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Analysis of thermostat models

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We present two new models describing the dynamic behavior of an automotive thermostat, involving delay-differential equations with hysteresis. Existence, uniqueness, and regularity of the solutions for both models are obtained by a continuation argument. We establish sufficient conditions for the models to exhibit intrinsic oscillations. We also present an algorithm for numerical approximations of the solutions and give some representative numerical simulations. These reveal a rather interesting dynamical behavior of the solutions.

1 Introduction

We present mathematical analysis of two new models for automotive thermostats. Each of the models is in the form of a delay-differential equation and a functional relation which represents the hysteresis behaviour of the system. The main interest of our investigation is to understand the oscillatory behaviour of the solutions to these models. These represent intrinsic oscillations which are observed in real systems. Understanding the source of such oscillations is the first step in controlling them, possibly reducing or removing them completely. In automotive applications, temperature oscillations are undesirable, and, by controlling them, better cooling systems may be designed. In this paper we obtain sufficient conditions for all the solutions to oscillate.

Thermostats in cars are devices that control the operating temperature of the engine. They are set to adjust the cooling so that an essentially constant and optimal operating temperature exists in the engine. The thermostat senses the coolant temperature and sends a larger or smaller flow of coolant through the radiator. In this way it keeps the coolant temperature at an almost constant value under normal operating conditions. They are very common devices, one per engine, and have been in operational use for many years; yet, there is little literature on their dynamic behaviour. Although they are conceptually simple, their dynamic behaviour is not, since they exhibit *hysteresis*, (see, e.g., [1]), i.e. the way they open when the temperature rises differs from the way they close when the temperature falls.

Hysteresis has recently received considerable attention in the mathematical literature, (see, e.g., [1–4] and the references there). Topics of delay-differential equations are well known and under current research. For reference, see for example References [5–10]. The mathematical novelty in this paper lies in the combination of the two. Moreover, our main interest lies in identifying the conditions for all the solutions to be oscillatory.

New models for the dynamic behaviour of automotive engine outlet or inlet thermostats, using systems of delay differential equations with hysteresis, have been derived [11, 12].

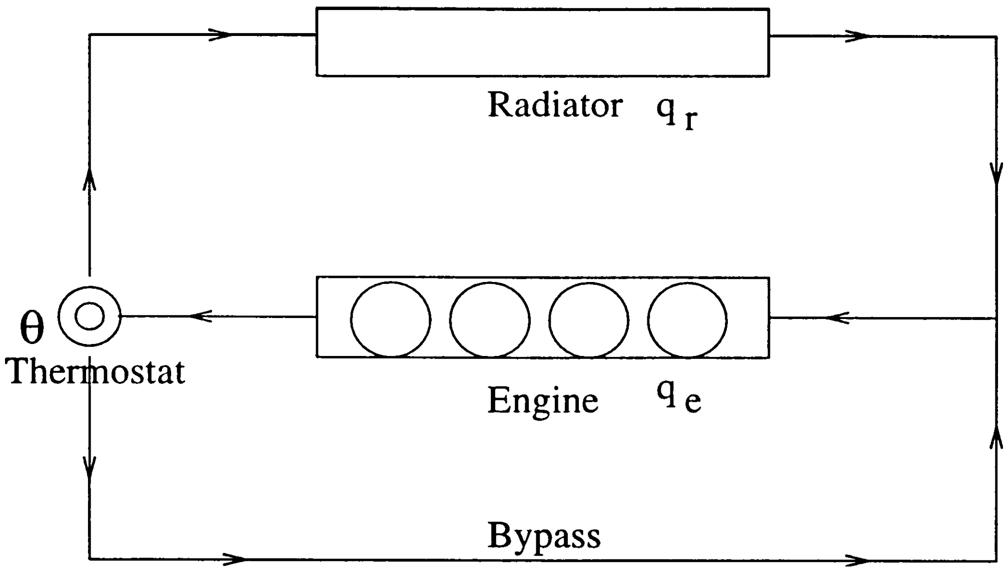

FIGURE 1. Schematic setting of the cooling system, thermostat, radiator, bypass, and engine.

Here we derive simpler models, which may be viewed as approximations, to better understand mathematically the interplay between hysteresis and delay.

We wish in particular to investigate the dependence of the appearance of oscillations on the hysteresis curves of the thermostat. It turns out that the mathematical analysis is fairly involved since we have both a delay and hysteresis in the model and their interplay is what characterizes the system behaviour.

We will employ Euler's explicit method to construct a numerical algorithm for solving the two models. Although we have not established any convergence results for the method, the numerical simulations indicate that it is well behaved. The simulations themselves give a very strong support for using both models to gain insight and understanding of the relationship among hysteresis, delays and oscillations.

The paper is organized as follows. The two models are derived in § 2. Both are in the form of a delay differential equation for the system temperature, and a functional relation for the hysteresis. All our mathematical results are summarized in § 3. The proofs of the theorems are given in § 4. In § 5 we present a numerical algorithm and some representative numerical solutions to both models. We conclude the paper in § 6.

2 The two models

In this section we present two versions of a model for the dynamic evolution of a thermostat. Other models can be found in [11, 12]. Here, as was mentioned above, our interest is in a simple setting so we can concentrate on the interplay between the delay and the hysteresis behaviour.

We consider a setting, depicted in Figure 1, where a thermostat is situated in the cooling loop at the outlet of the engine. Heat is being generated by the engine and carried by the coolant to the radiator or the bypass. The coolant from these two returns to the

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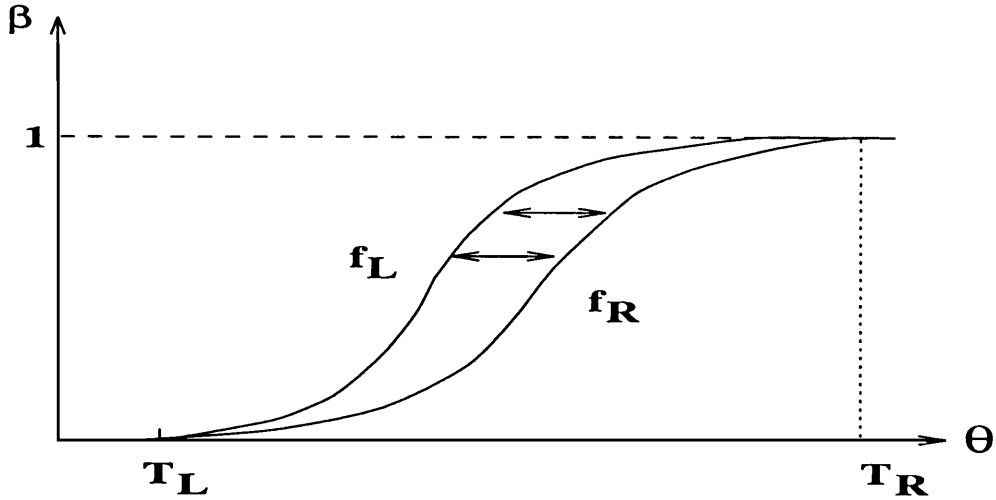

FIGURE 2. The hysteresis curves of β.

engine, the part which passed through the radiator being colder. In this manner the engine operating temperature is being controlled with the aim of having a steady and specified temperature. The model is based on many simplifying assumptions. We deal only with the thermal aspects of the situation by assuming that the coolant flow is known. We do not take into account any spatial behaviour, thus considering the various system elements as 'lumped masses'. We assume that the temperature of the fluid reaching the thermostat is the same as that of the thermostat itself, denoted by $\theta = \theta(t)$. The fractional thermostat opening is denoted by $\omega = \omega(t)$. The engine heat generation is denoted by q_e and the cooling power of the radiator by q_r, both assumed to be positive constants. Some of the results below hold when these are known functions of time. For convenience, we set the thermal capacity of the coolant and the total flow rate to be one.

We next describe the way the thermostat operates. When the engine is running, say starting from cold, it produces heat, part of which is carried away by the coolant. When the coolant reaches a prescribed temperature T_L, the thermostat starts opening, and a fraction ω of the flow is diverted to the radiator where it is cooled. The remaining fraction, $1-\omega$, flows via the bypass, without changing its temperature. At the entrance to the engine both flows merge again. If the engine outlet temperature, which is taken as the thermostat temperature, continues to rise, the thermostat continues to open until fully open, $\omega = 1$, when it reaches the temperature T_R. Then the full flow is via the radiator and the bypass is closed. When the engine coolant temperature falls below T_R, the thermostat starts closing and is fully closed when the temperature is below T_L.

We require that the fractional opening $\omega = \omega(t)$ satisfy

$$0 \leqslant \omega \leqslant 1.$$

When $\omega = 0$ the path to the radiator is closed, when $\omega = 1$ all the coolant flows through the radiator, and when $0 < \omega < 1$ the path to the radiator is partially open.

Our interest lies in the fact that the way the thermostat opens, as the temperature rises, is different from the way it closes, when the temperature drops. This is the *hysteresis*

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behaviour, which is depicted in Figure 2 where the *hysteresis graph* β is shown. Therefore, we require that the function $\omega = \omega(t)$ be a selection from the graph β. In turn β depends on the temperature θ and on the temperature rate of change $\dot{\theta} = \mathrm{d}\theta/\mathrm{d}t$; that is,

$$\beta = \beta(\theta, \dot{\theta}).$$

Actually, β depends only on the sign of $\dot{\theta}$.

Hysteresis affects the dynamic behaviour as follows. When the initial state of the system $\{\theta, \omega\}$ is on the curve f_R and the temperature is rising, i.e. $\dot{\theta} > 0$, the system will continue moving along the curve f_R. When the system is on the curve f_L and the temperature is decreasing, i.e. $\dot{\theta} < 0$, the system will continue moving along the curve f_L. When the temperature rises with the system on f_R and then reverses at a temperature between T_L and T_R, the system will move along the straight horizontal segment that connects f_R with f_L to the curve f_L at constant valve opening ω. Then it will continue moving down on f_L. The assumption that the motion is on horizontal segments represents the so-called 'generalized play' model (see, e.g., [1]). Other choices of families of curves that fill the area between f_L and f_R, the *hysteresis region* $\mathscr{H}$, are possible, leading to different models and possibly to different types of behaviour. Based on the description above we denote by H_β the 'generalized play hysteresis' operator so that for a given temperature function θ the valve opening is $w(t) = H_\beta(\theta(t))$. A more detailed description of H_β can be found in §4.

We employ energy balance considerations to model the dynamics of the system. The rate of change of the total energy of the coolant is

$$\text{rate of change} = cv\frac{\mathrm{d}\theta}{\mathrm{d}t},$$

where c is the total heat capacity of the coolant and v is the volume flow rate. Below we set both equal to one by rescaling the variables. The engine energy rate of heating is vq_e, and the radiator's cooling power, i.e. the rate of energy loss, is $\omega v q_r$.

Next we address the issue of the delay that exists in the system. Let τ be the time of flow from the radiator to the engine, assumed to be constant. We neglect the flow time from the engine to the thermostat. Then the cooling effect at the engine is felt τ units of time later, which means that at time t the cooling effect of the radiator is that of the fluid that was there at $t - \tau$.

Combining all the above yields the first model.

First model for the thermostat. Find a pair $\{\theta, \omega\}$ such that

$$\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} = q_e - q_r\omega(t-\tau) \qquad 0 \leqslant t, \tag{2.1}$$

$$\omega(t) = H_\beta(\theta(t)) \qquad -\tau \leqslant t. \tag{2.2}$$

To complete the model we have to impose the initial conditions

$$\theta(t) = \theta_0(t) \quad \text{for} \quad -\tau \leqslant t \leqslant 0, \tag{2.3}$$

$$\omega(-\tau) = \omega_0. \tag{2.4}$$

Here, due to the delay, the initial condition for the temperature over the interval $[-\tau, 0]$ is needed, which is typical for problems with delays. We assume that β is specified, which in the 'play model' means that f_L and f_R are given. In applications, these curves should be found experimentally. Moreover, we must supply the initial value ω_0 that is compatible with β and $\theta_0(-\tau)$.

The model consists of these three elements: the energy equation, the initial conditions, and the hysteresis domain and curves. From the mathematical point of view, we consider a system of differential equations (2.1) and (2.3) where the right-hand side is chosen from an infinite family of curves.

We assume that $q_e < q_r$, since otherwise the system does not have enough cooling power and thus the engine temperature will grow without bound as time goes on.

In the first model it was assumed that the cooling power of the radiator is fixed. In a real radiator it depends on the air flow and the air ambient temperature T_{amb}. For this reason, we assume that the cooling power is given by Newton's law of cooling

$$q_r = h(\theta - T_{\text{amb}}),$$

where h is the radiator coefficient of heat exchange. Substituting this expression in equation (2.1) and choosing T_{amb} as the reference temperature for the rescaled temperature yields the second model which we shall consider. We assume, as above, that $q_e < hT_R$, to guarantee enough cooling power. Also, we assume that $T_{\text{amb}} < T_L$, which is always satisfied in applications, so $0 < T_L$ after rescaling.

Second thermostat model. Find a pair $\{\theta, \omega\}$ such that

$$\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} = q_e - h\omega(t-\tau)\theta(t-\tau) \qquad 0 \leqslant t, \tag{2.5}$$

$$\omega(t) = H_\beta(\theta(t)) \qquad -\tau \leqslant t, \tag{2.6}$$

$$\theta(t) = \theta_0(t) \quad \text{for } -\tau \leqslant t \leqslant 0, \tag{2.7}$$

$$\omega(-\tau) = \omega_0. \tag{2.8}$$

The meaning of the initial conditions is the same as above.

Remark The hysteresis conditions (2.2) or (2.6) may be written in the form of a *variational inequality*, (see, e.g., [4, p. 65]). To this end we define, for each $\theta \in \mathbb{R}$, the vertical interval

$$J(\theta) = [f_R(\theta), f_L(\theta)], \tag{2.9}$$

and let $I_{J(\theta)}$ be the indicator function of this interval. Thus, $I_{J(\theta)}(\omega) = 0$ if $\omega \in J(\theta)$ and $I_{J(\theta)}(\omega) = +\infty$ if $\omega \notin J(\theta)$.

Then (2.2) or (2.6) may be written as the variational inequality

$$\omega(t) \in J(\theta), \quad \frac{\mathrm{d}\omega}{\mathrm{d}t}(\omega(t) - \xi) \leqslant 0 \quad \text{for all } \xi \in J(\theta(t)), \tag{2.10}$$

for each $0 \leqslant t$. Equivalently, we may write it as

$$\frac{\mathrm{d}\omega(t)}{\mathrm{d}t} \in -\partial I_{J(\theta(t))}(\omega(t)), \tag{2.11}$$

where the subdifferential of $I_{J(\theta)}$ is defined by

$$-\partial I_{J(\theta)}(\omega) = \begin{cases} \mathbb{R}^+ & \text{if } \omega = f_R(\theta) < f_L(\theta), \\ \{0\} & \text{if } f_R(\theta) < \omega < f_L(\theta), \\ \mathbb{R}^- & \text{if } \omega = f_L(\theta) > f_R(\theta), \\ \mathbb{R} & \text{if } \omega = f_L(\theta) = f_R(\theta). \end{cases} \tag{2.12}$$

3 Main results

In this section we present the results of our mathematical analysis of both models. We are concerned with existence, uniqueness, and regularity of solutions first. Then we concentrate on conditions for intrinsic oscillations. Here we state all the results; the proofs will be given in § 4.

We make the following assumptions on the data. θ_0 is continuous on $[-\tau, 0]$, and ω_0 is consistent with the hysteresis curves β and $\theta_0(-\tau)$, i.e. $f_R(\theta_0(-\tau)) \leqslant \omega_0 \leqslant f_L(\theta_0(-\tau))$. The system coefficients satisfy

$$q_e < q_r, \qquad \text{in the first model,} \tag{3.1}$$

$$q_e < hT_R, \qquad \text{in the second model.} \tag{3.2}$$

The hysteresis curves f_L and f_R are Lipschitz continuous functions (with uniformly bounded left and right derivatives) defined on $(-\infty, \infty)$, such that $f_L(r) = f_R(r) = 0$ for $-\infty < r \leqslant T_L$, $f_L(r) = f_R(r) = 1$ for $T_R \leqslant r < \infty$, and for $0 < T_L < r < T_R$ they are monotone increasing with $f_L(r) < f_R(r)$, as depicted in Figure 2. We begin with the first model.

Theorem 3.1 *Under the above assumptions there exists a unique solution $\{\theta, \omega\}$ to problem* (2.1)–(2.4) *satisfying*

$$\theta \in C^1((0,\infty)) \cap C([0,\infty)), \tag{3.3}$$

$$\omega \in \text{ Lip } ([0, \infty)). \tag{3.4}$$

Moreover, θ is bounded on $[0,\infty)$.

We turn next to system oscillations. Let $\tilde{\theta}_L$ and $\tilde{\theta}_R$ be the solutions of

$$f_L(\tilde{\theta}_L) = q_e/q_r; \qquad f_R(\tilde{\theta}_R) = q_e/q_r,$$

respectively, and let α be the minimum of the one-sided slopes of the hysteresis curves at these values, i.e.

$$\alpha = \min\{f_L'(\tilde{\theta}_L - 0), f_L'(\tilde{\theta}_L + 0), f_R'(\tilde{\theta}_R - 0), f_R'(\tilde{\theta}_R + 0)\}.$$

We have

Theorem 3.2 *Assume that*

$$q_r \tau\alpha > \frac{1}{e}. \tag{3.5}$$

Then, for every solution $\{\theta, \omega\}$ to (2.1)–(2.4) *such that θ is not eventually constant, θ oscillates about the interval $[\tilde{\theta}_L, \tilde{\theta}_R]$, and ω oscillates about $\omega_* = q_e/q_r$.*

To say that θ oscillates about the interval $[\tilde{\theta}_L, \tilde{\theta}_R]$ means that $\theta - \tilde{\theta}_L$ and $\theta - \tilde{\theta}_R$ both have zeros for arbitrarily large values of t. Moreover, there may be oscillatory solutions for some values of $q_r\, \tau\alpha$ which do not satisfy (3.5). θ is eventually constant if $\theta(t) = \theta_*$ for $t_0 \leqslant t$, for some $t_0 > 0$. We do not have a complete characterization of all the solutions that are eventually constant, but we conjecture that they are rare.

It will follow from the proof of Theorem 3.2 that

Corollary 3.3 *Even when* (3.5) *does not hold, every solution* $\{\theta, \omega\}$ *of* (2.1)–(2.4) *satisfies one of the following:*

(a) *θ and ω are eventually constant, and*

$$\theta(t) = \theta_* \in [\tilde{\theta}_L, \tilde{\theta}_R], \quad \omega(t) = \omega_* \quad \textit{for } t \geqslant t_0;$$

(b) *θ and ω are oscillatory, as in Theorem 3.2;*

(c) *θ converges to $\tilde{\theta}_R$ or $\tilde{\theta}_L$ as $t \to \infty$, and for t sufficiently large*

$$\theta \nearrow \tilde{\theta}_R, \qquad \textit{or} \qquad \theta \searrow \tilde{\theta}_L.$$

We turn to the second model.

Theorem 3.4 *Under the above assumptions there exists a unique solution* $\{\theta, \omega\}$ *to problem* (2.5)–(2.8) *satisfying*

$$\theta \in C^1((0,\infty)) \cap C([0,\infty)), \tag{3.6}$$

$$\omega \in \text{ Lip } ([0, \infty)), \tag{3.7}$$

and θ is bounded on $(0, \infty)$.

It is well known (see, e.g., [6] or [7]) that delay-differential equations of the type (2.5), without hysteresis, can have unbounded oscillating solutions. It turns out that the hysteresis structure in the second model prevents this from occurring in (2.5)–(2.8), which guarantees the boundedness assertion in the theorem.

To describe our next result, analogous to condition (3.5), let θ_L and θ_R be the unique solutions of

$$\frac{q_e}{h\theta} = f_L(\theta), \quad \text{and} \quad \frac{q_e}{h\theta} = f_R(\theta),$$

respectively. We note that $0 < T_L < \theta_R < T_R$, since $q_e < hT_R$. The *singular values* of (2.5)–(2.8) are all the elements of the interval $[\theta_L, \theta_R]$. We have

Theorem 3.5 *A sufficient condition that all the solutions of* (2.5)–(2.8) *oscillate about a singular value is*

$$f_R(\theta_R) > \frac{1}{h\tau e}. \tag{3.8}$$

Remark In the second model there are two possible types of oscillations. The first type involves changes in both θ and ω, in the second type only θ oscillates while ω

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remains constant. Moreover, for a certain choice of the parameters we have explicit periodic solutions of the second type to problem. Indeed, let $\tau = \pi/2h\omega_0$ and $\theta_0(t) = A\cos(h\omega_0 t + \gamma) + q_e/h\omega_0$; then it is easy to verify that

$$\theta(t) = A\cos(h\omega_0 t + \gamma) + \frac{q_e}{h\omega_0}, \tag{3.9}$$

$$\omega(t) = \omega_0, \tag{3.10}$$

is a solution of the second type for any constant γ.

The next theorem clarifies the asymptotic nature of solutions of (2.5)–(2.8) and serves as part of the proof of Theorem 3.5.

Theorem 3.6 *Let* $\{\theta, \omega\}$ *be a solution of* (2.5)–(2.8).

(a) *If* $\theta(t) \to \theta_R$ *as* $t \to \infty$, *then* $\omega(t) \leqslant \omega_R := \frac{q_e}{h\theta_R}$ *for all sufficiently large* t, *and* θ *is eventually increasing.*

(b) *If* $\theta(t) \to \theta_L$ *as* $t \to \infty$, *then* $\omega(t) \geqslant \omega_L := \frac{q_e}{h\theta_L}$ *for all sufficiently large* t, *and* θ *is eventually decreasing.*

(c) *If* $\theta \to \theta_*$ *as* $t \to \infty$ *where* $\theta_L < \theta_* < \theta_R$, *then* $\omega(t)$ *is eventually constant.*

4 Proofs of the theorems

We begin with defining the 'play hysteresis operator' and establishing an important and useful property.

Define $F : \mathbb{R}^2 \to [0,1]$ by

$$F(\theta,\gamma) = \text{median}\{f_L(\theta), f_R(\theta), \gamma\} = \begin{cases} f_L(\theta) & \text{if } \gamma \geqslant f_L(\theta), \\ \gamma & \text{if } f_R(\theta) < \gamma < f_L(\theta), \\ f_R(\theta) & \text{if } \gamma \leqslant f_R(\theta). \end{cases} \tag{4.1}$$

The *play hysteresis operator* $H_\beta : (\theta, \omega_0) \to \omega$ is defined as follows. The input consists of all pairs (θ, ω_0) where θ is a continuous piecewise monotone function defined on $[-\tau, \infty)$, and $f_R(\theta(-\tau)) \leqslant w_0 \leqslant f_L(\theta(-\tau))$. The output consists of continuous functions ω from $[-\tau, \infty)$ to $[0,1]$. Let

$$-\tau = t_0 < t_1 < \cdots \tag{4.2}$$

be the points where the monotonicity of θ changes. Define inductively

$$\omega(t) = F(\theta(t), \omega_0), \qquad t_0 \leqslant t \leqslant t_1, \tag{4.3}$$

$$\omega(t) = F(\theta(t), \omega(t_i)), \qquad t_i \leqslant t \leqslant t_{i+1}; \quad i = 1, 2, \cdots \tag{4.4}$$

Note that $f_R(\theta(t_0)) \leqslant \omega_0 \leqslant f_L(\theta(t_0))$ means that $\omega(t_0) = \omega_0$, and $f_R \leqslant F(\cdot,\gamma) \leqslant f_L$ implies that $\omega(t_i) = F(\theta(t_i), \omega(t_i))$ $(i = 1, 2, \cdots)$. We also note that the points in (4.3) and (4.4) need only be chosen so that θ is monotone on each interval $[t_i, t_{i+1}]$, $\quad (i = 0, 1, \cdots)$.

That is, we only require that the points in (4.3) and (4.4) include the points of (4.2). To see this, suppose that θ is monotone on $[u, v]$, $\omega(t) = F(\theta(t), \omega(u))$ for $u \leqslant t \leqslant v$, and that $u < s < v$. Without loss of generality, suppose that θ is increasing on $[u, v]$. If $\omega(s) = \omega(u)$, then clearly $\omega(t) = F(\theta(t), \omega(s))$ for $s \leqslant t \leqslant v$. Suppose $\omega(s) > \omega(u)$ ($F(\cdot, \omega(u))$ is increasing so $\omega(s) < \omega(u)$ would imply that θ is not increasing on $[u, v]$). Since $\omega(s) = \text{median}\{f_L(\theta(s)), f_R(\theta(s)), \omega(u)\}$, we have that $f_L(\theta(s)) \geqslant f_R(\theta(s)) = \omega(s) > \omega(u)$. Since θ is increasing, $f_L \geqslant f_R$, and f_R is increasing, $f_L(\theta(t)) \geqslant f_R(\theta(t)) \geqslant f_R(\theta(s)) = \omega(s) > \omega(u)$ for $s \leqslant t \leqslant u$, and so $\omega(t) = F(\theta(t), \omega(u)) = F(\theta(t), \omega(s))$ for $s \leqslant t \leqslant v$. The assertion now follows by inserting one point at a time in (4.2).

As is given in [4, p. 66] for $T > -\tau$

$$\|\omega_1 - \omega_2\|_{[-\tau,T]} \leqslant \max\{\|\theta_1 - \theta_2\|_{[-\tau,T]}, |\omega_0^1 - \omega_0^2|\} \tag{4.5}$$

where for $j = 1, 2$, θ_j is a continuous piecewise monotone function on $[-\tau, \infty)$, $f_R(\theta_j(-\tau)) \leqslant \omega_0^j \leqslant f_L(\theta_j(-\tau))$, and $H_\beta : (\theta_j, \omega_0^j) \mapsto \omega_j$. Here we have used the uniform norms. It follows that the hysteresis operator defined above is uniformly continuous on its domain, and thus extends continuously to a unique operator on the set of all (θ, ω_0) when $\theta \in C[-\tau, \infty)$ and $f_R(\theta(-\tau)) \leqslant \omega_0 \leqslant f_L(\theta(-\tau))$.

For $0 < \gamma < 1$, let

$$\begin{aligned} f_L^{-1}(\gamma) &= \inf\{\theta : \ f_L(\theta) = \gamma\}, \\ f_R^{-1}(\gamma) &= \sup\{\theta : \ f_R(\theta) = \gamma\}. \end{aligned} \tag{4.6}$$

Then we have the following:

Lemma 4.1 *Let* $H_\beta : (\theta, \omega_0) \mapsto \omega$.

(a) *If* $\omega(u) > \gamma > \omega(v)$ *where* $u < v$*, then* $\theta(t) = f_L^{-1}(\gamma)$ *for some* $u < t < v$.

(b) *If* $\omega(u) < \gamma < \omega(v)$ *where* $u < v$*, then* $\theta(t) = f_R^{-1}(\gamma)$ *for some* $u < t < v$.

Proof We only prove (a). Also, it suffices to assume that θ is piecewise monotone. Let s be the largest value less than v for which $\omega(s) = \gamma$. We may also assume that ω and θ are decreasing on $[s, v]$. By the preceding remarks, $\omega(t) = F(\theta(t), \omega(s))$ for $s \leqslant t \leqslant v$. Since $f_L(\theta(s)) \geqslant \omega(s) = \gamma$, $\theta(s) \geqslant f_L^{-1}(\gamma)$. Also, $\omega(v) < \gamma = \omega(s)$ implies that $f_R(\theta(v)) \leqslant f_L(\theta(v)) < \omega(s)$. So $\omega(v) = f_L(\theta(v)) < \gamma$. Thus $\theta(v) < f_L^{-1}(\gamma)$. Hence, $\theta(t) = f_L^{-1}(\gamma)$ for some $s < t < v$. □

We now prove Theorem 3.1.

Proof We proceed by using the method of time steps (see, e.g., [5] or [8]) and take advantage of the 'history dependence' of hysteresis. For $-\tau \leqslant t \leqslant 0$ the function θ is given and $\omega(t)$ is determined by the condition $\omega(-\tau) = \omega_0$ and $\theta(s)$ for $-\tau \leqslant s \leqslant t$.

Next, equation (2.1) is solved on the interval $0 \leqslant t \leqslant \tau$, where $\omega(t)$ is determined for $-\tau \leqslant t \leqslant 0$ from θ_0, (2.2) and (2.3). Clearly, θ has a continuous derivative on $(0, \tau]$. But it is easy to see that $\dot{\theta}$ may have a jump at $t = 0$. Then $\omega(t)$ is obtained for $0 < t \leqslant \tau$. We proceed with the construction step by step. Assume that θ and ω have been found on $[0, n\tau]$. Then $\omega(t-\tau)$ is known on the interval $[n\tau, (n+1)\tau]$, and we obtain a unique solution $\theta(t)$ such that $\theta(n\tau-) = \theta(n\tau+)$. Then $\omega(t)$ is obtained from the hysteresis operator. Note that now $\dot{\theta}(n\tau-) = \dot{\theta}(n\tau+)$. To complete the proofs of (3.3) and (3.4) we need to show that θ is bounded and that ω is Lipschitz continuous. We omit them as they are similar to, but more elementary than, the proofs for the model (2.5)–(2.8). □

Before proving the basic oscillation result, Theorem 3.2, for (2.1)–(2.4), we recall an oscillation theorem from the theory of delay differential equations. We say that a function θ, which is defined on $[-\tau, \infty)$, oscillates about θ_* if $\theta - \theta_*$ has zeros for arbitrarily large values of t. Then

Theorem 4.2 *Let f be a continuous decreasing function on $(-\infty, \infty)$ such that $f(\theta_*) = 0$. If the one-sided derivatives $f'(\theta_* + 0)$ and $f'(\theta_* - 0)$ exist and*

$$\min\{|f'(\theta_* + 0)|, |f'(\theta_* - 0)|\} > \frac{1}{\tau e}, \tag{4.7}$$

then all solutions of the delay differential equation

$$\frac{d\theta(t)}{dt} = f(\theta(t-\tau)) \tag{4.8}$$

oscillate about θ_*.

Theorem 4.2 can be obtained in a number of ways from existing oscillations theorems. One way is as a combination of Theorems 2.2.3 and 4.1.1 in Györi and Ladas [7].

We use this result in the proof of Theorem 3.2 below.

Proof If $\omega(t) = \omega_*$ eventually, then $\theta' = 0$ eventually, and θ is a constant solution of (2.1). Suppose now that $\omega(t_1) > \omega_*$. Then $\theta(t_1) > \theta_L$. We argue that for some $t_2 > t_1$, $\omega(t_2) < \omega_*$. In this case, Lemma 4.1 implies that $\theta(t) = \theta_L = f_L^{-1}(\omega_*)$ for some t in the interval (t_1, t_2). A similar argument yields that if $\omega(t_1) < \omega_*$, then there exits $t_2 > t_1$ such that $\omega(t_2) > \omega_*$ and there is a $t_1 < t < t_2$ so that $\theta(t) = \theta_R = f_R^{-1}(\omega_*)$.

Assume that $\omega(t) \geqslant \omega_*$ for all $t \geqslant t_1$. It follows from (2.1) that $\theta'(t) \leqslant 0$ for all $t \geqslant t_1 + \tau$. Then θ is decreasing on the interval $[t_1 + \tau, \infty)$, and thus $\omega(t) = F(\theta(t), \omega_1)$ for all $t \geqslant t_1 + \tau$, where $\omega_1 = \omega(t_1 + \tau)$. If $\omega(t_1 + \tau) = \omega_*$, then ω is eventually constant, contrary to the assumption. Thus $\omega_1 > \omega_*$, and therefore

$$\frac{d\theta(t)}{dt} = q_e - q_r F(\theta(t-\tau), \omega_1), \tag{4.9}$$

and θ_L is the singular value of the delay differential equation (4.9). Also note that in a neighborhood of θ_L, $F(\theta, \omega_1) = f_L(\theta)$. Thus the conditions on f_L imply the conditions of Theorem 4.2, and therefore θ oscillates about θ_L. We may select the smallest $s > t_1 + \tau$ such that $\theta(s) = \theta_L$. Now $\theta'(t) > 0$ for $s-\tau < t < s$ and thus θ is strictly decreasing on $[s, s+\tau]$. In particular, $\theta(s+\tau) < \theta_L$. But $\omega(s+\tau) = F(\theta(s+\tau), \omega_1) = f_L(\theta(s+\tau)) < f_L(\theta_L) = \omega_1$, a contradiction. □

We next prove Theorem 3.4.

Proof Existence and uniqueness for (2.5)–(2.8) can be established in the same manner as those for (2.1)–(2.4), and thus we shall omit these details. We need to prove the boundedness of θ and the Lipschitz property of ω. We consider now the boundedness of θ.

We first observe that the function $w(t)\theta(t)$ is bounded below since $\omega(t) = 0$ when $\theta(t) \leqslant T_L$.

We also observe that if $q_e - h\omega(t)\theta(t) = 0$ then $\theta(t) \in [T_L, T_R]$, since we assume that $q_e < hT_R$. Indeed, if $T_R < \theta(t)$ then $\omega(t) = 1$, so $q_e - h\theta(t) = 0$, thus $\theta(t) = q_e/h < T_R$, which is a contradiction. Similarly, assume that $\theta(t) < T_L$, then $\omega(t) = 0$, and thus $q_e = 0$, a contradiction since $q_e > 0$.

Assume that θ is unbounded from above. Then for $\theta(t)$ sufficiently large $\omega(t) = 1$ and

$$\theta'(t+\tau) = q_e - h\theta(t) < 0.$$

Therefore, θ cannot be a monotone increasing function, so we can find a sequence $(t_k, \theta(t_k))$ where $t_k \to \infty$ such that $\theta(t_k)$ is a local maximum, $\theta(t_k) < \theta(t_{k+1})$, and $\theta(t_k) \to \infty$ as $k \to \infty$.

It follows from the differential equation (2.5) that

$$q_e - h\omega(t_k - \tau))\theta(t_k - \tau)) = 0,$$

so $\theta(t_k - \tau) \in [T_L, T_R]$. By the Mean Value Theorem $\tau\theta'(\hat{t}_k) = \theta(t_k) - \theta(t_k - \tau)$ for some $\hat{t}_k \in (t_k - \tau, t_k)$. Then (2.5) yields

$$\begin{aligned} \frac{\theta(t_k) - \theta(t_k - \tau)}{\tau} &= \theta'(\hat{t}_k) \\ &= q_e - h\omega(\hat{t}_k - \tau)\theta(\hat{t}_k - \tau). \end{aligned}$$

Thus,

$$\omega(\hat{t}_k - \tau)\theta(\hat{t}_k - \tau) = \frac{1}{h}\left[q - \frac{\theta(t_k) - \theta(t_k - \tau)}{\tau}\right].$$

Now, $\theta(t_k) \to \infty$ and $\theta(t_k - \tau)$ remains bounded and therefore the right-hand side tends to $-\infty$, thus $\omega(\hat{t}_k - \tau)\theta(\hat{t}_k - \tau) \to -\infty$, contradicting the fact that $\omega(t)\theta(t)$ is bounded from below. Hence θ is bounded from above.

The proof that θ is bounded from below is similar. Assume that θ is unbounded from below. Equation (2.5) implies that when $\theta(t)$ is sufficiently negative, then $\theta'(t+\tau) > q_e > 0$.

Therefore, there exists a sequence $(s_k, \theta(s_k))$, where $\theta(s_k)$ are local minima of θ, $s_k \to \infty$, and $\theta(s_k) \to -\infty$. Now

$$0 = \theta'(s_k) = q_e - h\omega(s_k - \tau)\theta(s_k - \tau)).$$

Thus $\theta(s_k - \tau) \in [T_L, T_R]$.

We invoke the Mean Value Theorem again to obtain

$$\theta'(\hat{s}_k) = \frac{\theta(s_k) - \theta(s_k - \tau)}{\tau}$$

for some $\hat{s}_k \in (t_k - \tau, t_k)$. Then by (2.5),

$$\frac{\theta(s_k) - \theta(s_k - \tau)}{\tau} = q_e - h\omega(\hat{s}_k - \tau)\theta(\hat{s}_k - \tau).$$

Now, when k is sufficiently large the left-hand side is as negative as we wish while the right-hand side remains bounded. Thus θ is bounded.

We turn to the Lipschitz continuity and show that for $0 \leqslant s \leqslant t < \infty$,

$$|\omega(t) - \omega(s)| \leqslant M|t - s|, \tag{4.10}$$

for a positive constant M which we derive below. First, we assume that θ is piecewise monotone and thus ω is piecewise monotone too. For $s < t$, we select

$$s = t_0 < \cdots < t_k = t,$$

such that ω is either strictly monotone or is constant on $[t_i, t_{i+1}]$. On each of the intervals where ω is strictly monotone we may write $\omega(t) = f(\theta(t))$ where $f = f_L$ or $f = f_R$. In either case

$$\begin{aligned} |\omega(t_{i+1}) - \omega(t_i)| &= |f(\theta(t_{i+1})) - f(\theta(t_i))| \\ &= |f'(\theta(t + \eta_1 \Delta t_i))\theta'(t + \eta_1 \Delta t_i)|\Delta t_i \\ &\leqslant LB\Delta t_i, \end{aligned}$$

where L is the larger of the Lipschitz constants of f_L and f_R, B is an upperbound for $|\theta'|$, and $\Delta t_i = t_{i+1} - t_i$. We note that $\theta'(t) = q_e - h\theta(t - \tau)\omega(t - \tau)$ is bounded since θ is bounded.

When ω is constant on $[t_i, t_{i+1}]$ then $\omega(t + \Delta t) - \omega(t) = 0$. Then (4.10) follows.

When θ is not piecewise monotone, we can approximate θ, in the uniform norm by piecewise monotone functions with the same bounds, and then we apply the continuity of the hysteresis operator. □

We first prove Theorem 3.6 as it is used in the proof of Theorem 3.5.

To prove Theorem 3.6, we examine the convergence of a solution of (2.5)–(2.8) to a

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singular value as $t \to \infty$. The singular values for the differential equation and fixed γ correspond to

$$q_e - hF(\theta, \gamma)\theta = 0, \tag{4.11}$$

or

$$F(\theta, \gamma) = \frac{q_e}{h\theta}. \tag{4.12}$$

Recall that $q_e/hT_R < 1$.

The singular values are θ_L (the solution of $f_L(\theta) = q_e/h\theta$), θ_R (the solution of $f_R(\theta) = q_e/h\theta$), and all the values $\theta_L < \theta < \theta_R$. The value θ_L is the singular value for $\gamma \geqslant q_e/h\theta_L$, θ_R is the singular value for $\gamma \leqslant q_e/h\theta_R$, and $\theta_L < \theta < \theta_R$ are the singular values for $q_e/h\theta_R < \gamma < q_e/h\theta_L$.

Lemma 4.3 *Let θ be a solution of*

$$\theta'(t) = q_e - hF(\theta(t-\tau), \gamma)\theta(t-\tau). \tag{4.13}$$

If $\theta(t) \to \theta_*$ as $t \to \infty$, then θ_0 is the singular value for γ.

Proof If $\theta(t-\tau) \to \theta_*$ as $t \to \infty$, then by continuity of $F(\cdot, \gamma)$, $\theta'(t) \to q - hF(\theta_0, \gamma)\theta_*$. If $q - hF(\theta_0, \gamma)\theta_* \neq 0$, θ' would eventually be bounded away from 0 and $\theta(t) \to \pm\infty$, a contradiction. □

Now we are ready to prove Theorem 3.6.

Proof We first prove (c). Choose t_1 so that $\theta(t) \in (\theta_L, \theta_R)$ for $t \geqslant t_1$. Suppose that $\omega(t_2) > \omega_0 := q_e/h\theta_*$ for some $t_2 > t_1$. If $\omega(t_2) \leqslant \omega_L := q_e/h\theta_L$, then $\omega(t)$ would be constant on $[t_1, \infty)$. Otherwise, by Lemma 4.1, $\theta(t)$ would be $f_L^{-1}(\gamma)$ or $f_R^{-1}(\gamma)$ for some $\omega_0 < \gamma < \omega_L$. This lies outside (θ_L, θ_R). With $\omega(t)$ constant, θ would eventually be a solution of $\theta'(t) = q_e - hF(\theta(t-\tau), \gamma)\theta(t-\tau)$, $\omega_0 < \gamma < \omega_L$. By Lemma 4.3, $\theta(t)$ converges to the singular value for this differential equation which is different from θ_0. Now if $\omega(t_2) > \omega_L$, then either $\omega(t) > \omega_L$ for all $t > t_2$ or $\theta(t) = f_L^{-1}(\omega_L) \leqslant \theta_L$ for some $t > t_2$ using Lemma 4.1. If $\omega(t) > \omega_L$ for all $t \geqslant t_2$, then $\theta'(t) = q_e - h\omega(t-\tau)\theta(t-\tau) < 0$ for all $t \geqslant t_1 + \tau$. So θ is decreasing on $[t_1 + \tau, \infty)$, Thus θ eventually satisfies the differential equation $\theta'(t) = q_e - hF(\theta(t-\tau), \omega(t_1))\theta(t-\tau)$ and by Lemma 4.3, $\theta(t) \to \theta_L$, a contradiction. The remaining cases are similar.

We now prove (a). Choose t_1 so that if $t \geqslant t_1$, then $\theta(t) > \theta_L$. Suppose $\omega(t_2) > \omega_R := q_e/h\theta_R$ for some $t_2 \geqslant t_1$. If $\omega(t) > \omega_L$ for some $t \geqslant t_2$, we obtain contradiction as in the proof of (c) above. Suppose $\omega(t) \leqslant \omega_L$ for all $t \geqslant t_2$. If $\omega(t) < \omega(t_2)$ for some $t > t_2$, then Lemma 4.1 yields that $\theta(t) \leqslant \theta_L$ for some $t > t_2$. Thus $\omega(t_2) \leqslant \omega(t) \leqslant \omega_L$ for all $t > t_2$. Lemma 4.1 again implies that ω is increasing on $[t_0, \infty)$. If ω is eventually constant, then as above Lemma 4.3 implies that $\theta(t)$ converges to a singular value less than θ_R. If ω is not eventually constant, select $t_2 < u_1 < v_1 < u_2 < v_2 < \cdots$ so that $u_i \to \infty$ and $\omega(v_i) > \omega(u_i)$. Select $\omega(u_i) < \gamma_i < \omega(v_i)$. Then $\omega_R < \gamma_1 < \gamma_2 < \cdots$. By Lemma 4.3, there

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exists $u_i < s_i < v_i$ so that $\theta(s_i) = f_R^{-1}(\gamma_i)$. But $f_R^{-1}(\gamma_1) > \theta_R$ and $\{\theta(s_i)\}$ is an increasing sequence for which $\theta(s_i) \to \theta_R$. This is a contradiction.

The proof of (b) is similar to that of (a). □

Finally, we prove Theorem 3.5.

Proof Suppose $\liminf_{t\to\infty} \theta(t) < \limsup_{t\to\infty} \theta(t)$. Then θ oscillates about a nontrivial interval. Otherwise, $\lim_{t\to\infty} \theta(t) = \theta_*$ for some $\theta_L \leqslant \theta_* \leqslant \theta_R$ by Lemma 4.3. By Theorem 3.6, θ is eventually a solution of the delay differential equation

$$\theta'(t) = q - hF(\theta(t-\tau), \omega)\theta(t-\tau). \tag{4.14}$$

We examine the conditions of Theorem 4.2. If $\omega \leqslant \omega_R$, then the singular value is θ_R and

$$f(\theta) = q - hf_R(\theta)\theta.$$

Then f is decreasing and

$$\begin{aligned}
&\min\{-f'(\theta_R - 0), -f'(\theta_R + 0)\} \\
&= h\min\{f_R(\theta_R) + f_R'(\theta_R + 0)\theta_R, f_R(\theta_R) + f_R'(\theta_R - 0)\theta_R\} \\
&\geqslant hf_R(\theta_R) > \frac{1}{\tau e}.
\end{aligned}$$

If $\omega = \omega_0$ and $\theta = \theta_* \in (\theta_L, \theta_R)$, then $f(\theta) = q - h\omega_0\theta$, and $-f'(\theta_0) = h\omega_0 > hf_R(\theta_R) > 1/\tau e$.

All the other cases can be dealt with similarly, and by Theorem 4.2, θ oscillates about θ_*. □

5 Numerical scheme and examples

In this section we present the algorithm used to solve the two models numerically. Then we show a number of examples of numerical simulations. Since both models are based on autonomous differential equations, it suffices to use Euler's method of time marching (explicit) to solve them numerically. Let Δt be the time step, and let

$$\theta^j = \theta(j\Delta t); \qquad \omega^j = \omega(j\Delta t),$$

denote the approximations, at time $j\Delta t$, to the thermostat temperature and opening, respectively. Let n_τ be the closest integer to $\tau/\Delta t$. Then the first model, (2.1)–(2.4), is approximated as follows. Equation (2.1) is discretized as

$$\theta^{j+1} = \theta^j + \Delta t(q_e - q_r w^{j-n_\tau}). \tag{5.1}$$

Equation (2.5) is discretized as

$$\theta^{j+1} = \theta^j + \Delta t(q_e - h\theta^{j-n_\tau} w^{j-n_\tau}). \tag{5.2}$$

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Next, ω^{j+1} is determined by a discretized version of the hysteresis conditions (2.2) or (2.6). It consists of following the hysteresis curves: when $\omega^j = f_R(\theta^j)$ and θ is increasing, $\theta^{j+1} > \theta^j$ then ω is determined by f_R, $w^{j+1} = f_R(\theta^{j+1})$. But when θ is not increasing, $\theta^{j+1} \leqslant \theta^j$ then the opening remains constant, that is $w^{j+1} = \omega^j$. And similarly when $\omega^j = f_L(\theta^j)$ and θ is decreasing.

The full discretized hysteresis algorithm is

```
if ( w^j = f_R(θ^j) ) then
    if (θ^{j+1} > θ^j) then
        w^{j+1} = f_R(θ^{j+1})
    else
        w^{j+1} = w^j
    endif
else if ( w^j = f_L(θ^j) ) then
    if (θ^{j+1} < θ^j) then
        w^{j+1} = f_L(θ^j)
    else
        w^{j+1} = w^j
    endif
else if (f_L(θ^j) > w^j > f_R(θ^j) ) then
        w^{j+1} = w^j
    endif
if (w^{j+1} < f_R(θ^{j+1}) set w^{j+1} = f_R(θ^{j+1})
if (w^{j+1} > f_L(θ^{j+1}) set w^{j+1} = f_L(θ^{j+1})
```

Numerical approximations for the first model The θ^j are given by $\theta_0(j\Delta t)$ for $-n_\tau \leqslant j \leqslant 0$, and ω^j are computed from them and the hysteresis algorithm.

Then on each interval $nn_\tau \leqslant j \leqslant (n+1)n_\tau$ we use (5.1) to compute θ^j, since the values of ω^{j-n_τ} are known. Then we use the hysteresis algorithm to compute the ω^j.

Numerical approximations for the second model The procedure is the same as above, but using (5.2).

The question of accuracy and convergence analysis of the numerical schemes is left open. Clearly such an analysis is rather involved and needs special considerations since the models include delay and hysteresis. However, we have computed numerically the solution corresponding to the exact solution (3.9) and (3.10) and obtained good accuracy. Indeed, the maximal error is found to be less than 2.0×10^{-2}. The boundedness of solutions and their Lipschitz continuity in time indicate that the numerical scheme should be stable, a fact supported by our experience with the numerical solutions.

We present a number of representative numerical results for the two models. For simplicity, we have assumed that the hysteresis curves are simply straight parallel lines as depicted below.

In Figure 3 we show two numerical solutions to the first model. On the left of each pair of frames is the graph of the thermostat temperature vs. time, while on the right we present the fractional opening vs. time. In both cases, the system starts with ambient

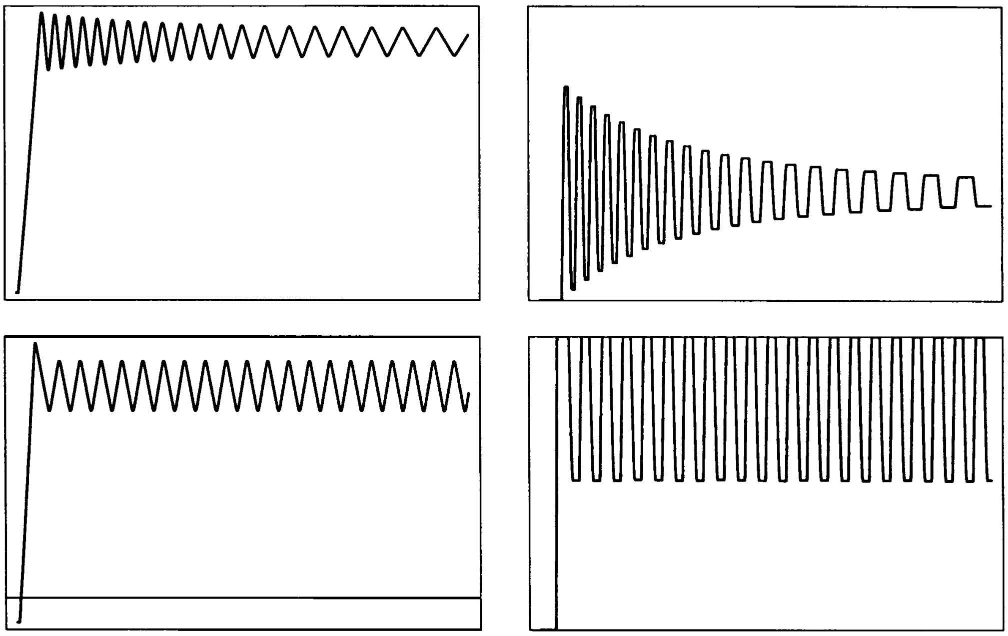

FIGURE 3. Two solutions to the first model. Temperature oscillations in time on the left, and opening as a function of time on the right.

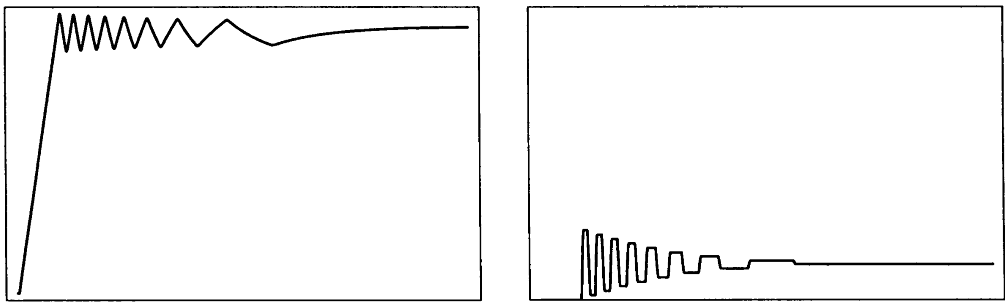

FIGURE 4. A decaying solution to the second model.

temperature 60 and 0 opening. In the first case the oscillations decay in time and their frequency decreases. In the second case the oscillations seem to persist. The parameters used in the two cases are as follows. Top: $q_e = 5.0$, $q_r = 13.5$, $\tau = 3.0$, the slope of the hysteresis curve is 0.036, $T_L = 180$, $T_R = 220$; Bottom: $q_e = 8.0$, $q_r = 10.5$, $\tau = 3.0$, the slope of the hysteresis curve is 0.025, T_R and T_L are the same. So in the first case $q_r\tau\alpha = 1.458 > 1/e$, and in the second case $q_r\tau\alpha = 0.7875 > 1/e$.

In Figure 4 we depict a solution of the second model. On the left we show θ as a function of time, and on the right the opening ω as a function of time. Clearly, the oscillations decay and their frequency decrease visibly. The parameters used are: $q_e = 5.0$, $h_r = 0.215$, $\tau = 1$, $T_L = 180$, $T_R = 220$, the slope of the hysteresis is 0.0357.

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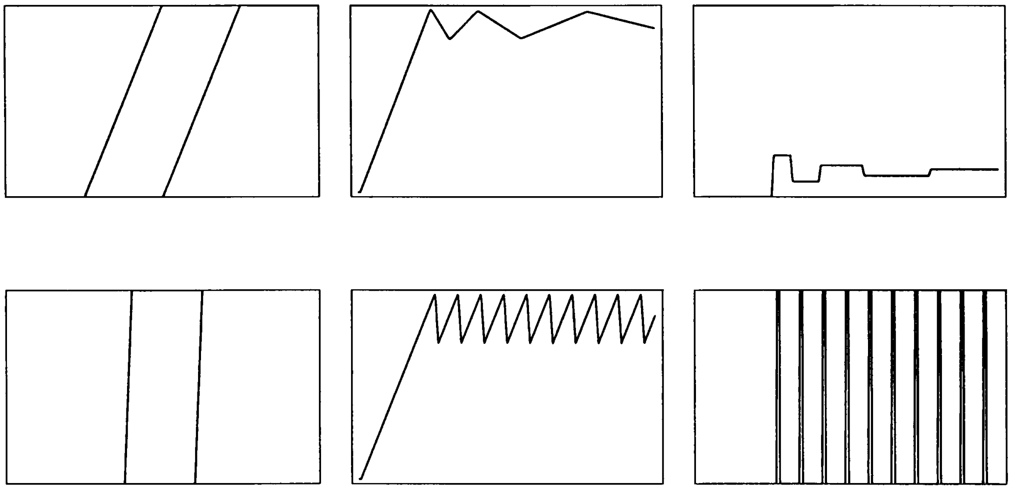

FIGURE 5. Comparison of two solutions to the first model with different slopes of the hysteresis graphs.

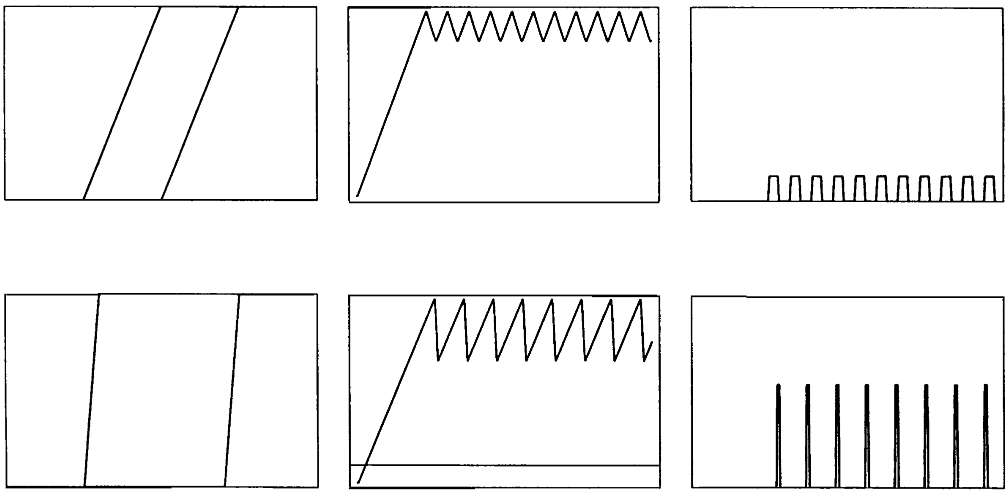

FIGURE 6. Comparison of two solutions to the second model with different slopes of the hysteresis graphs.

We next present our investigation of the effects of the slope of the hysteresis curves on the oscillations.

In Figure 5 we depict a comparison between two solutions of the first model in the cases when the slopes of f_L and f_R are increased. The left frame now represents the hysteresis graph, i.e. the fractional opening vs. temperature. The middle frame depicts the temperature, and the right one the opening. As can be seen in the second case, increasing the slope of the hysteresis curves causes very rapid oscillations in the temperature θ and the opening ω fluctuates between being fully closed and fully open. It seems as the oscillations in the upper case are moderate and decaying in time.

In Figure 6 we show a comparison for solutions of the second model. The display is as

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in Figure 5. The conclusions are similar, except that in both cases the oscillations seem to be steady. Moreover, the opening ω oscillates very sharply in Figures 5 and 6 for steep slopes of the hysteresis curves.

6 Conclusions

We have constructed two models describing the dynamics of an automotive thermostat situated in the engine cooling loop. Each of the models consists of a delay-differential equation and a hysteresis functional relation. The mathematical novelty is in the combination of these two in a system of equations. The existence, uniqueness, and regularity of solutions to both models have been established. Moreover, we have derived sufficient conditions for the appearance of oscillatory solutions. These, in addition to their theoretical interest, have considerable practical importance since it is the aim of engine designers to eliminate them if possible, and to minimize them if not.

It is seen that the sufficient conditions for all the solutions to be oscillatory, (3.5) and (3.8), have a somewhat different structure. Nevertheless, they both relate the delay, the slope of the hysteresis curve, and the cooling power in ways that are observed in real situations.

We conclude that even such ‘simple’ models for thermostats, which include hysteresis and delays show very interesting types of behaviour, leading to insight into the relationship among delay, hysteresis and system oscillations.

A number of interesting questions remain open. Indeed, what can be said about oscillatory solutions when (3.5) or (3.8) do not hold? Under which conditions will the oscillations be periodic, quasiperiodic, decaying, or possibly chaotic? Also, the numerical analysis of our algorithm remains to be investigated.

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