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TORSION UNITS IN INTEGRAL GROUP RINGS

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ABSTRACT. Special cases of Bovdi's conjecture are proved. In particular the conjecture is proved for supersolvable and Frobenius groups. We also prove that if $\exp(G/Z)$ is finite, $\alpha \in V\mathbb{Z}G$ a torsion unit and m the smallest positive integer such that $\alpha^m \in G$ then m divides $\exp(G/Z)$.

Let G be a group and let $V\mathbb{Z}G$ be the group of units of augmentation one of the integral group ring $\mathbb{Z}G$. Given an element $x = \sum x(g)g \in \mathbb{Z}G$ we set

$$T^{(k)}(x) = \sum_{g \in G(k)} x(g),$$

called the *k-generalized trace* of x. Here $G(k) = \{g \in G : o(g) = k\}$. We also set

$$\tilde{x}(g) = \sum_{h \sim g} x(h).$$

A. A. Bovdi proved the following [1]:

LEMMA 1. *If p is a prime, $x \in V\mathbb{Z}G$ and $o(x) = p^n$, then $T^{(p^n)}(x) \equiv 1 (\bmod p)$ and $T^{(p^j)}(x) \equiv 0 (\bmod p)$ for $j < n$. In particular there is an element $g \in G$ such that $o(x) = o(g)$.*

Considering these statements he conjectured that if x is as in Lemma 1 then

BC1: $T^{(p^n)}(x) = 1$ and $T^{(p^j)}(x) = 0$ for $j < n$.

In [4] **BC1** is proved for metabelian nilpotent groups and in [2] it is proved in general for nilpotent groups. Bovdi also conjectured the following [1]:

BC2: Let $n = \exp\big(G/Z(G)\big)$ be finite, where $Z(G)$ denotes the center of G. If $\alpha \in V\mathbb{Z}G$ is a torsion unit and m is the smallest positive integer such that $\alpha^m \in G$, then m divides n.

We recall that H. J. Zassenhaus had conjectured the following:

ZC1: Let G be a finite group and $\alpha \in V\mathbb{Z}G$ a torsion unit then α is conjugated in QG, to an element of G.

Lemma 1.1 below shows that **ZC1** implies **BC1**. In this paper we deal with the conjectures **BC1** and **BC2** and show that **BC1** holds for Frobenius groups and polycyclic groups whose commutator subgroup is nilpotent. In particular we re-obtain the result of [2] that **BC1** holds for nilpotent groups. Also, we show that **BC2** is true for all groups.

In the text we denote by δ_{nj} the Kronecker delta function which is 0 if $j \neq n$ and 1 if $j = n$.

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1. **Some technical lemmas.** First, we list some results which will be needed in our arguments.

LEMMA 1.1 [9, THEOREM 6]. *Let G be a finite group and $\alpha \in V\mathbb{Z}G$ a unit of finite order. Then $\beta^{-1}\alpha\beta \in G$ for some $\beta \in U(QG)$ if and only if there for every element γ in the subgroup generated by β there exist an element $g_0 \in G$, unique up to conjugacy, such that $\tilde{\gamma}(g_0) \neq 0$.*

LEMMA 1.2 [15, 41.12]. *Let $G = P \rtimes X$ where P is the Sylow p-subgroup of G. Let $H \subseteq U\big(1 + \Delta(G,P)\big)$ be finite. Then there exists $\alpha \in QG$ such that $H^{\alpha} \subseteq G$.*

LEMMA 1.3 [15, 47.5]. *Let G be a noetherian group and $u \in V\mathbb{Z}G$ a torsion element. Let $x \in G$ be of infinite order. Then $\tilde{u}(x) = 0$.*

We now prove some results that will be useful to produce an induction argument in the sequel.

LEMMA 1.4. *Let G be a finite group and $H \lhd G$ a normal subgroup of G. Let $\psi\colon \mathbb{Z}G \longrightarrow \mathbb{Z}(G/H)$ be the natural projection and $\alpha \in V\mathbb{Z}G$ such that $\big(o(\alpha), |H|\big) = 1$. If $\beta = \psi(\alpha)$ then $T^{(k)}(\alpha) = T^{(k)}(\beta)$ for every k such that $(k, |H|) = 1$ and $T^{(k)}(\alpha) = 0$ if $(k, |H|) \neq 1$.*

PROOF. Suppose that $(k, |H|) = 1$. Set

$$S = \big\{g \in G : o\big(\psi(g)\big) = k\big\}$$
$$S_1 = \{g \in S : o(g) > k\}$$

Note that if $g \in G$ is such that $\big(o(g), |H|\big) = 1$ then $o(g) = o\big(\psi(g)\big)$. Also if $\big(o(g), |H|\big) \neq 1$ then $\tilde{\alpha}(g) = 0$ by [9, Theorem 2.7]. Hence, $\tilde{\alpha}(g) = 0$ for all $g \in S_1$. Since S_1 is a normal subset of G we have that $\sum_{g\in S_1} \alpha(g) = 0$. Using these facts we have that:

$$T^{(k)}(\beta) = \sum_{o(\psi(g))=k} \alpha(g) = \sum_{g\in S} \alpha(g) = \sum_{o(g)=k} \alpha(g) + \sum_{g\in S_1} \alpha(g) = \sum_{o(g)=k} \alpha(g) = T^{(k)}(\alpha)$$

The second part follows by [9, Theorem 2.7] and the fact that $G(k)$ is a normal subset of G. ∎

LEMMA 1.5. *Let p be a prime and G a finite group. Suppose that G has a unique subgroup H of order p. Let $\alpha \in V\mathbb{Z}G$ be such that $o(\alpha) = p^n$. Then, with the notation of Lemma 1.4, we have that $T^{(p^{j+1})}(\alpha) = T^{(p^j)}(\beta)$ for $j \geq 1$ and $T^{(p^1)}(\alpha) \in \{0, 1\}$.*

In particular if **BC1** *holds for G/H then* **BC1** *holds for G.*

PROOF. Let $g \in G$ be an element of order p^{j+1}. If $j = 0$ then this is just Bermans' Lemma. So suppose that $j > 0$. Then $g^{p^j} \in H$, by the uniqueness of H. Hence $o\big(\psi(g)\big) = p^j$. Also if $o\big(\psi(g)\big) = p^j$ then $p^j \in H \setminus \{1\}$. Hence $o(g) = p^{j+1}$. Using these facts we have that

$$T^{(p^j)}(\beta) = \sum_{o(\psi(g))=p^j} \alpha(g) = \sum_{o(g)=p^{j+1}} \alpha(g) = T^{(p^{j+1})}(\alpha).$$

The second statement is a consequence of the first part and Lemma 1.4. ∎

LEMMA 1.6. *Let G be a noetherian group containing $H \lhd G$ with H torsion free. If $\alpha \in V\mathbb{Z}G$ is a torsion element then, with the notation of Lemma 1.4, we have that $T^{(k)}(\alpha) = T^{(k)}(\beta)$. In particular* **BC1** *holds for G if it holds for G/H.*

PROOF. Let $g \in G$ be an element of finite order. We set $\bar{g} = \psi(g)$ and $\bar{G} = \psi(G)$. Then, since H is torsion free, we have that $o(\bar{g}) = o(g)$. Hence we have that $\psi^{-1}\big(\bar{G}(k)\big) = G(k) \cup \{g \in G : o(g) = \infty, o(\bar{g}) = k\}$. Now $S = \{g \in G : o(g) = \infty, o(\bar{g}) = k\}$ is a normal subset of G and hence it is a disjoint union of conjugacy classes. So, by Lemma 1.3, $\sum_{g\in S} \alpha(g) = 0$ and hence we have that $T^{(k)}(\beta) = T^{(k)}(\alpha)$. ■

We now give a definition that will simplify some arguments we use in our proofs. Let G be a group and m a positive integer. We say that G is *m-absorbent* if the subgroup $\langle g \in G : o(g) \mid m^n \rangle$ has exponent divisible by m. If G is m-absorbent for all integers m then G is called *absorbent*. Clearly abelian groups, regular p-groups and K_8 are absorbent. Here K_8 denotes the quaternion group of order eight.

LEMMA 1.7. *Let G be group and $\alpha \in V\mathbb{Z}G$ an element such that $o(\alpha) = p^n$, p a prime. If G is (p,k)-absorbent for all $k \leq n$ then $T^{(p^j)}(\alpha) = \delta_{nj}$.*

PROOF. Since G is p^k-absorbent we have that $H_k = \{g \in G : o(g) \mid p^k\}$ is a normal subgroup of G. Consider the projection $\psi\colon \mathbb{Z}G \longrightarrow \mathbb{Z}(G/H_k)$. Since α is a torsion unit we have, by [14, III 1.3], that $\sum_{g\in H_k} \alpha(g) \in \{0,1\}$. Since $\sum_{g\in H_k} \alpha(g) = \sum_{0\leq j\leq k} T^{(p^j)}(\alpha)$ it follows that $\sum_{0\leq j\leq k} T^{(p^j)}(\alpha) \in \{0,1\}$ for all $0 \leq k \leq n$. Since [14, III 1.3] shows that $\alpha(1) \in \{0,1\}$ we have, inductively, that $T^{(p^j)}(\alpha) \in \{0,1\}$ for all $0 \leq j \leq n$. Lemma 1 now gives us the desired result. ■

The following result is well-known; we give its proof for the sake of completeness.

LEMMA 1.8. *Let H be an abelian Sylow p-subgroup of a finite solvable group G. Then one of the following holds:*

i) $H \lhd G$
ii) $O_{p'}(G) \neq 1$.

PROOF. Denote by F the Fitting subgroup of G. If F is a p-group then, since G is solvable and a Sylow p-subgroup of G is abelian we have, by [11, 5.4.4], that F is a Sylow p-subgroup of G. Since $F \lhd G$ we have that $H = F$.

If F is not a p-group we choose a prime $q \neq p$ and let N be a Sylow q-subgroup of F. Since $N \lhd F$ and F is characteristic, we obtain that $N \lhd G$ and the result follows. ■

LEMMA 1.9. *Let G be a group such that $\exp\big(G/\mathcal{Z}(G)\big)$ is finite. Let $\alpha \in V\mathbb{Z}G$ be a torsion unit and $\psi\colon \mathbb{Z}G \longrightarrow \mathbb{Z}\big(G/\mathcal{Z}(G)\big)$ the natural projection. Set $\beta = \psi(\alpha)$ and let m be the smallest positive integer such that $\alpha^m \in G$. If there exists an element $g \in G$ such that $o(\beta) = o\big(\psi(g)\big)$ then m is a divisor of $\exp\big(G/\mathcal{Z}(G)\big)$.*

PROOF. Let $k = o(\beta)$. Then by hypothesis we have that $k \mid \exp\big(G/\mathcal{Z}(G)\big)$. Also, $\alpha^k - 1 \in \Delta\big(G, \mathcal{Z}(G)\big)$. Since α is a torsion unit we have, by [2, Proposition 3], that $\alpha^k = g \in G$. By the minimality of m we must have that $m \mid k$ and hence $m \mid \exp\big(G/\mathcal{Z}(G)\big)$. ■

2. **BC1.** The following three results appeared in [6].

THEOREM 2.1. **BC1** *holds for any finite solvable group such that every Sylow subgroup of G is abelian.*

THEOREM 2.2. *Let G be a finite solvable group and* $\alpha \in V\mathbb{Z}G$ *an element of order* p^n. *Suppose that a Sylow p-subgroup of G is abelian. Then* $T^{(p^j)}(\alpha) = \delta_{nj}$.

THEOREM 2.3. *Let G be a finite solvable group and set* $L = \gamma_n(G)$ *as in the remark below. Furthermore, suppose that if a prime p is such that* $p \mid |L|$ *then* $p^4 \nmid |G|$. *Then* **BC1** *holds. In particular* **BC1** *holds if the order of G is not divisible by the fourth power of any prime.*

REMARK 2.4. Notice that if $\gamma_n(G)$ is the smallest nontrivial term of the lower central series of a group G, then the quotient $G/\gamma_n(G)$ is nilpotent and a result of A. Weiss [16] shows that **ZC1**, and hence **BC1**, holds for $G/\gamma_n(G)$. Thus Lemma 1.4 shows that $T^{(k)}(\alpha) \in \{0, 1\}$ for every $\alpha \in V\mathbb{Z}G$ such that $\big(o(\alpha), |\gamma_n(G)|\big) = 1$. In particular, by [4, pp. 431–433], there exist an element $g \in G$ such that $o(g) = o(\alpha)$. It also follows that **BC1** holds for finite solvable groups G such that if a prime p divides $|\gamma_n(G)$ then G contains a Sylow p-subgroup which is abelian, then **BC1** holds for G.

In this this section we shall prove the following:

THEOREM 2.5. **BC1** *holds for supersolvable groups.*

THEOREM 2.6. **BC1** *holds for finite Frobenius groups.*

If G is finite then Theorem 2.5 is a consequence of the following result.

THEOREM 2.7. *Let G be a finite group whose commutator subgroup is nilpotent. Then* **BC1** *holds for G.*

PROOF. Let G be a least counterexample to our statement and $\alpha \in V\mathbb{Z}G$ an element of order $o(\alpha) = p^n$. We first show that G' has to be a p-group. In fact if this is not true then, since G' is nilpotent, we may choose $H \lhd G$, $H \subset G'$, such that p does not divide $|H|$. Since G is a least counterexample we apply Lemma 1.4 to derive a contradiction. Hence G' is a p-group and thus G has a normal Sylow p-subgroup. It follows, by the Theorem of Schur-Zassenhaus [11, 9.1.2], that G is as in Lemma 1.1 and hence Lemma 1.1 and Lemma 1.2 give us that $\tilde{\alpha}(g_0) \neq 0$ for an element $g_0 \in G$ which is unique, up to conjugacy. Hence $T^{(p^j)}(\alpha) = \delta_{nj}$ by Lemma 1.2. So **BC1** holds for G, a final contradiction. ∎

PROOF OF THEOREM 2.5. Since G is supersolvable we have, by [11, 5.4.15], that G has a normal subgroup H, which is torsion free and of finite index. Hence G satisfies the condition of Lemma 1.6. Still by [11, 5.4.15], we have that G' is nilpotent. So the result follows from the previous theorem ∎

We now proceed towards the proof of Theorem 2.4. We shall first handle the case where G is solvable.

LEMMA 2.8. *Let G be a finite solvable group such that the Sylow subgroups of G are abelian or generalized quaternion groups. Then* **BC1** *holds for G.*

PROOF. If $p \neq 2$ then a Sylow p-subgroup of G is abelian and hence we may apply Theorem 2.2. So we need only to consider the case where $p = 2$. We use induction on $|G|$. Let $\alpha \in V\mathbb{Z}G$, be such that $o(\alpha) = 2^n$. By Theorem 2.2 we may suppose that a Sylow 2-subgroup of G is a generalized quaternion group. Assume first that $Fit(G)$ is not a 2-group. Then, it contains a subgroup H, of odd order, which is normal in G. Consider the projection $\psi: \mathbb{Z}G \longrightarrow \mathbb{Z}(G/H)$. Since G/H also satisfies the hypotheses of the theorem it follows, by induction, that **BC1** holds for G/H and, by Lemma 1.4, we have that $T^{(k)}(\alpha) = T^{(k)}(\beta)$.

So, we may suppose that $\mathrm{Fit}(G)$ is a 2-group. Since a Sylow 2-subgroup of G is a generalized quaternion group we have that either $\mathrm{Fit}(G)$ is cyclic or it is also a generalized quaternion group. Hence, by [11, p. 141], we have that either $\mathrm{Aut}\big(\mathrm{Fit}(G)\big)$ is a 2-group or it is isomorphic to S_4, where the last case occurs only if $\mathrm{Fit}(G) \cong K_8$. Recall that if H is a subgroup of G then the quotient group $N_G(H)/C_G(H)$ has a monomorphic image in $\mathrm{Aut}(H)$. Also, since G is solvable, it follows by [11, 5.4.4] that the centralizer of $\mathrm{Fit}(G)$ equals its centre. So if $\mathrm{Aut}\big(\mathrm{Fit}(G)\big)$ is a 2-group then, G is a 2-group and hence A. Weiss' result [16] applies. If $\mathrm{Fit}(G) \cong K_8$ then, $|G| = 48$. Set $H = \mathcal{Z}\big(\mathrm{Fit}(G)\big)$; then H is the unique subgroup of order 2 of G. By Theorem 2.3 the quotient group G/H satisfies **BC1**. Hence we may apply Lemma 1.5 to conclude that G satisfies **BC1**. ∎

If G is a finite solvable Frobenius group in Theorem 2.4, then the following result proves **BC1** for G.

LEMMA 2.9. *Let $G = A \rtimes X$, where A is nilpotent and* $(|A|, |X|) = 1$. *Suppose that* **BC1** *holds for X; then* **BC1** *also holds for G.*

PROOF. Let G be a least counterexample to the statement and $\alpha \in V\mathbb{Z}G$ an element of order $o(\alpha) = p^n$, p a prime. We first show that A has prime power order. In fact, suppose that two distinct primes divide $|A|$. Then we may choose a prime $q \neq p$ such that $q \bigm| |A|$. Let H be a Sylow q-subgroup of A; then $H \lhd G$. Consider the projection $\psi: \mathbb{Z}G \longrightarrow \mathbb{Z}(G/H)$ and set $\beta = \psi(\alpha)$. Then, by Lemma 1.4, we have that $T^{(p^j)}(\alpha) = T^{(p^j)}(\beta)$. Now, by the minimality of G, G/H satisfies **BC1** and hence we have a contradiction.

Now we shall show that the prime involved in $|A|$ is not p. In fact, if A is a p-group then, by our hypothesis, A is a Sylow p-subgroup of G and hence, by Lemma 1.2, **ZC1**, and hence **BC1**, holds, a contradiction.

So we must have that p divides $|X|$. In this case consider the projection $\psi: \mathbb{Z}G \longrightarrow \mathbb{Z}(G/A)$. Then, with the notation of Lemma 1.4, we have that $T^{(p^j)}(\alpha) = T^{(p^j)}(\beta)$. Since **BC1** holds for X, by our hypothesis, we have a final contradiction. ∎

PROOF OF THEOREM 2.6 (SOLVABLE CASE). By the results of Thompson and Burnside on finite Frobenius groups, [11, 10.5.6], we have that G is as in Lemma 2.9 and the Sylow subgroups of X are cyclic or generalized quaternion groups; hence, by Lemma 2.8, X satisfies **BC1**. The result then follows once more from Lemma 2.9. ∎

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Lemma 2.9 tells us that in order to prove the non-solvable case we only have to prove **BC1** for non-solvable Frobenius complements.

REMARKS. 1. In Lemma 2.8 we may change generalized quaternion by dihedral. The proof is the same if we use the classification of these groups [5, p. 462] and the classification of the groups of order 24 [3].

2. Let G be a group and p a prime such the Sylow p-subgroups of G are elementary abelian. Suppose that $\alpha \in \mathbb{Z}(G)$ is an element whose order is a power of p, say $o(\alpha) = p^n$. By [9, Theorem 2.7] we have that $\tilde{\alpha}(g) = 0$ if g is not a p-element. Since $\alpha(1) \in \{0, 1\}$ by [14, III.1.3], we have that $T^p \in \{0, 1\}$.

LEMMA 2.10. **BC1** *holds for* $G = \mathrm{SL}(2, 5)$.

PROOF. Let $G = \mathrm{SL}(2, 5)$. By [10, 18.6] we have that G is a Frobenius complement and hence a Sylow 2-subgroup of G is isomorphic to the quaternion group of order 8. Observe that $|G| = 120 = 2^3.3.5$. Hence, by item 2 of the remarks above, we may consider units $\alpha \in V\mathbb{Z}G$ such that $o(\alpha) = 2^n$. By the Theorem of Brauer-Suzuki [7, p. 102 Theorem 7.8], G has a unique subgroup H of order 2 and hence G/H has elementary abelian Sylow 2-subgroups. So Lemma 1.5 applies. ∎

We are now ready to prove:

LEMMA 2.11. *Let G be a non-solvable Frobenius complement. Then* **BC1** *holds for G.*

PROOF. By [10, 18.6] G has a normal subgroup H such that $H = \mathrm{SL}(2, 5) \times H_0$ where 2, 3 and 5 do not divide $|H_0|$ and hence all Sylow subgroup of H_0 must be cyclic, so H_0 satisfies **BC1**. Moreover we have that either:

i) $G = H$ or

ii) $[G : H] = 2$.

Note that if $p \in \{3, 5\}$ then a Sylow p-subgroup of G is elementary abelian and hence, as remarked above, we need only to consider units whose orders are powers of a prime p, with p distinct from 3 or 5. Recall also that a Sylow 2-subgroup of G is a generalized quaternion group so the Theorem of Brauer-Suzuki, [7, p. 102 Theorem 7.8], applies. We now discuss the two cases, mentioned above, separately.

CASE 1: $G = H$. In this case we may apply the Lemmas 1.4, 2.8 and 2.10 to obtain the result.

CASE 2: $[G : H] = 2$. Let $\alpha \in V\mathbb{Z}G$ be a torsion element such that $o(\alpha) = p^n$. We discuss two sub-cases.

CASE (i): $p \neq 2$. Note that $\mathrm{SL}(2, 5) \lhd G$, hence we may apply Lemma 1.4, with $H = \mathrm{SL}(2, 5)$, and then Lemma 2.8 to obtain that $T^{(p^j)}(\alpha) = \delta_{nj}$.

CASE (ii): $p = 2$. Note that $H_0 \lhd G$. Consider the quotient group $\bar{G} = G/H_0$. Then $|\bar{G}| = 240$. Now $\bar{G}$ has a unique subgroup of order 2, say H_1. So we may apply Lemma 1.5 for $\bar{G}$ and H_1. The quotient group, $\bar{G}/H_1$ is non-solvable, of order 120 and hence must be S_5, for which **ZC1** holds, (see [8]). So, by Lemma 1.5, we have that **BC1** holds for G/H_0. Hence, applying Lemma 1.4 for G and H_0, we obtain that $T^{(2^j)}(\alpha) = \delta_{nj}$. ∎

PROOF OF THEOREM 2.6 (NON-SOLVABLE CASE). The proof is the same as in the solvable case, using Lemma 2.11 instead of Lemma 2.8. ∎

The same proof of Lemma 2.8 together with Lemma 1.4 and the Remark 2.4 give us the following result:

THEOREM 2.12. *Let G be a finite solvable group such that if a prime p divides $|\gamma_n(G)|$ then a Sylow p-subgroup of G is cyclic or a generalized quaternion group. Then* **BC1** *holds for G.*

3. BC2.

In this section we shall prove that **BC2** holds. Some partial results already appeared in [6].

THEOREM 3.1. *Let $n = \exp\big(G/\mathcal{Z}(G)\big)$ be finite, where $\mathcal{Z}(G)$ denotes the center of G. If $\alpha \in V\mathbb{Z}G$ is a torsion unit and m is the smallest positive integer such that $\alpha^m \in G$, then m divides n, i.e.,* **BC2** *holds.*

PROOF. Let $\alpha \in V\mathbb{Z}G$ be a torsion unit. Write $o(\alpha) = p_1^{r_1}\cdots p_n^{r_n}$. Let $m_i = \prod_{j\neq i} p_i^{r_j}$ and set $\alpha_i = \alpha^{m_i}$. Then $o(\alpha_i) = p_i^{r_i}$. Denote by k_i the smallest positive integer such that $\alpha_i^{k_i} \in G$. Then, by Lemma 1 and Lemma 1.9, we have that $k_i | \exp\big(G/\mathcal{Z}(G)\big)$. Since the orders of the α_i are relatively prime, it follows that $k = \prod k_i$ divides $\exp\big(G/\mathcal{Z}(G)\big)$. Since $(m_1,\ldots,m_n) = 1$ we may choose integers $c_1,\ldots,c_n \in \mathbb{Z}$ such that $c_1m_1+\cdots+c_nm_n = 1$. So we have that $\alpha = \prod(\alpha_i)^{c_i}$. Thus $\alpha^k \in G$ and hence $m|k$. Consequently we have that $m|\exp\big(G/\mathcal{Z}(G)\big)$. ∎

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