Counting divisors

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The function $\tau(n)$

The 'divisors' of n are the positive integers (including 1 and n itself) that divide into n. We will denote by $\tau(n)$ the number of these divisors. (This notation owes its origin to the German 'Teiler'; English speakers often use $d(n)$, but in my view the letter d is in far too much demand to be tied up in this way!)

Here we outline a number of results about $\tau(n)$, culminating in Dirichlet's splendid theorem on the cumulative sum of its values. For any readers with the appetite for it, we then go on to show how the methods extend to give analogous results about the number of ways of expressing a number as a product of three factors.

Every divisor j comes with a natural partner, its 'codivisor' n/j. Another way to describe $\tau(n)$ is: the number of ordered pairs (j, k) of positive integers with $jk = n$ (just think of j as defining the divisors in turn). Really τ is τ_2 in a sequence of functions τ_k; we return to this later.

First, a pleasant fact that follows at once from this pairing idea:

Proposition 1: $\tau(n)$ is odd if, and only if, n is a square.

Proof: If n is not a square, then all the divisors can be listed in pairs (j, k), with $j < \sqrt{n}$ and $jk = n$, so the number of them is even. If n is a square, say $n = m^2$, then the divisors consist of these pairs together with m, so the number is odd.

It is quite easy to give an expression for $\tau(n)$ in terms of the prime factorisation of n. Note first that if p is prime, then $\tau(p) = 2$ and $\tau(p^a) = a + 1$, since the divisors of p^a are $1, p, \ldots, p^a$. For a general number n, we have the following expression for $\tau(n)$:

Proposition 2: Suppose that $n > 1$, with prime factorisation $n = \prod_{j=1}^{m} p_j^{a_j}$. Then $\tau(n) = \prod_{j=1}^{m} (a_j + 1)$.

Proof: Because of unique prime factorisation, the divisors are the numbers $p_1^{b_1} p_2^{b_2} \ldots p_m^{b_m}$, where $0 \leqslant b_j \leqslant a_j$ for each j. For each j, there are $a_j + 1$ possible values for b_j, and each combination gives a different divisor. (The divisor 1 is obtained by choosing $b_j = 0$ for each j, and the divisor n by choosing $b_j = a_j$; it would be strange to exclude these two.)

For example, since $60 = 2^2 \times 3 \times 5$, we have $\tau(60) = 3 \times 2 \times 2 = 12$.

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As a further illustration, we will describe, in terms of the possible patterns of prime factors, the numbers for which $\tau(n) = 8$, and find the smallest such number. The factorisations of 8 are 8, 4×2 and $2 \times 2 \times 2$. So $\tau(n) = 8$ if n is one of the forms p^7, $p^3 q$, pqr (where p, q, r are distinct primes). The smallest of each type is: $2^7 = 128$, $2^3 \times 3 = 24$, $2 \times 3 \times 5 = 30$. So the smallest is 24. (The reader might like to repeat this exercise with a different value for $\tau(n)$, for example 12.)

Of course, Proposition 1 follows easily from Proposition 2: if n is a square, then each a_j is even, so $\prod_{j=1}^{m} (a_j + 1)$ is odd.

A way in which the divisor function seems to appear out of the blue is as follows. Recall that the Riemann zeta function is defined by $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ for real $s > 1$ (or indeed for complex s with $\mathrm{Re}\, s > 1$). So

$$\zeta(s)^2 = \left(\sum_{j=1}^{\infty} \frac{1}{j^s}\right)\left(\sum_{k=1}^{\infty} \frac{1}{k^s}\right).$$

In this product, consider the terms that equate to $1/n^s$ for a fixed n. There is such a term for each ordered pair (j, k) with $jk = n$, so $1/n^s$ occurs $\tau(n)$ times. Hence

$$\zeta(s)^2 = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}.$$

(Again, it is clear that 1 and n must be counted as divisors.)

This is actually a special case of the following. Given arithmetic functions $a(n)$, $b(n)$, the *convolution* $a * b$ is defined by

$$(a * b)(n) = \sum_{j \mid n} a(j)\, b\left(\frac{n}{j}\right) = \sum_{jk = n} a(j)\, b(k).$$

Clearly $\tau = u * u$, where u is the 'unit function' defined by $u(n) = 1$ for all n. Convolution defines the coefficients when two Dirichlet series are multiplied: if we write $\sum_{n=1}^{\infty} a(n)\, n^{-s} = F_a(s)$, and similarly $F_b(s)$, then $F_a(s)\, F_b(s) = F_{a*b}(s)$. Convolutions are very useful in number theory, but they will not be used in this article.

Summation of $\tau(n)$

Individual values of $\tau(n)$ fluctuate wildly. However, the variation is smoothed out when the cumulative sums of these values are considered, and in fact it is possible to give a very satisfactory estimate of such sums. For all real $x > 0$, write

$$T(x) = \sum_{n \leqslant x} \tau(n).$$

We do not restrict x to integer values: the advantage of this will be seen in the applications below. The notation $\sum_{n \leqslant x}$ means that summation is over the integers n such that $1 \leqslant n \leqslant x$. Of course, as a function of a real variable, $T(x)$ is constant between integers and jumps by $\tau(n)$ at the integer n.

There are two ways, both obvious, in which $T(x)$ can be described as enumerating ordered pairs of positive integers:

(a) the number of ordered pairs (j, n) with $j \mid n$ and $n \leqslant x$;

(b) the number of ordered pairs (j, k) with $jk \leqslant x$.

Geometrically, ordered pairs (j, k) of integers are called 'lattice points'. Note that (b) can be described as the number of lattice points in the (s, t)-plane lying below the hyperbola $st = x$.

Denote by $[x]$ the largest integer not greater than x, and write $\{x\}$ for $x - [x]$, the fractional part of x. Clearly $0 \leqslant \{x\} < 1$.

From (a), by a neat example of 'double counting', we have at once the following expression for $T(x)$:

Proposition 3:

$$T(x) = \sum_{j \leqslant x} \left[\frac{x}{j}\right]. \tag{1}$$

Proof: Consider the pairs in (a). For a fixed j (instead of fixed n), the values of n allowed are the multiples kj not greater than x, so that $k \leqslant x/j$. The number of such k is clearly $[x/j]$. The stated expression follows.

This expression gives a way to evaluate $T(x)$ without calculating individual values of $\tau(n)$. However, a better alternative will be described shortly.

To derive a formula approximating $T(x)$, we need an estimate of the harmonic sum

$$H(x) = \sum_{n \leqslant x} \frac{1}{n}.$$

As the reader may be aware, comparison with the integral $\int_1^x (1/t)\,dt = \ln x$ shows that $H(x)$ is roughly $\ln x$. For greater precision, and for later application, we reiterate here how such estimates work in general. The basic underlying result for a decreasing function is:

Lemma 1: Let $f(t)$ be a decreasing, non-negative function for $t \geqslant 1$, and let

$$S(x) = \sum_{n \leqslant x} f(n), \qquad I(x) = \int_1^x f(t)\,dt.$$

Then for all $x \geqslant 1$,

$$I(x) \leqslant S(x) \leqslant I(x) + f(1).$$

When x is an integer, this result is obtained by combining the obvious inequalities

$$f(r) \leqslant \int_{r-1}^{r} f(t)\, dt \leqslant f(r-1)$$

for $2 \leqslant r \leqslant x$. The version for non-integer x follows quite easily; see for example [1, p. 19].

Applied with $f(t) = 1/t$, Lemma 1 gives:

Lemma 2: For all $x > 1$,

$$\ln x \leqslant H(x) \leqslant \ln x + 1. \qquad (2)$$

The following estimate of $T(x)$ now drops into our lap.

Theorem 1: For all $x > 1$,

$$x \ln x - x \leqslant T(x) \leqslant x \ln x + x. \qquad (3)$$

Proof: Since $x - 1 \leqslant [x] \leqslant x$, (1) gives

$$\sum_{j \leqslant x} \left(\frac{x}{j} - 1\right) \leqslant T(x) \leqslant \sum_{j \leqslant x} \frac{x}{j},$$

which equates to

$$xH(x) - [x] \leqslant T(x) \leqslant xH(x).$$

Inserting (2), we obtain (3).

We now show how to obtain a much more accurate estimate. Both the result and the method were presented by Dirichlet in 1841. Peter Gustav Lejeune Dirichlet (1805-1859) grew up in the German Rhineland, in a family of mixed German and French origins. He taught for most of his life in Berlin, and married a sister of the composer Mendelssohn. He made it his mission to make the awe-inspiring works of Gauss better known and understood, but he also made important contributions of his own in several different areas of mathematics.

The key step is to replace (1) by the following expression for $T(x)$, which is known as 'Dirichlet's hyperbola identity':

Proposition 4: For all $x > 1$,

$$T(x) = 2 \sum_{j \leqslant \sqrt{x}} \left[\frac{x}{j}\right] - \left[\sqrt{x}\right]^2. \qquad (4)$$

Proof: In the expression (b), let N_1, N_2 be the number of pairs (j, k) with $j \leqslant \sqrt{x}$ and $k \leqslant \sqrt{x}$ respectively (in the diagram, these are the points in $A \cup B$ and $A \cup C$). By symmetry, $N_1 = N_2$. For fixed $j \leqslant \sqrt{x}$, the number of k such that $jk \leqslant x$ is $[x/j]$. Hence $N_1 = \sum_{j \leqslant \sqrt{x}} [x/j]$.

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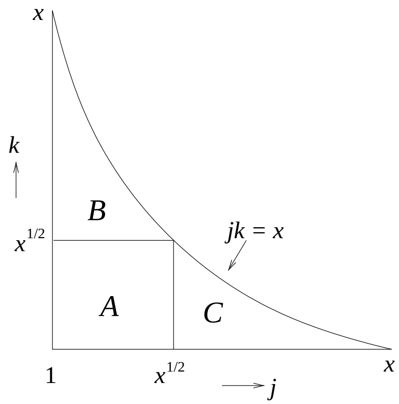

FIGURE 1

In the sum $N_1 + N_2$, the points satisfying both $j \leqslant \sqrt{x}$ and $k \leqslant \sqrt{x}$ are counted twice (in the diagram, these are the points in A). Hence $T(x) = 2N_1 - N_0$, where N_0 is the number of such points. Clearly $N_0 = [\sqrt{x}]^2$, so (4) follows.

Note that the summation in (4) is over the shorter range $j \leqslant \sqrt{x}$ instead of $j \leqslant x$. We illustrate this by using (4) to calculate $T(100)$. We tabulate the values of $[100/j]$ as follows:

j	1	2	3	4	5	6	7	8	9	10
$[100/j]$	100	50	33	25	20	16	14	12	11	10

Addition gives $\sum_{j=1}^{10} [100/j] = 291$. So $T(100) = 2 \times 291 - 10^2 = 482$. Using (1), we would have needed to consider $[100/j]$ for all $j \leqslant 100$.

At the same time, we deploy a better approximation than (2) for $H(x)$: such an approximation is $\ln x + \gamma$, where γ is Euler's constant. The exact statement, in the form that we require, is as follows; a detailed proof (for those who wish) can be seen in [1, pp. 24-25].

Lemma 3: For all $x \geqslant 1$

$$H(x) = \ln x + \gamma + q(x), \tag{5}$$

where $|q(x)| \leqslant 1/x$.

With (4) and (5), the path to Dirichlet's theorem is clear. It says:

Theorem 2: For all $x \geqslant 1$,

$$T(x) = x \ln x + (2\gamma - 1)x + \Delta(x), \tag{6}$$

where $|\Delta(x)| \leqslant 4\sqrt{x}$. In other words, $T(x)$ is approximated by $x \ln x + (2\gamma - 1)x$, with the error no greater than $4\sqrt{x}$.

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Proof: Let N_0, N_1 be as in Proposition 4. Then $N_0 = [\sqrt{x}]^2$. Since $\sqrt{x} - 1 \leqslant [\sqrt{x}] \leqslant \sqrt{x}$, we have $N_0 = x - q_1(x)$, where $0 \leqslant q_1(x) \leqslant 2\sqrt{x}$.

Recalling that $[x] = x - \{x\}$ and $0 \leqslant \{x\} < 1$, we have

$$N_1 = \sum_{j \leqslant \sqrt{x}} \left(\frac{x}{j} - \left\{ \frac{x}{j} \right\} \right) = xH(\sqrt{x}) - q_2(x),$$

where $0 \leqslant q_2(x) \leqslant \sqrt{x}$. By (5),

$$H(\sqrt{x}) = \tfrac{1}{2} \ln x + \gamma + q(\sqrt{x}),$$

where $|q(\sqrt{x})| \leqslant 1/\sqrt{x}$, so

$$xH(\sqrt{x}) = x\left(\tfrac{1}{2} \ln x + \gamma\right) + q_3(x),$$

where $|q_3(x)| \leqslant \sqrt{x}$. Put together, we obtain

$$T(x) = 2N_1 - N_0 = x \ln x + (2\gamma - 1)x + q_1(x) - 2q_2(x) + 2q_3(x).$$

Clearly $|q_1(x) - 2q_2(x)| \leqslant 2\sqrt{x}$, and hence $|\Delta(x)| \leqslant 4\sqrt{x}$.

Another striking example of the importance of Euler's constant!

Recall that the notation $O[g(x)]$ denotes a quantity $f(x)$ that satisfies $|f(x)| \leqslant Kg(x)$ for some constant K throughout the range of definition. In this notation, (6) can be stated as follows:

$$T(x) = x \ln x + (2\gamma - 1)x + O(\sqrt{x}).$$

The theorem can be interpreted as saying that $\tau(n)$ averages out as if it were $\ln 2 + 2\gamma$, because the method of integral estimation gives

$$\sum_{n \leqslant x} (\ln n + 2\gamma) = x \ln x + (2\gamma - 1)x + O(\ln x).$$

The following table compares some actual values of $T(x)$ with the estimate $x \ln x + (2\gamma - 1)x$.

x	100	1,000	10,000	100,000	1,000,000
actual	482	7,069	93,668	1,166,750	13,970,034
estimate	476	7,062	93,648	1,166,736	13,969,942

A table like this gives only a very partial picture, because the difference $\Delta(x)$ obviously has irregular fluctuations corresponding to those of $\tau(n)$, and can be positive or negative. To illustrate this, note that since $100{,}000 = 2^5 5^5$, we have $\tau(100{,}000) = 36$, so that $S_\tau(99{,}999) = 1{,}166{,}714$, while the estimate is 1,166,723. So $\Delta(99{,}999) \approx -9$ (to the nearest integer), while $\Delta(100{,}000) \approx 14$.

With this said, $\Delta(x)$ is visibly smaller than the stated estimate $4\sqrt{x}$ for the values listed. The problem of determining the true order of magnitude of $\Delta(x)$ is called the 'Dirichlet divisor problem'. It has been the subject of a great deal of study. The point of interest is not the factor 4 (which can in

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fact quite easily be reduced to 1) but the power of x. Denote by θ_0 the infimum of numbers θ such that $\Delta(x)$ is $O\left(x^{\theta}\right)$. It was already shown by Voronoi in 1903 that $\theta_0 \leqslant \frac{1}{3}$. A proof of this can be seen in [2, sect. 1.6.4]; it uses delicate estimates of 'exponential sums' of the form $\sum_{n \leqslant x} e^{if(n)}$. With very considerable effort, the estimate has been reduced slightly in a long succession of small steps, rather like the 10,000 metres record in athletics! The current record, held by M. N. Huxley, is $\theta_0 \leqslant \frac{131}{416} \approx 0.31490$. On the other hand, it was shown by Hardy and Landau in 1915 that θ_0 is at least $\frac{1}{4}$, and a bold conjecture is that this is the true value.

Combining Proposition 3 and Theorem 2, we can derive a rather striking application to sums of fractional parts (with the divisor function nowhere in sight):

Proposition 5: We have

$$\sum_{j \leqslant x} \left\{\frac{x}{j}\right\} = (1 - \gamma)x + \Delta_1(x).$$

where $\left|\Delta_1(x)\right| \leqslant 4\sqrt{x} + 1$.

Proof: Denote the sum by $V(x)$. By (1) and (5),

$$T(x) = \sum_{j \leqslant x} \left(\frac{x}{j} - \left\{\frac{x}{j}\right\}\right) = xH(x) - V(x)$$

$$= x\left(\ln x + \gamma\right) + xq(x) - V(x),$$

where $\left|xq(x)\right| \leqslant 1$. Now equating this to (6), we have

$$V(x) = (1 - \gamma)x - \Delta(x) + xq(x),$$

and the statement follows.

So the average of these fractional parts approximates to $1 - \gamma$; one might have expected it to approximate to $\frac{1}{2}$.

Expressions as products of three factors

We now widen our investigation to consider the number of ways of expressing n as a product of three factors. More exactly, define $\tau_3(n)$ to be the number of ordered triples (i, j, k) with $ijk = n$. We show how the previous methods and results extend quite naturally to this case.

For a prime p, we have $\tau_3(p) = 3$: the triples are $(p, 1, 1)$, $(1, p, 1)$ and $(1, 1, p)$.

Again, we will consider the cumulative sums: let $T_3(x) = \sum_{n \leqslant x} \tau_3(n)$. Clearly this is the number of (positive) triples (i, j, k) with $ijk \leqslant x$.

We can relate τ_3 and T_3 to $\tau\,(= \tau_2)$ and $T\,(= T_2)$ as follows.

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Proposition 6: We have

$$\tau_3(n) = \sum_{k \mid n} \tau(k), \tag{7}$$

$$T_3(x) = \sum_{k \leqslant x} T\left(\frac{x}{k}\right). \tag{8}$$

Proof: For a fixed divisor k of n, the number of triples with $ijk = n$ is the number of pairs (i, j) with $ij = n/k$, that is, $\tau(n/k)$. When k runs through the divisors of n, so does n/k. Hence (7).

For (8), take a fixed $k \leqslant x$. The number of triples with $ijk \leqslant x$ is the number of pairs (i, j) with $ij \leqslant x/k$. By (b), this is $T(x/k)$. Hence (8).

In terms of convolutions, (7) says that $\tau_3 = \tau * u = u * u * u$. Corresponding to Proposition 2, we have:

Proposition 7: Let n have prime factorisation $n = \prod_{j=1}^{m} p_j^{a_j}$. Then

$$\tau_3(n) = \prod_{j=1}^{m} \tfrac{1}{2}\left(a_j + 1\right)\left(a_j + 2\right),$$

Proof: First consider p^a for a prime p. By (7),

$$\tau_3\left(p^a\right) = \sum_{b=0}^{a} \tau\left(p^b\right) = \sum_{b=0}^{a} (b + 1) = \tfrac{1}{2}(a + 1)(a + 2).$$

When n is expressed as a product of three factors, these factors are of the form

$$\prod_{j=1}^{m} p_j^{b_j}, \qquad \prod_{j=1}^{m} p_j^{c_j}, \qquad \prod_{j=1}^{m} p_j^{d_j},$$

in which $b_j + c_j + d_j = a_j$ for each j. As just shown, for a fixed j, the number of choices of $\left(b_j, c_j, d_j\right)$ is $\frac{1}{2}(a + 1)(a + 2)$. These choices combine to give distinct factorisations of n, hence $\tau_3(n)$ is as stated.

Given this expression, the reader may care to try showing that $\tau_3(n)$ is a multiple of 3 unless n is a cube, in which case it is congruent to 1 mod 3.

Using (8) and Theorem 2, we can derive a corresponding estimate of $T_3(x)$. We now need an estimate of $\sum_{n \leqslant x} \left[(\ln n)/n\right]$. This is delivered by the following variant of Lemma 1, which is proved by a slight extension of the same method (for details, see [3, pp. 206-208]).

Lemma 4: Suppose that $f(t)$ is non-negative, increasing for $1 \leqslant t \leqslant x_0$ and decreasing for $x \geqslant x_0$, with maximum value $f\left(x_0\right) = M$. Define $S(x)$

and $I(x)$ as in Lemma 1. Then

$$I(x) - M \leqslant S(x) \leqslant I(x) + M.$$

Writing $\ln^2 x$ for $(\ln x)^2$, we deduce for our case:

Lemma 5: We have

$$\sum_{n \leqslant x} \frac{\ln x}{n} = \tfrac{1}{2} \ln^2 x + r(x), \tag{9}$$

where $|r(x)| \leqslant e^{-1}$.

Proof: The function $f(t) = (\ln t)/t$ increases for $1 \leqslant t \leqslant e$ and decreases for $t \geqslant e$, with $f(e) = e^{-1}$. Also, $\int_1^x f(t)\, dt = \tfrac{1}{2} \ln^2 x$.

Actually, $\sum_{n \leqslant x} [(\ln n)/n] - \tfrac{1}{2} \ln^2 x$ converges to the 'Stieltjes constant' $\gamma_1 \approx -0.072816$ as $x \to \infty$, but we don't need this.

Theorem 3: We have

$$T_3(x) = \tfrac{1}{2} \ln^2 x + (3\gamma - 1) x \ln x + O(x). \tag{10}$$

Proof: By (8) and (6),

$$T_3(x) = \sum_{n \leqslant x} \left[\frac{x}{n} (\ln x - \ln n) + (2\gamma - 1) \frac{x}{n} + \Delta\left(\frac{x}{n}\right) \right].$$

Write this as $J_1 - J_2 + J_3$, where

$$J_1 = x(\ln x + 2\gamma - 1) H(x),$$

$$J_2 = x \sum_{n \leqslant x} \frac{\ln n}{n},$$

$$J_3 = \sum_{n \leqslant x} \Delta\left(\frac{x}{n}\right).$$

We deal with J_2 and J_3 first. By (9),

$$J_2 = \tfrac{1}{2} x \ln^2 x + r_2(x),$$

where $|r_2(x)| \leqslant e^{-1} x$. Also,

$$|J_3| \leqslant 4 \sum_{n \leqslant x} \frac{\sqrt{x}}{\sqrt{n}}.$$

By Lemma 1, we see that $\sum_{n \leqslant x} (1/\sqrt{n}) \leqslant 2\sqrt{x}$. Hence $|J_3| \leqslant 8x$.

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Now consider J_1. Substituting for $H(x)$ by (5), we have

$$J_1 = x(\ln x + 2\gamma - 1)(\ln x + \gamma + q(x))$$
$$= x\left[\ln^2 x + (3\gamma - 1)\ln x + c\right] + r_1(x),$$

where $c = \gamma(2\gamma - 1)$ (so $c < \frac{1}{2}$) and $r_1(x) = xq(x)(\ln x + 2\gamma - 1)$. Since $|q(x)| \leqslant 1/x$ and $2\gamma - 1 < 1$, we have $|r_1(x)| \leqslant \ln x + 1$, hence $|r_1(x)| \leqslant x$. Combining the three pieces, we obtain (10), with the term '$O(x)$' no greater than $10x$.

With slightly less effort, we could have used (2) and (3), instead of (5) and (6), to derive the correspondingly less accurate estimate

$$T_3(x) = \tfrac{1}{2}x\ln^2 x + O(x\ln x).$$

Of course, $\tau_k(n)$ and $T_k(x)$ are defined similarly for all $k \geqslant 3$, enumerating k-tuples instead of triples. In the same way, one has

$$T_k(x) = \sum_{n \leqslant x} T_{k-1}\left(\frac{x}{n}\right),$$

and it can be shown by induction that

$$T_k(x) = \frac{x}{(k-1)!}\ln^{k-1} x + O\left(x\ln^{k-2} x\right).$$

The details can be seen in [3, chapters 6 and 7].

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