}_{t\in\mathbb{Z}}$ is an arbitrary sequence of complex numbers taking values in the unit disk ($|\zeta(t)| \le 1$).

Hence, Eq. (14) for the multidimensional PGF of the output process Y, combined with (C.1), yields

$$\mathbf{E}\left[\prod_{t\in\mathbb{Z}}\theta(t)^{Y(t)}\right] = \mathbf{E}\left[\prod_{t\in\mathbb{Z}}\alpha(t)^{X(t)}\right]$$
$$= \mathbf{E}\left[\mathbf{E}\left[\prod_{t\in\mathbb{Z}}\alpha(t)^{X(t)}|\Lambda\right]\right]$$
$$= \mathbf{E}\left[\exp\left\{-\sum_{t\in\mathbb{Z}}(1-\alpha(t))\Lambda(t)\right\}\right]$$
$$= \mathbf{E}\left[\exp\left\{-\sum_{t\in\mathbb{Z}}a(t)\Lambda(t)\right\}\right], \qquad \textbf{(C.2)}$$

where $a(t) = 1 - \alpha(t) = 1 - \mathbf{E}[\theta(t+\xi)]$.

Analogously, Eq. (15) for the multidimensional PGF of the queue process N, combined with (C.1), yields

$$\mathbf{E}\left[\prod_{t\in\mathbb{Z}}\theta(t)^{N(t)}\right] = \mathbf{E}\left[\exp\left\{-\sum_{t\in\mathbb{Z}}b(t)\Lambda(t)\right\}\right], \qquad \textbf{(C.3)}$$

where $b(t) = 1 - \beta(t) = 1 - \mathbf{E}[\theta(t)\theta(t+1)\cdots\theta(t+\xi-1)]$.

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C.2. THE EXAMPLE

Consider now the special case where the rate process Λ admits the moving-average representation of Eq. (42) and note that

$$\begin{aligned}\sum_{t\in\mathbb{Z}} \zeta(t)\Lambda(t) &= \sum_{t\in\mathbb{Z}} \zeta(t)\left(\sum_{s\in\mathbb{Z}} w(t-s)\eta(s)\right)\\ &= \sum_{s\in\mathbb{Z}}\left(\sum_{t\in\mathbb{Z}} \zeta(t)w(t-s)\right)\eta(s)\\ &= \sum_{s\in\mathbb{Z}}\left(\sum_{n\in\mathbb{Z}} \zeta(s+n)w(n)\right)\eta(s)\\ &= \sum_{s\in\mathbb{Z}} \mathbf{E}[\zeta(s+W)]\,\eta(s). \end{aligned} \tag{C.4}$$

Since $(\eta(t))_{t\in\mathbb{Z}}$ is a nonnegative "white noise" with Laplace transform $\Phi(\lambda) := \mathbf{E}\big[\exp\{-\lambda\eta(t)\}\big]$ ($\lambda \geq 0$, $t \in \mathbb{Z}$), (C.4) implies that

$$\begin{aligned}\mathbf{E}\left[\exp\left\{-\sum_{t\in\mathbb{Z}} \zeta(t)\Lambda(t)\right\}\right] &= \mathbf{E}\left[\exp\left\{-\sum_{t\in\mathbb{Z}} \mathbf{E}[\zeta(t+W)]\,\eta(t)\right\}\right]\\ &= \prod_{t\in\mathbb{Z}} \mathbf{E}\big[\exp\{-\mathbf{E}\,[\zeta(t+W)]\,\eta(t)\}\big]\\ &= \prod_{t\in\mathbb{Z}} \Phi\left(\mathbf{E}[\zeta(t+W)]\right), \end{aligned} \tag{C.5}$$

where $\{\zeta(t)\}_{t\in\mathbb{Z}}$ is an arbitrary sequence of complex numbers taking values in the unit disk ($|\zeta(t)| \leq 1$).

For the output process Y, we have $\zeta(t) = a(t)$ and, hence,

$$\begin{aligned}\mathbf{E}[\zeta(t+W)] &= \mathbf{E}[a(t+W)]\\ &= \mathbf{E}[1-\alpha(t+W)]\\ &= \mathbf{E}[1-\mathbf{E}[\theta(t+W+\xi)]]\\ &= 1-\mathbf{E}[\theta(t+W+\xi)]\,. \end{aligned} \tag{C.6}$$

Analogously, for the queue process N, we have $\zeta(t) = b(t)$ and, hence,

$$\mathbf{E}[\zeta(t+W)] = 1-\mathbf{E}[\theta(t+W)\cdots\theta(t+W+\xi-1)]\,. \tag{C.7}$$

Substituting (C.6) and (C.7) into equation (C.5)—while using, respectively, (C.2) and (C.3)—we conclude that

$$\mathbf{E}\left[\prod_{t\in\mathbb{Z}}\theta(t)^{Y(t)}\right]=\prod_{t\in\mathbb{Z}}\Phi\left(1-\mathbf{E}\left[\theta(t+W+\xi)\right]\right) \tag{C.8}$$

and

$$\mathbf{E}\left[\prod_{t\in\mathbb{Z}}\theta(t)^{N(t)}\right]=\prod_{t\in\mathbb{Z}}\Phi\left(1-\mathbf{E}\left[\theta(t+W)\cdots\theta(t+W+\xi-1)\right]\right). \tag{C.9}$$

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