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PAPER

Triposes as a generalization of localic geometric morphisms

Jonas Frey[1] and Thomas Streicher[2*]

[1]Department of Philosophy, Carnegie Mellon University, Pittsburgh, PA 15213, USA and [2]Fachbereich Mathematik, Technische Universität Darmstadt, Darmstadt, Germany
*Corresponding author. Email: streicher@mathematik.tu-darmstadt.de

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Abstract
In Hyland et al. (1980), Hyland, Johnstone and Pitts introduced the notion of *tripos* for the purpose of organizing the construction of realizability toposes in a way that generalizes the construction of localic toposes from complete Heyting algebras. In Pitts (2002), one finds a generalization of this notion eliminating an unnecessary assumption of Hyland et al. (1980). The aim of this paper is to characterize triposes over a base topos $\mathcal{S}$ in terms of so-called *constant objects* functors from $\mathcal{S}$ to some elementary topos. Our characterization is slightly different from the one in Pitts's PhD Thesis (Pitts, 1981) and motivated by the fibered view of geometric morphisms as described in Streicher (2020). In particular, we discuss the question whether triposes over **Set** giving rise to equivalent toposes are already equivalent as triposes.

Keywords: Fibered categories; triposes; toposes

Dedicated to the Memory of Martin Hofmann

1. Introduction

As described in Johnstone (1977), localic geometric morphisms to a topos $\mathcal{S}$ are given by functors F from $\mathcal{S}$ to some topos $\mathcal{E}$ such that

(1) F preserves finite limits,
(2) every object $A \in \mathcal{E}$ appears as subquotient of some FI, and
(3) F has a right adjoint.

In the appendix of Streicher (2020), one finds a proof of Jibladze's Theorem (Jibladze, 1988) saying that fibered toposes over $\mathcal{S}$ having internal sums correspond to finite limit preserving functors from $\mathcal{S}$ to some topos $\mathcal{E}$. In particular, a finite limit preserving functor $F : \mathcal{S} \to \mathcal{E}$ gives rise to the fibration $P_F = F^* P_{\mathcal{E}}$ over $\mathcal{S}$ obtained by change of base along F from the fundamental ("codomain") fibration $P_{\mathcal{E}} = \mathsf{cod} : \mathcal{E}^{\mathbb{2}} \to \mathcal{E}$ (where $\mathbb{2}$ is the small category $0 \to 1$ corresponding to the ordinal 2). But every fibered topos $P : \mathcal{X} \to \mathbf{Set}$ with internal sums is equivalent to P_Δ where $\mathcal{E}$ is the fiber of P over 1 and $\Delta : \mathcal{S} \to \mathcal{E}$ sends $I \in \mathcal{S}$ to $\Delta(I) = \coprod_I 1_I$ in $\mathcal{E}$.

Moreover, as also shown in Streicher (2020) for terminal object preserving $F : \mathcal{S} \to \mathcal{E}$, the fibration P_F is locally small iff F has a right adjoint. Thus, as observed by Bénabou already in his 1974 Montreal lectures, inverse image parts of geometric morphisms correspond to terminal object preserving functors F between toposes such that the fibration P_F has internal sums and is locally small.

Moreover, as also observed in Streicher (2020) for a finite limit preserving functor $F : \mathcal{S} \to \mathcal{E}$ between toposes condition (2) is equivalent to the requirement that every map $a : A \to FI$ in $\mathcal{E}$ fits into a commuting diagram

$$\begin{array}{ccc} C & \xrightarrow{e} & A \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle a} \\ FJ & \xrightarrow[Fu]{} & FI \end{array}$$

where e is epic and m is monic. Obviously, this condition entails (2) instantiating I by a terminal object. For the reverse direction, choose $m : C \rightarrowtail FJ$ and $e : C \twoheadrightarrow A$ (which exist by condition (2)) and observe that

$$\begin{array}{ccccc} & & C & \xrightarrow{e} & A \\ & \swarrow{\scriptstyle m} & \downarrow{\scriptstyle \langle m, ae\rangle} & & \downarrow{\scriptstyle a} \\ FJ & \xleftarrow[F\pi_1]{} & F(J \times I) & \xrightarrow[F\pi_2]{} & FI \end{array}$$

using the assumption that F preserves finite limits and thus finite products. Thus, condition (2) amounts to the requirement that every object of $\mathcal{E}$ can be covered by a(n internal) sum of subterminals (in the appropriate fibrational sense!). As observed in Streicher (2020) under assumption (3) this is equivalent to the requirement that g in

$$\begin{array}{ccc} G & \longrightarrow & 1_{\mathcal{E}} \\ {\scriptstyle g}\downarrow & \lrcorner & \downarrow{\scriptstyle \top_{\mathcal{E}}} \\ FU\Omega_{\mathcal{E}} & \xrightarrow[\varepsilon_{\Omega_{\mathcal{E}}}]{} & \Omega_{\mathcal{E}} \end{array}$$

is a generating family for the fibration P_F (where U is right adjoint to F).

2. A Fibrational Account of Triposes

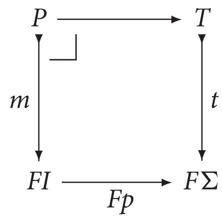
In Hyland et al. (1980) Hyland, Johnstone and Pitts have identified a notion of fibered preorder $\mathscr{P}$ over a base topos $\mathcal{S}$ giving rise to a topos $\mathcal{S}[\mathscr{P}]$ by "adding subquotients" related to the base topos via a *constant object functor* $\Delta_{\mathscr{P}} : \mathcal{S} \to \mathcal{S}[\mathscr{P}]$ satisfying conditions (1) and (2) of the previous section. As obvious from the considerations in *loc.cit.*, one may get back the fibered preorder $\mathscr{P}$ from the subobject fibration $\mathsf{Sub}_{\mathcal{S}[\mathscr{P}]}$ by change of base along $\Delta_{\mathscr{P}}$. For this reason, such $\mathscr{P}$ were called "topos representing indexed pre-ordered sets" suggesting the acronym "tripos" (echoing the traditional name for final exams at University of Cambridge).

The original definition in Hyland et al. (1980) required triposes to have a *generic family*, i.e. a $T \in \mathscr{P}(\Sigma)$ from which all objects in $\mathscr{P}(I)$ may be obtained (up to isomorphim) by reindexing along an appropriate (generally not unique) map $I \to \Sigma$. In Pitts (2002), Pitts observed that the requirement of a generic family can be replaced by a weaker "comprehension axiom" (Pitts, 2002, Axiom 4.1) which still implies – and is actually equivalent to – the fact that the category $\mathcal{S}[\mathscr{P}]$ of partial equivalence relations is a topos.[1]

In this work, we focus on triposes in this more general sense, and contrary to the literature, we use the word "tripos" not for the fibered preorder, but for the associated constant objects functor (from which the fibered preorder can be reconstructed as pointed out above). This way we obtain a definition that is a straightforward generalization of the notion of localic geometric morphism as presented in the introduction. Pitts's comprehension axiom is then presented as Lemma 2.2. We refer to (constant objects functors arising from) the more restrictive notion of Hyland et al. (1980) as "traditional triposes".

Definition 2.1. *A* tripos *over a topos $\mathcal{S}$ is a finite limit preserving functor F from $\mathcal{S}$ to a topos $\mathcal{E}$ such that every $A \in \mathcal{E}$ appears as subquotient of FI for some $I \in \mathcal{S}$.*

A tripos is called traditional *if the fibered preorder $\mathscr{P}_F = F^*\mathsf{Sub}_{\mathcal{E}}$ admits a generic family, i.e. there is a a mono $t : T \rightarrowtail F\Sigma$ such that every mono $m : P \rightarrowtail FI$ fits into a pullback diagram*



for some (typically not unique) $p : I \to \Sigma$.

The fibered preorder $\mathscr{P}_F$ is a first-order hyperdoctrine, i.e. a fibration of Heyting algebras with internal sums and products, since $\mathsf{Sub}_{\mathcal{E}}$ has and change of base along the finite limit preserving functor F preserves the required properties (see Streicher 2020). Recalling Pitts's proof of the comprehension axiom, we now show that $\mathscr{P}_F$ admits an interpretation higher order (intuitionistic) logic.

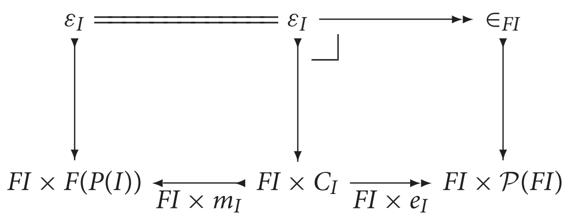
Lemma 2.2. *For every tripos $F : \mathcal{S} \to \mathcal{E}$, the fibered preorder $\mathscr{P}_F$ satisfies the following* comprehension axiom.

(CA) *For every object I in $\mathcal{S}$, there is are objects $P(I)$ in $\mathcal{S}$ and $\varepsilon_I \in \mathscr{P}_F(I \times P(I))$ such that for all J in $\mathcal{S}$ and $\rho \in \mathscr{P}_F(I \times J)$, the formula*

$$\forall j \in J.\exists p \in P(I).\forall i \in I.\ \rho(i,j) \leftrightarrow i\, \varepsilon_I\, p$$

holds in the internal logic of $\mathscr{P}_F$.

Proof. Let I be an object of $\mathcal{S}$. Then since $\mathscr{P}$ is a tripos there is an object $P(I)$ in $\mathcal{S}$ such that $\mathcal{P}(FI)$ appears as subquotient of $F(P(I))$, i.e. there is a subobject $m_I : C_I \rightarrowtail F(P(I))$ such that there exists an epi $e_I : C_I \twoheadrightarrow \mathcal{P}(FI)$. Consider



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giving rise to a subobject ε_I of $F(I \times P(I)) \cong FI \times F(P(I))$. Since the left square in the above diagram is a pullback, we have $(FI \times m_I)^* \varepsilon_I = (FI \times e_I)^* \in_{FI}$.

Suppose $\rho : R \rightarrowtail F(I \times J) \cong FI \times FJ$. Then

$$\begin{array}{ccc} R & \longrightarrow & \in_{FI} \\ {\scriptstyle \rho} \downarrow & \lrcorner & \downarrow \\ FI \times FJ & \xrightarrow[FI \times r]{} & FI \times \mathcal{P}(FI) \end{array}$$

for a unique $r : FJ \to \mathcal{P}(FI)$. Consider the pullback

$$\begin{array}{ccc} C & \xrightarrow{\widetilde{r}} & C_I \\ {\scriptstyle e} \downarrow & \lrcorner & \downarrow {\scriptstyle e_I} \\ FJ & \xrightarrow[r]{} & \mathcal{P}(FI) \end{array}$$

where e is epic since in a topos epis are stable under arbitrary pullbacks. Thus, we have

$$\begin{aligned} (FI \times e)^* \rho &= (FI \times e)^* (FI \times r)^* \in_{FI} \cong (FI \times er)^* \in_{FI} = \\ &= (FI \times e_I \widetilde{r})^* \in_{FI} \cong (FI \times \widetilde{r})^* (FI \times e_I)^* \in_{FI} = \\ &= (FI \times \widetilde{r})^* (FI \times m_I)^* \varepsilon_I \cong \\ &\cong (FI \times m_I \widetilde{r})^* \varepsilon_I \end{aligned}$$

from which it readily follows that

$$\forall j \in J. \exists p \in P(I). \forall i \in I.\ \rho(i, j) \leftrightarrow i\, \varepsilon_I\, p$$

holds in the internal logic of $\mathscr{P}_F$. □

From Lemma 2.2 and the results of Pitts (2002), it follows that fibered preorders of the form $F^*\mathsf{Sub}_{\mathcal{E}}$ for some tripos $F : \mathcal{S} \to \mathcal{E}$ may up to equivalence be characterized as Heyting algebras $\mathscr{P}$ fibered over $\mathcal{S}$ with internal sums $\exists$ and internal products $\forall$ satisfying the comprehension axiom (CA).

A tripos $F : \mathcal{S} \to \mathcal{E}$ is traditional iff $\mathscr{P}_F$ is a tripos in the sense of Hyland et al. (1980), i.e. there exists a $T \in \mathscr{P}(\Sigma)$ from which all $P \in \mathscr{P}(I)$ can be obtained by reindexing along some map $p : I \to \Sigma$.

If a tripos F has a right adjoint (and thus is the inverse image part of a localic geometric morphism) then it is always traditional since we can set $\Sigma = U\Omega_{\mathcal{E}}$, and in this case, the maps p are unique.

Finally, we note that F is the inverse image part of a localic geometric morphism iff $F^*\mathsf{Sub}_{\mathcal{E}}$ is locally small iff P_F is locally small (see Streicher 2020).

3. Constant Objects Functors are not Unique

For arbitrary base toposes $\mathcal{S}$ triposes $F, G : \mathcal{S} \to \mathcal{E}$ need not be equivalent since if $\mathcal{S}$ is $\mathsf{Sh}(X)$ and $\mathcal{E}$ is $\mathsf{Sh}(Y)$ for some sober spaces X and Y then there are at least as many triposes $\mathcal{S} \to \mathcal{E}$ (up to equivalence) as there are continuous maps from Y to X. But even if $\mathcal{S}$ is **Set**, there are in general many nonequivalent triposes over **Set** giving rise to the same topos as shown by the following simple counterexample.

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Theorem 3.1. *For every natural number $n > 0$ the functor $F_n : \mathbf{Set} \to \mathbf{Set} : I \mapsto I^n$ is a tripos. The triposes $F_n^*\mathsf{Sub}_{\mathbf{Set}}$ and $F_m^*\mathsf{Sub}_{\mathbf{Set}}$ are equivalent if and only if $n = m$.*

Proof. Obviously, the F_n preserve finite limits since they are right adjoints and every $I \in \mathbf{Set}$ appears as split subobject of $F_n(I)$. Thus, all F_n are triposes but $F_n^*\mathsf{Sub}_{\mathbf{Set}}$ and $F_m^*\mathsf{Sub}_{\mathbf{Set}}$ are equivalent as triposes if and only if $n = m$ since the latter is equivalent to $2^n = 2^m$ which, in turn, is equivalent to $F_n^*\mathsf{Sub}_{\mathbf{Set}}(2) \simeq F_m^*\mathsf{Sub}_{\mathbf{Set}}(2)$. □

Notice, however, that F_n is a traditional tripos if and only if $n = 1$. Thus, it may still be the case that there exist traditional triposes $F, G : \mathcal{S} \to \mathcal{E}$, which are not equivalent as triposes. Unfortunately, we have not been able so far to find examples of nonequivalent *traditional* triposes $\mathscr{P}_1$ and $\mathscr{P}_2$ over **Set** such that the ensuing toposes $\mathbf{Set}[\mathscr{P}_1]$ and $\mathbf{Set}[\mathscr{P}_2]$ are equivalent. However, though a bit annoying, we can't find this as a major problem since our weak notion of tripos is conceptually more adequate than the traditional one because from a logical point of view adding the comprehension axiom to first-order posetal hyperdoctrines appears much more natural than requiring that they are witnessed by Skolem functions in the base $\mathcal{S}$, i.e. requiring for all $\rho \in \mathscr{P}(I \times J)$ the existence of a function $r : J \to P(I)$ such that

$$\forall j \in J. \forall i \in I.\ \rho(i, j) \leftrightarrow i\, \varepsilon_I\, r(j)$$

holds in the logic of $\mathscr{P}$. At the end of Pitts (2002) the author expresses a similar view in a slightly more cautious way.

Finally, we observe that triposes over **Set** may give rise to non-localic Grothendieck toposes. Let $\mathcal{E}$ be the topos of reflexive graphs, i.e. presheaves over the three-element monoid $\Delta([1], [1])$ of monotone endomaps of the ordinal 2. As observed by Lawvere, the global elements functor $\Gamma : \mathcal{E} \to \mathbf{Set}$ fits into a sequence of adjoints $\Pi \dashv \Delta \dashv \Gamma \dashv \nabla : \mathbf{Set} \hookrightarrow \mathcal{E}$. The rightmost functor ∇ preserves all limits since it has a left adjoint. Subobjects of objects of the form $\nabla(I)$ are up to isomorphism precisely those reflexive graphs where between two nodes there is at most one edge (i.e. directed graphs as traditionally considered in combinatorics!). But since any reflexive graph can be covered by such a traditional directed graph every object of $\mathcal{E}$ appears as subquotient of some $\nabla(I)$ for which reason ∇ is a tripos over **Set** though it is not the inverse image part of a geometric morphism.

4. Regular Triposes

It is well known that a morphism $e : Y \to X$ in an elementary topos $\mathcal{E}$ is epic iff the pullback functor $e^* : \mathsf{Sub}_{\mathcal{E}}(X) \to \mathsf{Sub}_{\mathcal{E}}(Y)$ reflects maximal subobjects, i.e. a mono $m : P \rightarrowtail X$ in $\mathcal{E}$ is an iso already if e^*m is an iso. Recall that a preorder fibered over a regular category is a *prestack* (w.r.t. the regular cover topology) iff for all regular epis e reindexing along it (preserves and) reflects the order. Thus, for a tripos $F : \mathcal{S} \to \mathcal{E}$ the fibered preorder $F^*\mathsf{Sub}_{\mathcal{E}}$ is a prestack iff F preserves (regular) epis.

This observation strongly suggests to require that triposes $F : \mathcal{S} \to \mathcal{E}$ also preserve epis since it vacuously holds when $\mathcal{S}$ is **Set** (since in **Set** all epis are split as ensured by the axiom of choice!) and, moreover, by Lemma 6.1 ("Pitts's Iteration Theorem") of Pitts (1981) triposes preserving epis are closed under composition.

Definition 4.1. *A tripos $F : \mathcal{S} \to \mathcal{E}$ is called* regular *iff F preserves epis.*

Recall that a functor between regular categories is called *regular* iff it preserves finite limits and regular epis. Thus, regular triposes are regular functors $F : \mathcal{S} \to \mathcal{E}$ between toposes such that every $A \in \mathcal{E}$ appears as subquotient of FI for some $I \in \mathcal{S}$.

From Proposition 3.14 of Pitts (1981), it follows that a traditional tripos $F : \mathcal{S} \to \mathcal{E}$ is regular iff it has "fibrewise quantification", i.e. there are maps $\bigvee, \bigwedge : \Omega_{\mathcal{S}}^{\Sigma} \to \Sigma$ such that $\exists_{Fu}(Fp)^* t$ and $\forall_{Fu}(Fp)^* t$ appear as pullbacks of $t : T \rightarrowtail F\Sigma$ along $F(\lambda i{:}I.\ \bigvee\{p(j) \mid u(j) = i\})$ and $F(\lambda i{:}I.\ \bigwedge\{p(j) \mid u(j) = i\})$, respectively, for all $u : J \to I$ and $p : J \to \Sigma$.

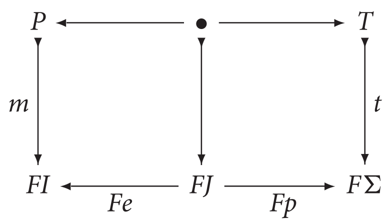
Theorem 4.2. *Let $F_1 : \mathcal{S} \to \mathcal{E}_1$ and $F_2 : \mathcal{S} \to \mathcal{E}_2$ be triposes and $H : F_1 \to F_2$, i.e. $H : \mathcal{E}_1 \to \mathcal{E}_2$ with $F_2 = HF_1$. Then, H is a tripos iff H preserves finite limits and H is a regular tripos iff H is a regular functor.*

Proof. The forward directions are trivial. For the backwards directions suppose $A \in \mathcal{E}_2$. Then, since F_2 is a tripos there exists a subobject $m : C \rightarrowtail F_2 I$ and an epi $e : C \twoheadrightarrow A$. Since $F_2 = HF_1$ we have $m : C \rightarrowtail H(F_1 I)$ and $e : C \twoheadrightarrow A$. Thus, we have shown that H validates the second condition required for a tripos. □

The previous theorem for regular triposes $F_1 : \mathcal{S} \to \mathcal{E}_1$ and $F_2 : \mathcal{S} \to \mathcal{E}_2$ suggests that the right notion of morphism from F_1 to F_2 is a functor $H : F_1 \to F_2$ such that $H : \mathcal{E}_1 \to \mathcal{E}_2$ is regular since for this definition morphisms to a regular tripos $F : \mathcal{S} \to \mathcal{E}$ coincide with regular triposes over $\mathcal{E}$.

In the subsequent Theorem 4.4, we will show that morphisms between traditional regular triposes are precisely the traditional regular triposes. But for this purpose, we need the following lemma characterizing traditional regular triposes among regular triposes in terms of a condition which at first sight looks weaker than the one given in Definition 2.1.

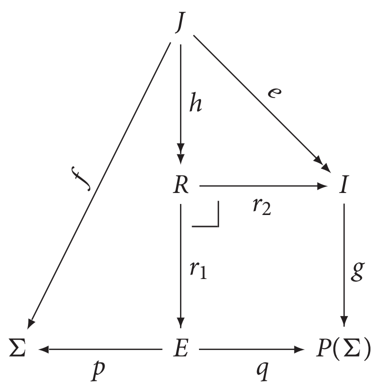
Lemma 4.3. *Let $F : \mathcal{S} \to \mathcal{E}$ be a regular tripos and $t : T \rightarrowtail F\Sigma$ be weakly generic for $F^*\mathsf{Sub}_{\mathcal{E}}$, i.e. every mono $m : P \rightarrowtail FI$ fits into a diagram*



where both squares are pullbacks and $e : J \to I$ is epic, then F is a traditional tripos.

Proof. Suppose $t : T \rightarrowtail F\Sigma$ is a weakly generic family for $F^*\mathsf{Sub}_{\mathcal{E}}$. Let $E = \{(u, U) \in \Sigma \times P(\Sigma) \mid u \in U\}$ and $p : E \to \Sigma$ and $q : E \to P(\Sigma)$ the respective projection maps. We will show that $\exists_q p^* t$ is a generic family for $F^*\mathsf{Sub}_{\mathcal{E}}$.

For this purpose suppose $m \in \mathsf{Sub}_{\mathcal{E}}(FI)$. By assumption there are $e : J \twoheadrightarrow I$ and $f : J \to \Sigma$ such that $e^* m \cong f^* t$. Since $F^*\mathsf{Sub}_{\mathcal{E}}$ is a prestack w.r.t. the regular cover topology we have $m \cong \exists_e e^* m \cong \exists_e f^* t$. Let $g : I \to P(\Sigma)$ with $g(i) = \{f(j) \mid e(j) = i\}$. Obviously, the map $\langle f, ge\rangle$ factors through $\langle p, q\rangle$ since $f(j) \in g(e(j))$. Consider the following diagram



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where $R = \{(u, U, i) \in \Sigma \times P(\Sigma) \times I \mid u \in U = g(i)\}$ with r_1 and r_2 the respective projections and $h(j) = (f(j), g(e(j)), e(j))$. Notice that h is onto since $g(i) = \{f(j) \mid e(j) = i\}$. We have

$$\begin{aligned} g^*\exists_q p^* t &\cong \exists_{r_2} r_1^* p^* t && \text{by Beck–Chevalley condition} \\ &\cong \exists_{r_2}\exists_h h^* r_1^* p^* t && \text{since } h \text{ is epic} \\ &\cong \exists_e f^* t && \text{since } e = r_2 h \text{ and } f = pr_1 h \\ &\cong m \end{aligned}$$

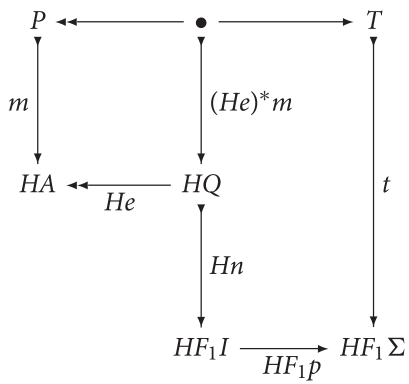
as desired. □

Theorem 4.4. *Let $F_1 : \mathcal{S} \to \mathcal{E}_1$ be a traditional regular tripos and $H : \mathcal{E}_1 \to \mathcal{E}_2$ a regular functor between toposes. Then H is a traditional regular tripos if and only if $F_2 = HF_1$ is a traditional regular tripos.*

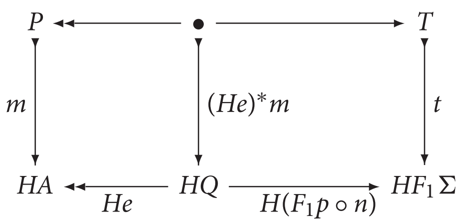
Proof. The forward direction is Pitts's Iteration Theorem.

For the backward direction suppose that $H : \mathcal{E}_1 \to \mathcal{E}_2$ is a regular functor such that $F_2 = HF_1$ is a traditional regular tripos. By Theorem 4.2 it is immediate that H is a regular tripos, too. Since F_2 has been assumed to be a traditional tripos there is a $t : T \rightarrowtail F_2\Sigma$ generic for $F_2^*\mathsf{Sub}_{\mathcal{E}_2}$. For showing that H is a traditional tripos it suffices by Lemma 4.3 to show that $t : T \rightarrowtail HF_1\Sigma$ is weakly generic for $H^*\mathsf{Sub}_{\mathcal{E}_2}$.

Suppose $m : P \rightarrowtail HA$ for some $A \in \mathcal{E}_1$. Since F_1 is a traditional tripos there exist $n : Q \rightarrowtail F_1 I$ and $e : Q \twoheadrightarrow A$ for some $I \in \mathcal{S}$. Since $F_2 = HF_1$ is a traditional tripos there exists $p : I \to \Sigma$ such that $Hn \circ (He)^* m$ arises as pullback of t along $HF_1 p$ for some $p : I \to \Sigma$. Thus we have



from which it follows that $(He)^* m$ arises as pullback of t along $H(F_1 p \circ n) = HF_1 p \circ Hn$. Thus, we have



where both squares are pullbacks as required. □

We conclude this section with some observations on the

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4.1 Preservation of assemblies by tripos morphisms

Following van Oosten (2008) for a tripos $F : \mathcal{S} \to \mathcal{E}$ one may define *assemblies* as those objects of $\mathcal{E}$, which appear as subobjects of some FI. If $G : \mathcal{S} \to \mathcal{F}$ is a tripos and $H : F \to G$ such that $H : \mathcal{E} \to \mathcal{F}$ preserves finite limits then H preserves assemblies, i.e. sends assemblies w.r.t. F to assemblies w.r.t. G, since $Hm : HP \rightarrowtail HFI = GI$ whenever $m : P \rightarrowtail FI$. It follows from the definition of tripos that every object A of $\mathcal{E}$ appears as subquotient of some FI, i.e. we have $A \overset{e}{\twoheadleftarrow} C \overset{m}{\rightarrowtail} FI$. If $H : F \to G$ is a regular functor between triposes then $HA \overset{He}{\twoheadleftarrow} HC \overset{Hm}{\rightarrowtail} HFI = GI$, i.e. H preserves coverings of objects by assemblies in a very strong sense.

5. Relation to Miquel's Implicative Algebras

In Miquel (2020b), Miquel has shown that traditional triposes over **Set** correspond to so-called *implicative algebras* (Miquel, 2020a).

Definition 5.1. *An* implicative structure *is a complete lattice $\mathcal{A}$ together with an operation $\to : \mathcal{A}^{\mathsf{op}} \times \mathcal{A} \to \mathcal{A}$ such that $x \to \bigwedge Y = \bigwedge_{y \in Y} (x \to y)$ for all $x \in \mathcal{A}$ and $Y \subseteq \mathcal{A}$. Then $K_{\mathcal{A}} = \bigwedge_{x,y \in \mathcal{A}} x \to y \to x$ and $S_{\mathcal{A}} = \bigwedge_{x,y,z \in \mathcal{A}} (x \to y \to z) \to (x \to y) \to x \to z$ are elements of $\mathcal{A}$.*

A separator *in an implicative structure $(\mathcal{A}, \to)$ is an upward closed subset $\mathcal{S}$ of $\mathcal{A}$ such that $K_{\mathcal{A}}, S_{\mathcal{A}} \in \mathcal{S}$ and $\mathcal{S}$ is closed under* modus ponens, *i.e. $b \in \mathcal{S}$ whenever $a \in \mathcal{S}$ and $a \to b \in \mathcal{S}$.*

An implicative algebra *is a triple $(\mathcal{A}, \to, \mathcal{S})$ such that $(\mathcal{A}, \to)$ is an implicative structure and $\mathcal{S}$ is a separator in $(\mathcal{A}, \to)$.*

With every implicative algebra $\mathcal{A}$ one associates a **Set**-based tripos $\mathscr{P}^{\mathcal{A}}$ where $\mathscr{P}^{\mathcal{A}}(I)$ is the preorder $\vdash_I$ on $\mathcal{A}^I$ defined as

$$\varphi \vdash_I \psi \quad \text{iff} \quad \bigwedge_{i \in I} (\varphi_i \to \psi_i) \in \mathcal{S}$$

and reindexing is given by precomposition.

In Miquel (2020b), Miquel has shown that every traditional regular tripos over **Set** is equivalent to $\mathscr{P}^{\mathcal{A}}$ for some implicative algebra $\mathcal{A}$.

For i=1, 2 let $F_i : \mathbf{Set} \to \mathcal{E}_i$ be the constant objects functor for the regular tripos induced by an implicative algebra $\mathcal{A}_i$ in **Set**, i.e. $\mathcal{E}_i = \mathbf{Set}[\mathscr{P}^{\mathcal{A}_i}]$. Due to the remark in Subsection 4.1 regular functors $G : \mathcal{E}_1 \to \mathcal{E}_2$ with $F_2 = GF_1$ correspond to cartesian functors $g : F_1^* \mathsf{Sub}_{\mathcal{E}_1} \to F_2^* \mathsf{Sub}_{\mathcal{E}_2}$ preserving regular logic, i.e. finite limits and existential quantification. Obviously, such g are uniquely determined by $h = g_{\mathcal{A}_1}(\mathrm{id}_{\mathcal{A}_1}) : \mathcal{A}_1 \to \mathcal{A}_2$ since $g_I(\varphi : I \to \mathcal{A}_1) = h \circ \varphi$. This suggests to define a morphism of implicative algebras from $\mathcal{A}_1$ to $\mathcal{A}_2$ as a function $h : \mathcal{A}_1 \to \mathcal{A}_2$ such that the cartesian functor $g : F_1^* \mathsf{Sub}_{\mathcal{E}_1} \to F_2^* \mathsf{Sub}_{\mathcal{E}_2}$ given by $g_I(\varphi : I \to \mathcal{A}_1) = h \circ \varphi$ preserves regular logic, i.e. finite limits and existential quantification.

Unfortunately, Miquel's result from Miquel (2020b) does not extend to arbitrary base toposes. The reason is that for a traditional regular tripos $F : \mathcal{S} \to \mathcal{E}$ there need not exist a subobject S of Σ such that

(1) its characteristic map $\chi_S : \Sigma \to \Omega_{\mathcal{S}}$ induces by postcomposition a cartesian functor $\gamma_S : F^* \mathsf{Sub}_{\mathcal{E}} \to \mathsf{Sub}_{\mathcal{S}}$ preserving finite meets in each fiber and
(2) $u : 1 \to \Sigma$ factors through S iff $(Fu)^* t$ is isomorphic to id_{F1}.

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Notice that the first condition means that $u^*S \leq v^*S$ whenever $F(u)^*t \leq F(v)^*t$ and that $t \in S$ and $\forall u, v : \Sigma.\, u \wedge v \in S \leftrightarrow (u \in S \wedge v \in S)$ hold in the internal logic of $\mathcal{S}$. For base toposes $\mathcal{S}$ which are not well-pointed such S need neither exist nor be unique (for the latter, see Example 4.12.12 of Frey 2013 for a counterexample[2]).

The related stronger condition that $u : I \to \Sigma$ factors through S iff $F(u)^*t$ is isomorphic to id_{FI} is known as "definability of truth", i.e. that the full subfibration of $F^*\mathsf{Sub}_{\mathcal{E}}$ on true predicates is definable in the sense of Bénabou (see Section 12 of Streicher 2020). This stronger condition, however, amounts to the requirement that the fibration $F^*\mathsf{Sub}_{\mathcal{E}}$ is locally small, i.e. equivalent to the externalization of a complete Heyting algebra internal to $\mathcal{S}$, which in turn is equivalent to the requirement that F is the inverse image part of a localic geometric morphism.

6. Summary and Conclusion

We have shown in which sense (generalized) triposes in the sense of Pitts (1981) may be understood as a generalizations of localic geometric morphisms. The traditional triposes of Hyland et al. (1980) can be characterized as those triposes $F : \mathcal{S} \to \mathcal{E}$ for which the fibered preorder $F^*\mathsf{Sub}_{\mathcal{E}}$ admits a generic family $t : T \rightarrowtail F\Sigma$.

We have defined regular triposes as triposes $F : \mathcal{S} \to \mathcal{E}$ where F preserves epis, i.e. $F^*\mathsf{Sub}_{\mathcal{E}}$ is a prestack. As opposed to ordinary triposes regular triposes are known to be closed under composition, i.e. are closed under iteration. A further advantage of regular triposes is that for a regular tripos $F : \mathcal{S} \to \mathcal{E}$ regular triposes over $\mathcal{E}$ correspond to morphisms of regular triposes from F to some regular tripos $G : \mathcal{S} \to \mathcal{F}$, i.e. $H : F \to G$ such that $H : \mathcal{E} \to \mathcal{F}$ is a regular functor. Somewhat surprisingly, an analogous result holds for traditional regular triposes as well.

Finally, we have recalled a theorem due to Miquel characterizing traditional regular triposes in terms of implicative algebras generalizing the notion of complete Heyting algebra and identified a notion of morphism between implicative algebras corresponding to regular morphisms of triposes over **Set**.

We think that the more general notion of tripos as introduced in Pitts (2002) is more natural since it corresponds to the class of first order posetal hyperdoctrines which give rise to toposes by "adding subquotients". Moreover, the comprehension axiom characterizing them is more natural than the Skolemized form postulated as an axiom in the definition of traditional triposes.

But restricting to regular triposes seems to be a good idea since the condition is most natural from the point of view of fibered categories and, moreover, allows one to identify regular tripos morphisms to $F : \mathcal{S} \to \mathcal{E}$ with regular triposes over $\mathcal{E}$ as shown in Theorem 4.2.

We have shown that triposes $F, G : \mathbf{Set} \to \mathcal{E}$ need not be equivalent. But we do not know whether such F and G are necessarily equivalent under the stronger assumption that both F and G are traditional triposes. There is no conceptual reason why this should hold in general but, alas, we have not been able to find a counterexample so far.

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Notes

1 The definition of tripos in Pitts (2002) generalizes that of Hyland et al. (1980) also in the sense that it admits arbitrary finite-product categories as base categories, and requires the Beck–Chevalley condition (BC) only for certain pullback squares definable from finite products. However, in the present work, we consider only toposes as base categories, and all triposes satisfy BC for arbitrary pullback squares.

2 Take for $\mathcal{S}$ the Sierpiński topos $\mathbf{Set}^{2^{\mathrm{op}}}$ and for F the functor $\mathrm{Id}_{\mathcal{S}}$. Then there are two possible choices for S, namely $\top : 1 \to \Omega_{\mathcal{S}}$ and the subobject S of $\Omega_{\mathcal{S}}$ with $S_0 = \Omega_0$ and $S_1 = \{\top\}$. In the first case, the corresponding γ_S is $\mathrm{id}_{\Omega_{\mathcal{S}}}$ and in the second case, it sends a subobject P of A in $\mathcal{S}$ to the subobject $\gamma_S(P)$ of A with $\gamma_S(P)_0 = A_0$ and $\gamma_S(P)_1 = P_1$.

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