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THE MODAL LOGIC OF σ-CENTERED FORCING AND RELATED FORCING CLASSES

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Abstract. We consider the modality "φ is true in every σ-centered forcing extension," denoted $\Box\varphi$, and its dual "φ is true in some σ-centered forcing extension," denoted $\Diamond\varphi$ (where φ is a statement in set theory), which give rise to the notion of a *principle of σ-centered forcing*. We prove that if ZFC is consistent, then the modal logic of σ-centered forcing, i.e., the ZFC-provable principles of σ-centered forcing, is exactly S4.2. We also generalize this result to other related classes of forcing.

§1. Introduction and preliminaries. In this work we continue the investigation of the *Modal Logic of Forcing*, initiated by Joel Hamkins and Benedikt Löwe in [7], where they consider the modal logic arising from considering a statement as necessary (respectively possible) if it is true in any (respectively some) forcing extension of the world. Here we restrict the modality only to extensions obtained by σ-centered forcing notions, and prove that the modal logic arising from this interpretation is S4.2 (see below). We then show that our techniques can be generalized to other related classes of forcing notions.

We begin by setting some preliminaries—first we cite common definitions and theorems of forcing and of modal logic; and then present the main tools developed in [5, 7] for the research of the modal logic of forcing; we add one new notion to this set of tools, the notion of an n-switch, and show its utility; and prove a general theorem (Theorem 1.15) which provides the framework for the main theorem (Theorem 3.14). In Section 2 we present the class of σ-centered forcing and some of its properties which give us the easy part of the theorem—that the modal logic of σ-centered forcing contains S4.2, and present the technique of coding subsets using σ-centered forcing. The hard part of the main theorem will be proved in Section 3, where we begin by defining a specific model of ZFC, and then present two forcing constructions that would allow us to establish that the modal logic of σ-centered forcing is contained in S4.2. We conclude with the above-mentioned generalizations and some open questions.

We begin by presenting some notations and background that will be used in this work. Our forcing notation is standard, and will usually follow Kunen's [11, Chapter 4].

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We work with propositional modal logic as presented in [3], in which we add to standard propositional logic two unary operators—$\Box$ and $\Diamond$, where $\Box\varphi$ is interpreted as "necessarily φ" and $\Diamond\varphi$ as "possibly φ."

The modal axioms which will be used are:

K	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$,
Dual	$\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$,
T	$\Box\varphi \to \varphi$,
4	$\Box\varphi \to \Box\Box\varphi$,
.2	$\Diamond\Box\varphi \to \Box\Diamond\varphi$,
.3	$(\Diamond\varphi \wedge \Diamond\psi) \to \Diamond[(\Diamond\varphi \wedge \psi) \vee (\varphi \wedge \Diamond\psi)]$,
5	$\Diamond\Box\varphi \to \varphi$,

and the modal theories discussed are: S4, axiomatized by K, Dual, T, and 4, S4.2, axiomatized by adding axiom .2, S4.3 by adding axiom .3, and S5 by adding axiom 5.

We assume the reader is familiar with *Kripke Semantics* for modal logic, where a *Kripke model* is a triplet $\mathcal{M} = \langle W, R, V \rangle$ such that W is a non-empty set (the set of worlds), R is a binary relation on W (the accessibility relation), and V is a function from the propositional variables to subsets of W (the valuation); $\mathcal{F} = \langle W, R \rangle$ is called the *frame* on which $\mathcal{M}$ is based; and the satisfaction relation $\mathcal{M}, w \vDash \varphi$ (for $w \in W$) is defined in the usual inductive way, using

$$\mathcal{M}, w \vDash \Box\varphi \text{ iff for every } u \in W \text{ such that } w\,R\,u,\ \mathcal{M}, u \vDash \varphi.$$

We say that φ is *valid in* $\mathcal{M}$ ($\mathcal{M} \vDash \varphi$) if $\mathcal{M}, w \vDash \varphi$ for every $w \in W$, and that φ is valid on a frame $\mathcal{F}$ ($\mathcal{F} \vDash \varphi$) if φ is valid in every model based on $\mathcal{F}$. A class of frames $\mathcal{C}$ *characterizes* a modal theory Λ if a formula is in Λ iff it is valid on every frame in $\mathcal{C}$.

We will use the following class of frames to characterize S4.2:

Definition 1.1. Let $\langle F, \leq \rangle$ such that $\leq$ is a reflexive and transitive binary relation on F. $\langle F, \leq \rangle$ is called a *pre-Boolean-algebra* (pBA) if $\langle F/\equiv, \leq \rangle$ is a Boolean-algebra (BA), where $\equiv$ is the natural equivalence relation on F defined by $x \equiv y$ iff $x \leq y \leq x$, and $\leq$ denotes also the induced order relation.

A pBA can be thought of as a BA where every element is replaced by a cluster of equivalent elements. We will use the following:

Theorem 1.2 ([7, Theorem 11]). *S4.2 is characterized by the class of all finite pBAs.*

1.1. The modal logic of forcing. We now review the framework of the modal logic of forcing, based on [5] and [7]. The reader who is familiar with these works may wish to skip to Definition 1.10 where we define the new notion of an n-switch.

In the context of set-theory, the possible world semantics suggest a connection between modal logic and forcing, as we can imagine all generic extensions of the universe (or of a specific model of ZFC) as an enormous Kripke model (called "*the generic multiverse*"). This leads naturally to the forcing interpretation of modal logic, in which we say that a sentence of set theory φ is necessary ($\Box\varphi$) if it is true in all forcing extensions, and possible ($\Diamond\varphi$) if it is true in some forcing extension. Given some definable class of forcing notions Γ, we can also restrict to posets belonging to that class, to get the operators $\Box_\Gamma$ and $\Diamond_\Gamma$. The following definitions, based on [5] and [7], allow us to formally ask the question—what statements are valid under this interpretation?

Definition 1.3.
1. Given a formula $\varphi = \varphi(q_0,\dots,q_n)$ in the language of modal logic, where $q_0,\dots,q_n$ are the only propositional variables, appearing in φ, and some set-theoretic sentences $\psi_0,\dots,\psi_n$, the *substitution instance* $\varphi(\psi_0,\dots,\psi_n)$ is the set-theoretic statement obtained recursively by replacing q_i with ψ_i and interpreting the modal operators according to the forcing interpretation (or the Γ-forcing interpretation).
2. Let Γ be a class of forcing notions. The ZFC-*provable principles of* Γ-*forcing* are all the modal formulas φ such that ZFC $\vdash \varphi(\psi_0,\dots,\psi_n)$ for every substitution $q_i \mapsto \psi_i$ under the Γ-forcing interpretation. This will also be called *the modal logics of* Γ-*forcing*, denoted $\mathsf{MLF}(\Gamma)$. If we discuss the class of all forcing notions we omit mention of Γ.

Theorem 1.4 (Hamkins and Löwe [7]). *If* ZFC *is consistent then the* ZFC-*provable principles of forcing are exactly* $\mathsf{S4.2}$.

We will now present the main tools which were developed to prove the theorem above, and which can be used to prove similar theorems. To prove such a theorem, we need to establish lower and upper bounds, i.e., find a modal theory Λ such that $\mathsf{MLF}(\Gamma) \supseteq \Lambda$ and $\mathsf{MLF}(\Gamma) \subseteq \Lambda$, respectively. Each type of bound require a different set of tools, which will be presented below.

1.1.1. Lower bounds. A simple observation is that the ZFC-provable principles of Γ-forcing are closed under the usual deduction rules for modal logic, so if a modal theory is given by some axioms, to show it is contained in $\mathsf{MLF}(\Gamma)$ it is enough to check that the axioms are valid principles of Γ-forcing. So, for example, axioms K and Dual are easily seen to be valid under the Γ-forcing interpretation for every class Γ. The validity of other axioms depends on specific properties of Γ:

Definition 1.5. A definable class of forcing notions Γ is said to be *reflexive* if it contains the trivial forcing; *transitive* if it is closed under finite iterations, i.e., if $\mathbb{P} \in \Gamma$ and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a poset such that $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}} \in \Gamma$, then $\mathbb{P} * \dot{\mathbb{Q}} \in \Gamma$; *persistent* if $\mathbb{P},\mathbb{Q} \in \Gamma$ implies $\mathbb{Q} \in \Gamma^{V^{\mathbb{P}}}$; and *directed* if $\mathbb{P},\mathbb{Q} \in \Gamma$ implies that there is some $\mathbb{R} \in \Gamma$ such that $\mathbb{R}$ is forcing equivalent to $\mathbb{P} * \dot{\mathbb{S}}$ and to $\mathbb{Q} * \dot{\mathbb{T}}$, where $\dot{\mathbb{S}} \in \Gamma^{V^{\mathbb{P}}}$ and $\dot{\mathbb{T}} \in \Gamma^{V^{\mathbb{Q}}}$.

Note that if a Γ is transitive and persistent, we can show it is directed by taking $\mathbb{R} = \mathbb{P} \times \mathbb{Q}$ for any $\mathbb{P},\mathbb{Q} \in \Gamma$.

Theorem 1.6 [5, Theorem 7]. *Axiom T is valid in every reflexive forcing class, axiom* 4 *in every transitive forcing class, and axiom* .2 *in every directed forcing class. Thus, if* Γ *is reflexive, transitive, and directed then* $\mathsf{MLF}(\Gamma) \supseteq \mathsf{S4.2}$.

1.1.2. Upper bounds. To establish that Λ is an upper bound for $\mathsf{MLF}(\Gamma)$, we need to show that every formula not in Λ is also not in $\mathsf{MLF}(\Gamma)$. To do so, we would need to find a model of ZFC and some substitution instance of φ that fails in this model. In the case that Λ is characterized by some class of frames $\mathcal{C}$, $\varphi \notin \Lambda$ means that there

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is some Kripke model based on a frame in $\mathcal{C}$ where φ fails. So our goal would be to find a suitable model of set theory W such that the Γ-generic multiverse generated by W (i.e., all Γ-forcing extensions of W) "looks like" the model where φ fails. The main tool for that is called a labeling:

DEFINITION 1.7. A Γ-*labeling of a frame* $\langle F, R\rangle$ *for a model of set theory* W is an assignment to each $w \in F$ a set-theoretic statement Φ_w such that:

1. The statements form a mutually exclusive partition of truth in the Γ-generic multiverse over W, i.e., every Γ-generic extension of W satisfies exactly one Φ_w.
2. The statements correspond to the relation, i.e., if $W[G]$ is a Γ-forcing extension of W that satisfies Φ_w, then $W[G] \vDash \Diamond\Phi_u$ iff $w\,R\,u$.
3. $W \vDash \Phi_{w_0}$ where w_0 is a given initial element of F.

LEMMA 1.8 (The labeling lemma—[5, Lemma 9]). *Suppose $w \mapsto \Phi_w$ is a Γ-labeling of a finite frame $\langle F, R\rangle$ for a model of set theory W with w_0 an initial world of F, and $\mathcal{M}$ a Kripke model based on F. Then there is an assignment of the propositional variables $p \mapsto \psi_p$ such that for every modal formula $\varphi(p_0,\dots,p_n)$,*

$$\mathcal{M}, w_0 \vDash \varphi(p_0,\dots,p_n) \iff W \vDash \varphi(\psi_{p_0},\dots,\psi_{p_n}).$$

COROLLARY 1.9. *If every finite pBA has a* Γ-*labeling over some model of* ZFC, *then* $\mathsf{MLF}(\Gamma) \subseteq \mathsf{S4.2}$.

PROOF. By Theorem 1.2, every modal formula $\varphi \notin \mathsf{S4.2}$ fails in a Kripke model based on some finite pBA. So, given a Γ-labeling for this frame over a model W, by the labeling lemma, there is a substitution instance of φ which fails at W under the Γ-forcing interpretation. So $\varphi \notin \mathsf{MLF}(\Gamma)$. ⊣

Hence to establish upper bounds, we try to find labelings for specific frames. Various labelings can be constructed using certain kinds of set-theoretic statements, called in general *control statements*.

DEFINITION 1.10 (Control statements). Let W be some model of set theory, and Γ some class of forcing notions.

1. A *switch for* Γ-*forcing over* W is a statement s such that necessarily, both s and $\neg s$ are possible. That is, over every Γ-extension of W one can force s or $\neg s$ as one chooses using Γ-forcing.
2. An *n-switch for* Γ- *forcing over* W is a set of statements $\{s_i \mid i < n\}$ (where $n > 1$) such that every Γ-generic extension W' of W satisfies exactly one s_i, and every s_j is necessarily possible, i.e., over every Γ-extension of W one can force s_j using Γ-forcing. The n-switch value in some $W[G]$ is the i such that $W[G] \vDash s_i$. Note that a 2-switch is essentially just a switch.[1]
3. A *button for* Γ- *forcing over* W is a statement b which is necessarily possibly necessary, i.e., $W \vDash \Box\Diamond\Box b$. This means that in every Γ-extension of W, we can force b to be true using Γ-forcing and to remain true in every further

[1]In [6], Hamkins and Linnebo independently define the notion of a "dial," which is essentially the same as an n-switch, and prove results similar to our Theorem 1.11 and Lemma 1.13. Our results were independent of these.

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Γ-extension. A button is called *pushed* if $\Box b$ holds, otherwise it is called *unpushed*. A *pure button* is a button b such that $\Box(b \to \Box b)$ (i.e., if it is true then it is pushed). If b is an unpushed button then $\Box b$ is an unpushed pure button.

4. A *ratchet for* Γ- *forcing over* W is a collection of pure buttons $\{r_i \mid i \in I\}$, possibly with i as a parameter, where I is well-ordered, such that pushing r_i pushes every r_j for $j < i$, and necessarily, every unpushed r_i can be pushed without pushing any r_j for $j > i$. An infinite ratchet $\{r_i \mid i \in I\}$ is called *strong* if there is no Γ-extension of W satisfying every r_i. The ratchet value in $W[G]$ is the first $i \in I$ such that $W[G] \vDash \neg r_i$.
5. A family of control statements (switches, n-switches, buttons, and ratchets) is called *independent over* W (for Γ-forcing) if in W, all buttons are unpushed (including the ones in any ratchet), and necessarily, using Γ-forcing, each button can be pushed, each switch can be turned on or off, the value of each n-switch can be changed, and the value of every ratchet can be increased, without affecting any other control statement in the family.

   Note the "necessarily"—the independence needs to be preserved in any Γ-forcing extension of W.

n-switches are less naturally occurring in set theory than the other notions, and indeed they were not explicitly defined in [5] and [7]. However, by examining the proofs of some of the main theorems there, one can see that what was implicitly used was an n-switch, which was constructed using switches (cf. [5, Theorems 10, 11, and 13]). Additionally, in some cases switches were constructed from ratchets and then transformed into n-switches (e.g., in [5, Theorems 12 and 15]). So, in the definition of some of the central labelings, n-switches turn out to be the more natural notion, and we will show how to construct them using either switches or a ratchet independently. Hence the following theorem, which gives sufficient conditions for the existence of labelings for finite pBAs, generalizes some of the above-mentioned theorems from [5], and they can be inferred from it. We will not be able to use the theorem as it is to prove our main theorem, but we will use its proof as a model, so it has instructive value in itself.

THEOREM 1.11. *Let Γ be some reflexive and transitive forcing class and W a model of set theory. If for every $m,n < \omega$ there is a family of m buttons mutually independent from an n'-switch for some $n' \geq n$ then there is a Γ-labeling over W for every frame which is a finite pBA.*

PROOF. Let $\langle F, \leq\rangle$ be a finite pBA. As noted earlier, it can be viewed a finite BA, where each element is replaced by a cluster of equivalent worlds. We can add dummy worlds to each cluster without changing satisfaction in the model, so we can assume that each cluster is of size n for some $1 < n < \omega$.[2] It is known that any finite BA is isomorphic to the BA $\langle \mathcal{P}(B), \subseteq\rangle$ for some finite set B. Let B be such that $\langle F/\equiv, \leq\rangle \cong \langle \mathcal{P}(B), \subseteq\rangle$, and set $m = |B|$. We can assume that in fact $B = \{0,...,m-1\}$. There is a correspondence between subsets $A \subseteq B$ and clusters in

[2]If every cluster has only one element then we actually don't need the n-switch, and we can label the BA only with the buttons.

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$\langle F, \leq\rangle$. Each cluster is of size n, so by enumerating each cluster, all the elements of F can be named w_i^A for $i < n$ and $A \subseteq B$, where $w_i^A \leq w_j^{A'}$ iff $A \subseteq A'$. An initial world in F must be in the bottom cluster, which corresponds to $\varnothing \subseteq B$ so without loss of generality it is enumerated as $w_0^{\varnothing}$.

By the assumption, adding more dummy worlds to each cluster if needed and increasing n, there are buttons $\{b_0,\dots,b_{m-1}\}$ and an n-switch $\{s_0,\dots,s_{n-1}\}$ all independent of each other over W. We can assume the buttons are pure. To define a labeling, each cluster, corresponding to some $A \subseteq B$, will be labeled by the statement that the only buttons pushed are the ones with indexes from A. Inside each cluster, each world will be labeled by the corresponding value of the n-switch. Formally, we set

$$\Phi(w_i^A) = \bigwedge_{j \in A} b_j \wedge \bigwedge_{j \notin A} \neg b_j \wedge s_i$$

and claim that this is a labeling as required by verifying the conditions:

1. If $W[G]$ is a Γ-generic extension of W, define $A = \{j < m \mid W[G] \vDash b_j\}$. By the definition of the n-switch, $W[G] \vDash s_i$ for some unique $i < n$. So it is clear that $W[G] \vDash \Phi(w_i^A)$, and that for any other pair $(A',i') \neq (A,i)$ with $A' \subseteq B$, and $i' < n$, $W[G] \nvDash \Phi(w_{i'}^{A'})$. So these statements indeed form a mutually exclusive partition of truth in the Γ-generic multiverse over W.
2. Assume $W[G]$ is a Γ-generic extension of W such that $W[G] \vDash \Phi(w_i^A)$. If $w_i^A \leq u$, then as we have seen, $u = w_{i'}^{A'}$ for some $i' < n$ and $A \subseteq A' \subseteq B$. By the assumption of independence of the control statements, we can, by Γ-forcing, push all the buttons in $A' \setminus A$ (and only them) and change the n-switch value to i' (if needed), to obtain an extension of $W[G]$ satisfying $\Phi(w_{i'}^{A'})$. Note that by the transitivity of Γ, the b_is are still independent pure buttons in $W[G]$, since every Γ-extension of $W[G]$ is also a Γ-extension of W. In particular, any button true in $W[G]$ remains true in the extension. So $W[G] \vDash \Diamond\Phi(w_{i'}^{A'})$ as required.

   If $W[G] \vDash \Diamond\Phi(w_{i'}^{A'})$, then there is some extension $W[G][H] \vDash \Phi(w_{i'}^{A'})$. By the definition of pure buttons and the reflexivity of Γ, $W[G] \vDash \bigwedge_{j \in A} b_j$ implies $W[G] \vDash \bigwedge_{j \in A} \Box b_j$, so $W[G][H] \vDash \bigwedge_{j \in A} b_j$. Therefore by the definition of $\Phi(w_{i'}^{A'})$, we must have $A \subseteq A'$, so $w_i^A \leq w_{i'}^{A'}$.
3. A part of the definition of independence is that no button is pushed in W (since they are pure and Γ reflexive, it is equivalent to saying none is true). We can assume without loss of generality that $W \vDash s_0$. So $W \vDash \Phi(w_0^{\varnothing})$. ⊣

Corollary 1.12. *Under the assumptions of Theorem 1.11*, $\mathsf{MLF}(\Gamma) \subseteq \mathsf{S4.2}$.

Proof. Apply Corollary 1.9. ⊣

Lemma 1.13. *An n-switch can be produced using the following control statements*:

1. *Independent switches $s_0,\dots,s_{m-1}$ if $n = 2^m$*;
2. *A strong ratchet $\{r_i \mid i \in I\}$ where I is either a limit ordinal or Ord, the class of all ordinals, and $i \in I$ is a parameter in r_i.*
3. *A family of independent buttons $\langle b_i \mid i \in I\rangle$ where I is as above, with no extensions where all of the buttons are pushed.*

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PROOF. For (1), if $j < 2^m$ let $\bar{s}_j$ be the statement that the pattern of switches corresponds to the binary digits of j, that is,

$$\bigwedge \{s_i \mid \text{ the } i \text{ th binary digit of } j \text{ is } 1\} \wedge \bigwedge \{\neg s_i \mid \text{ the } i \text{ th binary digit of } j \text{ is } 0\}.$$

Clearly in any extension exactly one pattern of the switches holds, so exactly one $\bar{s}_j$ holds. By the independence of the switches, any pattern can be forced over any extension.

For (2), every $i \in I$ is an ordinal, so of the form $\omega \cdot \alpha + k$ for some $\alpha \in Ord$ and $k < \omega$. Then we let $\bar{s}_j$ be the statement "if $i = \omega \cdot \alpha + k$ is the first such that $\neg r_i$ then $k \bmod n = j$." Since no extension satisfies all the r_is, there is always some i which is the first such that $\neg r_i$, and therefore there is some unique j such that $\bar{s}_j$ holds. Since it is a ratchet, in every extension, for every $j' < n$, we can increase its value to some $i' = \omega \cdot \alpha' + k'$ for some $k' > k$ such that $k' \bmod n = j$ (we use the assumption that if I is an ordinal then it is a limit).

(3) is similar to (2) by setting $r_i = (\forall j < i) b_j \wedge \neg b_i$. ⊣

So with our previous theorem, we get the following:

COROLLARY 1.14 ([5, Theorems 13 and 15][3]). *Let Γ be some reflexive and transitive forcing class and W a model of set theory. If there are arbitrarily large finite families of buttons mutually independent with arbitrarily large finite families of switches, with a strong ratchet as above or with another family of independent buttons as above, then there is a Γ-labeling for every frame which is a finite pBA over W. So in such cases,* $\mathsf{MLF}(\Gamma) \subseteq \mathsf{S4.2}$.

Note that in the above corollary, we must have independent buttons which are not all pushed. However, the following theorem shows that we can weaken this assumption, given that we have *some* n-switch—not necessarily independent. It is this modification which will eventually be used to find a σ-centered labeling for finite pBAs.

THEOREM 1.15. *Let Γ be some reflexive and transitive forcing class and W a model of set theory. If there is a family $\langle b_i \mid i \in \omega\rangle$ of independent buttons for Γ-forcing over W, where i is a parameter in b_i, and for every n there is an n-switch for Γ-forcing over W, then there is a Γ-labeling for every finite pBA, and thus* $\mathsf{MLF}(\Gamma) \subseteq \mathsf{S4.2}$.

PROOF. Let $\langle F, \leq\rangle$ be a pBA. As before Let $B = \{0,\dots,m-1\}$ be such that $\langle F/\equiv, \leq\rangle$ is isomorphic to $\langle \mathcal{P}(B), \subseteq\rangle$, and every cluster is without loss of generalization of size n or some $1 < n < \omega$. So every world in F is of the form w_j^C for some $j < n$ and $C \subseteq B$, and we have $w_j^C \leq w_{j'}^{C'}$ iff $C \subseteq C'$.

Let $\langle b_i \mid i \in \omega\rangle$ be as in the assumption, and using $i \mapsto i - m + 1$ we rename it as $\langle b_i \mid m-1 \leq i < \omega\rangle$. We will imitate the proof of Theorem 1.11 by using the statements b_i for $m-1 \leq i \leq 0$ as the buttons, and obtaining from $\langle b_i \mid 0 < i < \omega\rangle$ an "almost" n-switch as in Lemma 1.13. It might not be a real n-switch, if there

[3]In [5, Theorem 15] the authors have a slightly different convention, where the ratchet value is the last button which *is* pushed, and they use the notion of a *uniform ratchet*, but the theorem is essentially the same.

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is some Γ-extension in which unboundedly many buttons are pushed, and for that reason we need the additional n-switch from the assumption.

So let $\{s_j \mid j < n\}$ be the n-switch from the assumption. To define the "almost" n-switch, define the following statements:

$$R_0 = \text{“}\neg b_i \text{ holds for every } 0 < i < \omega.\text{”}$$
$$R_j = \text{“}j \text{ is the largest such that } b_j \text{ holds”(for } j < \omega).$$
$$R_\omega = \text{“}\sup\{n \mid b_n \text{ holds}\} = \omega.\text{”}$$

So R_0 holds iff no button is pushed, and if in some Γ-extension of W we have R_j for $0 < j < \omega$, then in particular we have $b_j \wedge \neg b_l$ for any $l > j$. So by the independence of the buttons, if some extension satisfies R_i ($i < \omega$), we can force with some Γ-forcing to push only b_l for any $l > i$ and obtain exactly R_l. Note that if some R_j for $j < \omega$ holds, it means in particular that the number of pushed buttons is bounded. Now, for every $j < n$ we define the statement:

$$t_j = \text{“There is some } k < \omega \text{ such that } k \equiv j \text{ mod } n \text{ and } R_k \text{ holds.”}$$

So in any Γ-extension of W, if t_j holds for some j, there is some k be such that R_k holds, and for every $j' < n$ we can find $k' > k$ with $k' \equiv j' \bmod n$ and then force to push only b_k' to obtain $R_{k'}$ and thus $t_{j'}$. It is also clear that no two distinct t_js can hold at the same time, and that if the number of is such that b_i holds is bounded, then some t_j holds. So, $\{t_j \mid j < n\}$ functions as an n-switch, but only as long as the number of pushed buttons is bounded. If in some Γ-extension there are unboundedly many buttons pushed (which we allow as a possibility), no R_k holds, so also no t_j holds. Hence this is "almost" an n-switch.

Now we are ready to define the labeling. For every $C \subseteq B$, define

$$\Psi_C = \bigwedge_{i \in C} b_{-i} \wedge \bigwedge_{i \notin C} \neg b_{-i},$$

which states that the pushed buttons out of $\{b_{-i} \mid i \in B\}$ are exactly the ones corresponding to the elements in C. These statements label the cluster we are in. To move within each cluster below the topmost one, we will use the "almost" n-switch $\{t_j \mid j < n\}$, and if we can no longer use it, that is, if there are unboundedly many b_is pushed, we put ourselves in the top cluster, and there we move using the n-switch $\{s_j \mid j < n\}$: for every $C \subseteq B$ and $j < n$ set

$$\Phi\big(w_j^C\big) = \begin{cases} \Psi_C \wedge t_j & C \neq B, \\ \big(\Psi_C \wedge t_j\big) \vee \big(R_\omega \wedge s_j\big) & C = B. \end{cases} \tag{1.1}$$

In this way, the fact that $\{s_j \mid j < n\}$ is not independent of the buttons will not affect us, as we will always stay in the top cluster anyway. We will now show that this is indeed a labeling as required.

The statements are mutually exclusive: It is clear that the statements $\{\Psi_C \mid C \subseteq B\}$ are mutually exclusive, so $\Phi\big(w_j^C\big), \Phi\big(w_{j'}^{C'}\big)$ for $C \neq C'$, both different than B, clearly exclude each other. If we look at $\Phi\big(w_j^C\big)$ and $\Phi\big(w_{j'}^B\big)$ for some $C \neq B$, they exclude each other since if $\Phi\big(w_{j'}^B\big)$ holds, then either Ψ_B holds which excludes Ψ_C, or we have $\sup\{n \mid b_n \text{ holds}\} = \omega$, which excludes t_j. Now for $j \neq j'$ if $C \subsetneq B$, $\Phi\big(w_j^C\big)$

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and $\Phi\big(w_{j'}^{C}\big)$ exclude each other since t_j and $t_{j'}$ exclude each other; and if $C = B$, if $\sup\{n \mid b_n \text{ holds}\} = \omega$ then s_j and $s_{j'}$ exclude each other, and otherwise again t_j and $t_{j'}$ exclude each other.

The statements exhaust the truth over Γ*-extensions of* W: Let $W[G]$ be some Γ-extension of W. If $W[G] \vDash \sup\{n \mid b_n \text{ holds}\} = \omega$, then there is some j such that $W[G] \vDash s_j$, and so $W[G] \vDash \Phi\big(w_j^B\big)$. Otherwise, the number of buttons pushed is finite, so there is some j such that $W[G] \vDash t_j$, and there is also some specific subset of the buttons $\{b_{-i} \mid i \in B\}$ which are pushed in $W[G]$, so there is some $C \subseteq B$ such that $W[G] \vDash \Psi_C$, and together we get $W[G] \vDash \Phi\big(w_j^C\big)$.

W satisfies $\Phi\big(w_0^{\varnothing}\big)$: In W we have $\neg b_i$ for all $m-1 \leq i < \omega$, so $W \vDash \Psi_{\varnothing}$ and also $W \vDash R_0$, and therefore also $W \vDash s_0$.

The statements correspond to the relation: Assume we are in U which is a Γ-extension of W where $\Phi\big(w_j^C\big)$ is true.

Assume first that $C \neq B$.

- Assume $\Diamond\Phi(w_{j'}^{C'})$—there is a Γ-extension U' of U satisfying $\Phi(w_{j'}^{C'})$. If $C' \neq B$ then

$$U' \vDash \Psi_{C'} = \bigwedge_{i \in C'} b_{-i} \wedge \bigwedge_{i \notin C'} \neg b_{-i}.$$

  But $U \vDash \bigwedge_{i \in C} b_{-i}$, which are buttons, so they remain pushed in U', i.e., $U' \vDash \bigwedge_{i \in C} b_{-i}$. So we must get $C \subseteq C'$, so $w_j^C \leq w_{j'}^{C'}$. If $C' = B$ then clearly we have $w_j^C \leq w_{j'}^{C'}$.
- Assume $w_j^C \leq w_{j'}^{C'}$, hence $C \subseteq C'$. We have

$$U \vDash \Psi_C = \bigwedge_{i \in C} b_{-i} \wedge \bigwedge_{i \notin C} \neg b_{-i},$$

  so for every $i \in C' \setminus C$, by the independence of the buttons we can force b_{-i}, to obtain an extension U' satisfying $\Psi_{C'}$ (the buttons from C will remain pushed). In U, which satisfies t_j, there is some k such that $k \bmod n = j$ and $U \vDash R_k$. In U' we still have R_k, since pushing the (finitely many) buttons corresponding to C' does not push any button b_i for $i > 0$. If $j' = j$ we are done, otherwise we can find some $k' > k$, such that $k' \bmod n = j'$, push $b_{k'}$, and thus obtain an extension U'' satisfying $t_{j'}$. Again this forcing does not affect the truth of b_{-i}s for $i \in B$, so U'' also satisfies $\Psi_{C'}$, so it satisfies $\Phi(w_{j'}^{C'})$. By transitivity of Γ, we get that indeed $U \vDash \Diamond\Phi(w_{j'}^{C'})$.

Now assume $C = B$, i.e., $U \vDash \Phi(w_j^B)$. We distinguish the two cases.

- $U \vDash (\Psi_B \wedge t_j)$:
  - Assume $\Diamond\Phi(w_{j'}^{C'})$. Since $U \vDash \Psi_B$, any extension of it also satisfies Ψ_B, so we cannot have $\Diamond\Phi(w_{j'}^{C'})$ for any $C' \neq B$. Therefore $C' = B$ and indeed $w_j^B \leq w_{j'}^B = w_{j'}^{C'}$.

– Assume $w_j^C \leq w_{j'}^{C'}$. So $C' = B$ as well. $U \models t_j$, so as we have seen before, we can force over U to obtain a generic extension satisfying $t_{j'}$. This extension will still satisfy Ψ_B since these are buttons, so it will satisfy $\Phi(w_{j'}^B)$ as required.

- $U \models \big(\sup\{n \mid b_n \text{ holds}\} = \omega\big) \wedge s_j$:
  - Assume $\Diamond\Phi(w_{j'}^{C'})$. Since $U \models \sup\{n \mid b_n \text{ holds}\} = \omega$, any extension of U also satisfies this, since these are buttons, so we cannot have $\Diamond\Phi(w_{j'}^{C'})$ for any $C' \neq B$. Therefore $C' = B$ and indeed $w_j^B \leq w_{j'}^{C'}$.
  - Assume $w_j^C \leq w_{j'}^{C'}$. So $C' = B$ as well. $U \models s_j$, so by the definition of an n-switch, we can force over U to obtain a generic extension satisfying $s_{j'}$. This extension will still satisfy $\sup\{n \mid b_n \text{ holds}\} = \omega$ since these are buttons, so it will satisfy $\Phi(w_{j'}^B)$ as required.

Hence we have defined a Γ labeling for the frame $\langle F, \leq\rangle$ over W. This was for every pBA, so by Corollary 1.9, $\mathsf{MLF}(\Gamma) \subseteq \mathsf{S4.2}$. ⊣

We end this section by citing another theorem of this sort, which we will use in Section 4.2:

Theorem 1.16 ([5, Theorem 12]). *If there is a long ratchet over a model of set theory W, i.e., a strong ratchet $\langle r(\alpha) | 0 < \alpha \in Ord\rangle$, where $r(\alpha)$ is obtained by a single formula with parameter α, then* $\mathsf{MLF}(\Gamma) \subseteq \mathsf{S4.3}$.

§2. σ-Centered forcing. We now proceed to the investigation of the modal logic of a specific class of forcing notions—the class of all σ-centered forcing notions.

Definition 2.1. Let $\mathbb{P}$ be any poset.

1. A subset $C \subseteq \mathbb{P}$ is called *centered* if any finite number of elements in C have a common extension in $\mathbb{P}$.
2. A poset is called *σ-centered* if it is the union of countably many centered subsets.

Remark 2.2. For convenience we will always assume that the top element $\mathbb{1}_{\mathbb{P}}$ is in each of the centered posets. This does not affect the generality since every element is compatible with it. It will also sometimes be convenient to assume that if $\mathbb{P} = \bigcup_{n\in\omega} \mathbb{P}_n$ where each $\mathbb{P}_n$ is centered, then each $\mathbb{P}_n$ is upward closed, i.e., if $q \in \mathbb{P}_n$ and $q \leq p$ then $p \in \mathbb{P}_n$. This also doesn't affect the generality since if $q_1,\dots,q_k \in \mathbb{P}_n$ and $q_i \leq p_i$ then a common extension for the q_is will also extend the p_is.

The following is a central example for a σ-centered forcing, versions of which will be used later on:

Definition 2.3. Let Y be a subset of $\mathcal{P}(\omega)$. We define a poset $\mathbb{P}_Y$ as follows:

- The elements are of the form $\langle s, t\rangle$ where s is a finite subset of ω and t a finite subset of Y.
- $\langle s, t\rangle$ is extended by $\langle s', t'\rangle$ if $s \subseteq s'$ and $t \subseteq t'$ and for every $A \in t$, $s \cap A = s' \cap A$.

So we think of the first component as finite approximations for a generic real $x \subseteq \omega$, while the second component limits our options in extending the approximation. A condition $p = \langle s,t\rangle$ tells us that $s \subseteq x$ and that for every $A \in t$, $x \cap A = s \cap A$, so that the intersection of x with any set in Y will turn out to be finite.

LEMMA 2.4. *For any $Y \subseteq \mathcal{P}(\omega)$, $\mathbb{P}_Y$ is σ-centered.*

PROOF. Note that if $t_1,\dots,t_n$ are finite subsets of Y, then for any $s \in [\omega]^{<\omega}$, the conditions $\langle s,t_1\rangle,\dots,\langle s,t_n\rangle$ are all extended by $\langle s,t_1 \cup \cdots \cup t_n\rangle$. So $\mathbb{P}$ is the union of the centered posets $\mathbb{P}_s = \{\langle s,t\rangle \mid t \subseteq Y \text{ finite}\}$. Since there are only countably many finite subsets of ω, we get that $\mathbb{P}$ is σ-centered. ⊣

We will explore the properties of this kind of posets in Section 2.1. The following lemma lists a few well-known properties of σ-centered forcing.

LEMMA 2.5. 1. *Every σ-centered poset has the c.c.c. and thus preserves cardinals and cofinalities.*
2. *Assume $\lambda \geq \aleph_0$, $2^\lambda = \kappa$, and let $\mathbb{P}$ be some σ-centered forcing notion. Then $V^{\mathbb{P}} \models 2^\lambda = \kappa$.*
3. *Let $\langle \mathbb{P}_\alpha \mid \alpha < \lambda\rangle$ for some $\lambda < \left(2^{\aleph_0}\right)^+$ be a collection of σ-centered posets. Let $\mathbb{P} = \prod_{\alpha<\lambda} \mathbb{P}_\alpha$ be the finite support product*[4] *of $\langle \mathbb{P}_\alpha \mid \alpha < \lambda\rangle$. Then $\mathbb{P}$ is also σ-centered.*
4. *If $\mathbb{P}$ is a σ-centered posets and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name such that $\mathbb{P}$ forces that $\dot{\mathbb{Q}}$ is a σ-centered posets, then also $\mathbb{P} * \dot{\mathbb{Q}}$ is σ-centered.*

Note that (2.5) shows that $\Gamma_{\sigma\text{-centered}}$, the class of all σ-centered forcing notions, is transitive. It is also reflexive since the trivial forcing is trivially σ-centered, and persistent since being the union of countably many centered subsets is an upward absolute notion. So, using Theorem 1.6, we have the following:

THEOREM 2.6. *The* ZFC*-provable principles of σ-centered forcing contain* S4.2.

Finally, it will be of use to know that essentially, σ-centered forcing notions are "small," so there aren't too many of them:

LEMMA 2.7. *Let $\mathbb{P}$ be a σ-centered forcing notion. Then the separative quotient of $\mathbb{P}$ is of size at most $2^{\aleph_0}$.*

PROOF. Let $\mathbb{P} = \bigcup_{n\in\omega} \mathbb{P}_n$ where each $\mathbb{P}_n$ is upward closed. Recall that the separative quotient is the quotient of $\mathbb{P}$ by the equivalence relation: $x,y \in \mathbb{P}$, $x \sim y$ iff $\{z \in \mathbb{P} \mid z \parallel x\} = \{z \in \mathbb{P} \mid z \parallel y\}$. We denote the equivalence class of x by $[x]$. Define for every $x \in \mathbb{P}$

$$A(x) = \{n \in \omega \mid x \in \mathbb{P}_n\}.$$

We claim that for every $x,y \in \mathbb{P}$, $[x] \neq [y]$ implies $A(x) \neq A(y)$. $[x] \neq [y]$ means that (without loss of generality) there is some $z \parallel x$ such that $z \perp y$. Let z' be a common extension of z and x. So $z' \perp y$ as well (otherwise z would be compatible with y). Let $n \in \omega$ such that $z' \in \mathbb{P}_n$. Since we assumed $\mathbb{P}_n$ is upward closed, also $x \in \mathbb{P}_n$,

[4]All products in this paper are of finite support.

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so $n \in A(x)$. Assume towards contradiction that $n \in A(y)$, i.e., $y \in \mathbb{P}_n$. But $\mathbb{P}_n$ is centered, so y and z' must be compatible, which is a contradiction.

So we get that $|\mathbb{P}/\sim| \leq |\{A(x) \mid x \in \mathbb{P}\}| \leq |\mathcal{P}(\omega)| = 2^{\aleph_0}$. ⊣

Corollary 2.8. *Up to forcing-equivalence, there are at most* $2^{2^{\aleph_0}}$ σ*-centered forcing notions.*

Proof. Every poset is forcing equivalent to its separative quotient and by the previous lemma there are at most $2^{2^{\aleph_0}}$ of those. ⊣

2.1. Almost disjoint forcing. In this section we present one of the tools for labeling frames with σ-centered forcing—almost disjoint forcing (a.d.), which is a version of the example introduced in the previous section. The results in this section are due to Jensen and Solovay in [10].

Two infinite sets are called a.d. if their intersection is finite. We would like to have a way to construct a.d. subsets of ω in a very definable and absolute way. For that, we fix some recursive enumeration $\mathfrak{t} = \langle \mathfrak{t}_i \mid i < \omega \rangle$ of all finite sequences of ω (which we will also use later on, note that $\mathfrak{t} \in L$ and is absolute), and define for every $f : \omega \to \omega$

$$\mathcal{S}(f) = \left\{ i < \omega \mid \mathfrak{t}_i \text{ is an initial segment of } f \right\}.$$

If f,g are distinct then $\mathcal{S}(f)$ and $\mathcal{S}(g)$ are a.d. Hence, $\{\mathcal{S}(f) \mid f : \omega \to \omega\}$ is a family of $2^{\aleph_0}$ pairwise a.d. subsets of ω.

From the discussion at Section 2.4 of [10] we have the following:

Theorem 2.9. *Let $\mathcal{F} \in M$ be a family of a.d. subsets of ω, $Y \subseteq \mathcal{F}$ (in M), and $\mathbb{P}_Y$ the forcing from* Definition 2.3. *Then forcing with $\mathbb{P}_Y$ adds a real x such that for every $y \in \mathcal{F}$, $x \cap y$ is finite iff $y \in Y$.*

So $\mathbb{P}_Y$ adds a generic real x which is a.d. from each member of Y. Furthermore, if x is obtained by the generic filter G, then clearly $M[G] = M[x]$. This gives us a method to code subsets of 2^ω using subsets of ω. Let M be some model of ZFC, set an enumeration $\{f_\alpha \mid \alpha < \kappa\} \in M$ of ω^ω (where $\kappa = \left(2^\omega\right)^M$), and define as before $\mathcal{F} = \{\mathcal{S}(f_\alpha) \mid \alpha < \kappa\}$ which are a.d. So for each $A \subseteq \kappa$, $A \in M$, we can define $Y = Y(A) = \{\mathcal{S}(f_\alpha) \mid \alpha \in A\}$, and force with $\mathbb{P}_Y$ to obtain a generic real $x = x_A$, and by the previous theorem, $\alpha \in A$ iff $\mathcal{S}(f_\alpha) \in Y$ iff $x \cap \mathcal{S}(f_\alpha)$ is finite. So, in $M[x]$, we get that

$$A = \left\{ \alpha < \kappa \mid \mathcal{S}(f_\alpha) \cap x \text{ is finite} \right\}$$

(note that $\mathbb{P}_Y$ preserves both cardinals and the continuum, so $\kappa = (2^\omega)^{M[x]}$, and if $\kappa = \aleph_\alpha^M$ for some α, then also $\kappa = \aleph_\alpha^{M[x]}$). In this case, we say that "x codes A."

§3. Control statements for σ-centered forcing. Our goal in this section is to prove that the modal logic of σ-centered forcing is contained in S4.2 by producing control statements for σ-centered forcing that will meet the requirements of Theorem 1.15. We begin by describing a specific model W which will be our ground model, and then construct the independent family of buttons and the n-switches required in the theorem.

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3.1. The ground model. We begin with the constructible universe L, and use Cohen forcing to obtain mutually generic reals $\langle a_{\alpha,i} \mid \alpha < \omega_1^L, i < \omega\rangle$, i.e., each $a_{\beta,j}$ is generic over $L[\langle a_{\alpha,i} \mid \alpha < \omega_1^L, i < \omega, (\alpha,i) \neq (\beta,j)\rangle]$. Let

$$Z = L\big[\langle a_{\alpha,i} \mid \alpha < \omega_1^L, i < \omega\rangle\big].$$

Our ground model W is a generic extension of Z, which preserves the mutual genericity of $\langle a_{\alpha,i} \mid \alpha < \omega_1^L, i < \omega\rangle$, such that these reals are ordinal-definable with a definition which is absolute for generic extensions of W by σ-centered forcing. This can be done, e.g., by using Easton forcing to code the reals in the power function above some large enough cardinal. Any extension of W for which the above definition is absolute will be called an *appropriate extension*. We will also require that in W we do not collapse cardinals and add no new subsets below $\aleph_\omega$, so, e.g., $\omega_1^L = \omega_1^W$. From now on we'll deal with forcings which do not collapse cardinals (by c.c.c.), and also do not change the continuum (by Lemma 2.5(2)), so we omit such superscripts.

3.2. The buttons. Now over W we can define T_i, for $i < \omega$ as the statement:

For every real x and for all but boundedly many $\alpha < \omega_1$,
$a_{\alpha,i}$ is Cohen generic over $L\big[x, \langle a_{\beta,j} \mid \beta < \omega_1, j \neq i\rangle\big]$.

Since the reals $a_{\alpha,i}$ and the sequences $\langle a_{\beta,j} \mid \beta < \omega_1, j \neq i\rangle$ are ordinal definable in W and its appropriate extensions, also $L[x, \langle a_{\beta,j} \mid \beta < \omega_1, j \neq i\rangle]$ is definable with x as a parameter. So, formally, T_i includes the definitions of these elements, which will be interpreted as we expect in all relevant models. The question whether a real r is generic over some definable submodel is also expressible in the language of set theory, as it just means that r is in every open dense subset of ω^ω which is in that model. So T_i is indeed a sentence in the language of set theory. Note that if we want, by slight abuse of notation we can treat i as a "variable" denoting a natural number, rather than a definable term; thus we would be able to phrase sentences such as $\forall i < \omega\, T_i$. This will be used in the next section. In this section when we talk about a specific T_i, we take i to be a fixed term.

REMARK 3.1. 1. $W \vDash T_i$ for every i: We required that we do not add any new subsets of ω^ω or any new real. So every real $x \in W$ is already in Z. Fix some $i < \omega$ and a real $x \in W$. This real was introduced by at most boundedly many $a_{\alpha,i}$, that is, there is some $\gamma < \omega_1$ such that

$$x \in L\big[\langle a_{\alpha,j} \mid \alpha < \omega_1, j \neq i\rangle \cup \langle a_{\alpha,i} \mid \alpha < \gamma\rangle\big].$$

All the reals $a_{\alpha,i}$ for $\alpha > \gamma$ are generic over the above model so also above $L[x, \langle a_{\beta,j} \mid \beta < \omega_1, j \neq i\rangle]$.

2. $\neg T_i$ is a pure button for appropriate extensions: if for some $a_{\alpha,i}$ there is some real x such that $a_{\alpha,i}$ is not generic over $L[x, \langle a_{\beta,j} \mid \beta < \omega_1, j \neq i\rangle]$, then it will never again be generic over this model. So, if we destroy T_i, we can never get it back as long as it keeps its above meaning. Note that if an extension is not appropriate, then T_i might have a completely different meaning than what is intended, as the definitions we use will give some different sets, so it is paramount we stick with appropriate extensions.

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We will now define forcing notions which will allow us to destroy T_i, by destroying the genericity of the relevant $a_{\alpha,i}$s.

DEFINITION 3.2. In W, we define $\mathbb{P}_i$ to be the forcing notion with conditions of the form $\{U_{s_1},\dots,U_{s_n},a_{\alpha_1,i},\dots,a_{\alpha_l,i}\}$ where $n,l<\omega$, $s_k\in\omega^{<\omega}$ and $U_{s_k}\subseteq\omega^\omega$ is the basic open set $\{x\in\omega^\omega\mid s_k\trianglelefteq x\}$; and for conditions $p,q\in\mathbb{P}_i$, $q\le p$ iff $p\subseteq q$ and whenever $a_{\alpha,i}\in p$ and $U_s\in q\setminus p$, $a_{\alpha,i}\notin U_s$. That is, to extend a condition, we can add any finite number of the reals, and we can add any finite number of basic open sets, as long as the new sets do not include any of the old reals.

We will show that the forcing $\mathbb{P}_i$ destroys the genericity of all the $a_{\alpha,i}$s, by adding dense open sets (approximated by the U_ss) that do not include them. So, intuitively, a condition $p=\{U_{s_1},\dots,U_{s_n},a_{\alpha_1,i},\dots,a_{\alpha_l,i}\}$ states which reals will be avoided in subsequent stages.

REMARK 3.3. Given some distinct $a_{\alpha_1,i},\dots,a_{\alpha_l,i}$ and $s\in\omega^{<\omega}$ we can always find some $s'\trianglerighteq s$ such that $a_{\alpha_1,i},\dots,a_{\alpha_l,i}\notin U_{s'}$: let $t=\bigcap_{k=1}^{l}a_{\alpha_k,i}$, i.e., the longest initial segment common to $a_{\alpha_1,i},\dots,a_{\alpha_l,i}$. If $s\trianglelefteq t$ or $t\trianglelefteq s$, let t' be the longer of the two and assume it is of length n. Take some $j\in\omega\setminus\{a_{\alpha_1,i}(n),\dots,a_{\alpha_l,i}(n)\}$ and set $s'=t'^\frown\langle j\rangle$, then $a_{\alpha_1,i},\dots,a_{\alpha_l,i}\notin U_{s'}$ and $s'\trianglerighteq s$. Otherwise $s'=s$ will do.

Let $G\subseteq\mathbb{P}_i$ be a generic filter. Note that by the former remark, the set of conditions having at least n basic-open sets in them is dense in $\mathbb{P}_i$ (given a condition p, we can find an s such that U_s does not contain any of the reals in p, and then add to p, e.g., $U_s,U_{s^\frown\langle 0\rangle},U_{s^\frown\langle 0,0\rangle}\dots$ to obtain an extension with at least n basic-open sets). So, the conditions in G give us an infinite sequence $\langle U_{s_k}\mid k<\omega\rangle$ of basic-open sets. We assume that $\langle s_k\mid k<\omega\rangle$ forms a subsequence of the recursive enumeration $\langle \mathfrak{t}_i\mid i<\omega\rangle$, i.e., there are indexes $\langle n_k\mid k<\omega\rangle$ such that $s_k=\mathfrak{t}_{n_k}$ for every k.

LEMMA 3.4. *For every $k<\omega$, the set $\bigcup_{n\ge k}U_{s_n}$ is open-dense in ω^ω.*

PROOF. It is clearly open as a union of open sets. To show it is dense, let $s\in\omega^{<\omega}$ and we need to find some $n\ge k$ such that $s_n\trianglerighteq s$. Note that as in Remark 3.3, for every $p\in\mathbb{P}_i$ we can extend s to some $s'\trianglerighteq s$ such that $p\cup\{U_t\}\le p$, and we can also make sure that $|s'|>N$ for any fixed N. So by genericity there is some $p\in G$ containing some $U_{s'}$ where $s'\trianglerighteq s$ and $|s'|>\max\{|s_l|\mid l<k\}$ so in particular $s'=s_n$ for $n\ge k$ as required. ⊣

LEMMA 3.5. *For every $\alpha<\omega_1$ there is some k such that $a_{\alpha,i}\notin\bigcup_{n\ge k}U_{s_n}$.*

PROOF. Fix $\alpha<\omega_1$ and let $D_\alpha=\{p\in\mathbb{P}_i\mid a_{\alpha,i}\in p\}$. So D_α is clearly open, and it is dense since for every p, $p\cup\{a_{\alpha,i}\}$ is a legitimate extension of p (we did not limit the addition of $a_{\beta,i}$s). So there is some $p\in G\cap D_\alpha$. Let k be larger than any n such that $U_{s_n}\in p$. We want to show that $a_{\alpha,i}\notin\bigcup_{n\ge k}U_{s_n}$. Otherwise, there is some $n\ge k$ such that $a_{\alpha,i}\in U_{s_n}$. So there is some $q\in G$ with $U_{s_n}\in q$, and by moving to a common extension we can assume $q\le p$. In fact, $q<p$, since $U_{s_n}\notin p$ by the choice of k and n. But $a_{\alpha,i}\in p$, $q<p$ and $U_{s_n}\in q\setminus p$ imply that $a_{\alpha,i}\notin U_{s_n}$, by contradiction. ⊣

So indeed, $\mathbb{P}_i$ adds open-dense sets which destroy the genericity of every $a_{\alpha,i}$. This will show that T_i is destroyed, once we show that T_i still means the same thing after forcing with $\mathbb{P}_i$.

LEMMA 3.6. *$\mathbb{P}_i$ is σ-centered.*

PROOF. For every $t_1,\dots,t_n \in \omega^{<\omega}$, let $\mathbb{P}(t_1,\dots,t_n)$ be the set of all conditions in $\mathbb{P}_i$ containing exactly the basic-open sets $U_{t_1},\dots,U_{t_n}$. Note that there are only ω such sets $\{t_1,\dots,t_n\}$, and that clearly $\mathbb{P}_i = \bigcup\{\mathbb{P}(t_1,\dots,t_n) \mid t_1,\dots,t_n \in \omega^{<\omega}\}$. Now notice that every $\mathbb{P}(t_1,\dots,t_n)$ is centered, since if $p_1,\dots,p_l \in \mathbb{P}(t_1,\dots,t_n)$, then $p_1 \cup \cdots \cup p_l$ is still a legitimate condition in $\mathbb{P}_i$, and it extends each p_j since the only limitation on extension concerned the basic-open sets, which we did not change. ⊣

COROLLARY 3.7. *Let W' be some appropriate extension of W. Let $G \subseteq \mathbb{P}_i$ be generic over W'. Then $W'[G] \vDash \neg T_i$.*

PROOF. By σ-centeredness, after forcing with $\mathbb{P}_i$ the meaning of all the definitions in T_i remain the same. So we will find a real $x \in W'[G]$ such that all the $a_{\alpha,i}$s are already not generic over $L[x]$, so surely T_i fails. Recall the enumeration $\langle \mathsf{t}_n \mid n < \omega\rangle$ we fixed earlier, and define $x = \{m \mid \exists p \in G(U_{\mathsf{t}_m} \in p)\}$. So, if as before $\langle U_{s_n} \mid n < \omega\rangle$ is the sequence of basic-open sets given by G, we assumed it is a subsequence of $\langle \mathsf{t}_n \mid n < \omega\rangle$, so in $L[x]$ we can already define each union $\bigcup_{n\geq k} U_{s_n}$. Hence, as we have shown above, each $a_{\alpha,i}$ is not in some dense-open set of ω^ω in $L[x]$, and therefore not generic over $L[x]$ as required. ⊣

Our next task will be to show that forcing with some $\mathbb{P}_j$ does not affect the truth of T_i for any $i \neq j$.

LEMMA 3.8. *Let W' be an appropriate extension of W, such that $W' \vDash T_i$. Let $G \subseteq \mathbb{P}_j$ be generic over W' for $j \neq i$. Then $W'[G] \vDash T_i$.*

PROOF. Assume otherwise. Then there is some $x \in W'[G]$ such that unboudedly many $a_{\alpha,i}$ are not generic over $L[x, \langle a_{\beta,k} \mid \beta < \omega_1, k \neq i\rangle]$. Let $\dot{x} \in W'$ be a $\mathbb{P}_j$-name for x. Since x is a real, we can assume that $\dot{x}$ is a name containing only elements of the form $\langle q, \check{n}\rangle$ for $n \in \omega$ and $q \in \mathbb{P}_j$. Furthermore, since $\mathbb{P}_j$ is c.c.c., we can assume that there are only countably many elements of the form $\langle q, \check{n}\rangle$ for each n. So $\dot{x}$ is a countable collection of elements of the form $\langle q, \check{n}\rangle$. We wish to "code" $\dot{x}$ by some real $y \in W'$. We do this in the usual way: Let γ be the supremum of all $\alpha < \omega_1$ such that $a_{\alpha,j} \in q$ for some $\langle q, \check{n}\rangle \in \dot{x}$. Since $\dot{x}$ is countable and each such q contains only finitely many $a_{\alpha,j}$s, $\gamma < \omega_1$. Each $q \in \mathbb{P}_j$ is of the form $\{U_{s_1},\dots,U_{s_n}, a_{\alpha_1,j},\dots,a_{\alpha_l,j}\}$, so it is determined by a finite subset of $\omega^{<\omega}$ and a finite subset of ordinals no larger than γ. This information can be coded by a finite sequence of natural numbers z_q. Each pair $\langle z_q, \check{n}\rangle$ can be coded by a natural number. So the entire $\dot{x}$ can be coded by a set of natural numbers y. All these codings are done in W' so $y \in W'$.

Now assume that $W'[G] \vDash$ "$a_{\alpha,i}$ is not generic over $M' := L[x, \langle a_{\beta,k} \mid \beta < \omega_1, k \neq i\rangle]$." We'll show that already $W' \vDash$ "$a_{\alpha,i}$ is not generic over $M := L[y, \langle a_{\beta,k} \mid \beta < \omega_1, k \neq i\rangle]$."

$M \subseteq W'$, and since the definition of $\mathbb{P}_j$ requires only the reals $\langle a_{\beta,j} \mid \beta < \omega_1\rangle$, $\mathbb{P}_j \in M$. In addition, since we can decode y in this model, we have $\dot{x} \in M$. Since $\mathbb{P}_j \in M \subseteq W'$, G is generic also over M. The fact that $a_{\alpha,i}$ is not generic over M' means that there is a dense open set $U \in M'$ such that in $W'[G]$, $a_{\alpha,i} \notin U$. From the perspective of M and M', $a_{\alpha,i}$ is merely a Cohen generic over $L[\langle a_{\beta,k} \mid \beta < \omega_1, k \neq i\rangle]$. Since $\dot{x} \in M$, $M' \subseteq M[G]$, so $U \in M[G] \subseteq W'[G]$. So there is some $p \in G$ and

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some $\mathbb{P}_j$-name $\dot{U} \in M$ such that $p \Vdash$ "$\dot{U}$ is an open-dense subset of ω^ω, which does not contain reals which are Cohen generics over $L[\langle a_{\beta,k} \mid \beta < \omega_1, k \neq i\rangle]$." Define

$$\bar{U} = \{r \in \omega^\omega \mid \exists p' \leq p(p' \Vdash \check{r} \in \dot{U})\}.$$

So $\bar{U} \in M$. We claim first that $\bar{U}$ is open-dense.

Open: let $r \in \bar{U}$, witnessed by $p' \leq p$ s.t. $p' \Vdash \check{r} \in \dot{U}$. Since p' also forces that $\dot{U}$ is open, there is some $p'' \leq p'$ and some $s \in \omega^{<\omega}$ such that $p'' \Vdash$"$\check{r} \in U_s \subseteq \dot{U}$," that is $p'' \Vdash \check{s} \trianglelefteq \check{r} \wedge (\check{s} \trianglelefteq \dot{r}' \to \dot{r}' \in \dot{U})$. The initial segment relation does not change, so $s \trianglelefteq r$. If $r' \in U_s$, then in particular we'll get $p'' \Vdash \check{r}' \in \dot{U}$, so by definition $r' \in \bar{U}$. So $U_s \subseteq \bar{U}$. So $\bar{U}$ is open.

Dense: Let $s \in \omega^{<\omega}$. Since p forces that $\dot{U}$ is open-dense, there is some $p' \leq p$ and some $t \trianglerighteq s$ such that $p' \Vdash (\check{t} \trianglelefteq \dot{r} \to \dot{r} \in \dot{U})$. So let some $t \trianglelefteq r \in \omega^\omega$, then in particular we get $p' \Vdash \check{r} \in \dot{U}$, so $r \in \bar{U}$ and $r \trianglerighteq s$ as required.

Second, we claim that in W', $a_{\alpha,i} \notin \bar{U}$. Otherwise, there is some $p' \leq p$ such that $p' \Vdash \check{a}_{\alpha,i} \in \dot{U}$. But p forced that no Cohen generic over $L[\langle a_{\beta,k} \mid \beta < \omega_1, k \neq i\rangle]$ is in $\dot{U}$, a contradiction.

So, we have found an open-dense set $\bar{U} \in M$ such that $a_{\alpha,i} \notin \bar{U}$, so $a_{\alpha,i}$ is not generic over M. This was for every $a_{\alpha,i}$ not generic over $L[x, \langle a_{\beta,k} \mid \beta < \omega_1, k \neq i\rangle]$, and we assumed there are unboundedly many of these. So there are unboundedly many $a_{\alpha,i}$s which are not generic over $M = L[y, \langle a_{\beta,k} \mid \beta < \omega_1, k \neq i\rangle]$, where $y \in W'$. But this contradicts the assumption that $W' \vDash T_i$. So, we indeed get that also $W'[G] \vDash T_i$. ⊣

To conclude, packing up what we have done in this section, we obtain the following:

Proposition 3.9. *$\{\neg T_i \mid i < \omega\}$ is a family of independent buttons over W for σ-centered forcing.*

Remark 3.10. In fact, we can replace "σ-centered" with any reflexive and transitive class of forcing notions, containing all the $\mathbb{P}_i$s, such that every extension of W with a forcing from the class yields an appropriate extension.

Note that if it were the case that in no extension of W by σ-centered forcing all these buttons are pushed, we could have finished the proof of our main Theorem using Corollary 1.14. However, by Lemma 2.5(3), $\prod_{i<\omega} \mathbb{P}_i$ is σ-centered, and it pushes all the buttons, so we will have to use Theorem 1.15.

3.3. The *n*-switches.

Proposition 3.11. *Let M be a model of* ZFC *such that $M \vDash 2^{\aleph_0} = \aleph_1 \wedge 2^{\aleph_1} = \aleph_2$ and every subset of ω_2 is ordinal-definable using a definition which is absolute to σ-centered forcing extensions. Then for every $n > 1$ there is an n-switch for σ-centered forcing over M.*

Proof. Let $\langle f_\alpha \mid \alpha < \omega_1\rangle \in M$ be a definable enumeration of all the functions $f : \omega \to \omega$ in M and define $y_\alpha = \mathcal{S}(f_\alpha)$ as in Section 2.1. Let $\langle A_\xi \mid \xi < \omega_2\rangle \in M$ be some fixed definable enumeration of all the subsets of ω_1 in M. Let $C(x, \xi)$ be the statement

$$x \subseteq \omega \text{ and } (\alpha \in A_\xi \leftrightarrow x \cap y_\alpha \text{ is finite}) \tag{3.1}$$

referred to as "x is a real coding A_ξ." By the discussion at the end of Section 2.1, for every ξ there is a σ-centered forcing notion $\mathbb{Q}_\xi \in M$ such that $\mathbb{Q}_\xi \Vdash \exists x C(x,\xi)$. We would like to define a ratchet by using the statements "$\alpha = \sup\{\xi < \omega_2 \mid \exists x C(x,\xi)\}$." By defining so, we can indeed always increase the value of alleged ratchet by forcing with $\mathbb{Q}_\alpha$, but in a certain extension, forcing with $\mathbb{Q}_\alpha$ might also add a real coding A_ξ some $\xi > \alpha$. To fix that, we will define an unbounded set $\mathcal{E}$ such that adding a code for A_α for some $\alpha \in \mathcal{E}$ doesn't add a code for any larger A_ξ.

We work now within M, and fix some σ-centered poset $Q \in M$. Let $\alpha < \omega_2$. We define by induction $\{\alpha_\zeta \mid \zeta < \omega_1\}$. Set $\alpha_0 = \alpha$. If $\alpha_\zeta < \omega_2$ is defined for $\zeta < \omega_1$, let

$$\alpha_{\zeta+1} = \sup\left\{\beta < \omega_2 \mid Q \times \prod_{\xi \leq \alpha_\zeta} \mathbb{Q}_\xi \nVdash \neg\exists x C\big(x,\check{\beta}\big)\right\} + 1.$$

The above set is not empty since $\mathbb{Q}_{\alpha_\zeta} \Vdash \exists x C(x,\check{\alpha}_\zeta)$. In particular, $\alpha_\zeta < \alpha_{\zeta+1}$.

Claim 3.12. $\alpha_{\zeta+1} < \omega_2$. ⊣

Proof. Let $P = Q \times \prod_{\xi \leq \alpha_\zeta} \mathbb{Q}_\xi$. Since $\alpha_\zeta < \omega_2 = (2^{\aleph_0})^+$, this is σ-centered by Lemma 2.5(3) so by Lemma 2.5 and our assumptions on M, $P \Vdash 2^{\aleph_1} = \aleph_2 > 2^{\aleph_0}$. In particular, $P \Vdash$" $\sup\{\beta \mid \exists x C(x,\beta)\} < \omega_2$," since there cannot be $\aleph_2$ reals each coding a different subset of ω_1. Note that by the c.c.c. of P there can only be $\aleph_0$ many possible values for $\sup\{\beta \mid \exists x C(x,\beta)\}$. Hence there is some $\beta_P < \omega_2$ bounding all these possible values, so it is forced by P that $\sup\{\beta \mid \exists x C(x,\beta)\} \leq \beta_P$. Now, if for some γ, $P \nVdash \neg\exists x C(x,\check{\gamma})$, then there is $p \in P$ such that $p \Vdash \exists x C(x,\check{\gamma})$, but also $p \Vdash$"$\sup\{\beta \mid \exists x C(x,\beta)\} \leq \beta_P$" so $\gamma \leq \beta_P$. So by the definition, $\alpha_{\zeta+1} \leq \beta_P + 1 < \omega_2$. ⊣

For $\zeta < \omega_1$ limit, set $\alpha_\zeta = \sup\{\alpha_\xi \mid \xi < \zeta\}$ (ω_2 is regular so also in this case $\alpha_\zeta < \omega_2$), and finally let $\alpha^* = \sup\{\alpha_\zeta \mid \zeta < \omega_1\}$. Again $\alpha^* < \omega_2$.

Claim 3.13. *Let G be generic for $\mathbb{Q} = Q \times \prod_{\xi < \alpha^*} \mathbb{Q}_\xi$ such that $M[G] \vDash \exists x C(x,\beta)$. Then $\beta < \alpha^*$.*

Proof. Let $x \in M[G]$ such that $M[G] \vDash C(x,\beta)$. So there is a $\mathbb{Q}$-name τ and some $p_* \in G$ which forces $C(\tau,\check{\beta})$. For every n, let $C_n \subseteq \mathbb{Q}$ be a maximal antichain below p_* of conditions deciding the statement $\check{n} \in \tau$. By the c.c.c. each C_n is countable, so also $C = \bigcup_{n\in\omega} C_n$ is countable. Every element of $\mathbb{Q}$ is of the form $\langle q, (p_\gamma)_{\gamma<\alpha^*}\rangle$ where only for finitely many γs $p_\gamma \neq \mathbb{1}_{\mathbb{Q}_\gamma}$. So for each $p \in C$, denote this finite set of ordinals by F_p, and let $\gamma^* = \sup \bigcup_{p\in C} F_p$. Each F_p is a set of ordinals less than α^*, so $\gamma^* \leq \alpha^*$. But since C is countable and each F_p is finite, γ^* has at most countable cofinality, while α^* is the limit of an increasing ω_1 sequence $\langle \alpha_\zeta \mid \zeta < \omega_1\rangle$, so $\gamma^* < \alpha^*$, and furthermore, there is some $\zeta < \omega_1$ such that $\gamma^* \leq \alpha_\zeta$. Let $\bar{\mathbb{Q}} = Q \times \prod_{\xi \leq \alpha_\zeta} \mathbb{Q}_\xi$. For every $p \in \mathbb{Q}$, if $p = \langle q, (p_\gamma)_{\gamma<\alpha^*}\rangle$, let $\bar{p} = \langle q, (p_\gamma)_{\gamma \leq \alpha_\zeta}\rangle$, and let $\bar{G} = \{\bar{p} \mid p \in G\}$. $\bar{G}$ is $\bar{\mathbb{Q}}$-generic over M. We claim that $x \in M[\bar{G}]$ and $M[\bar{G}] \vDash C(x,\beta)$. Define the

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$\bar{\mathbb{Q}}$-name

$$\bar{\sigma} = \bigcup_{n\in\omega} \left(\left\{\bar{p} \mid p \in C_n, p \Vdash \check{n} \in \tau\right\} \times \left\{\check{n}\right\}\right).$$

REMARK. By the choice of $\alpha_\zeta \geq \gamma^*$, for every $p \in C_n$, if it is of the form $\langle q, (p_\gamma)_{\gamma<\alpha^*}\rangle$, then $p_\gamma = \mathbb{1}$ for every $\gamma > \alpha_\zeta$. ⊣

If $n \in x$ then there is some $p \in G$, $p \leq p_*$ that forces $\check{n} \in \tau$. By the maximality of C_n, it intersects G, which is a filter, so a condition in the intersection must also force $\check{n} \in \tau$ (and not $\check{n} \notin \tau$). So we can choose such $p \in C_n \cap G$, and by definition $\langle \bar{p}, \check{n}\rangle \in \bar{\sigma}$. In addition, $\bar{p} \in \bar{G}$, so $n \in \bar{\sigma}_{\bar{G}}$.

If $n \in \bar{\sigma}_{\bar{G}}$ there is some $p \in C_n$, $p \Vdash \check{n} \in \tau$ such that $\bar{p} \in \bar{G}$. So, there is some $r \in G$ such that $\bar{r} = \bar{p}$. Note that by the remark r and p are equal in every coordinate where p is not trivial, so $r \leq p$. Therefore also $r \Vdash \check{n} \in \tau$, and $r \in G$ so $n \in x$.

So, we get that $\bar{\sigma}_{\bar{G}} = x$, so $x \in M[\bar{G}]$. $M[\bar{G}] \subseteq M[G]$, and in $M[G]$ we have $C(x,\beta)$, i.e.,

$$A_\beta = \left\{\alpha \mid x \cap y_\alpha \text{ is finite}\right\},$$

so we can already have this equation in $M[\bar{G}]$ (since the y_αs don't change), so $M[\bar{G}] \vDash C(x,\beta)$. Therefore, we have that $Q \times \prod_{\xi\leq\alpha_\zeta} \mathbb{Q}_\xi \nVdash \neg\exists x C(x,\beta)$, so by the definition of $\alpha_{\zeta+1}$, $\beta \leq \alpha_{\zeta+1} < \alpha^*$, as required.

So we have defined an operation $\alpha \mapsto \alpha^*$ for every $\alpha < \omega_2$ (note that this operation was relative to Q). Since for every $\alpha < \omega_2$ we have $\alpha < \alpha^* < \omega_2$, the set $\{\alpha^* \mid \alpha < \omega_2\}$ is unbounded in ω_2, so the set $\mathcal{C}_Q$ consisting of all limit points of this set is a club.

By Corollary 2.8, in M there are at most $(2^{2^{\aleph_0}})^M$ σ-centered forcing notions up to equivalence, or to be exact, at most $(2^{2^{\aleph_0}})^M$ separative σ-centered forcing notions. By our assumptions on M, this cardinal is $\aleph_2$. Note that $\left\{\prod_{\alpha<\xi} \mathbb{Q}_\alpha \mid \xi \in \mathcal{C}_Q\right\}$ where Q is the trivial forcing are non-equivalent σ-centered posets, so we get exactly $\aleph_2$ posets. Since every separative σ-centered poset has size at most $\aleph_1$, each can be coded as binary relations on ω_1, so by our assumption we can have a definable enumeration $\langle Q_\zeta \mid \zeta < \omega_2\rangle$ of all the separative σ-centered forcing notions in M. Define $\mathcal{C}$ as be the diagonal intersection of $\langle \mathcal{C}_{Q_\zeta} \mid \zeta < \omega_2\rangle$:

$$\mathcal{C} := \mathop{\triangle}_{\zeta<\omega_2} \mathcal{C}_{Q_\zeta} = \left\{\alpha < \omega_2 \mid \alpha \in \bigcap_{\zeta<\alpha} \mathcal{C}_{Q_\zeta}\right\},$$

which is also a club in ω_2. Now we let

$$\mathcal{E} = \mathcal{C} \cap \left\{\alpha < \omega_2 \mid \mathrm{cf}(\alpha) = \omega_1\right\}.$$

$\mathcal{E}$ is unbounded: it is known that the set $\{\alpha < \omega_2 \mid \mathrm{cf}(\alpha) = \omega_1\}$ is stationary in ω_2 so it intersects the club $\{\alpha \in \mathcal{C} \mid \alpha > \gamma\}$ for each $\gamma < \omega_2$. Let $\langle e_\alpha \mid \alpha < \omega_2\rangle$ be a (definable) ascending enumeration of $\mathcal{E}$.

REMARK. We would have preferred to work with $\mathcal{C}$ rather than $\mathcal{E}$. The problem is that the * operation may not be continuous at limits of countable cofinality—to prove continuity, we would like to imitate the proof of Claim 3.13, but it requires that the limit is of uncountable cofinality. If the length of the product is of countable cofinality, there might be a real that is not introduced in any bounded product.

Now we can define the following statements

$$r(\alpha) = \text{“}\alpha = \min\{\beta < \omega_2 \mid \neg\exists x C(x, e_\beta)\}.\text{”}$$

These are indeed statements (with ordinal parameters perhaps) invoking the definition of $\langle e_\alpha \mid \alpha < \omega_2\rangle$, which retains its intended meaning in every extension of M by σ-centered forcing. Now given $n > 1$, define for every $j < n$

$$\Phi_j = \text{“}r(\omega \cdot \alpha + k) \to (k \bmod n = j).\text{”}$$

We claim that $\{\Phi_j \mid j < n\}$ is an n-switch for σ-centered forcing over W.

Let $Q \in M$ be some σ-centered poset, and $G \subseteq Q$ generic over M. By Lemma 2.5(1) $M[G]$ satisfies $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$, so in particular the set $\{\beta < \omega_2 \mid \exists x C(x, e_\beta)\}^{M[G]}$ is bounded, since there are only ω_1 reals, and therefore there cannot be unboundedly many subsets of ω_1 coded by them. So there is some unique $\gamma < \omega_2$ such that $M[G] \vDash r(\gamma)$. There are some unique $j, k < n$ such that $\gamma = \omega \cdot \alpha + k$ and $k \bmod n = j$, so $M[G] \vDash \Phi_j$. Hence every σ-centered extension of M satisfies exactly one Φ_j. Now we need to show that for every $j' \neq j$ there is some σ-centered extension of $M[G]$ satisfying $\Phi_{j'}$. Recall the club $\mathcal{C}_Q$ from the above construction. $Q = Q_\xi$ for some $\xi < \omega_2$. By the unboundedness of $\mathcal{E}$, we can find some γ' such that $e_{\gamma'} > \xi$ and $\gamma' = \omega \cdot \alpha + j'$ for some α. We want to show that we have a generic extension of $M[G]$ satisfying $r(\gamma')$.

Let $H \subseteq \prod_{\zeta < e_{\gamma'}} \mathbb{Q}_\zeta$ be generic over $M[G]$. By the product lemma, $M[G \times H] = M[G][H]$ and $G \times H$ is $Q \times \prod_{\zeta < e_{\gamma'}} \mathbb{Q}_\zeta$ generic over M. So for every $\beta < e_{\gamma'}$, $\prod_{\zeta < e_{\gamma'}} \mathbb{Q}_\zeta \Vdash \exists x C(x, \beta)$, so $M[G][H] \vDash \exists x C(x, \beta)$. On the other hand, recall that $e_{\gamma'}$ is in the diagonal intersection of the clubs $\mathcal{C}_{Q_\zeta}$, so by definition, and since $e_{\gamma'} > \xi$, $e_{\gamma'} \in \bigcap_{\zeta < e_{\gamma'}} \mathcal{C}_{Q_\zeta} \subseteq \mathcal{C}_{Q_\xi}$. So, by the definition of $\mathcal{C}_{Q_\xi}$, $e_{\gamma'}$ is either of the form δ^* for some δ and the * operation corresponding to Q_ξ, or a limit point of such points. In the first case, we can just apply Claim 3.13. In the second case, since $e_{\gamma'} \in \mathcal{E}$ is of uncountable cofinality, we can repeat the proof of Claim 3.13 with a sequence $\langle \delta^*_\zeta \mid \zeta < \omega_1\rangle$ that witnesses $e_{\gamma'} \in \mathcal{C}_{Q_\xi}$, and get that the statement in Claim 3.13 is true in this case as well. That is, in both cases, we get that if $M[G \times H] \vDash \exists x C(x, \beta)$ then $\beta < e_{\gamma'}$. So $e_{\gamma'}$ is the first β such that $M[G \times H] \vDash \neg\exists x C(x, \beta)$. Since the enumeration of $\mathcal{E}$ is increasing, we get that $\gamma' = \min\{\beta < \omega_2 \mid \neg\exists x C(x, e_\beta)\}$. So $M[G \times H] \vDash r(\gamma')$, and since $\gamma' = \omega \cdot \alpha + j'$, $M[G \times H] \vDash \Phi_{j'}$ as required.

The forcing notions used in this n-switch add real numbers in a rather uncontrollable way, so it is indeed likely that they might add some real which destroys the genericity of the $a_{\alpha,i}$s, therefore it is unlikely that this n-switch is independent of the buttons $\neg T_i$. However, by using both constructions presented in this section, we can overcome the drawbacks each of them has, and obtain our main result:

THEOREM 3.14. *If* ZFC *is consistent, then the* ZFC*-provable principles of σ-centered forcing are exactly* S4.2.

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PROOF. If ZFC is consistent then we can obtain the model W and by Propositions 3.9 and 3.11 we obtain the buttons and n-switches required for using Theorerm 1.15. So the ZFC-provable principles of σ-centered forcing are contained in S4.2, and by Theorem 2.6 we get equality. ⊣

§4. Generalizations and open questions.

4.1. Forcing over L. In most of the calculations of the modal logic of a certain set-theoretic construction, the upper bound was obtained using control statement over the constructible universe L (e.g., all the calculations in [5]). A notable exception is the upper bound for the modal logic of *grounds* obtained in [8] using Reitz's model, which is a class-forcing extension of L. Similarly, for our construction we also first had to build a special model of ZFC—a set-forcing extension of L, over which we could obtain the desired control statements. Whether we could have avoided this and work over L itself remains an open question, which leads to the following more exact one:

QUESTION 4.1. *What is the modal logic of σ-centered forcing over L?*

This question relates to a second line of inquiry introduced in [7]—the calculation of the valid principles of forcing over a specific model. In the case of all forcing notions, it has been shown in [7] that these principles always contain S4.2 and are contained in S5, where both bounds can be realized. Models having other validities have been recently announced by Block and Hamkins (see discussion in [12, p. 32]). It has also been shown (see [6, 12]) that any model has a ground whose valid principles of forcing are S4.2. However, in the case of σ-centered forcing we have only limited results: S4.2 is still a lower bound over any model, as this class is reflexive, transitive, and persistent. As for an upper bound—note that any model satisfying the assumptions of Proposition 3.11, L in particular, has n-switches for σ-centered forcing for every n. Such n-switches can be used, as in [5, Theorem 10], to show that the valid principles of σ-centered forcing for such a model are contained in the modal theory S5. So we do have:

PROPOSITION 4.2. *The modal logic of σ-centered forcing over L (or any model satisfying the assumptions of Proposition 3.11) is between* S4.2 *and* S5.

However in the case of L we do not expect to be able to raise the upper bound to S5, as not even axiom .3:

$$(\Diamond\varphi \wedge \Diamond\psi) \rightarrow \Diamond[(\Diamond\varphi \wedge \psi) \vee (\varphi \wedge \Diamond\psi)],$$

which corresponds to the linearity of the forcing class (see [5]), holds over L: by a result of Błaszczyk and Shelah [4], in L there is a σ-centered forcing notion which does not add a Cohen real. So let $\mathbb{P}$ be the L-least such forcing notion, consider the statements:

$$\varphi' = \text{“There is a Cohen real over } L\text{”},$$
$$\psi' = \text{“There is a } \mathbb{P}\text{-generic filter over } L\text{”}$$

and set $\varphi = \varphi' \wedge \neg\psi'$ and $\psi = \psi' \wedge \neg\varphi'$. So clearly $L \vDash \Diamond\varphi \wedge \Diamond\psi$ but $(\Diamond\varphi \wedge \psi) \vee (\varphi \wedge \Diamond\psi)$ is not forceable since a generic for one of these forcings will stay generic

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in further extensions. Hence Question 4.1 is still open, as well as the more general question:

QUESTION 4.3. *What modal theories can be realized as the valid principles of σ-centered forcing over some model of* ZFC?

4.2. Related forcing classes. Throughout this work, we have focused on σ-centered forcing notions. However, by examining the proofs, one can see that we have not used the full strength of σ-centeredness. To obtain the lower bound, we used the reflexivity, transitivity, and persistence of σ-centered posets. And to obtain the upper bound, we defined labelings using two main ingredients—the posets constructed Section 3.2, giving us the independent buttons, and the n-switch of Proposition 3.11. To work with the buttons, we also required that all extensions of W will be appropriate. Assuming this, once we had an n-switch, we did not use its specific construction in defining the labeling. So in fact we have the following:

THEOREM 4.4. *Let W the model constructed in Section 3.1 and Γ a class of forcing notions with the following properties*:

1. *Γ is reflexive, transitive, and persistent.*
2. *Every extension of W by a Γ-forcing is appropriate.*
3. *All posets constructed in Section 3.2 are in Γ.*
4. *There is an n-switch for Γ-forcing over W for any n.*

Then $\mathsf{MLF}(\Gamma) = \mathsf{S4.2}$.

Now let's see what was needed to obtain the n-switch of Proposition 3.11. We relied heavily on the c.c.c. of all posets in question; we used all posets coding subsets of ω_1, as well as products of at most $\aleph_1$ of them; we relied on the fact that σ-centered posets cannot enlarge $2^{\aleph_0}$ or $2^{\aleph_1}$; we used the fact that there were (in W) only $\aleph_2$ σ-centered posets up to equivalence, and that they were all already in the smaller model Z. So, this construction can be carried with any class of forcing notions satisfying these requirements. To conclude:

THEOREM 4.5. *Let Γ be a class of forcing notions with the following properties*:

1. *Γ is reflexive, transitive, and persistent.*
2. *Every extension of W by a Γ-forcing is appropriate.*
3. *All posets constructed in Section 3.2 are in Γ.*

4.1. *Each poset in Γ has the c.c.c., and does not enlarge $2^{\aleph_0}$ or $2^{\aleph_1}$.*
4.2. $|\Gamma| \leq 2^{2^{\aleph_0}}$ (*where the size is measured up to equivalence of forcing*).
4.3. $\Gamma^W \subseteq Z$.
4.4. *All posets which are used to code subsets of ω_1, and products of at most $\aleph_1$ of them, are in Γ.*

Then $\mathsf{MLF}(\Gamma) = \mathsf{S4.2}$.

REMARK. Conditions 3 and 4.4 will hold for any class containing every σ-centered forcing.

DEFINITION 4.6. A subset $C \subseteq \mathbb{P}$ is called *n-linked* if any n elements of C are compatible, i.e., have a common extension (perhaps not in C itself). 2-linked is also

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called simply *linked*. A poset is called *σ-n-linked* if it is the union of ω many n-linked subsets. Again, σ-linked means σ-2-linked.

It is clear that we have the following implications:

$$\sigma\text{-centered} \rightarrow \sigma\text{-}n\text{-linked for every } n \rightarrow \cdots \rightarrow \sigma\text{-}n\text{-linked} \rightarrow \sigma\text{-linked},$$

and it is known that the other directions do not hold (cf. [1]).

Corollary 4.7. *Let Γ be either the class of all σ-n-linked posets (for some fixed n), or the class of all posets which are σ-n-linked for every n. Then* $\mathsf{MLF}(\Gamma) = \mathsf{S4.2}$.

Proof. Lemmas 2.5 and 2.7 hold for these classes as well, so they are all transitive, preserve cardinals and the continuum function, and have size at most $2^{2^{\aleph_0}}$ (up to equivalence). These classes are also clearly reflexive and persistent, and they contain the class of all σ-centered forcings, so they satisfy all the conditions of Theorem 4.5. ⊣

Parallel to this hierarchy of properties, we can define the following hierarchy (cf. [1]):

Definition 4.8. 1. Given $n \in \omega$, $\mathbb{P}$ has property K_n if every $A \in [\mathbb{P}]^{\aleph_1}$ contains an uncountable n-linked subset. K_2 is also called the *Knaster property*.
2. $\mathbb{P}$ has *pre-caliber* ω_1 if every $A \in [\mathbb{P}]^{\aleph_1}$ contains an uncountable centered subset.

Note that pre-caliber ω_1 implies property K_n, and K_n implies K_m for $m \leq n$. So these form a hierarchy of properties. Furthermore, if $\mathbb{P}$ is σ-centered then it has pre-caliber ω_1, and if it is σ-n-linked then it has property K_n. So we get the following implications:

$$\begin{array}{ccccccc}
\sigma\text{-centered} & \longrightarrow & \forall n(\sigma\text{-}n\text{-linked}) & \dashrightarrow & \sigma\text{-}n\text{-linked} & \dashrightarrow & \sigma\text{-2-linked} & & \\
 & \searrow & & & \downarrow & & \downarrow & & \\
 & & \text{pre-caliber } \omega_1 & \dashrightarrow & \mathrm{K}_n & \dashrightarrow & \mathrm{K}_2 & \longrightarrow & \text{c.c.c.}
\end{array}.$$

Let P be either pre-caliber ω_1 or K_n for some n.

Corollary 4.9. *Let $\Gamma_{\mathsf{P}_{<\delta}}$ the class of all P-forcing notions of size $< \delta$, for some regular $\delta > 2^{\aleph_0^{\aleph_0}}$. Then* $\mathsf{MLF}(\Gamma_{\mathsf{P}_{<\delta}}) = \mathsf{S4.2}$.

Proof. We verify the conditions of Theorem 4.4. It is standard to verify that $\Gamma_{\mathsf{P}_{<\delta}}$ is reflexive, transitive, and persistent (hence directed), so condition (1) holds. Note that the coding of the reals $a_{\alpha,i}$ can be started as high as we want, so by limiting ourselves to forcing notions of a bounded size, we can do this coding somewhere high enough that will not be affected by these forcings, and therefore obtain condition (2) (note that by c.c.c. cardinals are not changed). Since $\delta > 2^{\aleph_0^{\aleph_0}}$ these classes contain all σ-centered forcings, so condition (3) holds.

Since these forcings do not preserve the continuum, we cannot obtain an n-switch for these classes using the same methods. However, note that for every I, the poset $\mathrm{Fn}(I,2)$ consisting of finite functions from I to 2, ordered by reverse inclusion, has pre-caliber ω_1 (cf. [11, p. 181]), and forces $2^{\aleph_0} > |I|$, which by c.c.c. is a button for

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P-forcing. Therefore the statements $\langle 2^{\aleph_0} \geq \aleph_\alpha \mid \alpha < \delta \rangle$ form a strong ratchet, and we can use it to construct an n-switch as in Lemma 1.13, to obtain condition (4). So all the conditions of Theorem 4.4 can be met, giving us the result. ⊣

Note that bounding the size of the forcings was essential, since otherwise the classes contain all posets of the form $\mathrm{Fn}(I,2)$, so extensions of W using such forcings may not be appropriate. We can however get some result on the full classes of P-forcings:

THEOREM 4.10. *Let Γ_{P} be the class of all P-forcings. Then* $\mathsf{S4.2} \subseteq \mathsf{MLF}(\Gamma_{\mathsf{P}}) \subseteq \mathsf{S4.3}$.

PROOF. As we noted, these classes are reflexive, transitive, and persistent, hence the left inclusion. Now, the statements $r(\alpha) = 2^{\aleph_0} \geq \aleph_\alpha$ for any $\alpha \in \mathrm{Ord}$ such that $\mathrm{cf}(\alpha) > \omega$ form a long ratchet over L in the sense of [5]: these are indeed statements obtained by a single formula using parameter α; each of them is a pure button for every class of c.c.c. forcings, since once $2^{\aleph_0} \geq \aleph_\alpha$ is true, it cannot be made false without collapsing cardinals; clearly $r(\alpha) \to r(\beta)$ for any $\beta < \alpha$; we can push $r(\alpha)$ without pushing any further $r(\beta)$ using the c.c.c. poset $\mathrm{Fn}(\aleph_\alpha \times \omega, 2)$ of all finite functions from $\aleph_\alpha \times \omega$ to $\{0,1\}$, which forces $2^{\aleph_0} = \aleph_\alpha$ (whenever $\mathrm{cf}(\alpha) > \omega$, see [11, Theorem IV. 3.13]. Note that the requirement there is $\aleph_\alpha^{\aleph_0} = \aleph_\alpha$, which is implied by $\mathrm{cf}(\alpha) > \omega$ so will not change by c.c.c. forcing); and clearly no forcing extension can satisfy every $r(\alpha)$. So by [5, Theorem 12] (Theorem 1.16) we get the inclusion on the right. ⊣

To get an exact result would require a different method, so the following is open:

QUESTION 4.11. *Let* P *be either pre-caliber ω_1 or K_n for some n. What is the modal logic of all* P*-forcing notions*?

Finally, the only property in the above diagram we did not discuss yet is c.c.c.

QUESTION 4.12. *What is the modal logic of all c.c.c. forcing notions*?

This natural question was already raised in [7]. The difficulty in answering it is that the class of all c.c.c. forcing notions is *not* directed, so it does not contain S4.2. It is reflexive and transitive, so Hamkins and Löwe conjectured that the answer is S4. To prove this, one would probably need to find a labeling for models based on trees, as the class of all trees is a class of simple frames characterizing S4. It should be mentioned that in [9], a labeling of models based frames which are "spiked pBAs" (denoted S4sBA, cf. [9] for exact definition) was provided, thus establishing an upper bound which is strictly between S4 and S4.2. However it is not known whether this modal theory is finitely axiomatizable, so it is not yet clear whether this can be shown to be a lower bound as well by the current methods. So, this question remains open.

To conclude, we list the current knowledge concerning all the classes in the diagram.

THEOREM 4.13. 1. *The modal logic of the following classes of forcing is exactly* S4.2:
(a) *σ-centered*;
(b) *σ-n-linked for all n*;
(c) *σ-n-linked for some n*;

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   (d) *Pre-caliber* ω_1 *of bounded size*;
   (e) K_n (*for some* n) *of bounded size.*
2. *The modal logic of the following classes contains* S4.2 *and is contained in* S4.3:
   (a) *Pre-caliber* ω_1;
   (b) K_n *for some* n.
3. *The modal logic of c.c.c. forcing contains* S4 *and is contained in* S4sBA.

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