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COMMUTING PROBABILITY OF COMPACT GROUPS

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Abstract

For any (Hausdorff) compact group G, denote by $\mathrm{cp}(G)$ the probability that a randomly chosen pair of elements of G commute. We prove that there exists a finite group H such that $\mathrm{cp}(G) = \mathrm{cp}(H)/|G : F|^2$, where F is the FC-centre of G and H is isoclinic to F with $\mathrm{cp}(F) = \mathrm{cp}(H)$ whenever $\mathrm{cp}(G) > 0$. In addition, we prove that a compact group G with $\mathrm{cp}(G) > \frac{3}{40}$ is either solvable or isomorphic to $A_5 \times Z(G)$, where A_5 denotes the alternating group of degree five and the centre $Z(G)$ of G contains the identity component of G.

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1. Introduction and results

Let G be a (Hausdorff) compact group. Then G has a unique normalised Haar measure which we denote by $\mathbf{m}_G$. The set $\mathcal{A}(G) := \{(x, y) \in G \times G \mid xy = yx\}$ is a measurable set (actually a closed set) of $G \times G$ and its measure in $G \times G$ will be denoted by $\mathrm{cp}(G)$. For G finite, $\mathrm{cp}(G)$ has been extensively studied (see [2, 7] and the references therein). It is well known that if G is finite, $\mathrm{cp}(G) \le \frac{5}{8}$ whenever G is nonabelian and this also holds for any nonabelian compact group [3]. The rationality of $\mathrm{cp}(G)$ is clear in the case of finite G and is proved for compact groups in [5, Theorem 3.11]. It is also shown in [5, Theorems 1.1 and 1.2] that if $\mathrm{cp}(G) > 0$, then $Z(\mathrm{FC}(G))$ is open in G. Here $Z(\mathrm{FC}(G))$ is the centre of the FC-centre $\mathrm{FC}(G)$ of G and $\mathrm{FC}(G)$ consists of the elements of G whose conjugacy classes are finite. In this case, the derived subgroup of $\mathrm{FC}(G)$ is also finite. We will use these results in what follows.

We prove a formula relating the commuting probability of compact groups to that of finite groups.

THEOREM 1.1. *For any compact group G there exists a finite group H such that* $\mathrm{cp}(G) = \mathrm{cp}(H)/|G : \mathrm{FC}(G)|^2$. *In particular, if* $\mathrm{cp}(G) > 0$*, the finite group H can be chosen so that it is isoclinic to* $\mathrm{FC}(G)$ *and* $\mathrm{cp}(\mathrm{FC}(G)) = \mathrm{cp}(H)$.

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Here we are using the usual convention that if $|G : \mathrm{FC}(G)|$ is infinite, then $1/|G : \mathrm{FC}(G)| = 0$. Following [4], two groups H and K are called isoclinic if there exist group isomorphisms $\alpha : H/Z(H) \to K/Z(K)$ and $\beta : H' \to K'$ such that $[x', y'] = [x, y]^\beta$ for all $x, y \in H$ and all $x' \in (xZ(H))^\alpha$ and $y' \in (yZ(H))^\alpha$. For a group T, we denote by $Z(T)$ and T' the centre and the derived subgroup of T, respectively.

Theorem 1.1 is a reduction result for the commuting probability $\mathrm{cp}(G)$ of a compact group G in terms of the commuting probability $\mathrm{cp}(H)$ of a finite group H, up to a multiplicative factor of the form $1/|G : \mathrm{FC}(G)|$. In particular, this yields the asymptotic behaviour of $\mathrm{cp}(G)$ close to zero when $\mathrm{FC}(G)$ is a large nonopen subgroup in a profinite group G.

In view of Theorem 1.1 it is possible to generalise results for commuting probability in the finite case to compact groups. As an example, we give the following result. The alternating group of degree five is denoted by A_5, as usual.

THEOREM 1.2. *Let G be a compact group.*

(1) *If* $\mathrm{cp}(G) > \frac{1}{4}$, *then both* G' *and* $G/Z(G)$ *are finite.*
(2) *If* $\mathrm{cp}(G) > \frac{3}{40}$, *then either* G *is solvable or* $G \cong A_5 \times Z(G)$ *and the identity component of* G *is contained in* $Z(G)$, *in which case* $\mathrm{cp}(G) = \frac{1}{12}$.

The second part of Theorem 1.1 has already been proved in [2, Theorem 12] for finite groups and we will use it in our proof for the compact case.

Some long-standing conjectures of Joseph on the number theoretical behaviour of the commutating probability in finite groups are resolved in [1]. A generalisation of the commuting probability to the probability that x^m and y^n commute in a compact group (for fixed positive integers m and n) is studied in [6].

2. Proof of the main results

PROOF OF THEOREM 1.1. Let $n = |G : Z|$ and $m = |F : Z|$, where F is the FC-centre of G and Z is its centre. By [5, Theorem 1.2], we may assume that $\mathrm{cp}(G) > 0$ so that n and m are both finite. Suppose that $F = \cup_{i=1}^{m} x_i Z$ and $G = \cup_{j=1}^{n} y_j Z$, where $x_i \in F$ and $y_j \in G$. Let $c_{\ell,k} = 1$ if $x_\ell x_k = x_k x_\ell$ and $c_{\ell,k} = 0$ otherwise.

We use the following observations in what follows.

(1) If $y \in G \backslash F$, then $\mathbf{m}_G(C_G(y)) = 0$, where $C_G(y) = \{a \in G \mid ay = ya\}$.
(2) By (1), if $y_j \in G \backslash F$, then $\mathbf{m}_G(x_i Z \cap C_G(y_j)) = 0$. If $y_j \in F$, we may assume, without loss of generality, that $y_j = x_\ell$ for some ℓ.
(3) We have $x_i Z \cap C_G(x_\ell) = x_i Z$ if $c_{\ell,i} = 1$ and $= \varnothing$ otherwise.
(4) For any $x \in F$,

$$C_G(x) \cap y_j Z = \begin{cases} y_j Z & \text{if } x \in C_G(y_j), \\ \varnothing & \text{otherwise.} \end{cases}$$

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Let $\mathbf{1}_{\mathcal{A}(G)}$ be the characteristic function of $\mathcal{A}(G) = \{(x, y) \in G \times G \mid xy = yx\}$. The first two equalities in the following calculation are from [3].

$$\begin{aligned}
\mathrm{cp}(G) &= \int_{G\times G} \mathbf{1}_{\mathcal{A}(G)}(x, y)\, d(x, y) = \int_G \mathbf{m}_G(C_G(x))\, dx \quad \text{(by Fubini's theorem)} \\
&\overset{(1)}{=} \int_F \mathbf{m}_G(C_G(x))\, dx \\
&= \sum_{i=1}^{m} \int_{x_iZ} \mathbf{m}_G(C_G(x))\, dx = \sum_{i=1}^{m}\sum_{j=1}^{n} \int_{x_iZ} \mathbf{m}_G(C_G(x) \cap y_jZ)\, dx \\
&\overset{(2)}{=} \sum_{i=1}^{m}\sum_{j=1}^{n} \int_{x_iZ\cap C_G(y_j)} \mathbf{m}_G(y_jZ)\, dx = \sum_{i=1}^{m}\sum_{j=1}^{n} \int_{x_iZ\cap C_G(y_j)} \mathbf{m}_G(Z)\, dx \\
&= \mathbf{m}_G(Z) \sum_{i=1}^{m}\sum_{j=1}^{n} \int_{x_iZ\cap C_G(y_j)} \mathbf{1}_{x_iZ\cap C_G(y_j)}(x)\, dx = \mathbf{m}_G(Z) \sum_{i=1}^{m}\sum_{j=1}^{n} \mathbf{m}_G(x_iZ \cap C_G(y_j)) \\
&\overset{(2)}{=} \mathbf{m}_G(Z) \sum_{i,j=1}^{m} \mathbf{m}_G(x_iZ \cap C_G(x_j)) \overset{(3)}{=} \mathbf{m}_G(Z) \sum_{i,j=1}^{m} \mathbf{m}_G(Z) c_{i,j} \\
&= \mathbf{m}_G(Z)^2 \sum_{i,j=1}^{m} c_{i,j} = \frac{1}{|G : Z|^2} \sum_{i,j=1}^{m} c_{i,j}.
\end{aligned}$$

The final equality, in particular, shows the rationality of $\mathrm{cp}(G)$. On the other hand,

$$\begin{aligned}
\mathrm{cp}(F) &= \mathbf{m}_{F\times F}\Big(\bigcup_{i,j=1}^{m} \{x_iZ \times x_jZ \mid x_ix_j = x_jx_i\}\Big) \\
&= \sum_{i,j=1}^{m} \mathbf{m}_{F\times F}(x_iZ \times x_jZ) c_{i,j} = \sum_{i,j=1}^{m} \mathbf{m}_F(Z)^2 c_{i,j} = \mathbf{m}_F(Z)^2 \sum_{i,j=1}^{m} c_{i,j}.
\end{aligned}$$

It follows that

$$\sum_{i,j=1}^{m} c_{i,j} = \frac{\mathrm{cp}(F)}{\mathbf{m}_F(Z)^2}. \tag{$*$}$$

Thus we also find the relation

$$\mathrm{cp}(G) = \frac{|F : Z|^2}{|G : Z|^2} \mathrm{cp}(F) = \frac{\mathrm{cp}(F)}{|G : F|^2}.$$

By [4, p. 135, paragraph 4], there exists a group H such that $Z(H) \le H'$ and H is isoclinic to F. Note that, since $F/Z(F)$ and F' are finite, H is a finite group. If $d_{i,j}$ are defined similarly to $c_{i,j}$ for H, then the isoclinism of H and F implies that

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$\sum_{i,j=1}^{m} c_{i,j} = \sum_{i,j=1}^{m} d_{i,j}$. The corresponding relation to (*) holds for H, and thus

$$\frac{\mathrm{cp}(F)}{\mathbf{m}_F(Z)^2} = \frac{\mathrm{cp}(H)}{\mathbf{m}_H(Z(H))^2}.$$

Since $\mathbf{m}_F(Z) = |F : Z|$ and $\mathbf{m}_H(Z(H)) = |H : Z(H)|$, the isoclinism again implies that $\mathrm{cp}(F) = \mathrm{cp}(H)$. This completes the proof of Theorem 1.1. □

PROOF OF THEOREM 1.2. By Theorem 1.1 there exists a finite group H isoclinic to $F := FC(G)$ such that $\mathrm{cp}(G) = \mathrm{cp}(H)/|G : F|^2$.

(1) Suppose that $\mathrm{cp}(G) > \frac{1}{4}$. If $G \neq F$, then $1/|G : F|^2 \leq \frac{1}{4}$ and so $\mathrm{cp}(H) > 1$, which is impossible. Thus $G = F$. From [5, Theorem 3.11], both G' and $G/Z(G)$ are finite.

(2) Suppose that $\mathrm{cp}(G) > \frac{3}{40}$. It follows that $\mathrm{cp}(H) > \frac{3}{40}$ and $|G : F| \in \{1, 2, 3\}$ so that $G' \leq F$. Now [2, Theorem 12] implies that H is solvable or $H \cong A_5 \times T$ for some abelian group T. If H is solvable, by isoclinism, $H' \cong F'$ and so F is solvable and so is G, since G/F is cyclic. Now assume that $H \cong A_5 \times T$ for some abelian group T so that $\mathrm{cp}(H) = \frac{1}{12}$. Now (1) implies that $G = F$. By isoclinism, $G/Z(G) \cong A_5$ and $G' \cong A_5$. Therefore $G = G'Z(G)$ and so $G \cong A_5 \times Z(G)$. Consider the identity component G_0 of G. Since G_0 is the largest connected closed subgroup of G containing the trivial element, the decomposition $G \cong A_5 \times Z(G)$ forces $Z(G) \supseteq G_0$. □

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