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# On the quasi-ergodicity of absorbing Markov chains with unbounded transition densities, including random logistic maps with escape

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*Abstract.* In this paper, we consider absorbing Markov chains $X_n$ admitting a quasi-stationary measure $\mu$ on $M$ where the transition kernel $\mathcal{P}$ admits an eigenfunction $0 \leq \eta \in L^1(M, \mu)$. We find conditions on the transition densities of $\mathcal{P}$ with respect to $\mu$ which ensure that $\eta(x)\mu(\mathrm{d}x)$ is a quasi-ergodic measure for $X_n$ and that the Yaglom limit converges to the quasi-stationary measure $\mu$-almost surely. We apply this result to the random logistic map $X_{n+1} = \omega_n X_n(1 - X_n)$ absorbed at $\mathbb{R} \setminus [0, 1]$, where $\omega_n$ is an independent and identically distributed sequence of random variables uniformly distributed in $[a, b]$, for $1 \leq a < 4$ and $b > 4$.

Key words: Markov chains with absorption, random dynamical systems, quasi-stationary measure, quasi-ergodic measure, Yaglom limit
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## 1. *Introduction*

Consider a family of transformations $\mathcal{F} = \{f_\omega : E \to E\}_{\omega \in \Delta}$, where $E$ is a metric space. Given a subset $M \subset E$ and endowing $\mathcal{F}$ with a probability measure, we aim to understand the statistical behaviour of the random dynamical system

$$f^n(x; \omega_1, \ldots, \omega_n) := f_{\omega_n} \circ \cdots \circ f_{\omega_1}(x)$$

conditioned upon remaining in $M$.

Such a problem can be naturally modelled via the Markov chain $X_n = f_n(X_{n-1})$ with $f_n \in \mathcal{F}$ absorbed at $\partial := E \setminus M$, that is, $X_n \in \partial$ implies $X_{n+1} \in \partial$. Statistical information for the above (conditioned) random dynamical system is then obtained by certain limiting distributions for the paths $X_n$. In the literature, such limiting distributions appear mainly in two forms. The first is the so-called *Yaglom limit*

$$\lim_{n\to\infty} \mathbb{P}[X_n \in A \mid X_0 = x, \tau > n], \tag{1.1}$$

where $x \in M$, $\tau = \min\{n \in \mathbb{N} \mid X_n \in \partial\}$ and $A$ is a measurable subset of $M$. The second one is the so-called *quasi-ergodic limit*

$$\lim_{n\to\infty} \mathbb{E}\bigg[\frac{1}{n}\sum_{i=0}^{n-1} \mathbb{1}_A \circ X_n \,\bigg|\, X_0 = x, \tau > n\bigg]. \tag{1.2}$$

There are several contexts in which the Yaglom limit converges to a quasi-stationary measure. We recall that a probability measure $\mu$ on $M$ is called a *quasi-stationary* measure for $X_n$ on $M$ if for every $n \in \mathbb{N}$, $\mu(\mathrm{d}x) := \mathbb{P}[X_n \in \mathrm{d}x \mid X_0 \sim \mu, \tau > n]$. However, the limit in equation (1.2) is related to the existence of a quasi-ergodic measure for $X_n$ on $M$. A measure $\nu$ on $M$ is called a *quasi-ergodic measure* for $X_n$ on $M$ if for every measurable subset $A$ of $M$, equation (1.2) converges to $\nu(A)$ for $\nu$-almost every $x \in M$.

In the literature [3, 4, 7–9, 29], various sufficient conditions are presented for the existence and uniqueness of quasi-stationary measures $\mu$ and quasi-ergodic measures $\nu$. These conditions imply that $\nu \ll \mu$ and that the Radon–Nikodym derivative $\eta(x) = \nu(\mathrm{d}x)/\mu(\mathrm{d}x)$ is an eigenfunction of $\mathcal{P}$, where $\mathcal{P}$ is the transition kernel of $X_n$. The uniform convergence of the sequence $\{\mathcal{P}^n(\cdot, M)/\lambda^n\}_{n\in\mathbb{N}}$, where $\lambda := \int_M \mathcal{P}(y, M)\mu(\mathrm{d}y)$, to the eigenfunction $\eta$ plays a crucial role in the proofs.

In this paper, we take a different approach. We set out to derive a quasi-ergodic measure starting from a quasi-stationary measure. The existence of quasi-stationary measures is a well-established problem (see [26] for a bibliography). Quasi-stationary measures arise as positive eigenmeasures of the operator $\mu \mapsto \int_M \mathcal{P}(x, \cdot)\mu(\mathrm{d}x)$ and extensive literature exists on how to solve such eigenvalue problems [22, 24, 25]. Since quasi-ergodic measures do not admit such an approach, they are less well understood. Quasi-ergodic measures are important in the analysis of random dynamical systems, for instance, in the context of the recently established conditioned Lyapunov spectrum [6, 11].

Inspired by these results, where quasi-ergodic measures can be expressed as a density over the quasi-stationary measures, we obtain natural conditions on the transition kernel $\mathcal{P}$ such that the existence of a quasi-ergodic measure becomes equivalent to solving an eigenvalue problem for $\mathcal{P}$ in $L^1$. As a result, we considerably simplify the procedure of finding a quasi-ergodic measure. Furthermore, we also obtain, under aperiodicity conditions, that the Yaglom limit in equation (1.1) converges to the quasi-stationary measure $\mu$-almost surely (a.s.)

As an application of our results, we characterize the limits in equations (1.1) and (1.2) for the random logistic map $Y_{n+1} = \omega_n Y_n(1 - Y_n)$ absorbed at $\mathbb{R} \setminus [0, 1]$, with $\{\omega_n\}_{n\in\mathbb{N}}$ an independent and identically distributed (i.i.d.) sequence of random variables such that $\omega_0 \sim \mathrm{Unif}([a, b])$, with $1 \le a < 4$ and $b > 4$, where $\mathrm{Unif}([a, b])$ denotes the continuous

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uniform distribution in $[a, b]$. The analysis of this system is challenging since its transition kernel presents a change of behaviour on the points 0 and 1. In particular, for every $x \in (0, 1)$,

$$\mathcal{P}(x, \mathrm{d}y) = \mathbb{P}[Y_n \in \mathrm{d}y \mid Y_0 = x] \ll \mathrm{Leb}(\mathrm{d}y),$$

while $\mathcal{P}(0, \mathrm{d}y) = \mathcal{P}(1, \mathrm{d}y) = \delta_0(\mathrm{d}y)$. This implies that the transition densities $\mathcal{P}(x, \mathrm{d}y)$ explode when $x$ approaches the points 0 and 1. Consequently, the results in the literature [**3**, **4**, **7**, **8**, **29**] cannot be applied, since $\mathcal{P}$ does not act as a compact operator on $\mathcal{C}^0(M)$ and $L^p(M)$, with $p \geq 1$. Hence, a more refined analysis is needed.

To overcome this issue, we consider $AM$-compact operators (see [**16**, Appendix A]), a generalization of compact operators. Inspired by the novel results on positive integral operators in [**14**–**16**], we analyse the action of $\mathcal{P}$ on $L^1(M, \mu)$. Since $\mathcal{P}$ is an integral operator, it is $AM$-compact, and we can establish its peripheral spectrum from which the asymptotic behaviour of $Y_n$ follows.

This paper is divided into six sections. In §2, the basic concepts of the theory of absorbing Markov chains are briefly recalled, the main underlying hypotheses of this paper are defined (Hypotheses H1 and H2) and the main results of this paper are stated (Theorems 2.1, 2.2, 2.3 and 2.4). In §3, it is shown that Hypothesis H1 implies that $\mathcal{P}/\lambda$ is a mean ergodic operator. Section 4 is dedicated to a brief presentation of Banach lattice theory, the definition of an $AM$-compact operator and the proof of Theorem 2.2. In §5, we combine the results of the previous sections to prove Theorems 2.3 and 2.4. Finally, in §6, we analyse the asymptotic behaviour of the random logistic map $Y_n$, introduced above, and prove Theorem 2.1.

## 2. *Main results*

Let $E$ be a metric space and $M$ a subspace of $E$. We aim to study Markov chains on $E$ conditioned upon remaining in the set $M$. With this objective in mind, we denote as $E_M$ the topological space $M \sqcup \partial$ generated by the topological basis

$$\mathcal{T} = \{A \cap M;\ A \text{ is an open set of } E\} \sqcup \partial,$$

where $\sqcup$ denotes disjoint union. In this paper, we assume that

$$X := (\Omega, \{\mathcal{F}_n\}_{n \in \mathbb{N}_0}, \{X_n\}_{n \in \mathbb{N}_0}, \{\mathcal{P}^n\}_{n \in \mathbb{N}_0}, \{\mathbb{P}_x\}_{x \in E_M})$$

is a Markov chain with state space $E_M$, in the sense of [**27**, Definition III.1.1], that is, the pair $(\Omega, \{\mathcal{F}_n\}_{n \in \mathbb{N}})$ is a filtered space; $X_n$ is an $\mathcal{F}_n$-adapted process with state space $E_M$; $\mathcal{P}^n$ a time-homogeneous transition probability function of the process $X_n$ satisfying the usual measurability assumptions and Chapman–Kolmogorov equation; $\{\mathbb{P}_x\}_{x \in E_M}$ is a family of probability function satisfying $\mathbb{P}_x[X_0 = x] = 1$ for every $x \in E_M$; and for all $m, n \in \mathbb{N}_0$, $x \in E_M$, and every bounded measurable function $f$ on $E_M$,

$$\mathbb{E}_x[f \circ X_{m+n} \mid \mathcal{F}_n] = (\mathcal{P}^m f)(X_n) \quad \mathbb{P}_x\text{-a.s.}$$

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We assume that $X_n$ is a Markov chain that is absorbed at $\partial$, meaning that $\mathcal{P}(\partial, \partial) = 1$. In view of the above definitions, it is natural to define the stopping time

$$\tau(\omega) := \inf\{n \in \mathbb{N};\, X_n(\omega) \notin M\}.$$

Throughout the paper, the following notation is used.

*Notation 2.1.* Given a probability measure $\mu$ on $M$ and $p \in [1, \infty]$, we denote $L^p(M, \mathscr{B}(M), \mu)$ as $L^p(M, \mu)$ and $\mathcal{M}(M)$ as the set of Borel signed-measures on $M$. Moreover, we denote $\mathbb{P}_\mu(\cdot) := \int_M \mathbb{P}_x(\cdot)\mu(\mathrm{d}x)$.

We denote as $\mathcal{C}^0(M) := \{f : M \to \mathbb{R};\ f \text{ is continuous}\}$ and $\mathcal{F}_b(M)$ as the set of bounded Borel measurable functions on $M$. Given $f \in \mathcal{F}_b(M)$, write

$$\mathcal{P}^n f(x) := \mathcal{P}^n(\mathbb{1}_M f)(x) = \int_M f(y)\mathcal{P}^n(x, \mathrm{d}y),$$

and by abuse of notation, denote

$$f \circ X_n := \begin{cases} f \circ X_n & \text{if } X_n \in M, \\ 0 & \text{if } X_n \notin M. \end{cases}$$

Given a sub $\sigma$-algebra $\mathscr{F}$ of $\mathscr{B}(M)$ and $f \in L^1(M, \mu)$, we denote $\mathbb{E}_\mu[f \mid \mathscr{F}] \in L^1(M, \mathcal{F}, \mu)$ as the conditioned conditional expectation of $f$ given $\mathscr{F}$, that is, the unique function in $L^1(M, \mathscr{F}, \mu)$ such that

$$\int_F f(x)\mu(\mathrm{d}x) = \int_F \mathbb{E}_\mu[f \mid \mathscr{F}]\mu(\mathrm{d}x) \quad \text{for every } F \in \mathscr{F}.$$

We define the sets

$$\mathcal{M}_+(M) = \{\mu \in \mathcal{M}(M);\, \mu(A) \geq 0 \text{ for every } A \in \mathscr{B}(M)\},$$

and

$$L^p_+(M, \mu) = \{f \in L^p(M, \mu);\ f \geq 0\ \mu\text{-a.s.}\} \quad \text{for every } p \in [1, \infty].$$

Finally, given a Banach space $E$, we say that the sequence $\{x_n\}_{n\in\mathbb{N}} \subset E$ converges in the weak topology to $x \in E$ if for every bounded linear functional $\phi \in E^*$, $\lim_{n\to\infty} \phi(x_n) = \phi(x)$. Moreover, we say that the sequence $\{\phi_n\}_{n\in\mathbb{N}} \subset E^*$ converges in the weak* topology to $\phi$ if $\lim_{n\to\infty} \phi_n(x) = \phi(x)$ for every $x \in E$.

Since stationary measures do not capture the behaviour of $X_n$ before absorption, they become irrelevant when dealing with absorbing Markov chains. Due to this issue, it is necessary to extend the concept of stationary measures to quasi-stationary measures. Below, we recall the definition of a quasi-stationary measure.

*Definition 2.2.* A Borel measure $\mu$ on a metric space $M$ is to be a *quasi-stationary measure* for the Markov chain $X_n$ if

$$\mathbb{P}_\mu[X_n \in \cdot \mid \tau > n] = \mu(\cdot) \quad \text{for all } n \in \mathbb{N}.$$

We call $\lambda = \int_M \mathcal{P}(x, M)\mu(\mathrm{d}x)$ the *survival rate* of $\mu$.

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Observe that if $X_n$ admits a quasi-stationary measure $\mu$ on $M$ with survival rate $\lambda$, then $\mathcal{P}$ may be seen as a bounded linear operator in $L^\infty(M, \mu)$. Moreover, since

$$\int_M \mathcal{P}(x, A)\mu(\mathrm{d}x) = \lambda\mu(A) \quad \text{for every } A \in \mathscr{B}(M), \tag{2.1}$$

and $L^\infty(M, \mu)$ is dense in $L^1(M, \mu)$, the operator $\mathcal{P}$ can be naturally extended as a bounded linear operator in $L^1(M, \mu)$.

While we have that ergodic stationary measures can be described in terms of Birkhoff averages for classical Markov chains, this is not true any longer when dealing with absorbing Markov chains, meaning that quasi-stationary measures cannot be described in terms of conditioned Birkhoff averages. This obstruction motivates the definition of quasi-ergodic measures.

*Definition 2.3.* A measure $\nu$ on $M$ is called a *quasi-ergodic measure* if for every $f \in \mathcal{F}_b(M)$,

$$\lim_{n\to\infty} \mathbb{E}_x\left[\frac{1}{n}\sum_{i=0}^{n-1} f \circ X_i \,\middle|\, \tau > n\right] = \int_M f(y)\nu(\mathrm{d}y) \quad \text{for } \nu\text{-almost every } x \in M.$$

One of the main objectives of this paper is to study the statistical asymptotic behaviour of the Markov chain $Y_{n+1}^{a,b} := \omega_n Y_n^{a,b}(1 - Y_n^{a,b})$ absorbed at $\mathbb{R} \setminus [0, 1]$, where $\{\omega_n\}_n$ is an i.i.d. sequence of random variables such that $\omega_0 \sim \mathrm{Unif}([a, b])$ with $1 \le a < 4$ and $b > 4$.

We mention that in the case where $Y_n^{a,b}$ does not escape from the interval $[0, 1]$, that is, $1 \le a < b \le 4$, [2, Theorem 2] and [30, Proposition 9.5] show that $Y_n^{a,b}$ admits a unique stationary measure $\mu_{a,b}$ for $Y_n^{a,b}$ on $[0, 1]$ such that $\mu_{a,b}((0, 1)) = 1$. For dynamical considerations of random logistic maps and an analysis of the case were the sample space is finite, see [1].

The following theorem describes the asymptotic distribution of $Y_n^{a,b}$ conditioned upon survival when $1 \le a < 4 < b$, also establishing the existence of quasi-stationary and quasi-ergodic measures for $Y_n^{a,b}$ on $[0, 1]$.

THEOREM 2.1. *Consider $M = [0, 1]$, $1 \le a < 4 < b$, the Markov chain $Y_n^{a,b}$ on $\mathbb{R}_M$ absorbed at $\partial = \mathbb{R} \setminus M$ and $\tau^{a,b}(\omega) = \min\{n \in \mathbb{N}; Y_n^{a,b} \in \mathbb{R} \setminus [0, 1]\}$. Then we have the following.*

(i) *$Y_n^{a,b}$ admits a quasi-stationary measure $\mu_{a,b}$ with survival rate $\lambda_{a,b}$ such that $\mathrm{supp}(\mu_{a,b}) = [0, 1]$ and $\mu_{a,b} \ll \mathrm{Leb}$, where Leb denotes the Lebesgue measure on $[0, 1]$.*

(ii) *There exists $\eta_{a,b} \in L^1(M, \mu)$ such that $\mathcal{P}\eta_{a,b} = \lambda_{a,b}\eta_{a,b}$, $\|\eta_{a,b}\|_{L^1(M,\mu)} = 1$ and $\eta_{a,b} > 0$ $\mu_{a,b}$-a.s.*

(iii) *For every $h \in L^\infty(M, \mathrm{Leb})$ and $x \in (0, 1)$,*

$$\lim_{n\to\infty} \mathbb{E}_x\left[\frac{1}{n}\sum_{i=0}^{n-1} h \circ Y_i^{a,b} \,\middle|\, \tau^{a,b} > n\right] = \int_M h(y)\eta_{a,b}(y)\mu_{a,b}(\mathrm{d}y).$$

(iv) *For every* $h \in L^\infty(M, \mathrm{Leb})$ *and* $x \in (0, 1)$,

$$\lim_{n\to\infty} \mathbb{E}_x[h \circ Y_i^{a,b} \mid \tau^{a,b} > n] = \int h(y)\mu_{a,b}(\mathrm{d}y).$$

Theorem 2.1 is proved in §6.1. Later, we generalize the above result allowing values of $a$ in the interval $(0, 1)$ (see Theorem 6.15). However, this result relies on the technical assumption of $(a, b)$ being an admissible pair (see Definition 6.1). We were not able to show the existence of quasi-stationary and quasi-ergodic measures for all values of $a \in [0, 1)$, which can be seen from technical details in the inequalities of Proposition 6.17 that is used in Step 2 of Lemma 6.7. This technical obstruction is explained in Remark 6.10.

We use a more general setup for the proof of the above theorem. We present two incrementally restrictive hypotheses, Hypothesis H1 and H2, which are satisfied by $Y_n^{a,b}$ for every $(a, b) \in [1, 4) \times (4, \infty)$, and implies results similar to Theorem 2.1 in different modes of convergence (see Theorems 2.2, 2.3 and 2.4).

In the following, we recall the definition of an integral operator.

*Definition 2.4.* Let $p, q \in [1, \infty)$ and $(\Omega_1, \mathcal{F}_1, \mu_1)$, $(\Omega_2, \mathcal{F}_2, \mu_2)$ be measure spaces. We say that the bounded linear map $T : L^p(\Omega_1, \mathcal{F}_1, \mu_1) \to L^q(\Omega_2, \mathcal{F}_2, \mu_2)$ is an integral operator if there exists a measurable function $\kappa : \Omega_2 \times \Omega_1 \to \mathbb{R}$, called *kernel function*, such that for every $f \in L^p(\Omega_1, \mathcal{F}_1, \mu_1)$,

$$\kappa(x, \cdot) f(\cdot) \in L^1(\Omega_1, \mu_1) \quad \text{for } \mu_2\text{-almost every } x \in \Omega_2,$$

and

$$Tf(x) = \int_{\Omega_1} f(y)\kappa(x, y)\mu_1(\mathrm{d}y) \quad \text{for } \mu_2\text{-almost every } x \in \Omega_2.$$

For a large class of Markov processes, it is common for the existence of a probability function $\rho$ on $M$ such that

$$\mathcal{P}(x, \mathrm{d}y) \ll \rho(\mathrm{d}y) \quad \text{for } \rho\text{-almost every } x \in M.$$

In such systems, it is natural to seek quasi-stationary measures that are absolutely continuous with respect to $\rho$. In this situation and assuming that $\mu \ll \rho$, we have from equation (2.1) that $\mathcal{P} : L^1(M, \mu) \to L^1(M, \mu)$ is an integral operator.

It is also natural to assume that the absorbing Markov chain $X_n$ is irreducible, that is, if there exists $A \in \mathscr{B}(M)$ such that $\mu(\{\mathcal{P}(\cdot, A) > 0\} \triangle A) = 0$, then either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$, where $\triangle$ denotes the symmetric difference of sets. In cases where $X_n$ is not irreducible, it is always possible to separate the state space into irreducible regions and analyse each region separately.

The conditions discussed above are summarized in Hypothesis H1.

HYPOTHESIS H1. *Let $X_n$ be an absorbing Markov chain on $E_M$ absorbed at $\partial$. We say that $X_n$ fulfils Hypothesis H1 if the following conditions are met.*

(H1a) *There exists a quasi-stationary measure $\mu \in \mathcal{M}_+(M)$ for the Markov chain $X_n$ with survival rate $\lambda$.*

(H1b) *There exists $\eta \in L^1_+(M, \mu)$ such that $\mathcal{P}\eta = \lambda\eta$ and $\|\eta\|_{L^1(M,\mu)} = 1$.*

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(H1c) *The transition kernel* $\mathcal{P} : L^1(M, \mu) \to L^1(M, \mu)$ *is an integral operator with kernel function* $\kappa : M \times M \to \mathbb{R}_+$.

(H1d) *For every* $A \in \mathcal{B}(M)$ *such that* $0 < \mu(A) < 1$,

$$\int_{M \setminus A} \int_A \kappa(x, y)\mu(\mathrm{d}y)\mu(\mathrm{d}x) > 0,$$

*that is, if there exists* $A \in \mathscr{B}(M)$ *such that*

$$\mu(\{\mathcal{P}\mathbb{1}_A > 0\} \triangle A) = 0,$$

*then either* $\mu(M \setminus A) = 0$ *or* $\mu(A) = 0$.

We mention that given an absorbing Markov chain $X_n$ satisfying Hypothesis H1, we obtain from [22, Lemma 4.2.9 and Example (i) on p. 262] that $\eta(x) > 0$ for $\mu$-almost every $x \in M$.

The theorem below implies that under Hypothesis H1, $\eta(x)\mu(\mathrm{d}x)$ is the only candidate for the quasi-ergodic measure for $X_n$ on $M$. Moreover, it is also shown that such a hypothesis implies the existence of a maximal $m \in \mathbb{N}$, with the following properties:

- there exists measurable sets $C_0, \ldots, C_{m-1} \subset M$ such that $M = C_0 \sqcup \cdots \sqcup C_{m-1}$;
- for every $n \in \mathbb{N}$, $X_n \in C_{k \pmod m}$, then $X_{n+1} \in C_{k+1 \pmod m}$.

THEOREM 2.2. *Let* $X_n$ *be an absorbing Markov chain fulfilling Hypothesis H1 then the following assertions hold.*

(i) *There exist a natural number* $m \in \mathbb{N}$ *and sets* $C_m := C_0, C_1, \ldots, C_{m-1} \in \mathscr{B}(M)$ *such that* $\mu(C_i) = 1/m$ *for every* $i \in \{0, 1 \ldots, m-1\}$ *and*

$$\{\mathcal{P}\mathbb{1}_{C_i} > 0\} \subset C_{i+1} \text{ for every } i \in \{0, 1, \ldots, m-1\}.$$

(ii) *For every* $f \in L^1(M, \mu)$,

$$\frac{1}{n}\sum_{i=0}^{n-1} \frac{1}{\lambda^n}\mathcal{P}^n f \xrightarrow{n\to\infty} \eta \int_M f(y)\eta(y)\mu(\mathrm{d}y),$$

*in* $L^1(M, \mu)$ *and* $\mu$*-a.s.*

(iii) *The following limit holds*

$$\frac{1}{\lambda^n}\mathcal{P}^n(x, M) \xrightarrow{n\to\infty} \eta(x) \text{ in } L^1(M, \mu).$$

(iv) *If, in addition, we assume that M is a Polish space, then for every* $h \in L^\infty(M, \mu)$,

$$\left(x \mapsto \mathbb{E}_x\left[\frac{1}{n}\sum_{i=0}^{n-1} h \circ X_i \mid \tau > n\right]\right) \xrightarrow{n\to\infty} \int_M h(y)\eta(y)\mu(\mathrm{d}y) \qquad (2.2)$$

*in the* $L^\infty(M, \mu)$*-weak* topology (see* [5, Ch. 3.4] *for the definition and the main properties of the weak* topology), in particular, we obtain that (2.2) also converges weakly in* $L^1(M, \mu)$.

Theorem 2.2 is proved in §4.

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It is observed that Theorem 2.2(iv) gives us $L^\infty(M,\mu)$-weak* convergence of equation (2.2), and to guarantee such convergence in $L^\infty(M,\mu)$, we require an additional regularity hypothesis (Hypothesis H2) on the kernel functions of the operator $\mathcal{P}$.

HYPOTHESIS H2. *Let $X_n$ be a Markov chain $X_n$ on $E_M$ absorbed at $\partial$. We say that $X_n$ fulfils Hypothesis H2 if:*

(1) *$X_n$ fulfils Hypothesis H1; and*

(2) *for $\mu$-almost every point $x \in M$, $\kappa(x,\cdot) \in L^\infty(M,\mu)$. Equivalently, since $\mu$ is an inner regular measure* [**28**, Proposition A.3.2]*, there exists a sequence of nested compact sets $\{K_i\}_{i\in\mathbb{N}}$ such that $\mu(\bigcup_{i\in\mathbb{N}} K_i) = 1$, and for every $i \in \mathbb{N}$,*

$$\operatorname*{ess\,sup}^{\mu\otimes\mu}_{(x,y)\in K_i\times M} \|\mathbb{1}_{K_i}(x)\kappa(x,y)\| < \infty.$$

We mention that, in practice, once (H1a) and (H1b) are verified, then (H1c), (H1d) and Hypothesis H2 can be readily verified. We exemplify this in §6 considering the absorbing Markov chain $Y_n^{a,b}$ (see the proof Theorem 6.15).

In addition to quasi-stationary measures, the so-called *Yaglom limit*

$$\lim_{n\to\infty} \frac{\mathcal{P}^n(x,A)}{\mathcal{P}^n(x,M)} \text{ for } x \in M \text{ and } A \in \mathscr{B}(M),$$

provides an alternative perspective on the asymptotic behaviour of the paths $X_n$ conditioned on survival. Observe that for the Yaglom limit to exist, it is necessary that $M$ does not exhibit a cyclic decomposition under $X_n$, that is, that $m = 1$ on item (i) of Theorem 2.2.

The following two results provide conditions that ensure the existence of a quasi-ergodic measure for $X_n$ on $M$ and the convergence of the Yaglom limit.

THEOREM 2.3. *Let $X_n$ be an absorbing Markov chain fulfilling Hypothesis H1. If any of the following items hold:*

(a) *there exists $K > 0$ such that $\mu(\{K < \eta\}) = 1$ a.s.;*

(b) *there exists $g \in L^1(M,\mu)$ such that*

$$\frac{1}{\lambda^n}\mathcal{P}^n(\cdot,M) \le g \quad \text{for every } n \in \mathbb{N};$$

(c) *the absorbing Markov chain $X_n$ fulfils Hypothesis H2,*
*then for every $h \in L^\infty(M,\mu)$,*

$$\lim_{n\to\infty} \mathbb{E}_x\left[\frac{1}{n}\sum_{i=0}^{n-1} h\circ X_i \,\middle|\, \tau > n\right] = \int_M h(y)\eta(y)\mu(\mathrm{d}y) \text{ for } \mu\text{-almost every } x \in M. \tag{2.3}$$

*If, additionally, $m = 1$ in Theorem 2.2(i), then*

$$\lim_{n\to\infty} \mathbb{E}_x[h\circ X_n \mid \tau > n] = \lim_{n\to\infty} \frac{\mathcal{P}^n h(x)}{\mathcal{P}^n(x,M)} = \int_M h(y)\mu(\mathrm{d}y) \text{ for } \mu\text{-almost every } x \in M. \tag{2.4}$$

Theorem 2.3 is proved in §5.

The following theorem is a refinement of the previous theorem, allowing us to characterize the set where the convergence of equations (2.3) and (2.4) hold.

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THEOREM 2.4. *Let $X_n$ be an absorbing Markov chain fulfilling Hypothesis H2. Then, given $h \in L^\infty(M, \mu)$, equation (2.3) holds for every $x \in \bigcup_{i\in\mathbb{N}} K_i$, where $\{K_i\}_{i\in\mathbb{N}}$ is the nested sequence of compact sets given by the second part of Hypothesis H2.*

*In the case where $m = 1$ in Theorem 2.2(i), equation (2.4) holds for every $x \in \bigcup_{i\in\mathbb{N}} K_i$.*

Theorem 2.4 is proved in §5.

*Remark 2.5.* Notice that Theorems 2.3 and 2.4 also hold in a non-escape context. This means that if $X_n$ is a Markov chain on the metric space $M$ without absorption, satisfying the following properties:

- $\mu$ is ergodic stationary measure for $X_n$ on $M$;
- the transition kernel $\mathcal{P} : L^1(M, \mu) \to L^1(M, \mu)$ is an integral operator; and
- $X_n$ is aperiodic, that is, $m = 1$ in Theorem 2.2(i),

then for every $h \in L^\infty(M, \mu)$, $\lim_{n\to\infty} \mathcal{P}^n h = \int h \, \mathrm{d}\mu$, $\mu$-a.s. In particular, from [20, Theorem 1(ii)], we obtain that $X_n$ is a weak-mixing Markov chain.

## 3. *Mean-ergodic operators*

For classical dynamical systems and Markov processes, mean-ergodic operators provide a vast array of tools and techniques for analysing their statistical properties [10, Chs. 7, 8 and 10]. This section shows that this is also true for absorbing Markov chains.

In the following, we recall the definition of a mean ergodic operator.

*Definition 3.1.* Let $(E, \|\cdot\|)$ be a Banach space, we say that $T : E \to E$ is a *mean-ergodic operator* if there exists a projection $P : E \to E$ such that

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{i=0}^{n-1} T^i x - Px \right\| = 0 \quad \text{for every } x \in E.$$

Let $M$ be a metric space, $\rho$ a Borel probability measure on $X$, and $T : L^1(M, \rho) \to L^1(M, \rho)$. We denote

$$\mathcal{I}(T, \rho) = \sigma(A \in \mathscr{B}(M); T^* \mathbb{1}_A = \mathbb{1}_A),$$

where $T^* : L^\infty(M, \rho) \to L^\infty(M, \rho)$ is the dual operator of $T$, that is, the unique bounded automorphism on $L^\infty(M, \rho)$ such that

$$\int_M (Tf)(x) h(x) \, \rho(\mathrm{d}x) = \int f(x)(T^*h)(x) \, \rho(\mathrm{d}x) \quad \text{for every } f \in L^1(M, \rho) \text{ and } h \in L^\infty(M, \rho).$$

Our results are highly dependent on the following two propositions.

PROPOSITION 3.1. ([18, Theorem 3.3.5] and [25, Corollary V.8.1]) *Let $M$ be a metric space and $\rho$ be a probability measure on $M$, and $T : L^1(M, \rho) \to L^1(M, \rho)$ be a linear operator such that $\|T\| = 1$. Assume that there exists $\eta \in L^1(M, \rho)$ satisfying $T\eta = \eta$ and $\rho(\{\eta > 0\}) = 1$. Then, we have the following.*

(i) *For every* $f \in L^1(M, \rho)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} T^i f = \eta \frac{\mathbb{E}_\rho[f\eta \mid \mathcal{I}(T, \rho)]}{\mathbb{E}_\rho[\eta \mid \mathcal{I}(T, \rho)]} \quad \mu\text{-a.s.}$$

(ii) *The operator* $T$ *is mean-ergodic.*

While Hypothesis H1 does not imply that $\mathcal{P}$ is a compact operator, the proposition shows that given $f \in L^1(M, \mu)$, the orbit $\{(1/\lambda^n)\mathcal{P}^n f\}_{n\in\mathbb{N}}$ is weakly precompact.

PROPOSITION 3.2. *Suppose that the Markov process* $X_n$ *satisfies Hypothesis H1, then for every* $f \in L^1(M, \mu)$ *the sequence*

$$\left\{\frac{1}{\lambda^i}\mathcal{P}^i f\right\}_{i\in\mathbb{N}} \quad \textit{is weakly-}L^1(M, \mu)\textit{ precompact.}$$

*Proof.* Let $f \in L^1_+(M, \mu)$. Note that for every $i, m \in \mathbb{N}$,

$$0 \le \frac{1}{\lambda^i}\mathcal{P}^i f \le \frac{1}{\lambda^i}\mathcal{P}^i(f - f \wedge m\eta) + \frac{1}{\lambda^i}\mathcal{P}^i(f \wedge m\eta) \le \frac{1}{\lambda^i}\mathcal{P}^i(f - f \wedge m\eta) + m\eta,$$

where given two functions $f_1, f_2$, we define $f_1 \wedge f_2 := \min\{f_1, f_2\}$. Since $\|(1/\lambda^i)\mathcal{P}^i f\|_{L^1(M,\mu)} = \|f\|_{L^1(M,\mu)}$ for every $i \in \mathbb{N}$ and $m\eta \wedge f \xrightarrow{m\to\infty} f$ in $L^1(M, \mu)$ and $\mu$-a.s., we can obtain that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\frac{1}{\lambda^i}\int_A \mathcal{P}^i f(x)\mu(\mathrm{d}x) < \varepsilon \text{ if } \mu(A) < \delta \quad \text{for every } i \in \mathbb{N}.$$

From [19, p. 87, item 3], we conclude that $\{(1/\lambda^n)\mathcal{P}^n f\}_{n\in\mathbb{N}}$ is weakly $L^1(M, \mu)$ precompact. □

## 4. *AM-compact operators*

Observe that under Hypothesis H1, the operator $(1/\lambda)\mathcal{P} : L^1(M, \mu) \to L^1(M, \mu)$, is well behaved in a functional analytical point of view. Namely, $\frac{1}{\lambda}\mathcal{P}$ is a positive integral operator whose orbits are weakly compact. The theory of Banach lattices provides powerful tools for studying the spectrum of such operators. In the following two paragraphs, we recall the definition of a Banach lattice (we follow the definitions provided in [22, Ch. 2] and [25, Ch. 2]).

Given $(L, \le)$ a partially ordered set and a set $B \subset L$, we define, if exists,

$$\sup B = \min\{\ell \in L;\ b \le \ell \text{ for all } b \in B\}$$

and

$$\inf B = \max\{\ell \in L;\ \ell \le b \text{ for all } b \in B\}.$$

With the above definitions, we say that $L$ is a *lattice* if for every $f_1, f_2 \in L$,

$$f_1 \vee f_2 := \sup\{f_1, f_2\},\ \ f_1 \wedge f_2 := \inf\{f_1, f_2\}$$

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exists. Additionally, in the case where $L$ is a vector space and the lattice $(L, \leq)$ satisfies

$$f_1 \leq f_2 \Rightarrow f_1 + f_3 \leq f_2 + f_3 \quad \text{for all } f_3 \in L, \text{ and}$$

$$f_1 \leq f_2 \Rightarrow \alpha f_1 \leq \alpha f_2 \quad \text{for all } \alpha > 0,$$

then $(L, \leq)$ is called a *vector lattice*. Finally, if $(L, \|\cdot\|)$ is a Banach space and the vector lattice $(L, \leq)$ satisfies

$$|f_1| \leq |f_2| \Rightarrow \|f_1\| \leq \|f_2\|,$$

where $|f_1| := f_1 \vee (-f_1)$, then the triple $(L, \leq, \|\cdot\|)$ is called a *Banach lattice*. When the context is clear, we denote the Banach lattice $(L, \leq, \|\cdot\|)$ simply by $L$.

In this paper, we use two fundamental notions from Banach lattice theory. The first one is that of an *ideal* of a Banach lattice and the second one is that of an *irreducible operator* on a Banach lattice. A vector subspace $I \subset L$ is called an *ideal* if for every $f_1, f_2 \in L$ such that $f_2 \in I$ and $|f_1| \leq |f_2|$, we have $f_1 \in I$. Finally, a positive linear operator $T : L \to L$ is called *irreducible* if $\{0\}$ and $L$ are the only $T$-invariant closed ideals of $T$.

The theory of $AM$-compact operators provides a generalization to the theory of compact operators. $AM$-compact operators are considerably more general than compact operators and possess a sufficient degree of regularity. In the following, we recall the definition of an $AM$-compact operator.

*Definition 4.1.* Let $E$ be a Banach lattice and $Y$ a Banach space. A linear operator $T : E \to Y$ is called $AM$-compact if for every $x_1, x_2 \in E$, $T([x_1, x_2])$ is precompact in $Y$, where $[x_1, x_2] := \{y \in E;\ x_1 \leq y \leq x_2\}$.

The following result shows us that all positive integral operators are $AM$-compact.

THEOREM 4.1. [**14**, Proposition A.5] *Let $(\Omega_1, \mu_1)$ and $(\Omega_2, \mu_2)$ be the σ-finite measure spaces and $p, q \in [1, \infty)$. Let $T : L^p(\Omega_1, \mu_1) \to L^q(\Omega_2, \mu_2)$ be a positive bounded integral operator, then T is an $AM$-compact operator.*

Given $f \in L^1(M, \mu)$, the key to our results is to understand the asymptotic behaviour of the sequence $\{(1/\lambda^n)\mathcal{P}^n f\}_{n\in\mathbb{N}}$. It turns out that the behaviour of this sequence has a strong connection with the peripheral spectrum of $\mathcal{P}$. In this way, we denote $L^1(M, \mu; \mathbb{C}) := L^1(M, \mu) \oplus i L^1(M, \mu)$ and linearly extend the operator $\mathcal{P}$ to the Banach space $L^1(M, \mu; \mathbb{C})$.

Here, we summarize the spectral properties implied by Hypothesis H1.

PROPOSITION 4.2. *Let $X_n$ be an absorbing Markov chain satisfying Hypothesis H1. Then:*

(i) *for every $f \in L^1(M, \mu)$,*

$$\frac{1}{n}\sum_{i=0}^{n-1} \frac{1}{\lambda^i}\mathcal{P}^i f \xrightarrow{n\to\infty} \eta \int_M f(y)\eta(y)\mu(\mathrm{d}y),$$

*in $L^1(M, \mu)$ and μ-a.s.;*

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(ii) *there exists a decomposition* $L^1(M, \mu; \mathbb{C}) = E_{\mathrm{rev}} \oplus E_{\mathrm{aws}}$, *such that* $E_{\mathrm{rev}}$ *and* $E_{\mathrm{aws}}$ *are* $\mathcal{P}$*-invariant,*

$$E_{\mathrm{rev}} = \mathrm{span}\left\{ f \in L^1(M, \mu; \mathbb{C});\ \frac{1}{\lambda}\mathcal{P} f = e^{2\pi j i/m} f \text{ for some } j \in \{0, 1, \ldots, m-1\} \right\},$$

*and*

$$E_{\mathrm{aws}} = \left\{ f \in L^1(M, \mu; \mathbb{C});\ \frac{1}{\lambda^i}\mathcal{P}^i f \xrightarrow{n\to\infty} 0, \text{ in } L^1(M, \mu) \right\}.$$

*Moreover,*

$$\dim \ker \left( \frac{1}{\lambda}\mathcal{P} - e^{2\pi j i/m}\mathrm{Id} \right) = 1 \quad \text{for every } j \in \{0, 1, \ldots, m-1\}.$$

*Proof.* (i) From Proposition 3.1, it is enough to show that if $A \in \mathcal{I}(\mathcal{P}/\lambda, \mu)$, then either $\mu(A) = 0$ or $\mu(A) = 1$. To see this, let $A \in \mathscr{B}(M)$ such that $\frac{1}{\lambda}\mathcal{P}^*\mathbb{1}_A = \mathbb{1}_A$, then

$$\begin{aligned} 0 = \mu(A \cap (M \setminus A)) &= \int_M \mathbb{1}_A(x)\mathbb{1}_{M\setminus A}(x)\mu(\mathrm{d}x) = \frac{1}{\lambda}\int_M \mathcal{P}^*\mathbb{1}_A(x)\mathbb{1}_{M\setminus A}(x)\mu(\mathrm{d}x) \\ &= \frac{1}{\lambda}\int_M \mathbb{1}_A(x)\mathcal{P}\mathbb{1}_{M\setminus A}(x)\mu(\mathrm{d}x) = \frac{1}{\lambda}\int_A \int_{M\setminus A} \kappa(x, y)\mu(\mathrm{d}y)\mu(\mathrm{d}x). \end{aligned}$$

From Hypothesis H1, we obtain that either $\mu(A) = 1$ or $\mu(A) = 0$.

(ii) From Propositions 3.2 and 4.1, we have that the semigroup $\{(1/\lambda^n)\mathcal{P}^n\}_{n\in\mathbb{N}}$ fulfils the standard assumptions of [**16**, §6]. Combining [**16**, Proposition 4.3, Theorem 2.2] and [**10**, Proposition 16.27 and Corollary 16.32], we obtain that

$$E_{\mathrm{rev}} = \overline{\left\{ f \in L^1(M, \mu; C);\ \frac{1}{\lambda}\mathcal{P} f = e^{2i\pi\theta} f \text{ for some } \theta \in \mathbb{R} \right\}},$$

and

$$E_{\mathrm{aws}} = \{ f \in L^1(M, \mu; \mathbb{C});\ \frac{1}{\lambda^i}\mathcal{P}^i f \to 0 \text{ in } L^1(M, \mu) \}.$$

Applying [**16**, Theorem 6.1(a)], we obtain that if $\lambda e^{2i\pi\theta} \in \sigma_{\mathrm{pnt}}(\mathcal{P})$, then $\theta \in \mathbb{Q}$. Observe Hypothesis H1 implies that $\mathcal{P}/\lambda$ is an irreducible operator [**22**, Example (i), p. 262]. From [**16**, Theorem 6.1(b)], we obtain that $\mathcal{P}/\lambda$ has only finitely many unimodular eigenvalues. Finally, from [**22**, Theorem 4.2.13(iii)] (taking $x' = 1$), the proof is finished. □

Let $\sigma_{\mathrm{pnt}}((1/\lambda)\mathcal{P}) := \{\widetilde{\lambda} \in \mathbb{C};$ there exists $h \in L^1(M)$, $(1/\lambda)\mathcal{P}h = \widetilde{\lambda}h\}$ be the point spectrum of the operator $(1/\lambda)\mathcal{P}$. In [**7**, **23**], it is shown that the cardinality of $\mathbb{S}^1 \cap \sigma_{\mathrm{pnt}}((1/\lambda)\mathcal{P})$ is intrinsically connected with the existence a possible periodic behaviour of $X_n$ in a suitable partition of $M$. This remains true under Hypothesis H1, and such periodic behaviour is established in Lemmas 4.3 and 4.4.

*Definition 4.2.* Given an absorbing Markov chain $X_n$ that satisfies Hypothesis H1, we define $m(X_n) := \#(\mathbb{S}^1 \cap \sigma_{\mathrm{pnt}}(\frac{1}{\lambda}\mathcal{P}))$, which is finite from Proposition 4.2.

From now on, we denote $m(X_n)$ simply as $m$.

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LEMMA 4.3. *Let $X_n$ be a Markov chain satisfying Hypothesis H1. Then there exist eigenfunctions $g_1, \ldots, g_{m-1} \in L^1_+(M, \mu)$ of $\mathcal{P}^m$ such that $\|g_j\|_{L^1(M,\mu)} = 1$ for every $j \in \{0, 1, \ldots, m-1\}$, and* $\operatorname{span}_{\mathbb{C}}(\{g_i\}_{i=0}^{m-1}) = \ker(\mathcal{P}^m - \lambda^m \mathrm{Id})$.

*Moreover, the eigenfunctions $g_0, g_1, \ldots, g_{m-1}$ can be chosen in a way such that they have disjoint support, that is, defining $C_i = \{g_i > 0\}$ for all $i \in \{0, \ldots, m-1\}$, then $\mu(C_i \cap C_j) = 0$ for all $i \neq j$.*

*Furthermore, the family of sets $\{C_i\}_{i=0}^{m-1}$ satisfies*

$$\mu(M \setminus (C_0 \sqcup C_1 \sqcup \cdots \sqcup C_{m-1})) = 0.$$

*Proof.* The proof follows from similar arguments and computations laid out in [7, Proposition 6.9] with the following two adaptations:

(1) the space $\mathcal{C}^0(M)$ is replaced by $L^1(M, \mu)$; and

(2) the set equalities are replaced by the relation $\sim$. Namely, the given $A, B \in \mathscr{B}(M)$ are said to be equivalent, that is, $A \sim B$ if $\mu(A \triangle B) = 0$, where $A \triangle B := (A \setminus B) \cup (B \setminus A)$. □

The proof of the following lemma is analogous to the proof [7, Lemma 6.15].

LEMMA 4.4. *Let $\{g_i\}_{i=0}^{m-1} \subset L^1_+(M, \mu)$, as in Proposition 4.3. Then, there exists a cyclic permutation $\sigma : \{0, 1, \ldots, m-1\} \to \{0, 1, \ldots, m-1\}$ of order $m$ such that for every $i \in \{0, 1, \ldots, m-1\}$, $\mathcal{P} g_i = \lambda g_{\sigma(i)}$. In particular, this implies that*

$$\{\mathcal{P}(x, C_i) > 0\} \subset C_{\sigma(i)} \text{ for every } i \in \{0, 1, \ldots, m-1\}.$$

The following two lemmas are the last ingredients needed for the proof of Theorem 2.2.

LEMMA 4.5. *Suppose the absorbing Markov chain $X_n$ satisfies Hypothesis H1. Then,*

$$\frac{1}{\lambda^n} \mathcal{P}^n(x, M) \xrightarrow{n \to \infty} \eta(x) \text{ in } L^1(M, \mu). \tag{4.1}$$

*Proof.* We divide the proof into three steps.

*Step 1*. We show that $\mu(C_i) = 1/m$ for every $i \in \{0, 1 \ldots, m-1\}$.

Observe that Proposition 3.1 implies that for every $i \in \mathbb{N}$,

$$\frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{\lambda^j} \mathcal{P}^j \mathbb{1}_{C_i} \xrightarrow{n \to \infty} \mu(C_i) \eta$$

and

$$\frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{\lambda^{mj}} (\mathcal{P}^m)^j \left( \frac{1}{\lambda^i} \mathcal{P}^i \mathbb{1}_M \right) \xrightarrow{n \to \infty} \eta \frac{\mathbb{E}_\mu[(1/\lambda^i) \mathcal{P}^i(\cdot, M) \mid \mathcal{I}((1/\lambda^m) \mathcal{P}^i, \mu)]}{\mathbb{E}_\mu[\eta \mid \mathcal{I}((1/\lambda^m) \mathcal{P}^i, \mu)]},$$

in $L^1(M, \mu)$ and $\mu$-a.s.

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However, from [12, Theorem E, p. 29], we obtain that for $\mu$-almost every $x \in M$,

$$\mu(C_i) = \lim_{n\to\infty} \frac{\sum_{j=0}^{n} \mathcal{P}^j(x, C_i)}{\sum_{j=0}^{n} \mathcal{P}^j(x, M)} \leq \lim_{n\to\infty} \frac{\sum_{j=0}^{n} (1/\lambda^{i+mj})\mathcal{P}^{i+mj}(x, M)}{\sum_{j=0}^{nm} (1/\lambda^{j})\mathcal{P}^{j}(x, M)}$$
$$= \frac{\lim_{n\to\infty} (1/nm) \sum_{j=0}^{n} (1/\lambda^{i+mj})\mathcal{P}^{i+mj}(x, M)}{\lim_{n\to\infty} (1/mn) \sum_{j=0}^{nm} (1/\lambda^{j})\mathcal{P}^{i}(x, M)}$$
$$= \frac{1}{m} \frac{\mathbb{E}_\mu[(1/\lambda^i)\mathcal{P}^i(\cdot, M) \mid \mathcal{I}((1/\lambda^m)\mathcal{P}^i, \mu)]}{\mathbb{E}_\mu[\eta \mid \mathcal{I}((1/\lambda^m)\mathcal{P}^i, \mu)]}.$$

Therefore,

$$\mu(C_i) = \mu(C_i) \int_M \mathbb{E}_\mu\bigg[\eta \mid \mathcal{I}\bigg(\frac{1}{\lambda^m}\mathcal{P}^i, \mu\bigg)\bigg]\mu(\mathrm{d}x)$$
$$\leq \frac{1}{m} \int_M \mathbb{E}_\mu\bigg[\frac{1}{\lambda^i}\mathcal{P}^i(\cdot, M) \mid \mathcal{I}\bigg(\frac{1}{\lambda^m}\mathcal{P}^i, \mu\bigg)\bigg](x)\mu(\mathrm{d}x)$$
$$\leq \frac{1}{m} \quad \text{for all } i \in \{0, 1, \ldots, m-1\}.$$

Since

$$1 = \mu(M) = \mu(C_0) + \cdots + \mu(C_{m-1}) \leq \frac{1}{m} + \cdots + \frac{1}{m} = 1,$$

we obtain that

$$\mu(C_i) = \frac{1}{m} \quad \text{for every } i \in \{0, 1, \ldots, m-1\}.$$

*Step 2*. We show that for every $i \in \{0, 1, \ldots, m-1\}$, there exists $f_i \in E_{\mathrm{aws}}$ such that

$$\mathbb{1}_{C_i} = \frac{1}{m} g_i + f_i.$$

From the decomposition $L^1(M, \mu) = E_{\mathrm{rev}} \oplus E_{\mathrm{aws}}$ (see Proposition 4.2(ii)), there exist $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ and $f_i \in E_{\mathrm{aws}}$ such that

$$\mathbb{1}_{C_i} = \sum_{i=0}^{m-1} \alpha_i g_i + f_i.$$

Since $f_i \in E_{\mathrm{aws}}$, it follows that $\int_M f_i(y)\mu(\mathrm{d}y) = 0$. Therefore, $\alpha_0, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_m = 0$ and $\mathbb{1}_{C_i} = \alpha_i g_i + f_i$. Since $\mu(C_i) = 1/m$, we obtain that $\alpha_i = 1/m$.

*Step 3*. We conclude the proof of the proposition.

From Step 2, we obtain that

$$\mathbb{1}_M = \sum_{i=0}^{m-1} \mathbb{1}_{C_i} = \frac{1}{m}(g_1 + \cdots + g_m) + (f_1 + \cdots + f_m) = \eta + (f_1 + \cdots + f_m).$$

Since $f := f_1 + \cdots + f_m \in E_{\mathrm{aws}}$, this shows that $(1/\lambda^n)\mathcal{P}^n(\cdot, M) \xrightarrow{n\to\infty} \eta$ in $L^1(M, \mu)$. □

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LEMMA 4.6. *Let $X_n$ be an absorbing Markov chain satisfying Hypothesis H1, then for every $h \in L^\infty(M,\mu)$,*

$$\frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{\lambda^i}\mathcal{P}^i\left(h(\cdot)\frac{1}{\lambda^{n-i}}\mathcal{P}^{n-i}(\cdot,M)\right)\xrightarrow{n\to\infty}\eta\int_M h(y)\eta(y)\mu(\mathrm{d}y),\ \textit{in } L^1(M,\mu). \quad (4.2)$$

*Proof.* Let $h \in L^\infty(M,\mu)$, from a direct computation

$$\begin{aligned}
I_h^n &:= \left\|\frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{\lambda^i}\mathcal{P}^i\left(h(\cdot)\frac{1}{\lambda^{n-i}}\mathcal{P}^{n-i}(\cdot,M)\right)-\frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{\lambda^i}\mathcal{P}^i(h\eta)\right\|_{L^1(M,\mu)}\\
&\le \frac{1}{n}\sum_{i=0}^{n-1}\left\|\frac{1}{\lambda^i}\mathcal{P}^i\left(h\cdot\left(\frac{1}{\lambda^{n-i}}\mathcal{P}^{n-i}(\cdot,M)-\eta\right)\right)\right\|_{L^1(M,\mu)}\\
&\le \frac{\|h\|_\infty}{n}\sum_{i=0}^{n-1}\left\|\frac{1}{\lambda^i}\mathcal{P}^i\left|\frac{1}{\lambda^{n-i}}\mathcal{P}^{n-i}(\cdot,M)-\eta\right|\right\|_{L^1(M,\mu)}\\
&\le \frac{\|h\|_\infty}{n}\sum_{i=0}^{n-1}\left\|\frac{1}{\lambda^{n-i}}\mathcal{P}^{n-i}(\cdot,M)-\eta\right\|_{L^1(M,\mu)}\xrightarrow[\text{Lem } 4.5]{n\to\infty}0.
\end{aligned}$$

The theorem follows by combining the above equation with Proposition 4.2 (i). □

Now, we prove Theorem 2.2.

*Proof of Theorem 2.2.* Items (i), (ii) and (iii) follows directly from respectively Propositions 4.4, 4.2(i) and Lemma 4.5.

In the following, we prove item (iv). Given $h \in L^\infty(M,\mu)$, define

$$g_n(x) := \mathbb{E}_x\left[\frac{1}{n}\sum_{i=0}^{n-1}h\circ X_i \mid \tau > n\right]=\frac{\lambda^n}{\mathcal{P}^n(x,M)}\frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{\lambda^i}\mathcal{P}^i\left(h(\cdot)\frac{1}{\lambda^{n-i}}\mathcal{P}^{n-i}(\cdot,M)\right)(x).$$

It is clear that $\|g_n\|_{L^\infty(M,\mu)} \le \|h\|_{L^\infty(M,\mu)}$ for every $n \in \mathbb{N}$. Since $M$ is a Polish space, from the Banach–Alaoglu theorem, we obtain that the space

$$B_{\|\cdot\|_{L^\infty(M,\mu)}}(0,\|h\|_\infty) := \{g \in L^\infty(M,\mu);\ \|g\|_{L^\infty(M,\mu)} \le \|h\|_{L^\infty(M,\mu)}\}$$

is a compact metric space when endowed with the $L^\infty(M,\mu)$-weak* topology. Let $\{g_{n_k}\}_{n\in\mathbb{N}}$ be a $L^\infty(M,\mu)$-weak* convergent subsequence of $\{g_n\}_{n\in\mathbb{N}}$, and denote its limit as $g$.

We show that $g = \int_M h\eta\,\mathrm{d}\mu$ $\mu$-a.s., which implies item (iv). Observe that given $A \in \mathscr{B}(M)$, from Lemmas 4.5 and 4.6, we obtain that

$$\begin{aligned}
\int_A g(x)\eta(x)\mu(\mathrm{d}x) &= \lim_{k\to\infty}\int_M g_{n_k}(x)\frac{1}{\lambda^{n_k}}\mathcal{P}^{n_k}(x,M)\mathbb{1}_A(x)\mu(\mathrm{d}x)\\
&= \lim_{n\to\infty}\int_M \frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{\lambda^i}\mathcal{P}^i\left(h(\cdot)\frac{1}{\lambda^{n-i}}\mathcal{P}^{n-i}(\cdot,M)\right)(x)\mathbb{1}_A(x)\mu(\mathrm{d}x)\\
&= \int_A \eta(x)\mu(\mathrm{d}x)\int_M h(x)\eta(x)\mu(\mathrm{d}x).
\end{aligned}$$

Since $\mu(\{\eta > 0\}) = 1$, it follows that $g = \int_M h(x)\eta(x)\mu(\mathrm{d}x)$ $\mu$-a.s. □

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## 5. *Almost-sure convergence*

In this section, we strengthen the $L^\infty(M,\mu)$-weak* convergence given in Theorem 2.2 to $L^\infty(M,\mu)$ convergence.

Note that for every $n\in\mathbb{N}$, $x\in M$ and $A\in\mathscr{B}(M)$,

$$\mathbb{E}_x\left[\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{1}_A\circ X_i \,\middle|\, \tau>n\right]=\frac{\lambda^n}{\mathcal{P}^n(x,M)}\frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{\lambda^i}\mathcal{P}^i\left(\mathbb{1}_A(\cdot)\frac{1}{\lambda^{n-i}}\mathcal{P}^{n-i}(\cdot,M)\right)(x).$$

Therefore, to prove Theorem 2.3, it is enough to find conditions where equations (4.1) and (4.2) converge almost surely.

To prove Theorem 2.3, we need the following three propositions.

PROPOSITION 5.1. [**22**, Proposition 3.3.3] *Let $T: L^1(M,\mu)\to L^1(M,\mu)$ be a positive bounded integral operator. Then if $\{f_n\}_{n\in\mathbb{N}}\subset L^1(M,\mu)$ is a $L^1(M,\mu)$-order bounded sequence satisfying $f_n\to 0$ in $\mu$-measure as $n\to\infty$, then $Tf_n\to 0$ as $n\to\infty$ $\mu$-almost everywhere.*

PROPOSITION 5.2. *Let $X_n$ be an absorbing Markov chain satisfying Hypothesis H1. Suppose that one of the following items holds:*

(a) *there exists $K>0$ such that $\mu(\{K<\eta\})=1$ almost surely;*

(b) *there exists $g\in L^1(M,\mu)$ such that*

$$\frac{1}{\lambda^n}\mathcal{P}^n(x,M)\leq g\quad \textit{for every } n\in\mathbb{N};$$

(c) *the absorbing Markov chain $X_n$ fulfils Hypothesis H2.*

Then for every $h\in L^\infty(M,\mu)$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{\lambda^i}\mathcal{P}^i\left(h(\cdot)\frac{1}{\lambda^{n-i}}\mathcal{P}^{(n-i)}(\cdot,M)\right)(x)\xrightarrow{n\to\infty}\eta(x)\int_M h(y)\eta(y)\mu(\mathrm{d}y)\quad \mu\text{-a.s.}\tag{5.1}$$

In addition, if

$$\frac{1}{\lambda^n}\mathcal{P}^n h\xrightarrow{n\to\infty}\eta\int_M h(x)\mu(\mathrm{d}x)\text{ in }L^1(M,\mu),\tag{5.2}$$

then $(1/\lambda^n)\mathcal{P}^n h\xrightarrow{n\to\infty}\eta\int_M h(x)\mu(\mathrm{d}x)$ $\mu$-a.s.

*Proof.* Observe that item (a) is a particular case of item (b). In fact, note that for every $x\in M$, $(1/\lambda^n)\mathcal{P}(x,M)\leq 1/K(1/\lambda^n)\mathcal{P}^n\eta(x)=\eta(x)/K$ which correspond to item (b) when setting $g:=\eta/K$. Now, we assume item (b). From Lemmas 4.2 and 4.5, we obtain that equations (5.1) and (5.2) converge in probability. Moreover, item (b) implies that for every $n\in\mathbb{N}$ and for $\mu$-almost every $x\in M$,

$$-\|h\|_{L^\infty(M,\mu)}g(x)\leq\frac{1}{\lambda^n}\mathcal{P}^n h(x)\leq\|h\|_{L^\infty(M,\mu)}g(x)$$

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and

$$-\|h\|_{L^\infty(M,\mu)}g(x) \le \frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{\lambda^i}\mathcal{P}^i\bigg(h(\cdot)\frac{1}{\lambda^{n-i}}\mathcal{P}^{(n-i)}(\cdot, M)\bigg)(x) \le \|h\|_\infty g(x).$$

Therefore, Proposition 5.1 implies the result.

Now, we assume item (c). Let us consider the set

$$K_m = \{x \in M; \|k(x,\cdot)\|_{L^\infty(M,\mu)} \le m\}.$$

It is clear that the

$$\mathcal{G}^m : L^1(M,\mu) \to L^\infty(K_m,\mu),$$
$$f \mapsto \frac{1}{\lambda}\mathbb{1}_{K_m}\mathcal{P}^i(f)$$

is a bounded linear operator. Therefore, we have for every $h \in L^\infty(M,\mu)$,

$$\lim_{n\to\infty}\bigg\|\mathbb{1}_{K_i}\bigg(\eta\int_M h(y)\mu(\mathrm{d}y) - \frac{1}{\lambda^i}\mathcal{P}^i h\bigg)\bigg\|_{L^\infty(K_i,\mu)} = 0$$

and

$$\lim_{n\to\infty}\bigg\|\mathbb{1}_{K_m}\frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{\lambda^i}\mathcal{P}^i\bigg(h(\cdot)\frac{1}{\lambda^{n-i}}\mathcal{P}^{(n-i)}(\cdot, M)\bigg) - \mathbb{1}_{K_n}\eta\int_M h(y)\eta(y)\mu(\mathrm{d}y)\bigg\|_{L^\infty(K_m,\mu)} = 0.$$

Since Hypothesis H2 implies that $\mu(\bigcup_{m=1} K_m) = 1$, we obtain the result. □

PROPOSITION 5.3. *Let $X_n$ be an absorbing Markov chain satisfying Hypothesis H2 and $m = 1$ in Theorem 2.2(i). Then for every $h \in L^\infty(M,\mu)$,*

$$\frac{1}{\lambda^n}\mathcal{P}^n h \to \eta\int_M h(x)\mu(\mathrm{d}x) \quad \text{in } L^1(M,\mu).$$

*Proof.* In the case where $m = 1$ in Theorem 2.2(i), we obtain that $\dim(E_{\mathrm{rev}}) = 1$. Then, given $f \in L^1(M,\mu)$, there exists $\alpha \in \mathbb{C}$ and $g \in \mathbb{E}_{\mathrm{aws}}$ such that

$$f = \alpha\eta + g.$$

From Proposition 4.2, we obtain that

$$\frac{1}{\lambda^n}\mathcal{P}^n f \xrightarrow{n\to\infty} \alpha\eta \quad \text{in } L^1(M,\mu).$$

Finally, integrating over $\mu$ in the above limit, we obtain that $\alpha = \int_M f(x)\mu(\mathrm{d}x)$. □

Finally, we prove Theorems 2.3 and 2.4.

*Proof of Theorem 2.3.* Observe that given $h \in L^\infty(M,\mu)$, for every $n \in \mathbb{N}$ and $x \in M$, we obtain

$$\mathbb{E}_x\bigg[\frac{1}{n}\sum_{i=0}^{n-1} h\circ X_i \mid \tau > n\bigg] = \frac{1/n\sum_{i=0}^{n-1}(1/\lambda^i)\mathcal{P}^i(h(\cdot)(1/\lambda^{n-i})\mathcal{P}^{n-i}(\cdot, M))(x)}{(1/\lambda^n)\mathcal{P}^n(x,M)}. \quad (5.3)$$

From Proposition 5.2, we obtain that for $\mu$-almost every $x \in M$,

$$\lim_{n\to\infty} \mathbb{E}_x[h \circ X_i \mid \tau > n] = \int_M \eta(x)h(x)\mu(\mathrm{d}x),$$

which proves the first part of the theorem.

In the case where $m = 1$ in Theorem 2.2, then $\#(\sigma_{\mathrm{pnt}}(\mathcal{P}/\lambda) \cap \mathbb{S}^1) = 1$, then combining Propositions 5.2 and 5.3, we obtain that for $\mu$-almost every $x \in M$,

$$\lim_{n\to\infty} \frac{\mathcal{P}^n h(x)}{\mathcal{P}^n(x, M)} = \int_M h(y)\mu(\mathrm{d}y). \qquad \square$$

*Proof of Theorem 2.4.* Observe that under the conditions of Theorem 2.4, we obtain that for every $i \in \mathbb{N}$, the operators

$$\mathcal{G}^i : L^1(M, \mu) \to \mathcal{C}^0(K_i, )$$
$$g \mapsto \mathbb{1}_{K_i}\mathcal{P}g$$

are bounded linear operators, since

$$\frac{1}{n}\sum_{i=0}^{n-1} \frac{1}{\lambda^i}\mathcal{P}^i\bigg(h(\cdot)\frac{1}{\lambda^{n-i}}\mathcal{P}^{(n-i)}(\cdot, M)\bigg)(x) \xrightarrow[L^1(M,\mu)]{n\to\infty} \eta(x)\int_M h(y)\eta(y)\mu(\mathrm{d}y) \tag{5.4}$$

and

$$\lim_{n\to\infty} \frac{1}{\lambda^n}\mathcal{P}^n(x, M) \xrightarrow[L^1(M,\mu)]{n\to\infty} \eta(x). \tag{5.5}$$

Composing $\mathcal{G}^i$ on both sides, we obtain that equations (5.4) and (5.5) hold pointwise in $K^i$ for every $i \in \mathbb{N}$. From equation (5.3), we obtain that for every $x \in \bigcup_{x\in\mathbb{N}} K_i$ and $h \in L^\infty(M, \mu)$,

$$\lim_{n\to\infty} \mathbb{E}_x\bigg[\frac{1}{n}\sum_{i=0}^{n-1} h \circ X_i \mid \tau > n\bigg] = \int_M h(y)\eta(y)\mu(\mathrm{d}y).$$

Note that if $m = 1$ in Theorem 2.2, we obtain that $\#(\sigma_{\mathrm{pnt}\mathcal{P}/\lambda} \cap \mathbb{S}^1) = 1$. Therefore, $h \in L^\infty(M, \mu)$

$$\lim_{n\to\infty} \frac{\mathcal{P}^n h}{\mathcal{P}^n(x, M)} = \int_M h(y)\mu(\mathrm{d}y) \quad \text{for every } x \in \bigcup_{m\in\mathbb{N}} K_m. \qquad \square$$

## 6. *Random logistic map with escape*

In this section, we analyse the Markov chain $Y_{n+1}^{a,b} = \omega_n Y_n^{a,b}(1 - Y_n^{a,b})$ absorbed at $\partial = \mathbb{R} \setminus M$, where $\{\omega_n\}_{n\in\mathbb{N}}$ is an i.i.d. sequence of random variables such that $\omega_n \sim \mathrm{Unif}([a, b])$, where $0 < a < 4 < b$ and $M = [0, 1]$. As before, for every $A \in \mathscr{B}(M)$ and $x \in M$, we denote

$$\mathcal{P}(x, A) := \mathbb{P}[Y_1^{a,b} \in A \mid Y_0^{a,b} = x].$$

Clearly, $\delta_0$ is a stationary measure for $Y_n^{a,b}$ on [0, 1]. In the following, we provide conditions to show that $Y_n^{a,b}$ admits a non-trivial quasi-stationary measure on [0, 1], which we define as a quasi-stationary measure for $Y_n^{a,b}$ different from $\delta_0$. For the sake

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of simplicity and in the interest of readability, we denote $Y_n^{a,b}$ simply as $Y_n$. Similarly, when the context is clear, we omit the $a, b$ superscript from future objects that depend on $a$ and $b$.

In the following proposition, we explicitly compute the transition functions of $Y_n$.

PROPOSITION 6.1. *Let $0 \le a \le b$, and consider the absorbing Markov chain $Y_n^{a,b}$. Moreover, given $f \in L^1(M, \mathrm{Leb})$,*

$$\mathcal{P}f(x) = \frac{1}{(b-a)x(1-x)} \int_{ax(1-x)}^{bx(1-x)\wedge 1} f(y)\mathrm{d}y.$$

*In the case where $f \in \mathcal{C}^0(M)$, then $\mathcal{P}f \in \mathcal{C}^0(M)$ and $\mathcal{P}f(0) = \mathcal{P}f(1) = f(0)$.*

*Proof.* Let $f \in L^1(M, \mathrm{Leb})$ by a direct computation,

$$\begin{aligned}\mathcal{P}f(x) &= \frac{1}{b-a}\int_a^b \mathbb{1}_{[0,1]}(\omega x(1-x)) f(\omega x(1-x))\mathrm{d}\omega \\ &= \frac{1}{(b-a)x(1-x)} \int_{ax(1-x)}^{bx(1-x)} \mathbb{1}_{[0,1]}(y) f(y)\mathrm{d}y \\ &= \frac{1}{(b-a)x(1-x)} \int_{ax(1-x)}^{bx(1-x)\wedge 1} f(y)\mathrm{d}y.\end{aligned}$$

Now, consider $f \in \mathcal{C}^0(M)$. The above equation implies that $\mathcal{P}f$ is continuous in (0, 1). For every $x \in (0, 1)$, let us define the interval $J_x := [ax(1-x), bx(1-x) \wedge 1]$. It follows that for every $x \in (0, 1/b)$, $\min_{y\in J_x} f(y) \le \mathcal{P}f(x) \le \max_{y \in J_x} f(y)$. From the continuity of $f$, we obtain that $\lim_{x\to 0} \mathcal{P}f = f(0)$. Since $\mathcal{P}(x) = \mathcal{P}(1-x)$ for every $x \in (0, 1/2)$, it follows that $\lim_{x\to 1} \mathcal{P}(x) = f(0)$, implying that $\mathcal{P}f \in \mathcal{C}^0(M)$. □

The first step to apply Theorem 2.4 to $Y_n$ on [0, 1] is to show that $Y_n$ admits a quasi-stationary measure different from $\delta_0$ on [0, 1].

Consider a measure $\mu \in \mathcal{M}(M)$ such that $\mu \ll \mathrm{Leb}(\mathrm{d}x)$ and define $g := \mu(\mathrm{d}x)/\mathrm{Leb}(\mathrm{d}x)$. Note that

$$\begin{aligned}\mathcal{P}^*(\mu)(A) &= \int_M \mathcal{P}(x, A) g(x)\,\mathrm{d}x \\ &= \int_M \frac{1}{(b-a)x(1-x)} \int_0^1 \mathbb{1}_{[ax(1-x),bx(1-x)]}(y)\mathbb{1}_A(y) g(x)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_A\int_M \mathbb{1}_{[ax(1-x),bx(1-x)]}(y)\frac{g(x)}{(b-a)x(1-x)}\,\mathrm{d}x\mathrm{d}y \\ &= \int_A \left(\int_{\alpha_-(y)}^{\alpha_+(y)} \frac{g(x)}{(b-a)x(1-x)}\,\mathrm{d}x - \int_{\beta_-(y\wedge a/4)}^{\beta_+(y\wedge a/4)} \frac{g(x)}{(b-a)x(1-x)}\,\mathrm{d}x\right)\mathrm{d}y,\end{aligned}$$

where

$$\alpha_\pm(x) := \frac{1}{2} \pm \frac{1}{2}\sqrt{1-\frac{4}{b}x} \quad \text{and} \quad \beta_\pm(x) := \frac{1}{2} \pm \frac{1}{2}\sqrt{1-\frac{4}{a}x}.$$

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The above observation motivates the definition of the stochastic transfer operator,

$$\begin{aligned}\mathcal{L} : L^1([0,1],\mathrm{Leb}) &\to L^1([0,1],\mathrm{Leb}), \\ g &\mapsto \bigg(x \mapsto \int_{\alpha_-(x)}^{\alpha_+(x)} \frac{g(y)}{(b-a)y(1-y)}\,\mathrm{d}y \\ &\quad - \int_{\beta_-(x\wedge a/4)}^{\beta_+(x\wedge a/4)} \frac{g(y)}{(b-a)y(1-y)}\,\mathrm{d}y\bigg),\end{aligned} \tag{6.1}$$

note that $\mathcal{L}$ is a well-defined linear operator since for every $g \in L^1([0,1],\mathrm{Leb})$,

$$\|\mathcal{L}(g)\|_{L^1(M,\mathrm{Leb})} := \int_0^1 |\mathcal{L}g(x)|\,\mathrm{d}x \le \int_M \mathcal{P}(x,M)|g(y)|\mathrm{d}y \le \|g\|_{L^1(M,\mathrm{Leb})}.$$

The following two propositions summarize the above comments and show that $\mathcal{L}$ is well defined as an automorphism in $L^p(M,\mathrm{Leb})$ for every $p \in [1,\infty]$. For the following result, see for instance [30, §5].

PROPOSITION 6.2. *A probability measure* $\mu \in \mathcal{M}_+(M)\setminus\{\delta_0\}$ *on* $[0,1]$ *is a quasi-stationary measure for* $Y_n$ *if and only if* $\mu(\mathrm{d}x) \ll \mathrm{Leb}(\mathrm{d}x)$ *and there exists* $0<\lambda<1$, *such that*

$$\mathcal{L}\frac{\mu(\mathrm{d}x)}{\mathrm{Leb}(\mathrm{d}x)} = \lambda\frac{\mu(\mathrm{d}x)}{\mathrm{Leb}(\mathrm{d}x)}.$$

PROPOSITION 6.3. *For every* $p \in [1,\infty]$, *the operators*

$$\mathcal{P}|_{L^p([0,1],\mathrm{Leb})}, \mathcal{L}|_{L^p([0,1],\mathrm{Leb})} : L^p([0,1]) \to L^p([0,1])$$

*are well defined and bounded.*

*Proof.* By a direct computation, one can check that

$$\mathcal{L}\mathbb{1}_M(x) = \begin{cases} \dfrac{4}{b-a}\bigg(\tanh^{-1}\bigg(\sqrt{1-\dfrac{4x}{b}}\bigg) - \tanh^{-1}\bigg(\sqrt{1-\dfrac{4x}{a}}\bigg)\bigg) & \text{if } 0 \le x \le \dfrac{a}{4}, \\ \dfrac{4}{b-a}\tanh^{-1}\bigg(\sqrt{1-\dfrac{4x}{b}}\bigg) & \text{if } \dfrac{a}{4} \le x \le 1,\end{cases}$$

implying that

$$\|\mathcal{L}\|_{L^\infty(M,\mathrm{Leb})} = \frac{4}{b-a}\tanh^{-1}\bigg(\sqrt{1-\frac{a}{b}}\bigg).$$

Since $\|\mathcal{L}\|_{L^1(M,\mathrm{Leb})} \le 1$, by the Riesz–Thorin interpolation theorem [13, Theorem 6.27],

$$\|\mathcal{L}\|_{L^p(M,\mathrm{Leb})} < \infty \quad \text{for all } p \in [1,\infty].$$

For the operator $\mathcal{P}$, note that for every $0 \le f \in L^1([0,1])$ $(1 \le p \le \infty)$,

$$\int_0^1 \mathcal{P}f(x)\,\mathrm{d}x = \int_0^1 f(x)\mathcal{L}\mathbb{1}_M(x)\,\mathrm{d}x \le \|\mathcal{L}\|_{L^\infty(M,\mathrm{Leb})}\|f\|_{L^1(M,\mathrm{Leb})},$$

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showing that $\|\mathcal{P}\|_{L^1(M,\text{Leb})} \leq \|\mathcal{L}\|_{L^\infty(M,\text{Leb})} < \infty$. Using that $\|\mathcal{P}\|_{L^\infty(M,\text{Leb})} \leq 1$, we have again by the Riesz–Thorin interpolation theorem that $\|\mathcal{P}\|_{L^p(M,\text{Leb})} < \infty$ for all $p \in [1, \infty]$. □

For every $a \in (0, 4)$ and $0 < \varepsilon < 3/8$, let us define $M_\varepsilon := [4\varepsilon(1-\varepsilon)^2, 1-\varepsilon]$ and the Markov chain $Y_{n+1}^{a,b,\varepsilon} := Y_{n+1}^{\varepsilon} = \omega_n Y_n^\varepsilon(1 - Y_n^\varepsilon)$ absorbed at $\partial^\varepsilon = \mathbb{R} \setminus M_\varepsilon$, where $\{\omega_n\}_{n\in\mathbb{N}}$ is an i.i.d. sequence of random variables and $\omega_n \sim \text{Unif}([a, b])$. Moreover, for every $\varepsilon \in (0, 3/8)$, we denote the transition kernels and transfer operator for the absorbing Markov chain $Y_n^\varepsilon$ respectively as

$$\mathcal{P}_\varepsilon f(x) := \mathbb{1}_{M_\varepsilon}(x)\mathcal{P}(\mathbb{1}_{M_\varepsilon} f)(x) \quad \text{and} \quad \mathcal{L}_\varepsilon f(x) := \mathbb{1}_{M_\varepsilon}(x)\mathcal{L}(\mathbb{1}_{M_\varepsilon} f)(x). \tag{6.2}$$

In the next proposition, we show the existence of a sequence of positive real numbers $\{\varepsilon_i\}_{i\in\mathbb{N}}$ converging to 0, such that for every $i \in \mathbb{N}$, the absorbing Markov chain $Y_n^{\varepsilon_i}$ admits a unique quasi-stationary measure $\mu_{\varepsilon_i}$ supported on $M_{\varepsilon_i}$. Moreover, these measures will play an important role in constructing a non-trivial quasi-stationary measure for $Y_n$ on $M$.

PROPOSITION 6.4. *Let $(a, b) \in [1, 4) \times (4, \infty)$ and $Y_n^{a,b,\varepsilon}$ be the Markov chain absorbed at $\partial^\varepsilon$ defined above. Then, there exists a sequence of positive numbers $\{\varepsilon_i\}_{i\in\mathbb{N}}$ converging to 0 such that, for every $i \in \mathbb{N}$, the following items hold:*

(a) *$Y_n^{a,b,\varepsilon_i}$ admits a unique quasi-stationary measure $\mu_{a,b,\varepsilon_i} := \mu_\varepsilon$ on $M_\varepsilon$ with survival rate $\lambda_{\varepsilon_i} > 0$;*
(b) *there exists a continuous function $g_\varepsilon^{a,b} := g_{\varepsilon_i} \in \mathcal{C}^0(M_{\varepsilon_i})$ such that $\mu_{\varepsilon_i}(\mathrm{d}x) = g_{\varepsilon_i}(x)\,\mathrm{d}x$; and*
(c) *$\text{supp}(\mu_{\varepsilon_i}) = M_{\varepsilon_i}$.*

*Proof.* From [17, Theorem B and Remark XIII/5], there exists a sequence $\{r_i\}_{i\in\mathbb{N}} \subset [a, 4)$ converging to 4 such that for every $i \in \mathbb{N}$, the logistic map $f_{r_i} : [0, 1] \to [0, 1]$, $f_{r_i}(x) = r_i x(1 - x)$ admits an invariant ergodic measure $\rho_{r_i} \ll \text{Leb}$ and $\text{supp}(\rho_{r_i}) = [f_{r_i}^2(1/2), f_{r_i}(1/2)]$.

Consider the sequence $\{\varepsilon_i = (4 - r_i)/4\}_{i\in\mathbb{N}}$. Combining equation (6.2) and Proposition 6.1, we obtain that

$$\frac{\mathcal{P}_{\varepsilon_i}(x, \mathrm{d}y)}{\text{Leb}(\mathrm{d}y)} = \frac{\mathbb{1}_{M_{\varepsilon_i}}(y)}{(b-a)x(1-a)}\mathbb{1}_{[ax(1-x),bx(1-x)]}(y) \quad \text{for every } i \in \mathbb{N}. \tag{6.3}$$

In the following, we show that for every $i \in \mathbb{N}$, given $x \in M_{\varepsilon_i}$ and open interval $I \subset M_{\varepsilon_i} = [f_{r_i}^2(1/2), f_{r_i}(1/2)]$, there exists $n_0 = n_0(x, I) \in \mathbb{N}$ such that $\mathcal{P}_{\varepsilon_i}^{n_0}(x, I) > 0$.

Consider the set $J := \{y \in M_{\varepsilon_i}; \omega x(1 - x) = y$ for some $\omega \in [a, b]\}$. Since $J$ has non-empty interior, we obtain that $\rho_{r_i}(J) > 0$. Since $\rho_{r_i}$ is an invariant ergodic measure, there exists $\omega_0 \in [a, b]$ such that $y := \omega_0 x(1 - x) \in J$ and $n_1 > 0 \in \mathbb{N}$ such that $f_{r_i}^{n_1}(y) \in I$.

Consider the natural number $n_0 \in \mathbb{N}$ and the continuous function $F^{x,n_0} : [a, b]^{n_0} \to \mathbb{R}$, $F^{x,n_0}(c_1, \ldots, c_{n_0}) := f_{c_1} \circ f_{c_2} \circ \ldots f_{c_{n_0}}(x)$. From the last paragraph, we obtain that $F^{x,n_0}(\omega_0, r_{\varepsilon_i}, \ldots, r_{\varepsilon_i}) \in I$. Finally, since $F^{x,n_0}$ is a continuous function, we obtain that

$$\begin{aligned}\mathcal{P}^{n_0}(x, I) &= \mathbb{P}[Y_{n_0}^{a,b,\varepsilon_i} \in I \mid X_0 = x] \\ &= \frac{1}{(b-a)^{n_0}} \text{Leb}^{\otimes n_0}(\{p \in [a,b]^{n_0};\ F^{x,n_0}(p) \in I\}) > 0. \end{aligned} \tag{6.4}$$

From equations (6.2) and (6.4), we conclude that [7, Hypothesis (H)] is fulfilled and therefore items (a), (b) and (c) follows directly from [7, Theorem A]. □

Observe that the family of measures given by the previous proposition $\{\mu_{\varepsilon_i}\}_{i\in\mathbb{N}}$ can be naturally extended on [0, 1] by imposing that $\mu_{\varepsilon_i}([0,1] \setminus M_{\varepsilon_i}) = 0$ for every $i \in \mathbb{N}$. To construct a quasi-stationary measure for the Markov process $Y_n$ on [0, 1], we use that $\{\mu_{\varepsilon_i}\}_{i\in\mathbb{N}}$ is precompact in the weak* of $\mathcal{M}([0, 1])$, that is,

$$\bigcap_{i\in\mathbb{N}} \overline{\{\mu_{\varepsilon_{k+i}}\}_{k\in\mathbb{N}}}^{\text{w}^*\text{-}\mathcal{M}(M)} \neq \emptyset, \tag{6.5}$$

where w*-$\mathcal{M}(M)$ denotes the weak* topology of $\mathcal{M}(M)$.

The proposition below shows that the elements of equation (6.5) are natural candidates for quasi-stationary measures for $Y_n$ on [0, 1].

PROPOSITION 6.5. *Assume that there exists a probability measure $\mu_{a,b} := \mu$ on $M$, $\lambda > 0$, and subsequences*

$$\{\mu_{\delta_n}\}_{n\in\mathbb{N}} \subset \{\mu_{\varepsilon_n}\}_{n\in\mathbb{N}},\ \{\lambda_{\delta_n}\}_{n\in\mathbb{N}} \subset \{\lambda_{\varepsilon_n}\}_{n\in\mathbb{N}},$$

*such that*

$$\mu_{\delta_n} \to \mu, \text{ in the weak}^*\text{-topology as } n \to \infty,$$

$$\lim_{n\to\infty} \lambda_{\delta_n} = \lambda \quad \text{and} \quad \lim_{n\to\infty} \delta_n = 0.$$

*Then $\mu$ is a quasi-stationary measure for $Y_n$ on* [0, 1].

*Proof.* Let

$$E = \{x \in [0,1], \mu(\{x\}) > 0\},$$

note that $E$ is, at most, countable. Consider the set

$$\mathcal{A} = \{I \in \mathcal{B}(M);\ I \text{ is an interval, } \overline{I} \subset (0,1) \text{ and } \sup I, \inf I \notin E\}.$$

It is clear that $\sigma(\mathcal{A}) = \mathcal{B}(M)$. Note that for every $I \in \mathcal{A}$, there exists $n_0 = n_0(I)$ such that

$$A \subset M_{\delta_n} \quad \text{for all } n > n_0.$$

This implies that for every $n > n_0$,

$$\int_M \mathcal{P}(x, I)\mu_{\delta_n}(\mathrm{d}x) = \int_{M_\varepsilon} \mathcal{P}(x, I)\mu_{\varepsilon_n}(\mathrm{d}x) = \lambda_{\delta_n}\mu_{\delta_n}(I).$$

Since $\mathcal{P}(x, I)$ is a continuous function, we obtain

$$\int_M \mathcal{P}(x, I)\mu(\mathrm{d}x) = \lambda\mu(I) \quad \text{for every } I \in \mathcal{A}.$$

Since $(4-b)/(b-a) \leq \mathcal{P}(x, M)$ for all $x \in [0, 1]$, it follows that $\lambda > 0$.

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Applying the monotone class theorem, we obtain that $\mu$ is a quasi-stationary measure for $Y_n$ on [0, 1]. □

In light of Proposition 6.5, to construct a non-trivial quasi-stationary measure for $Y_n$ on [0, 1], it remains to show that

$$\bigcap_{i\in\mathbb{N}} \overline{\{\mu_{\varepsilon_{i+k}}\}_{i\in\mathbb{N}}}^{w^*\text{-}\mathcal{M}(M)} \setminus \{\delta_0\} \neq \emptyset. \tag{6.6}$$

Note that for every $i \in \mathbb{N}$, $\mu_{\varepsilon_i}(\mathrm{d}x) \ll \mathrm{Leb}(\mathrm{d}x)$. To show that equation (6.6) holds, we study the behaviour of the distributions of $\mu_{\varepsilon_i}$ with respect to the Lebesgue measure.

The definition below provides conditions on $a$ and $b$, which implies that equation (6.6) holds (see Theorem 6.9).

*Definition 6.1.* A pair $(a, b) \in (0, 4) \times (4, \infty)$ is called an *admissible pair* if either

- $a \geq 2$; or
- for every $x \in [(4a^2 - a^3)/16, a/4]$,

$$0 \leq \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{2}{b}\bigg(1 - \sqrt{1 - \frac{4x}{a}}\bigg)} \leq \frac{a}{4} \tag{6.7}$$

and

$$\frac{2\big(\tanh^{-1}\big(\sqrt{2\sqrt{1-(4x/b)}+b-2/b}\big) - \tanh^{-1}\big(\sqrt{a+2\sqrt{1-(4x/b)}-2/a}\big)\big)}{2\tanh^{-1}\big(\sqrt{2\sqrt{1-(4x/a)}+b-2/b}\big) + \log(a/4-a)} \leq \frac{\sqrt{1-4x/b}}{\sqrt{1-4x/a}}. \tag{6.8}$$

In Theorem 6.18, we show that if $(a, b) \in [1, 4) \times (4, \infty)$, then $(a, b)$ is an admissible pair. Assuming that $(a, b)$ is an admissible pair, it is possible to show that $Y_n^{a,b}$ admits a non-trivial quasi-stationary measure on [0, 1]. To accomplish this goal, we need the following three technical lemmas.

LEMMA 6.6. *Let $(a, b)$ be an admissible pair, with $a < 2$, and $f : [0, 1] \to \mathbb{R}$ be a function continuous by parts with a finite number of discontinuities, such that:*

(1) $0 \leq f(x)$ *for every* $x \in [0, 1]$;
(2) *$f$ is non-decreasing in the interval* $[0, a/4]$; *and*
(3) *$f$ is non-increasing in the interval* $[a/4, 1]$.

*Then $\mathcal{L}f$ is a continuous function such that:*

(1) $0 \leq \mathcal{L}f(x)$ *for every* $x \in [0, 1]$;
(2) *$\mathcal{L}f$ is non-decreasing in the interval* $[0, (4a^2 - a^3)/16]$; *and*
(3) *$\mathcal{L}f$ is non-increasing in the interval* $[a/4, 1]$.

*Proof.* Recall that

$$\mathcal{L}f(x) = \int_{\alpha_-(x)}^{\alpha_+(x)} \frac{f(y)}{(b-a)y(1-y)}\,\mathrm{d}y - \int_{\beta_-(x\wedge a/4)}^{\beta_+(x\wedge a/4)} \frac{f(y)}{(b-a)y(1-y)}\,\mathrm{d}y, \tag{6.9}$$

where

$$\alpha_{\pm}(x) = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{4}{b}x} \quad \text{and} \quad \beta_{\pm}(x) = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{4}{a}x}.$$

It is clear that $\mathcal{L}f$ is continuous and a non-negative function. Observe that $\mathcal{L}f$ is differentiable except for finitely many points. In fact, the derivative of $\mathcal{L}f$ on the points where the derivative exists is given by

$$\frac{\mathrm{d}\mathcal{L}f}{\mathrm{d}x}(x) = -\frac{f(\alpha_+(x)) + f(\alpha_-(x))}{(b-a)x\sqrt{1-4/bx}} + \mathbb{1}_{[0,a/4]}(x)\frac{f(\beta_+(x)) + f(\beta_-(x))}{(b-a)x\sqrt{1-4/ax}}. \tag{6.10}$$

Since for every $x \in [a/4, 1]$,

$$\frac{\mathrm{d}\mathcal{L}f}{\mathrm{d}x}(x) = -\frac{1}{b-a}\frac{f(\alpha_+(x)) + f(\alpha_-(x))}{x\sqrt{1-4/bx}} \leq 0,$$

if follows that $\mathcal{L}f$ is non-increasing in $[a/4, 1]$.

Observe that for every $x \in [0, a/4]$, we obtain

$$\frac{1}{(b-a)x\sqrt{1-4/bx}} < \frac{1}{(b-a)x\sqrt{1-4/ax}} \quad \text{and} \quad \frac{a}{4} \leq \beta_+(x) \leq \alpha_+(x).$$

Since $f$ is non-increasing in $[a/4, 1]$, we conclude that

$$-\frac{f(\alpha_+(x))}{(b-a)x\sqrt{1-4/bx}} + \frac{f(\beta_+(x))}{(b-a)x\sqrt{1-4/ax}} \geq 0.$$

To finish the proof, it is enough to show that $f(\beta_-(x)) \geq f(\alpha_-(x))$ for every $x \in [0, (4a^2 - a^3)/16]$. Observe that since $f$ is non-decreasing on $[0, a/4]$, we obtain that for every $x \in [0, (4a^2 - a^3)/16]$,

$$\beta_-(x) \leq \alpha_-(x) \leq a/4,$$

implying that

$$f(\beta_-(x)) - f(\alpha_-(x)) \geq 0. \qquad \square$$

LEMMA 6.7. *Let* $(a, b)$ *be an admissible pair and* $\varepsilon \in (0, 3/8)$ *such that* $[(4a^2 - a^3)/16, a/4] \subset M_\varepsilon$. *Consider that sequence of functions* $\{\mathcal{L}_\varepsilon^n \mathbb{1}_{M_\varepsilon}\}_{n\in\mathbb{N}}$, *then for every* $n \in \mathbb{N}$, *the following assertions hold:*

(1) $0 \leq \mathcal{L}_\varepsilon^n \mathbb{1}_{M_\varepsilon}(x)$ *for every* $x \in [0, 1]$;

(2) $\mathcal{L}_\varepsilon^n \mathbb{1}_{M_\varepsilon}(x)$ *is non-decreasing in the interval* $[0, (4a^2 - a^3)/16]$; *and*

(3) $\mathcal{L}_\varepsilon^n \mathbb{1}_{M_\varepsilon}(x)$ *is non-increasing in the interval* $[a/4, 1]$.

*Proof.* Recall that for every $\varepsilon \in (0, 3/8)$ and $f \in \mathcal{C}^0(M_\varepsilon)$, $\mathcal{L}_\varepsilon f = \mathbb{1}_{M_\varepsilon}\mathcal{L}(\mathbb{1}_{M_\varepsilon} f)$. We divide the proof into two steps.

*Step 1*. We show the result for the case where $a \geq 2$.

We show the above result by induction on $n$. The case $n = 0$ is immediately verified. Suppose that items (1), (2) and (3) are true for $\mathcal{L}_\varepsilon^n \mathbb{1}_{M_\varepsilon}$. We will show that the same is true for $\mathcal{L}_\varepsilon^{n+1} \mathbb{1}_{M_\varepsilon}$.

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Item (1) is trivially fulfilled since $\mathcal{L}_\varepsilon$ is a positive operator. Additionally items (2) and (3) follow from equation (6.10) and realizing that for every $(a, b) \in [2, 4) \times (4, \infty)$,

$$\alpha_-(x) \le \beta_-(x) \le \frac{4a^2 - a^3}{16} \le \frac{a}{4} \le \beta_+(x) \le \alpha_+(x) \quad \text{for every } x \in \left[0, \frac{4a^2 - a^3}{16}\right].$$

This proves Step 1.

*Step 2*. We show that if $(a, b)$ is an admissible pair and $a \in (0, 2)$, then:

(1) $0 \le \mathcal{L}_\varepsilon^n \mathbb{1}_{M_\varepsilon}(x)$ for every $x \in [0, 1]$;

(2) $\mathcal{L}_\varepsilon^n \mathbb{1}_{M_\varepsilon}(x)$ is non-decreasing in the interval $[0, a/4]$; and

(3) $\mathcal{L}_\varepsilon^n \mathbb{1}_{M_\varepsilon}(x)$ is non-increasing in the interval $[a/4, 1]$.

We will prove that the above items hold by strong induction on $n$. For the cases $n = 0$ and $n = 1$, the computations can explicitly be done and such a conclusion is achieved.

Now, suppose that the conclusions of Step 2 are true for

$$\mathbb{1}_{M_\varepsilon}, \mathcal{L}_\varepsilon^1 \mathbb{1}_{M_\varepsilon}, \ldots, \mathcal{L}_\varepsilon^n \mathbb{1}_{M_\varepsilon} \text{ with } n \ge 1$$

and we will show that it is also true for $\mathcal{L}_\varepsilon^{n+1} \mathbb{1}_{M_\varepsilon}$.

From Lemma 6.6, it follows that:

(1) $0 \le \mathcal{L}_\varepsilon^{n+1} \mathbb{1}_{M_\varepsilon}(x)$ for every $x \in [0, 1]$;

(2) $\mathcal{L}_\varepsilon^{n+1} \mathbb{1}_{M_\varepsilon}$ is non-decreasing in the interval $[0, (4a^2 - a^3)/16]$; and

(3) $\mathcal{L}_\varepsilon^{n+1} \mathbb{1}_{M_\varepsilon}$ is non-increasing in the interval $[a/4, 1]$.

It remains to show that $\mathcal{L}_\varepsilon^{n+1} M_\varepsilon$ is non-decreasing in $[(4a^2 - a^3)/16, a/4]$. From the proof of the previous theorem, it is enough to show that

$$\frac{\mathcal{L}_\varepsilon^n \mathbb{1}_{M_\varepsilon}(\alpha_-(x))}{\sqrt{1 - 4x/b}} \le \frac{\mathcal{L}_\varepsilon^n \mathbb{1}_{M_\varepsilon}(\beta_-(x))}{\sqrt{1 - 4x/a}} \quad \text{for every } x \in \left[\frac{4a^2 - a^3}{16}, \frac{a}{4}\right]. \tag{6.11}$$

Observe that

$$\alpha_-(x) < \frac{a}{4} < \beta_-(x) \quad \text{for every } x \in \left[\frac{4a^2 - a^3}{16}, \frac{a}{4}\right].$$

Therefore,

$$\mathcal{L}_\varepsilon^n \mathbb{1}_{M_\varepsilon}(\beta_-(x)) = \int_{\alpha_- \circ \beta_-(x)}^{\alpha_+ \circ \beta_-(x)} \frac{\mathcal{L}_\varepsilon^{n-1} \mathbb{1}_{M_\varepsilon}(y)}{(b - a)y(1 - y)} \, dy \tag{6.12}$$

and

$$\begin{aligned}
\mathcal{L}_\varepsilon^n \mathbb{1}_{M_\varepsilon}(\alpha_-(x)) &= \int_{\alpha_- \circ \alpha_-(x)}^{\alpha_+ \circ \alpha_-(x)} \frac{\mathcal{L}_\varepsilon^{n-1} \mathbb{1}_{M_\varepsilon}(y)}{(b - a)y(1 - y)} \, dy - \int_{\beta_- \circ \alpha_-(x)}^{\beta_+ \circ \alpha_-(x)} \frac{\mathcal{L}_\varepsilon^{n-1} \mathbb{1}_{M_\varepsilon}(y)}{(b - a)y(1 - y)} \, dy \\
&= \int_{\alpha_- \circ \alpha_-(x)}^{\beta_- \circ \alpha_-(x)} \frac{\mathcal{L}_\varepsilon^{n-1} \mathbb{1}_{M_\varepsilon}(y)}{(b - a)y(1 - y)} \, dy + \int_{\beta_+ \circ \alpha_-(x)}^{\alpha_+ \circ \alpha_-(x)} \frac{\mathcal{L}_\varepsilon^{n-1} \mathbb{1}_{M_\varepsilon}(y)}{(b - a)y(1 - y)} \, dy.
\end{aligned}$$

Since $(a, b)$ is an admissible pair, equation (6.7) implies that for every $x \in [(4a^2 - a^3)/16, a/4]$,

$$\beta_- \circ \alpha_-(x) < \alpha_- \circ \beta_-(x) < \frac{a}{4} < \alpha_+ \circ \beta_-(x) \le \beta_+ \circ \alpha_-(x).$$

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This implies that

$$\begin{aligned}\mathcal{L}_\varepsilon^n \mathbb{1}_{M_\varepsilon}(\alpha_-(x)) \leq& \frac{\mathcal{L}_\varepsilon^{n-1}\mathbb{1}_{M_\varepsilon}(\beta_- \circ \alpha_-(x))}{b-a} \int_{\alpha_-\circ\alpha_-(x)}^{\beta_-\circ\alpha_-(x)} \frac{1}{y(1-y)}\,\mathrm{d}y \\ &+ \frac{\mathcal{L}_\varepsilon^{n-1}\mathbb{1}_{M_\varepsilon}(\beta_+ \circ \alpha_-(x))}{b-a} \int_{\beta_+\circ\alpha_-(x)}^{\alpha_+\circ\alpha_-(x)} \frac{1}{y(1-y)}\mathrm{d}y \\ \leq& \frac{\mathcal{L}_\varepsilon^{n-1}\mathbb{1}_{M_\varepsilon}(\beta_- \circ \alpha_-(x)) + \mathcal{L}_\varepsilon^{n-1}\mathbb{1}_{M_\varepsilon}(\beta_+ \circ \alpha_-(x))}{b-a} I_1^{(a,b)}(x),\end{aligned}$$

where

$$I_1^{(a,b)}(x) := 2\big(\tanh^{-1}\big(\sqrt{2\sqrt{1-4x/b}+b-2/b}\big) - \tanh^{-1}\big(\sqrt{a+2\sqrt{1-4x/b}-2/a}\big)\big).$$

However, from the induction hypothesis and equation (6.12),

$$\begin{aligned}\mathcal{L}_\varepsilon^n \mathbb{1}_{M_\varepsilon}(\beta_-(x)) \geq& \frac{\mathcal{L}_\varepsilon^{n-1}\mathbb{1}_{M_\varepsilon}(\beta_- \circ \alpha_-(x))}{b-a} \int_{\alpha_-\circ\beta_-(x)}^{a/4} \frac{1}{y(1-y)}\,\mathrm{d}y \\ &+ \frac{\mathcal{L}_\varepsilon^{n-1}\mathbb{1}_{M_\varepsilon}(\beta_+ \circ \alpha_-(x))}{b-a} \int_{a/4}^{\alpha_+\circ\beta_-(x)} \frac{1}{y(1-y)}\mathrm{d}y \\ \geq& \frac{\mathcal{L}_\varepsilon^{n-1}\mathbb{1}_{M_\varepsilon}(\beta_- \circ \alpha_-(x)) + \mathcal{L}_\varepsilon^{n-1}\mathbb{1}_{M_\varepsilon}(\beta_+ \circ \alpha_-(x))}{b-a} I_2^{(a,b)}(x),\end{aligned}$$

where

$$I_2^{(a,b)}(x) := \left(2\tanh^{-1}\left(\sqrt{2\sqrt{1-4x/a}+b-2/b}\right) + \log\left(\frac{a}{4-a}\right)\right).$$

Combining the above three equations, equation (6.8) and using the definition of admissible pair, we obtain that equation (6.11) holds. This proves Step 2.

Observe the above two steps imply the proof of the lemma. □

Recall from Proposition 6.4, for every $i \in \mathbb{N}$,

$$g_{\varepsilon_i}^{a,b} := g_{\varepsilon_i} = \frac{\mu_{\varepsilon_i}(\mathrm{d}x)}{\mathrm{Leb}(\mathrm{d}x)} \in L^1([0,1], \mathrm{Leb}),$$

where we set $g_{\varepsilon_i}(x) = 0$ for every $x \in M \setminus M_{\varepsilon_i}$.

LEMMA 6.8. *Let $(a, b)$ be an admissible pair. Then, for every $i \in \mathbb{N}$:*

(1) $0 \leq g_{\varepsilon_i}(x)$ *for every* $x \in [0, 1]$;

(2) $g_{\varepsilon_i}(x)$ *is non-decreasing in the interval* $[0, (4a^2 - a^3)/16]$; *and*

(3) $g_{\varepsilon_i}(x)$ *is non-increasing in the interval* $[a/4, 1]$.

*Proof.* Recall that for every $i \in \mathbb{N}$ and $f \in \mathcal{C}^0(M_{\varepsilon_i})$, $\mathcal{L}_{\varepsilon_i} f = \mathbb{1}_{M_{\varepsilon_i}}\mathcal{L}(\mathbb{1}_{M_{\varepsilon_i}} f)$. Observe that if $(a, b)$ is an admissible pair and $i \in \mathbb{N}$, then $\mathcal{L}_{\varepsilon_i} : \mathcal{C}^0(M_{\varepsilon_i}) \to \mathcal{C}^0(M_{\varepsilon_i})$ is an irreducible compact operator. Moreover, it is readily verified that $\mathcal{L}_{\varepsilon_i}$ admits a single eigenvalue in its peripheral spectrum, implying that

$$\left\| \frac{1}{\lambda_{\varepsilon_i}^n}\mathcal{L}_{\varepsilon_i}^n \mathbb{1}_M - \alpha_{\varepsilon_i} g_{\varepsilon_i} \right\|_{\mathcal{C}^0(M_{\varepsilon_i})} \xrightarrow{n\to\infty} 0$$

for some $\alpha_{\varepsilon_i} > 0$.

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The lemma follows directly from the above equation in combination with Lemma 6.7. □

Combining Lemmas 6.6, 6.7 and 6.8, we obtain the following result.

THEOREM 6.9. *Let* $(a, b)$ *be an admissible pair. Then the absorbing Markov chain* $Y_n^{a,b}$ *admits a quasi-stationary measure* $\mu$ *on* $[0, 1]$ *different from* $\delta_0$.

*Proof.* For every $i \in \mathbb{N}$, let $\mu_{\varepsilon_i}(\mathrm{d}x) = g_{\varepsilon_i}(x)\mathrm{d}x$ be the unique quasi-stationary measures for $Y_n^{\varepsilon_i}$ on $M_{\varepsilon_i}$ given by Proposition 6.4 and extend it to $[0, 1]$ in a way that $\mu_{\varepsilon_i}(M \setminus M_{\varepsilon_i}) = 0$.

Since $\mathcal{M}_1([0, 1])$ is sequentially compact in the weak* topology, we can assume without loss of generality (passing to a subsequence if necessary) that the sequence of real numbers $\{\varepsilon_i\}_{i\in\mathbb{N}}$ is such that $\lim_{i\to\infty} \varepsilon_i = 0$, $\mu_{\varepsilon_i} \to \mu$ in the weak* topology and $\lim_{i\to\infty} \lambda_{\varepsilon_i} = \lambda \geq (4 - a)/(b - a)$.

From Proposition 6.5, the probability measure $\mu$ is a quasi-stationary measure of $\mathcal{P}$. It remains to show that $\mu \neq \delta_0$. Suppose by contradiction that $\mu = \delta_0$. Then, $\lim_{i\to\infty} \mu_{\varepsilon_i}([0, (4a^2 - a^3)/32]) = 1$. However, from Lemma 6.8, it follows that

$$\mu_{\varepsilon_i}([0, (4a^2 - a^3)/32]) \leq \mu_{\varepsilon_i}([(4a^2 - a^3)/32, 4a^2 - a^3)/16]) \quad \text{for every } i \in \mathbb{N}.$$

Taking the limit as $i \to \infty$, we obtain that $1 \leq \mu([4a^2 - a^3)/32, (4a^2 - a^3)/16])$, which is a contradiction, implying that $\mu \neq \delta_0$. □

*Remark 6.10.* Observe that without assuming that $(a, b)$ is an admissible pair, the inductive step presented in Step 2 of Lemma 6.7 no longer holds. Without this lemma, the core argument in the proof of Theorem 6.9 cannot be applied, and the existence of a non-trivial quasi-stationary measure for $Y_n^{a,b}$ becomes unclear.

From now on, we define $\mu_{a,b} = \mu$ as a non-trivial quasi-stationary measure for $Y_n$ on $[0, 1]$ and $\lambda_{a,b} = \lambda$ its associated survival rate (given by Theorem 6.9). The next proposition shows that $\mu$ is absolutely continuous to the Lebesgue measure.

PROPOSITION 6.11. *Let* $(a, b)$ *be an admissible pair. Then,* $\mu \ll \mathrm{Leb}(\mathrm{d}x)$ *and* $0 < \lambda < 1$.

*Proof.* We can decompose $\mu(\mathrm{d}x) = \mu(\{0\})\delta_0(\mathrm{d}x) + \mu'(\mathrm{d}x) + \mu(\{1\})\delta_1(\mathrm{d}x)$.

Since $\delta_0 \neq \mu$, we obtain that $\mu(\{0\}) \neq 1$. Observe that

$$\begin{aligned}\lambda\mu(\{0\})\delta_0(\mathrm{d}x) + \lambda\mu'(\mathrm{d}x) + \lambda\mu(\{1\})\delta_1(\mathrm{d}x) &= \lambda\mu(\mathrm{d}x)\\ &= \mathcal{P}^*(\mu)(\mathrm{d}x)\\ &= (\mu(\{1\}) + \mu\{0\})\delta_0(\mathrm{d}x) + \mathcal{P}^*(\mu').\end{aligned}$$

Since $\mathcal{P}^*(\mu') \ll \mathrm{Leb}(\mathrm{d}x)$, it follows that $\mathcal{P}^*(\mu')(\{1\}) = 0$, implying that

$$\mu(\{1\}) = 0$$

and

$$\lambda\mu(\{0\}) = \mu(\{0\}).$$

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We claim that $\lambda < 1$. Suppose, by contradiction, the opposite that $\lambda = 1$, then

$$\mu = (\mathcal{P}^*)^n \mu = \mu(\{0\})\delta_1(\mathrm{d}x) + (\mathcal{P}^*)^n \mu'.$$

Since $\mathcal{P}^n 1(x) \to \mathbb{1}_{\{0\}\sqcup\{1\}}(x)$ pointwise as $n \to \infty$, we have, by the Lebesgue dominated convergence theorem,

$$\mu = \lim_{n\to\infty} (\mathcal{P}^*)^n \mu = \mu\{0\}\delta_1(\mathrm{d}x) + \lim_{n\to\infty} \mathcal{P}^{*n} \mu' = \mu(\{0\})\delta_0,$$

which is contradiction since $\mu(\{0\}) \neq 1$.

This implies that $\lambda < 1$ and therefore $\mu(\{0\}) = 0$. Therefore, $\mu(\{0\} \cup \{1\}) = 0$ and

$$\lambda\mu = \mathcal{P}^*\mu \ll \mathrm{Leb}(\mathrm{d}x). \qquad \square$$

From now on, we define

$$\frac{\mu_{a,b}(\mathrm{d}x)}{\mathrm{Leb}(\mathrm{d}x)} =: g^{a,b} = g \in L^1([0, 1], \mu).$$

The next result summarizes the properties of $g$.

PROPOSITION 6.12. *Let $(a, b)$ be an admissible pair. Then the function $g$ fulfils the following properties:*

(i) $g \in \mathcal{C}^0(M)$;

(ii) *$g$ is non-decreasing in the interval* $[0, 4a^2 - a^3)/16]$;

(iii) *$g$ is non-increasing in the interval* $[a/4, 1]$;

(iv) *there exists $k > 0$ such that $k < g(x)$ for every $x \in M$.*

*Proof.* We divide this proof into 3 steps.

*Step 1.* We show that $g(x) > 0$ for every $x \in (0, 1]$.

Suppose that there exists $x \in (0, 1]$ such that $g(x) = 0$. Therefore,

$$0 = \lambda g(x) = \int_{\alpha_-(x)}^{\alpha_+(x)} \frac{g(y)}{(b-a)y(1-y)}\mathrm{d}y - \int_{\beta_-(x\wedge a/4)}^{\beta_+(x\wedge a/4)} \frac{g(y)}{(b-a)y(1-y)}\mathrm{d}y.$$

This implies that

$$g(y) = 0 \quad \text{for all } y \in I_1 := [\alpha_-(x), \beta_-(x \wedge a/4)] \cup [\beta_+(x \wedge a/4), \alpha_+(x)] \subset (0, 1).$$

Let $x_0 \in \mathrm{supp}(\mu) \cap (0, 1)$. By the same arguments presented in the proof of Proposition 6.4, we can show that there exist $n_0 = n_0(x, I_1)$ such that $\mathcal{P}^{n_0}(x_0, I_1) > 0$. Since $\mathcal{P}^{n_0}(x_0, I_1)$ is a continuous function, there exists an open neighbourhood $B \subset (0, 1)$ of $x$ such that

$$\inf_{y\in B} \mathcal{P}^{n_0}(y, I_1) \geq \tfrac{1}{2}\mathcal{P}^{n_0}(x_0, I_1) > 0.$$

Therefore,

$$0 = \mu(I_1) = \frac{1}{\lambda^n}\int_M \mathcal{P}^{n_0}(y, I)g(y)\,\mathrm{d}x \geq \frac{\mathcal{P}^{n_0}(x, I_1)}{2}\mu(B) > 0,$$

which is a contradiction. Therefore, $g(x) > 0$ for every $x \in (0, 1]$.

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*Step 2*. We show $(i)$, $(ii)$ and $(iii)$.

Recall that for every $i \in \mathbb{N}$, $g_{\varepsilon_i}(x) = (1/\lambda_{\varepsilon_i})\mathbb{1}_{M_{\varepsilon_i}}\mathcal{L}g_{\varepsilon_i}(x)$. This observation, combined with Theorem 6.8 and Lemma 6.6, implies that

$$\|\mathcal{L}g_{\varepsilon_i}\|_{L^\infty} = \sup_{y\in[(4a^2-a^3)/16,a/4]} \mathcal{L}g_{\varepsilon_i}(y) \quad \text{for every } i \in \mathbb{N}.$$

Let

$$J := \bigcup_{x\in[(4a^2-a^3)/16,a/4]} ([\alpha_-(x), \beta_-(x) \wedge a/4] \cup [\beta_-(x) \wedge a/4, \alpha_+(x) \wedge a/4]) \subset (0, 1),$$

and observe that $J$ is a compact set. Finally,

$$\begin{aligned} 0 \le g_{\varepsilon_i}(x) &\le \frac{1}{\lambda_{\varepsilon_i}}\mathcal{L}g_{\varepsilon_i}(x) \le \frac{1}{\lambda_\varepsilon} \sup_{y\in[(4a^2-a^3)/16,a/4]} \mathcal{L}g_{\varepsilon_i}(y) \le \frac{1}{\lambda_{\varepsilon_i}}\int_J \frac{g_{\varepsilon_i}(y)}{(b-a)y(1-y)}\mathrm{d}y \\ &\le \sup_{y\in J} \frac{1}{4y(1-y)} \sup_{i\in\mathbb{N}} \frac{1}{\lambda_{\varepsilon_i}} =: C < \infty. \end{aligned} \tag{6.13}$$

Therefore, we obtain a uniform bound for $\{g_{\varepsilon_i}\}_{i\in\mathbb{N}}$ on $L^\infty(M)$ for $n$ big enough. For every $\delta > 0$, consider the map

$$\begin{aligned} T_\delta : L^\infty(M) &\to \mathcal{C}^0([\delta, 1-\delta]), \\ f &\to \mathbb{1}_{[\delta,1-\delta]}\mathcal{L}f. \end{aligned}$$

From the Arzelà–Ascoli theorem, it is readily verified that $T_\delta$ is a compact operator for every $0 < \delta < 1/2$. From equation (6.13), we obtain that there exists a subsequence $\{\mathcal{L}g_{\varepsilon_{i_n}}\}_{n\in\mathbb{N}} \subset \{\mathcal{L}g_{\varepsilon_i}\}_{i\in\mathbb{N}}$ and $f_\delta \in \mathcal{C}^0([\delta, 1-\delta])$ such that

$$\lim_{n\to\infty} \|T_\delta\mathcal{L}g_{\varepsilon_{i_n}} - f_\delta\|_\infty = 0.$$

Choosing an interval $I_\delta \subset [\delta, 1-\delta]$, observe that

$$\mathcal{P}(x, I_\delta) \text{ is continuous on } x \in [0, 1],$$

it follows that

$$\int_{I_\delta} f_\delta(y)\,\mathrm{d}x = \lim_{n\to\infty}\int_{I_\delta} \mathcal{L}g_{\varepsilon_{i_n}}(x)\,\mathrm{d}x = \lim_{n\to\infty}\int_0^1 \mathcal{P}(x, I_\delta)g_{\varepsilon_{i_n}}(x)\,\mathrm{d}x = \int_{I_\delta} \lambda g(x)\,\mathrm{d}x.$$

Since $I_\delta$ is an arbitrary interval subset of $[\delta, 1-\delta]$, we obtain $f_\delta = \lambda g|_{[\delta,1-\delta]}$. Using that for every subsequence of $\{\mathbb{1}_{[\delta,1-\delta]}\mathcal{L}g_{\varepsilon_i}\}_{i\in\mathbb{N}}$ there exists subsubsequence converging to $\mathbb{1}_{[\delta,1-\delta]}\lambda g$, we obtain that

$$\lim_{i\to\infty} \|\mathbb{1}_{[\delta,1-\delta]}(\mathcal{L}g_{\varepsilon_i} - \lambda g)\|_\infty = 0.$$

Since $\{g_{\varepsilon_i}\}_{i\in\mathbb{N}}$ is bounded $L^\infty(M, \mu)$ and $g$ lies in $L^1(M, \mu)$, the above equation implies that

$$\mathcal{L}g_{\varepsilon_i} \to \lambda g \text{ in } L^1([0, 1]).$$

Thus, there exists a subsequence $\{g_{\varepsilon_{n_i}}\}_{i\in\mathbb{N}} \subset \{g_{\varepsilon_n}\}_{n\in\mathbb{N}}$ such that

$$\lim_{i\to\infty} \mathcal{L}g_{\varepsilon_{n_i}} = \lambda g \quad \mu\text{-a.s.} \tag{6.14}$$

Therefore, for $\mu$-almost every $x \in M$,

$$0 \le g(x) \le \frac{1}{\lambda} \lim_{i\to\infty} \mathcal{L}g_{\varepsilon_{n_i}}(x) \le \frac{C}{\lambda},$$

which implies that $g$ is $L^\infty([0,1])$. Since for every $i \in \mathbb{N}$:

(1) $\mathcal{L}g_{\varepsilon_{n_i}}$ is non-decreasing in the interval $[0, (4a^2 - a^3)/16]$; and

(2) $\mathcal{L}g_{\varepsilon_{n_i}}$ is non-increasing in the interval $[a/4, 1]$,

from equation (6.14) and the continuity of $g$ on $(0, 1]$, we obtain that:

(1) $g$ is non-decreasing in the interval $[0, (4a^2 - a^3)/16]$; and

(2) $g$ is non-increasing in the interval $[a/4, 1]$.

The proof is finished observing that $g \in \mathcal{C}^0([0,1])$ when imposing $g(0) := \inf_{x\in(0,a/4)} g(x)$.

*Step 3*. We show item (iv).

Observe that in virtue of Step 2, it is enough to show that $g(0) > 0$. Since $g$ is continuous, it follows that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^\varepsilon g(y)\, \mathrm{d}y = g(0).$$

Since $g(x)\,\mathrm{d}x$ is a quasi-stationary measure of $Y_n$ on $[0, 1]$, we obtain that

$$\int_0^\varepsilon g(y)\mathrm{d}y = \frac{1}{\lambda} \int_0^1 \mathcal{P}(y, [0,\varepsilon])g(y)\, \mathrm{d}y.$$

It is clear that

$$\mathcal{P}(x, [0,\varepsilon]) = 1 \quad \text{for every } x \in [\alpha_+(\varepsilon), 1] \subset [a/4, 1].$$

Since $g$ is decreasing in $[a/4, 1]$ and $g(1) > 0$, it follows that

$$\int_0^\varepsilon g(y)\, \mathrm{d}y = \frac{1}{\lambda} \int_0^1 \mathcal{P}(y, [0,\varepsilon])g(y)\mathrm{d}y \ge \frac{g(1)}{\lambda}\Big(1 - \alpha_+(\varepsilon)\Big) = \frac{g(1)}{\lambda}\bigg(\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4\varepsilon}{b}}\bigg).$$

Finally,

$$g(0) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^\varepsilon g(y)\mathrm{d}y \ge \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\frac{g(1)}{\lambda}\bigg(\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4\varepsilon}{b}}\bigg) = \frac{g(1)}{b\lambda} > 0.$$

Combining Steps 1–3, we conclude the proof of the theorem. □

To apply Theorem 2.4, we need to show that $\mathcal{P}$ admits an eigenfunction lying in $L^1([0,1], \mathrm{Leb})$. To do this, consider the operator

$$T : \mathcal{C}^0([0,1]) \to \mathcal{C}^0([0,1]),$$
$$f \mapsto \frac{\mathcal{L}(gf)}{\lambda g}.$$

It is clear that $T$ is a Markov operator that is:

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(1) $T : \mathcal{C}^0([0,1]) \to \mathcal{C}^0([0,1])$ is a bounded positive linear operator;
(2) $T1 = 1$.

PROPOSITION 6.13. *Let $(a, b)$ be an admissible pair. Then there exists a probability $\nu \ll$ Leb such that $\nu$ is a fixed point of the operator $T^* : \mathcal{M}(M) \to \mathcal{M}(M)$.*

*Proof.* Since $T$ is a Markov operator, it is well known that there exists a probability measure $\nu$ such that $T^*\nu = \nu$ (see [10, Ch. 10]).

Let us decompose $\nu$ as

$$\nu = \alpha_1\delta_0 + \alpha_2\nu' + \alpha_3\delta_1,$$

where $\nu \in \mathcal{M}_1(M)$ and $\nu'(\{0\} \cup \{1\}) = 0$.

Since

$$\begin{aligned}\mathcal{L}(fg)(0) &= \frac{1}{b-a}\lim_{x\to 0}\bigg(\int_{\alpha_-(x)}^{\beta_-(x)} \frac{f(y)g(y)}{y(1-y)}\mathrm{d}y + \int_{\beta_+(x)}^{\alpha_+(x)} \frac{f(y)g(y)}{y(1-y)}\mathrm{d}y\bigg)\\ &= \frac{\log(b/a)}{b-a}f(0)g(0) + \frac{\log(b/a)}{b-a}f(1)g(1),\end{aligned}$$

we obtain that

$$Tf(0) = \frac{\log(b/a)}{(b-a)\lambda}f(0) + \frac{\log(b/a)}{(b-a)\lambda}\frac{g(1)}{g(0)}f(1). \tag{6.15}$$

From a similar computation, we obtain that

$$Tf(1) = \frac{1}{\lambda g(1)}\int_{\alpha_-(1)}^{\alpha_+(1)} f(x)g(x)\,\mathrm{d}x. \tag{6.16}$$

Note that given $A \in \mathscr{B}([0,1])$ such that $\mathrm{Leb}(A) = 0$ and $A \subset [\delta, 1-\delta]$ for some $\delta > 0$, then $T(\mathbb{1}_A) = 0$. This implies that

$$T^*\nu'(A) = \int_0^1 T^*\mathbb{1}_A(x)\nu'(\mathrm{d}x) = 0,$$

since $\nu'(\{0\} \cup \{1\}) = 0$, we obtain

$$T^*\nu'(\mathrm{d}x) \ll \mathrm{Leb}(\mathrm{d}x). \tag{6.17}$$

Combining $T^*\nu = \nu$, and equations (6.15), (6.16) and (6.17), we obtain that $\nu' \ll \mathrm{Leb}(\mathrm{d}x)$.

Let $\{f_n\}_{n\in\mathbb{N}} \in \mathcal{C}^0(M)$ be a sequence of continuous functions such that:
(1) $0 \le f_n(x) \le 1$ for every $n \in \mathbb{N}$ and $x \in [0,1]$;
(2) $f_n(1) = 1$; and
(3) $f_n(x) = 0$ for every $x \in [0, 1 - 1/n]$.

Since $T^*\nu = \nu$ and $f_n$ is continuous, it follows that

$$\int_M f_n(x)\nu(\mathrm{d}x) = \int_M Tf_n(x)\nu(\mathrm{d}x) \quad \text{for every } n \in \mathbb{N}. \tag{6.18}$$

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The left-hand side of equation (6.18) is equal to

$$\int_M f_n(x)\nu(\mathrm{d}x) = \alpha_2 \int f_n(x)\nu(\mathrm{d}x) + \alpha_3 f_n(1) = \alpha_2 \int f_n(x)\nu(\mathrm{d}x) + \alpha_3,$$

and the right-hand side of equation (6.18) is equal to

$$\begin{aligned}
&\int_M f_n(x)T^*\nu(\mathrm{d}x)\\
&\quad = \alpha_1\bigg(\frac{\log(b/a)}{(b-a)\lambda} f_n(0) + \frac{\log(b/a)g(1)}{(b-a)g(0)\lambda} f_n(1)\bigg) + \alpha_2 \int_0^1 f_n(x)T^*\nu(\mathrm{d}x)\\
&\qquad + \alpha_3 \frac{1}{\lambda g(1)} \int_{\alpha_-(1)}^{\alpha_+(1)} f_n(x)g(x)\,\mathrm{d}x\\
&\quad = \alpha_1 \frac{\log(b/a)}{b-a}\frac{g(1)}{\lambda g(0)} + \alpha_2 \int_0^1 f_n(x)T^*\nu(\mathrm{d}x) + \alpha_3 \frac{1}{\lambda g(1)} \int_{\alpha_-(1)}^{\alpha_+(1)} f_n(x)g(x)\,\mathrm{d}x.
\end{aligned}$$

Taking the limit as $n \to \infty$ in equation (6.18), we obtain that

$$\alpha_3 = \alpha_1 \frac{\log(b/a)}{b-a}\frac{g(1)}{\lambda g(0)}.$$

Repeating the same argument with the sequence $\{f_n(1-x)\}_{n\in\mathbb{N}} \subset \mathcal{C}^0([0,1])$, we obtain that

$$\alpha_1 = \frac{\log(b/a)}{(b-a)\lambda}\alpha_1.$$

If $\alpha_1 = 0$, then $\alpha_3 = 0$ and the proof is finished. Suppose by contradiction that $\alpha_1 > 0$, the above equation shows that $1 = \log(b/a)\lambda^{-1}(b-a)^{-1}$. However, we obtain that

$$g(0) = \frac{1}{\lambda}\mathcal{L}g(0) = \frac{1}{\lambda}\frac{\log(b/a)}{b-a} g(0) + \frac{1}{\lambda}\frac{\log(b/a)}{b-a} g(1) = g(0) + \frac{1}{\lambda}\frac{\log(b/a)}{b-a} g(1),$$

therefore, $g(1) = 0$, contradicting Proposition 6.12. □

With the above results, we can prove the following two theorems.

THEOREM 6.14. *Let $(a, b)$ be an admissible pair. Then the operator $\mathcal{P} : L^1([0, 1], \mu) \to L^1([0, 1], \mu)$ admits eigenvalue $\eta$ with respect to eigenvalue $\lambda$ such that $\mu(\{\eta > 0\}) = 1$ and $\|\eta\|_{L^1(M,\mu)} = 1$. In particular, $Y_n^{a,b}$ fulfils Hypothesis H1.*

*Proof.* From Proposition 6.13, there exists an eigenmeasure $\nu(\mathrm{d}x) = h(x)\,\mathrm{d}x$ of $T^*$ with $h \in L^1([0, 1], \mu)$. This implies that for every $f \in \mathcal{C}^0(M)$,

$$\int_0^1 T(f)(x)h(x)\,\mathrm{d}x = \int_0^1 f(x)h(x)\,\mathrm{d}x.$$

However, since $fg \in L^\infty([0, 1])$, we obtain that

$$\begin{aligned}
\int_0^1 f(x)h(x)\,\mathrm{d}x = \int_0^1 Tf(x)h(x)\,\mathrm{d}x &= \int_0^1 \frac{\mathcal{L}(fg)(x)}{\lambda g(x)}h(x)\,\mathrm{d}x\\
&= \int_0^1 f(x)\frac{g(x)}{\lambda}\mathcal{P}\bigg(\frac{h}{g}\bigg)\,\mathrm{d}x.
\end{aligned}$$

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Finally, defining $\eta(x) = h(x)/g(x)$, it follows that $\mathcal{P}\eta = \lambda\eta$. Since

$$\eta(x) = \frac{1}{\lambda(b-a)x(1-x)} \int_{ax(1-x)}^{bx(1-x)\wedge 1} \eta(y)\,\mathrm{d}y,$$

we clearly have that $\eta \in \mathcal{C}^0((0,1))$. Moreover, it is easy to see that if there exists $x_0 \in (0,1)$ such that $\eta(x_0) = 0$, then $\eta(x) = 0$ Leb-a.s. in $(0,1)$, which is a contradiction. □

THEOREM 6.15. *Let $(a,b)$ be an admissible pair. Consider $M = [0,1]$ and the Markov chain $Y_{n+1}^{(a,b)} = \omega_n Y_n^{(a,b)}(1 - Y_n^{(a,b)})$ absorbed at $\partial = \mathbb{R} \setminus M$, with $\{\omega_n\}_{n\in\mathbb{N}}$ an i.i.d sequence of random variables such that $\omega_n \sim \mathrm{Unif}([a,b])$ on $\mathbb{R}_M$ with absorption $\partial$. Then we have the following.*

(i) *$Y_n^{a,b}$ admits a quasi-stationary measure $\mu_{a,b}$ with survival rate $\lambda_{a,b}$ such that $\mathrm{supp}(\mu) = [0,1]$ and $\mu \ll \mathrm{Leb}$, where Leb denotes the Lebesgue measure on $[0,1]$.*

(ii) *There exists $\eta^{a,b} \in L^1(M,\mu)$ such that $\mathcal{P}\eta^{a,b} = \lambda_{a,b}\eta^{a,b}$, $\|\eta^{a,b}\|_{L^1(M,\mu)} = 1$ and $\eta^{a,b} > 0$ $\mu_{a,b}$-a.s.*

(iii) *For every $h \in L^\infty(M, \mathrm{Leb})$,*

$$\lim_{n\to\infty} \mathbb{E}\bigg[\frac{1}{n}\sum_{i=0}^{n-1} h \circ Y_n^{a,b} \mid \tau > n\bigg] = \int_M h(y)\eta^{a,b}(y)\mu_{a,b}(\mathrm{d}y) \quad \textit{for every } x \in (0,1).$$

(iv) *For every $h \in L^\infty(M,\mu)$,*

$$\lim_{n\to\infty} \mathbb{E}_x[h \circ X_i \mid \tau > n] = \int h(y)\mu(\mathrm{d}y) \quad \textit{for every } x \in (0,1).$$

*Proof.* Note that Theorem 6.14 implies that $Y_n^{(a,b)}$ satisfies items (H1a) and (H1b) of Hypothesis H1, also items (H1c) and (H1d) of Hypothesis H1 follow from Propositions 6.1 and 6.12.

Once again, from Propositions 6.1 and 6.12, we obtain that $Y_n^{a,b}$ satisfies Hypothesis H2 defining $K_i := [1/i, 1-1/i]$ for every $i \in \mathbb{N}$. Also, since the logistic map $4x(1-x)$ is chaotic in $[0,1]$ and $f : \mathbb{R} \times [a,b] \to \mathbb{R}$, $f(x,\omega) = \omega x(1-x)$ is a continuous function, so we conclude that $m = 1$ in Theorem 2.2. Therefore, the conclusions of the theorem follow directly from Theorem 2.4. □

6.1. *Analysis of the admissible pairs.* Fixing a pair $(a,b) \in (0,4) \times (4,\infty)$, it is relatively easy to check if $(a,b)$ is an admissible pair or not. However, it is complicated to solve inequality equations (6.7) and (6.8) in terms of $(a,b)$. In this section, we prove that every $(a,b) \in [1,4) \times (4,\infty)$ is an admissible pair.

We start showing that for every $(a,b) \in [1,2) \times (4,\infty)$, inequality equation (6.7) is fulfilled.

PROPOSITION 6.16. *For every $(a,b) \in [1,2) \times (4,\infty)$, we have that*

$$0 \le \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{2}{b}\bigg(1 - \sqrt{1 - \frac{4x}{a}}\bigg)} \le \frac{a}{4} \quad \textit{for every } x \in \bigg[\frac{4a^2 - a^3}{16}, \frac{a}{4}\bigg].$$

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*Proof.* Note that $\frac{1}{2} - \frac{1}{2}\sqrt{1 - 2/b(1 - \sqrt{1 - 4x/a})}$ is an increasing function in $x$. Therefore, for every $x \in [(4a^2 - a^3)/16, a/4]$, we obtain

$$0 \leq \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{2}{b}\left(1 - \sqrt{1 - \frac{4x}{a}}\right)} \leq \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{2}{b}} \leq \frac{1}{4} \leq \frac{a}{4}. \qquad \square$$

PROPOSITION 6.17. *For every $b > 4$, the following maps*

$$F_1^{a,b}(x) := \frac{2\big(\tanh^{-1}\big(\sqrt{1 - 2/b + 2/b\sqrt{1 - (4x/b)}}\big) - \tanh^{-1}\big(\sqrt{1 - 2/a + 2/a\sqrt{1 - (4x/b)}}\big)\big)}{\sqrt{1 - (4x/b)}}$$

*and*

$$F_2^{a,b}(x) := \frac{2\tanh^{-1}\big(\sqrt{-2 + 2\sqrt{1 - (4x/a)} + b/b}\big) + \log\big(a/(4 - a)\big)}{\sqrt{1 - (4x/a)}}$$

*are increasing in $x$ in the interval* $[(4a^2 - a^3)/16, a/4]$.

*Proof.* It is readily verified that

$$x \mapsto \tanh^{-1}\left(\sqrt{1 - \frac{2}{b} + \frac{2}{b}\sqrt{1 - \frac{4x}{b}}}\right) - \tanh^{-1}\left(\sqrt{1 - \frac{2}{a} + \frac{2}{a}\sqrt{1 - \frac{4x}{b}}}\right) \quad \text{and}$$
$$x \mapsto \sqrt{1 - \frac{4x}{b}}$$

are respectively increasing and decreasing for $x \in [(4a^2 - a^3)/16, a/4]$, implying that $F_1^{a,b}(x)$ is increasing in $x$ in the interval $[(4a^2 - a^3)/16, a/4]$.

In the following, we prove that $F_2^{a,b}$ is an increasing function in $[(4a^2 - a^3)/16, a/4]$. Through the change of coordinates $y = \sqrt{(-2 + 2\sqrt{1 - 4x/a} + b)/b}$, we obtain that to show that $F_2^{a,b}$ is an increasing function, it is enough to show that

$$F_3^{a,b}(y) = \frac{\log((1 + y)/(1 - y)) + \log(a/(4 - a))}{by^2 - b + 2}$$

is decreasing in $x$ in the interval $[\sqrt{(b-2)/b}, \sqrt{(b-1)/b}] \supset [\sqrt{(-2+b)/b}, \sqrt{(b-a)/b}]$. Since

$$\frac{\mathrm{d}F_3^b}{\mathrm{d}y}(y) = \frac{2(b(y^2 - 1)y(\log(a/(4 - a)) + \log((1 + y)/(1 - y))) + by^2 - b + 2)}{(1 - y^2)(b(y^2 - 1) + 2)^2},$$

it is enough to show that

$$b(y^2 - 1)\left(y \log\left(\frac{a}{4 - a} \cdot \frac{1 + y}{1 - y}\right) + 1\right) + 2 \leq b(y^2 - 1)\left(y \log\left(\frac{1 + y}{3 - 3y}\right) + 1\right) + 2$$
$$\leq 0 \quad \text{for every } y \in \left[\sqrt{\frac{b-2}{b}}, \sqrt{\frac{b-1}{b}}\right].$$

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Observe that given $y \in [\sqrt{(-2+b)/b}, \sqrt{(-1+b)/b}] \subset [\sqrt{2}/2, 1]$, we obtain that

$$by(y^2-1)\log\left(\frac{1+y}{3-3y}\right) \leq 0 \quad \text{and} \quad \frac{1+y}{3-3y} - 1 \geq 0.$$

From [**21**, equation (2)], it follows that $\log(1+x) \geq x/(1+x/2)$ for every $x \geq 0$. Therefore,

$$\begin{aligned} 2 + b(y^2-1) + by(y^2-1)\log\left(\frac{y+1}{3-3y}\right) \\ \leq 2 + b(y^2-1) + \frac{by((y+1)/(3-3y)-1)(y^2-1)}{1/2((y+1)/(3-3y)-1)+1} \\ = 2 - \frac{b(1-y^2)(4y^2-3y+2)}{2-y}. \end{aligned} \tag{6.19}$$

Using standard techniques, one can check that $b > 4$ and $y \in [\sqrt{(-2+b)/b}, \sqrt{(-1+b/b}]$ then equation (6.19) is less than or equal to 0, implying that $F_3^{a,b}$ is decreasing in the interval $[\sqrt{(-2+b)/b}, \sqrt{(-1+b)/b}]$ for every $(a, b) \in [1, 2] \times (4, \infty)$ and therefore $F_2^{a,b}$ is increasing in $[(4a^2 - a^3)/16, a/4]$ for every $(a, b) \in [1, 2] \times (4, \infty)$. □

Using the above proposition, we show that if $(a, b) \in [1, 4) \times (4, \infty)$, then $(a, b)$ is an admissible pair.

THEOREM 6.18. *If $(a, b) \in [1, 4) \times (4, \infty)$, then $(a, b)$ is an admissible pair.*

*Proof.* From the definition of admissible pair, we just need to consider the case $(a, b) \in [1, 2] \times (4, \infty)$. From Proposition 6.16, we obtain that the pair $(1, b)$ satisfies equation (6.7).

In the following, we show that the pair $(1, b)$ satisfies equation (6.8). Observe that equation (6.8) is equivalent of showing that $F_1^{a,b}(x) \leq F_2^{a,b}(x)$ for every $x \in [(4a^2 - a^3)/16, a/4]$, where $F_1^{a,b}$ and $F_2^{a,b}$ are defined in Proposition 6.17. From Proposition 6.17, it is enough to show that

$$F_1^{a,b}\left(\frac{a}{4}\right) \leq F_2^b\left(\frac{4a^2-a^3}{16}\right) \quad \text{for every } (a, b) \in [1, 4) \times (4, \infty).$$

We divide the proof into two steps.

*Step 1*. 1 We show that for every $b > 4$, $(1, b)$ is an admissible pair.

Note that for every $b > 4$,

$$\begin{aligned} F_1^{1,b}\left(\frac{1}{4}\right) &= \frac{2\big(\tanh^{-1}\big(\sqrt{(b+2\sqrt{1-1/b}-2)/b}\big) - \tanh^{-1}\big(\sqrt{2\sqrt{1-1/b}-1}\big)\big)}{\sqrt{1-1/b}} \\ &\leq 4\tanh^{-1}\left(\frac{\sqrt{2\sqrt{(b-1)/b}-1} - \sqrt{(b+2\sqrt{(b-1)/b}-2)/b}}{1 - \sqrt{\big(2\sqrt{(b-1)/b}-1\big)\big((b+2\sqrt{(b-1)/b}-2)/b\big)}}\right) \end{aligned}$$

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and

$$F_2^{1,b}\left(\frac{3}{16}\right) = 4\tanh^{-1}\left(\frac{\sqrt{1-(1/b)}-1/2}{1-1/2\sqrt{1-(1/b)}}\right) = 4\tanh^{-1}\left(\frac{3\sqrt{b-1}\sqrt{b}-2}{3b+1}\right).$$

Since the function $x \mapsto 4\tanh^{-1}(x)$ is increasing, to finish the proof of this step it is enough to show that

$$\frac{\sqrt{-1+2\sqrt{(b-1)/b}}-\sqrt{(b+2\sqrt{(b-1)/b}-2)/b}}{1-\sqrt{\big(-1+2\sqrt{(b-1)/b}\big)\big((b+2\sqrt{(b-1)/b}-2)/b\big)}} \leq \frac{3\sqrt{b-1}\sqrt{b}-2}{3b+1} \quad \text{for every } b>4. \tag{6.20}$$

Using standard methods, one can show that the above equation simplifies in showing that

$$p(b) := 4239b^6 - 23868b^5 + 31482b^4 + 8964b^3 - 40401b^2 + 23424b - 4096 \geq 0 \quad \text{for every } b>4.$$

However, since for every $\delta > 0$,

$$p(4+\delta) = 4239\delta^6 + 77868\delta^5 + 571482\delta^4 + 2119716\delta^3 + 4091679\delta^2 + 3683256\delta + 998384 > 0,$$

we obtain that equation (6.20) holds. This completes Step 1.

*Step 2*. 2 We show that $(a, b)$ is an admissible pair for every $(a, b) \in [1, 2) \times (4, \infty)$. Fixing $b > 4$, observe that

$$(2-a)F_1^{a,b}\left(\frac{a}{4}\right) = 2\frac{2-a}{\sqrt{1-(a/b)}}\left(\tanh^{-1}\left(\sqrt{\frac{2\sqrt{1-(a/b)}+b-2}{b}}\right)-\tanh^{-1}\left(\sqrt{\frac{2\sqrt{1-(a/b)}+a-2}{a}}\right)\right)$$

and

$$(2-a)F_2^{a,b}\left(\frac{4a^2-a^4}{16}\right) = 2\left(2\tanh^{-1}\left(\sqrt{\frac{b-a}{b}}\right)+\log\left(\frac{a}{4-a}\right)\right).$$

It is readily verified that

$$\frac{2-a}{\sqrt{1-(a/b)}} \quad \text{and} \quad \tanh^{-1}\left(\sqrt{\frac{2\sqrt{1-(a/b)}+b-2}{b}}\right)-\tanh^{-1}\left(\sqrt{\frac{2\sqrt{1-(a/b)}+a-2}{a}}\right)$$

are decreasing functions in $a \in [1, 2)$, implying that $(2-a)F_1^{a,b}(a/4)$ is a decreasing function in $a \in [0, 1]$ and

$$(a-2)F_2^{a,b}((4a^2-a^3)/16) = 2\left(2\tanh^{-1}\left(\sqrt{\frac{b-a}{b}}\right)+\log\left(\frac{a}{4-a}\right)\right)$$

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is an increasing function in $a \in [1, 2)$. From Step 1, we obtain that for every $a \in [1, 2)$,

$$F_1^{a,b}\left(\frac{a}{4}\right) = \frac{(2-a)F_1^{a,b}(a/4)}{2-a} \leq \frac{F_1^{a,b}(1/4)}{2-a} \leq \frac{F_2^{a,b}(3/16)}{2-a} \leq F_2^{a,b}\left(\frac{4a^2-a^3}{16}\right).$$

This completes the proof of the theorem. □

We finish the paper proving Theorem 2.1.

*Proof of Theorem 2.1.* The theorem follows directly from Theorems 6.15 and 6.18. □

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