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List Colourings of Regular Hypergraphs

DAVID SAXTON† and ANDREW THOMASON

DPMMS, Centre for Mathematical Sciences, Wilberforce Road, Cambridge, UK
(e-mail: D.W.Saxton@dpmms.cam.ac.uk, A.G.Thomason@dpmms.cam.ac.uk)

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In fond memory of Dick Schelp

We show that the list chromatic number of a simple d-regular r-uniform hypergraph is at least $(1/2r\log(2r^2)+o(1))\log d$ if d is large.

1. Introduction

A hypergraph G is said to be k-*choosable* if, whenever we assign to each vertex v a list L_v of k colours, there is a proper vertex colouring of G in which the colour of v is chosen from L_v. (Here, as usual, *proper* means that no edge has vertices of just one colour.) The *list chromatic number* $\chi_\ell(G)$ (also called the *choice number*) is the smallest k such that G is k-choosable. Clearly $\chi_\ell(G)$ is at least $\chi(G)$, the ordinary chromatic number of G.

The notion of choosability for graphs is due to Vizing [9] and Erdős, Rubin and Taylor [5]. One of the main discoveries of [5] is that $\chi_\ell(G)$ can be much larger than $\chi(G)$; it is shown that $\chi_\ell(K_{d,d}) = (1+o(1))\log_2 d$, whereas of course $\chi(K_{d,d}) = 2$.

In fact, unlike $\chi(G)$, $\chi_\ell(G)$ must grow with the minimum degree of the graph G. Alon [2], improving on an earlier result [1], showed that $\chi_\ell(G) \geqslant (1/2+o(1))\log_2 d$ holds for any graph G of minimum degree d. It is natural to ask whether a similar phenomenon holds for r-uniform hypergraphs. Very little is known about the answer, though, except in certain special cases. A hypergraph is said to be *simple*, sometimes called *linear*, if no two edges share more than one vertex. Haxell and Pei [6] showed that the list chromatic number of a Steiner triple system on n vertices (this is a simple 3-uniform $(n-1)/2$-regular hypergraph) is of order at least $\log n/\log\log n$, and Haxell and Verstraëte [7] proved that $\chi_\ell(G) \geqslant (\log d/5\log\log d)^{1/2}$ for every simple, d-regular 3-uniform hypergraph G when d is large. Alon and Kostochka [4] showed that if at least half the $(r-1)$-tuples of vertices

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of G lie in at least m edges then $\chi_\ell(G) \geqslant c_r \log m$. The same authors [3] showed that the list chromatic number of a simple d-regular hypergraph is of order at least $(\log d)^{1/(r-1)}$.

We show that $\chi_\ell(G)$ must be of order at least $\log d$ for any simple uniform d-regular hypergraph G.

Theorem 1.1. *Let G be a simple d-regular r-uniform hypergraph. Then*

$$\chi_\ell(G) \geqslant \left(\frac{1}{2r\log(2r^2)} + o(1)\right)\log d,$$

where the $o(1)$ term is as $d \to \infty$.

Note that this theorem is best possible (up to a constant factor). Haxell and Verstraëte [7] showed that $\chi_\ell(K(r \times n)) = (1 + o(1))\log_r n$, where $K(r \times n)$ is the complete r-uniform r-partite hypergraph with n vertices in each class. It is not difficult to construct a simple d-regular sub-hypergraph G of $K(r \times n)$ with n not much larger than d, and therefore $\chi_\ell(G) = O(\log d)$.

The proof gives more than is stated in the theorem. Indeed if the ratio of the minimum to maximum degree is bounded then the result holds (but with a smaller constant). Moreover the proof readily extends to more general colourings, not just proper colourings. It is also possible to give a good bound for regular hypergraphs even if they are not simple. However, the method does not give much information about hypergraphs which are not close to being regular. These matters are discussed in Section 4.

The proof method is different in nature to those in [1], [2], [3], [4], [6] and [7]. The central idea is that if there can be found some small collection $\mathcal{C}$ of sets, each of size say $(1 - 1/r)n$, where n is the number of vertices of G, such that every independent set is a subset of one of the sets in the collection, then $\chi_\ell(G)$ must be large. This idea is made explicit in Theorem 2.1 in Section 2.

Given Theorem 2.1, what remains to be done to prove Theorem 1.1 is to find a collection $\mathcal{C}$ that satisfies the conditions of Theorem 2.1. Of course, each independent set is contained in a maximal independent set which, in a regular hypergraph, is of size at most $(1 - 1/r)n$. However, the collection of maximal independent sets does not in general satisfy the conditions of Theorem 2.1 because it is too large: it is necessary that $|\mathcal{C}| = 2^{o(n)}$, whereas there can be many more maximal independent sets than this. For example, if $r = 2$ and d is odd, let F be $K_{d-1,d-1}$ with $(d-1)/2$ independent edges added to each vertex class. Then the graph consisting of $n/2(d-1)$ disjoint copies of F is d-regular and has more than $2^{n/4}$ maximal independent sets.

So, in order to make use of Theorem 2.1, it is necessary to find a suitable collection $\mathcal{C}$ by some other means. This is the subject of Section 3.

2. Small covers for independent sets

We use standard notation, such as $[n] = \{1, \ldots, n\}$, $[n]^{(l)} = \{A : A \subset [n], |A| = l\}$ and $2^{[n]} = \{A : A \subset [n]\}$. Throughout the paper, we shall assume that the hypergraph G has n vertices and that its vertex set is $[n]$.

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A vertex colouring of a hypergraph is a partition $A_1, \ldots, A_t$ of its vertex set. The colour given to the vertex v is i, where $v \in A_i$. Often we are interested in proper colourings, namely those in which each set A_i is an independent set, but we do not always make this restriction. Colourings allowed by vertex lists are defined as follows.

Definition. A collection of lists $\{L_v : v \in [n]\}$ *admits* a partition $A_1, \ldots, A_t$ if the colour given to each $v \in [n]$ belongs to L_v, *i.e.*, for each $i \in [t]$, $A_i \subset \{v : i \in L_v\}$.

The next definition relates a colouring, or indeed any partition of the vertices, to a covering collection $\mathcal{C}$.

Definition. Let $\mathcal{C} \subset 2^{[n]}$ be a collection of subsets of $[n]$ and let $A_1, \ldots, A_t$ be a partition of $[n]$. The partition is *$\mathcal{C}$-compatible* if, for $1 \leqslant i \leqslant t$, $A_i \subset C$ for some $C \in \mathcal{C}$.

The result of this section shows that, provided $\mathcal{C}$ is not large and no set in $\mathcal{C}$ is close to $[n]$, then there is a collection of quite large lists that do not admit a $\mathcal{C}$-compatible colouring.

Theorem 2.1. *For $c > 0$ and $k < n$, let $\mathcal{C} \subset 2^{[n]}$ satisfy*
(i) $|\mathcal{C}| \leqslant 2^{n/k}$,
(ii) $|C| \leqslant (1-c)n$ *for all* $C \in \mathcal{C}$.
Then there exists a collection of lists $\{L_v : v \in [n]\}$, each of size

$$|L_v| \geqslant (1 + o(1)) \log k / \log(1/c)$$

(*where $o(1) \to 0$ as $k \to \infty$ with c fixed*), *which does not admit a $\mathcal{C}$-compatible partition.*

Proof. Let $\epsilon > 0$, let $l = \lfloor (1-\epsilon) \log k / \log(1/c) \rfloor$ and let $t = \lfloor 2l^2/c \rfloor$. For each $v \in [n]$, let $L_v \in [t]^{(l)}$ be a subset of $[t]$ of size l chosen uniformly and independently at random. It suffices to show that, if k is sufficiently large (depending on ϵ and c), then the probability of the lists $\{L_v : v \in [n]\}$ admitting any $\mathcal{C}$-compatible partition is less than one.

Suppose that the collection of lists $\{L_v : v \in [n]\}$ does admit some $\mathcal{C}$-compatible partition $A_1, \ldots, A_t$. Then there exists a tuple $(C_1, \ldots, C_t) \in \mathcal{C}^t$ such that $A_i \subset C_i$ for each $i \in [t]$. Given a tuple $(C_1, \ldots, C_t)$, define

$$B_v = B_v(C_1, \ldots, C_t) = \{i \in [t] : v \in C_i\}.$$

Since $\{L_v : v \in [n]\}$ admits $A_1, \ldots, A_t$, it must be that, for each $v \in [n]$, there exists $i \in L_v$ with $v \in A_i \subset C_i$; that is, $B_v \cap L_v \neq \emptyset$. We aim to show that, with positive probability, this does not happen for any tuple $(C_1, \ldots, C_t) \in \mathcal{C}^t$; that is, with positive probability, for every tuple $(C_1, \ldots, C_t)$ there is some $v \in [n]$ with $B_v \cap L_v = \emptyset$. This will complete the proof of the lemma.

Given a tuple $(C_1, \ldots, C_t)$, let p_v be the probability that $B_v \cap L_v = \emptyset$, which is to say, $L_v \subset [t] \setminus B_v$. Putting $z_v = \max\{l-1, t - |B_v|\}$, we observe that

$$p_v = \Pr(B_v \cap L_v = \emptyset) = \binom{z_v}{l} \binom{t}{l}^{-1}.$$

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Let z be the average of the values z_v; then we have

$$nz = \sum_v z_v \geqslant \sum_v t - |B_v| = nt - \sum_v |B_v| = nt - \sum_{i=1}^{t} |C_i| \geqslant nct.$$

So, since the function $\binom{z_v}{l}$ is convex for $z_v \geqslant l-1$, we have

$$\sum_v p_v = \sum_v \binom{z_v}{l}\binom{t}{l}^{-1} \geqslant n\binom{z}{l}\binom{t}{l}^{-1} \geqslant n\binom{ct}{l}\binom{t}{l}^{-1} \geqslant n(c-(l-1)/t)^l \geqslant nc^l/2,$$

using the fact that when k is large then l is also large and $(l-1)/t \leqslant c/2l$. Hence

$$\Pr(B_v \cap L_v \neq \emptyset \text{ for all } v \in [n]) = \prod_v (1-p_v) \leqslant \exp\Big\{-\sum_v p_v\Big\} \leqslant \exp\{-nc^l/2\}.$$

Finally, summing this probability over all tuples $(C_1, \ldots, C_t)$, the probability that some tuple $(C_1, \ldots, C_t)$ satisfies $B_v \cap L_v \neq \emptyset$ for all v is at most

$$|\mathcal{C}|^t e^{-nc^l/2} = \exp\{(nt/k)\log 2 - nc^l/2\} \leqslant \exp\left\{\frac{n}{2k}\left[\frac{4\log 2}{c}\left(\frac{(1-\epsilon)\log k}{\log 1/c}\right)^2 - k^\epsilon\right]\right\},$$

which is less than one for k sufficiently large. Hence with positive probability the collection of lists does not admit a $\mathcal{C}$-compatible colouring. □

3. Finding a small cover

We turn our attention now to the question of finding a collection $\mathcal{C}$ of sets covering the independent subsets of the hypergraph G, suitable for the application of Theorem 2.1. Our aim is to describe how to construct, for each small set T, a set $C(T)$ with $|C(T)| \leqslant (1-c)n$, so that for every independent set I there is some T with $I \subset C(T)$. Provided T is small, the number of such sets $C(T)$ is not large, so we can apply Theorem 2.1.

In fact, our construction is most easily described in terms of three small sets R, S and T, but the principle is the same.

Theorem 3.1. *For $r \geqslant 2$, $c > 0$ and b sufficiently large (depending on c and r), let G be a simple r-graph of order n such that every set $A \subset V(G) = [n]$ of size $|A| \geqslant (1-2rc)n$ contains at least nb edges of G. Then there exists a collection of sets $\mathcal{C} \subset 2^{[n]}$ satisfying:*

(i) $|\mathcal{C}| \leqslant 2^{n/b^{1/2r}}$,
(ii) $|C| \leqslant (1-c)n$ *for every* $C \in \mathcal{C}$,
(iii) *for every independent set* $I \subset V(G)$, *there is some* $C \in \mathcal{C}$ *with* $I \subset C$.

Proof. Let $V = V(G) = [n]$ be the vertex set of G and let $E = E(G)$ be the edge set. For sets $R, S \subset V$ and $0 \leqslant j \leqslant r-1$, let

$$\Gamma_j(R,S) = \{v \in V : \text{ there exist } f \in R^{(j)} \text{ and } g \in S^{(r-j-1)} \text{ with } \{v\} \cup f \cup g \in E\}.$$

For example, when $r = 2$, we have $\Gamma_1(R,S) = \Gamma(R)$, the usual neighbourhood of R, and $\Gamma_0(R,S) = \Gamma(S)$.

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Given subsets $R, S, T \subset V$, let

$$C_j(R,S,T) = \begin{cases} (V \setminus \Gamma_j(R,S)) \cup T & \text{if } n - |\Gamma_j(R,S)| + |T| \leqslant (1-c)n, \\ \emptyset & \text{otherwise.} \end{cases}$$

Note that $|C_j(R,S,T)| \leqslant (1-c)n$ by definition. We will show that for every independent set I, there are small subsets $R, S, T \subset V$ such that $I \subset C_j(R,S,T)$. Letting $\mathcal{C}$ be the collection of all such sets $C_j(R,S,T)$, the size of $\mathcal{C}$ will be small because each of R, S and T is small. Specifically, let

$$u = \frac{1}{\sqrt{3r2^{r-1}}}\left(\frac{3}{b}\right)^{1/2(r-1)} \quad \text{and} \quad q = 15r2^{r-1}u.$$

Let $\mathcal{C} = \{C_j(R,S,T) : 0 \leqslant j \leqslant r-1,\ |R|,|S|,|T| \leqslant qn\}$. Then

$$|\mathcal{C}| \leqslant r(qn)^3 \binom{n}{qn}^3 \leqslant r(qn)^3\left(\frac{ne}{qn}\right)^{3qn} \leqslant 2^{n/b^{1/2r}}$$

for b sufficiently large (depending on c and r). This collection $\mathcal{C}$ will satisfy the conditions of the lemma.

Fix an independent set I. For $A \subset V$ and $0 \leqslant j \leqslant r$, say that an edge $e \in E$ is a (j,A)-*edge* if it has a partition $e = f \cup g$, $f \in I^{(j)}$, $g \in A^{(r-j)}$. Let $P(j)$ be the statement

for all $A \subset V$ with $|A| \geqslant (1-2(r-j)c)n$, there are at least nbu^j (j,A)-edges.

The statement $P(r)$ is false since an (r,A)-edge is one contained inside I, of which there are none. Statement $P(0)$ is true by assumption on G, since a $(0,A)$-edge is one contained inside A.

There must therefore exist $j \in \{0,1,\ldots,r-1\}$ such that $P(j)$ is true and $P(j+1)$ is false. Fix a set A witnessing the falsity of $P(j+1)$; thus $|A| \geqslant (1-2(r-j-1)c)n$ and there are fewer than nbu^{j+1} $(j+1,A)$-edges.

By analogy with $\Gamma_j(R,S)$, for a vertex $v \in V$, let

$$d_j(v) = |\{e \in E : \text{ there exist } f \in I^{(j)} \text{ and } g \in A^{(r-j-1)} \text{ with } e = \{v\} \cup f \cup g\}|.$$

We define a set $D = \{v_1, \ldots, v_{\lfloor 2cn \rfloor}\} \subset A$ in the following way. Given $v_1, \ldots, v_{i-1}$ with $i \leqslant 2cn$, let $A_i = A \setminus \{v_1, \ldots, v_{i-1}\}$. Since $|A_i| \geqslant (1-2(r-j)c)n$, $P(j)$ implies that A_i contains at least nbu^j (j, A_i)-edges, and these edges are all (j,A)-edges. Therefore

$$\sum_{v \in A_i} d_j(v) \geqslant (r-j)nbu^j \geqslant nbu^j.$$

Let $v_i \in A_i$ be a vertex with $d_j(v_i) \geqslant bu^j$.

Let $p = (3/bu^j)^{1/(r-1)}$, so $p^{r-1}bu^j = 3$. Since $j \leqslant r-1$, we observe that

$$p \leqslant \left(\frac{3}{b}\right)^{1/(r-1)}\frac{1}{u} = \sqrt{3r2^{r-1}}\left(\frac{3}{b}\right)^{1/2(r-1)} = 3r2^{r-1}u = \frac{q}{5}.$$

Let $R \subset I$ and $S \subset A$ be random sets where each vertex (of I and A respectively) is included independently with probability p. By Markov's inequality, $|R|,|S| \leqslant 5pn \leqslant qn$ with probability at least $3/5$. Let $T = \Gamma_j(R,S) \cap I$. Then clearly, $I \subset C_j(R,S,T)$ provided

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$n - |\Gamma_j(R,S)| + |T| \leqslant (1-c)n$. So to complete the proof, it is enough to show that the inequalities $|T| \leqslant qn$ and $|\Gamma_j(R,S)| \geqslant (c+q)n$ each hold with probability at least 4/5.

A vertex $v \in I$ will be included in $\Gamma_j(R,S)$ (*i.e.*, in T) if it lies in a $(j+1,A)$-edge e with $e = \{v\} \cup f \cup g$, $f \in R^{(j)}$, $g \in S^{(r-j-1)}$. The number of $(j+1,A)$-edges is at most nbu^{j+1}. For each such edge e and each vertex $v \in e$ there are are most 2^{r-1} ways to partition $e - \{v\}$ as $f \cup g$, and for each such partition, the probability that both $f \in R^{(j)}$ and $g \in S^{(r-j-1)}$ is p^{r-1}. So the expected size of T is at most

$$r2^{r-1}p^{r-1}nbu^{j+1} = 3r2^{r-1}un = qn/5.$$

Applying Markov's inequality again implies that $|T| \leqslant qn$ with probability at least 4/5.

Now fix a vertex $d \in D$. This lies in at least bu^j edges e of the form $e = \{d\} \cup f \cup g$ with $f \in I^{(j)}$ and $g \in A^{(r-j-1)}$. For each e with $d \in e$, fix such a partition $e \setminus \{d\} = f \cup g$. The probability that $f \subset R$ and $g \subset S$ is p^{r-1} and, because G is simple, these events over all e are independent. Hence the probability that $d \notin \Gamma_j(R,S)$ is at most

$$(1-p^{r-1})^{bu^j} \leqslant \exp\{-p^{r-1}bu^j\} = \exp\{-3\} < 1/20.$$

Markov's inequality implies that with probability at least 4/5, the number of vertices from D not in $\Gamma_j(R,S)$ is at most $|D|/4$. Hence with probability at least 4/5, $|\Gamma_j(R,S)| \geqslant 3|D|/4 \geqslant \lfloor 3cn/2 \rfloor \geqslant (c+q)n$ if b is large. This completes the proof. □

Now we can easily derive Theorem 1.1.

Proof of Theorem 1.1. Let $\epsilon > 0$ and $c = 1/2r^2 - \epsilon$. Let G be a simple d-regular r-uniform hypergraph and let $A \subset V(G)$ satisfy $|A| \geqslant (1-2rc)n$. Let e be the number of edges inside A and let f be the number meeting both A and $V(G) \setminus A$. Then $er + f(r-1) \geqslant |A|d \geqslant (1-2rc)nd$, because G is d-regular. On the other hand, $f \leqslant |V(G) \setminus A|d \leqslant 2rcnd$. Thus $er + 2r(r-1)cnd \geqslant (1-2rc)nd$, so $er \geqslant (1-2r^2c)nd = 2r^2\epsilon nd$.

Hence, provided d is large enough (depending on r and ϵ) we may apply Theorem 3.1 to G with $b = 2\epsilon rd$ to obtain a collection $\mathcal{C}$ covering the independent sets of G, with $|\mathcal{C}| \leqslant 2^{n/k}$ where $k = b^{1/2r}$. Any vertex colouring $A_1, \ldots, A_t$ with t colours is $\mathcal{C}$-compatible. Thus Theorem 2.1 now shows that there are lists $\{L_v : v \in V(G)\}$, each of size $(1+o(1))\log k/\log(1/c)$, admitting no proper vertex colouring.

Hence

$$\chi_\ell(G) \geqslant (1+o(1))\log k/\log(1/c) = (1/2r + o(1))\log d/[\log(2r^2) - \log(1-2\epsilon r^2)]$$

if d is large. This is true for every $\epsilon > 0$, and the theorem follows. □

4. Further remarks

We finish with some remarks about extensions to Theorem 1.1.

4.1. Hypergraphs that are not regular

It is easy to see that the proof of Theorem 1.1 works just as well for simple hypergraphs whose degrees are all in the range d/C to Cd for some constant C, at the expense of reducing the value of the constant $1/2r\log(2r^2)$ in the theorem. Hence the list chromatic number of such a graph is $\Omega(\log d)$. The same is true for any graph containing a subgraph whose degrees are in the range d/C to Cd, or in the range d^δ/C to Cd^δ for any constant δ. However, so far as we can tell, not all simple hypergraphs of average degree d contain such a subgraph. Indeed it appears that extending Theorem 1.1 to hypergraphs of average degree d is not at all straightforward. A similar difficulty was noted in [3]. A proof for hypergraphs of average degree d is expected to be much more complicated than the one here; we hope to attempt it elsewhere [8].

4.2. Colouring by hereditary properties

The role of independent sets in the proof of Theorem 3.1 is very slight: indeed, the fact that I is an independent set is used only once in the proof, which is to guarantee that $P(r)$ is false. Now $P(r)$ is the claim that I contains at least nbu^r edges, and this is false even if I contains $o(nb^{(r-2)/2(r-1)})$ edges.

A *hereditary property* $\mathcal{P}$ of r-uniform hypergraphs is one closed under isomorphism and under taking induced subgraphs. Every hypergraph G has a *$\mathcal{P}$-chromatic number* $\chi^{\mathcal{P}}(G)$, which is the smallest value of t for which G has a partition $A_1,\ldots,A_t$ such that each set A_i induces a subgraph having property $\mathcal{P}$. The *$\mathcal{P}$-list chromatic number* $\chi_\ell^{\mathcal{P}}(G)$ is defined similarly.

It follows from the proof of Theorem 1.1 that if $\mathcal{P}$ is a property such that the average degree of a hypergraph having $\mathcal{P}$ must be, say, $o(d^{(r-2)/2(r-1)})$, then $\chi_\ell^{\mathcal{P}}(G) = \Omega(\log d)$ for simple r-uniform d-regular hypergraphs G. There is plenty of room in the proof of Theorem 3.1 to permit a similar result even if $\mathcal{P}$ allows larger average degrees. Again, we hope to be able to say more in [8].

4.3. Hypergraphs that are not simple

The *co-degree* of two vertices of a hypergraph is the number of edges containing them both, and the *maximum co-degree* $D(G)$ is the maximum co-degree taken over all pairs of vertices. Thus G is simple if and only if $D(G) = 1$.

Examples given in [3] show that the list chromatic number of a d-regular hypergraph can be as low as $O(\log(d/D(G)))$. We show that it cannot be smaller.

Corollary 4.1. *Let G be an r-uniform d-regular hypergraph. If $d/D(G)$ is large then $\chi_\ell(G) = \Omega(\log(d/D(G)))$.*

Proof. We only sketch the proof, because once again we hope to give a stronger result in [8]. Let $p = 1/8r^2D$ and let G' be a sub-hypergraph of G formed by choosing edges independently with probability p. Let $c = 1/4r^2$. Given $A \subset V(G)$ with $|A| \geqslant (1-1/c)n$ then, as in the proof of Theorem 1.1, there must be at least $nd/2r$ edges of G within A. The expected number of edges G' within A is thus at least $pnd/2r$, and the probability that there are fewer than half this many is at most $\exp\{-pnd/16r\}$, by standard estimates. Hence the probability that some subset of A contains fewer than $pnd/4r$ edges of G' is at most $2^n\exp\{-pnd/16r\} < 1/2$, since $pd/16r = d/128r^3D$ is large.

A *butterfly* is two edges having more than one vertex in common. Each edge is in at most $\binom{r}{2}D$ butterflies so the number of butterflies in G is at most $ndDr/2$. Let B be the number of butterflies in G'; then the expected value of B is at most $p^2 ndDr/2 = pnd/16r$. Hence $B \leqslant pnd/8r$ holds with probability at least 1/2. From this and the previous paragraph it follows that there is a choice of G' with $B \leqslant pnd/8r$ and for which every subset A with $|A| \geqslant (1-1/c)n$ contains at least $pnd/4r$ edges of G'.

Take such a G' and remove an edge from each butterfly to form a simple sub-hypergraph G''. Each subset A contains at least $pnd/8r = nd/64r^3D$ edges of G''. By Theorems 3.1 and 2.1 it follows that $\chi_\ell(G) \geqslant \chi_\ell(G'') = \Omega(\log(d/D))$. □

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